Module 2 Solved Problems

Complex numbers: a review

Useful facts and identities:

- It is well worth memorizing a few commonly seen θ's (with $r = 1$):

$$\begin{aligned}
e^{i\pi} &= \cos\pi + i\sin\pi &&= -1 \\
e^{i2\pi} &= \cos 2\pi + i\sin 2\pi &&= 1 \\
e^{i\frac{\pi}{2}} &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} &&= i \\
e^{i\frac{\pi}{4}} &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} &&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \\
e^{i\frac{\pi}{3}} &= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} &&= \frac{1}{2} + \frac{\sqrt{3}}{2}i \\
e^{i\frac{\pi}{6}} &= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} &&= \frac{\sqrt{3}}{2} + \frac{1}{2}i
\end{aligned}$$

- Useful trig identities for quantum computing:

$$\begin{aligned}
\sin(-\theta) &= -\sin\theta \\
\cos(-\theta) &= \cos\theta \\
\sin^2\theta + \cos^2\theta &= 1 \\
\sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \\
\cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2
\end{aligned}$$

- From which

$$\begin{aligned}
\cos(2\theta) &= 2\cos^2\theta - 1 \\
\sin(2\theta) &= 2\sin\theta\cos\theta
\end{aligned}$$

Problem:
Simplify i^{15}

Solution:

$$i^{15} = i \cdot i^{14} = i \cdot (i^2)^7 = i(-1)^7 = -i$$

Problem:
Simplify $\frac{i-1}{3-2i}$ into $a + ib$ form.

Solution:
Use the division trick of multiplying and dividing by the conjugate of the denominator:

$$\begin{aligned}
\frac{i-1}{3-2i} &= \frac{i-1}{3-2i}\,\frac{3+2i}{3+2i} \\
&= \frac{3i + 2i^2 - 3 - 2i}{3^2 + 2^2} \\
&= \frac{-5+i}{13} \\
&= -\frac{5}{13} + \frac{1}{13}i
\end{aligned}$$

Problem:
Calculate Re $((3+2i)(1-i))$

Solution:

$$(3+2i)(1-i) \quad = \quad 3-3i+2i-2i^2 \quad = \quad 5-i$$

Thus, $\mathrm{Re}\left((3+2i)(1-i)\right) = 5$

Problem:
Compute $(1+i)^8$

Solution:
Instead of laboriously multiplying, we'll write $1+i$ in polar form, where powers are straightforward:

$$\begin{aligned} 1+i &= \sqrt{2}(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) \\ &= \sqrt{2}e^{i\frac{\pi}{4}} \end{aligned}$$

Thus,

$$\begin{aligned} (1+i)^8 &= (\sqrt{2}e^{i\frac{\pi}{4}})^8 \\ &= (\sqrt{2})^8 e^{2\pi i} \\ &= 2^4 \\ &= 16 \end{aligned}$$

Problem:
Prove, for complex numbers z_1, z_2:

1. $(z_1 + z_2)^* = z_1^* + z_2^*$
2. $(z_1 z_2)^* = z_1^* z_2^*$
3. $|z_1 z_2| = |z_1||z_2|$

Solution:
Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$.

1. $$(z_1+z_2)^* \quad = \quad (a_1+ib_1+a_2+ib_2)^* \quad = \quad a_1+a_2-i(b_1+b_2) \quad = \quad (a_1-ib_1)+(a_2-ib_2) \quad = \quad z_1^*+z_2^*$$
2. $$(z_1z_2)^* \quad = \quad ((a_1+ib_1)(a_2+ib_2))^* \quad = \quad (a_1a_2-b_1b_2+(a_1b_1+a_2b_1)i)^* \quad = \quad a_1a_2-b_1b_2-(a_1b_1+a_2b_1)i) \quad = \quad a_1a_2-b_1b_2-(a_1b_1+a_2b_1)i) \quad = \quad ($$
3. $$|z_1z_2|^2 \quad = \quad (z_1z_2)(z_1z_2)^* \quad = \quad (z_1z_2)(z_1^*z_2^*) \quad = \quad (z_1z_z^*)(z_2z_2^*) \quad = \quad |z_1|^2|z_2|^2$$

   Since magnitudes are positive real numbers, $|z_1z_2| = |z_1||z_2|$.

