Module 3 Solved Problems

Problem:
Show that $\sqrt{X}\sqrt{X} = I$

Solution:
First note that

$$(1+i)^2 + (1-i)^2 \quad = \quad 1 + i^2 + 2i + 1 + i^{-}2i \quad = \quad 0$$

and

$$(1+i)(1-i) + (1+i)(1-i) \quad = \quad 1^2 + 1^2 + 1^2 + 1^2 \quad = \quad 4$$

Next,

$$\sqrt{X}\,\sqrt{X} \quad = \quad \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} \quad = \quad \frac{1}{4}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \frac{1}{4}\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \quad = \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad = \quad X$$

Problem:
Consider the matrix

$$A \quad = \quad \begin{bmatrix} a^* & b^* \\ -b & a \end{bmatrix}$$

Show that A is unitary. Then, show how the numbers a, b can be selected so that $A|0\rangle = |\psi\rangle$ for any arbitary $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. That is, A can be used to generate any desired state from $|0\rangle$.

Solution:
First,

$$A^\dagger \quad = \quad \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}$$

(The diagonal entries conjugate. The others reflect about the diagonal and conjugate.) Thus,

$$A^\dagger A \quad = \quad \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix} \begin{bmatrix} a^* & b^* \\ -b & a \end{bmatrix} \quad = \quad \begin{bmatrix} aa^* + bb^* & ab^* - ab^* \\ a^*b - a^*b & bb^* + aa^* \end{bmatrix} \quad = \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

because $aa^* + bb^* = |a|^2 + |b|^2 = 1$. The squared magnitudes add to 1 because the columns are orthonormal (hence, unit-length).

Next,

$$A|0\rangle \quad = \quad \begin{bmatrix} a^* & b^* \\ -b & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad = \quad \begin{bmatrix} a^* \\ -b \end{bmatrix}$$

Thus, by equating $\alpha = a^*, \beta = -b$, we can generate any vector.