Complex vectors with Dirac notation

Problem:
With $|u\rangle = (1, 2-i, 3i)$, $|v\rangle = (2, 1, i)$, $|w\rangle = (1, -2, 3)$, and $\alpha = (2-3i), \beta = -1$, calculate

1. the vectors $\alpha\,|v\rangle\,, |\alpha v\rangle\,, \langle\alpha u|\,,\ \alpha^*\,\langle u|$
2. the numbers $\langle u|v\rangle\,, \langle v|u\rangle\,, \langle v|u\rangle^*$, and compare the first and third.
3. the matrix $|u\rangle\langle v|$, the matrix-vector product $|u\rangle\langle v|\,|w\rangle$, and the scalar-vector product $\langle v|w\rangle\,|u\rangle$, and compare the latter two results.
4. $||u\rangle|^2, \langle u|u\rangle$

Solution:

1. Since $|v\rangle$ is already a column vector

   $$|\alpha v\rangle \quad = \quad \alpha\,|v\rangle \quad = \quad (2-3i)\begin{bmatrix} 2 \\ 1 \\ i \end{bmatrix} \quad = \quad \begin{bmatrix} -3i \\ -8i \\ 9+5i \end{bmatrix}$$

   Next,

   $$\alpha u \quad = \quad (2-3i)\begin{bmatrix} 2 \\ 1 \\ i \end{bmatrix} \quad = \quad \begin{bmatrix} 4-6i \\ 2-3i \\ 3+2i \end{bmatrix}$$

   and so

   $$\langle\alpha u| \quad = \quad \begin{bmatrix} 4+6i & 2+3i & 3-2i \end{bmatrix}$$

If, on the other hand, we conjugate and then scalar-multiply:

$$\alpha^* \langle u| \;\; = \;\; (2+3i)\begin{bmatrix} 2 & 1 & -i \end{bmatrix} \;\; = \;\; \begin{bmatrix} 4+6i & 2+3i & 3-2i \end{bmatrix}$$

as expected.

2. First,

$$\langle u|v\rangle \;\; = \;\; \begin{bmatrix} 1 & 2+i & -3i \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ i \end{bmatrix} \;\; = \;\; 2 + (2+i) - 3i^2 \;\; = \;\; 7+i$$

Next

$$\langle v|u\rangle \;\; = \;\; \begin{bmatrix} 2 & 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 2-i \\ 3i \end{bmatrix} \;\; = \;\; 7-i$$

and so

$$\langle v|u\rangle^* \;\; = \;\; 7+i \;\; = \;\; \langle u|v\rangle$$

3. First

$$|u\rangle\langle v| \;\; = \;\; \begin{bmatrix} 2 & 1 & -i \\ 4-2i & 2-i & -1-2i \\ 6i & 3i & 3 \end{bmatrix}$$

Multiplying this matrix into $|w\rangle$ gives

$$|u\rangle\langle v| \;\; |w\rangle \;\; = \;\; \begin{bmatrix} 2 & 1 & -i \\ 4-2i & 2-i & -1-2i \\ 6i & 3i & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \;\; = \;\; \begin{bmatrix} -3i \\ -3-6i \\ 9 \end{bmatrix}$$

Next,

$$\langle v|w\rangle \;\; = \;\; \begin{bmatrix} 2 & 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \;\; = \;\; -3i$$

Multiplying into $|u\rangle$ gives

$$\langle v|w\rangle \;\; |u\rangle \;\; = \;\; -3i \begin{bmatrix} 1 \\ 2-i \\ 3i \end{bmatrix} \;\; = \;\; \begin{bmatrix} -3i \\ -3-6i \\ 9 \end{bmatrix}$$

Thus, we see that

$$(|u\rangle\langle v|)\,|w\rangle \;\; = \;\; |u\rangle \;\; (\langle v|w\rangle) \;\; = \;\; (\langle v|w\rangle)\,|u\rangle$$

since $\langle v|w\rangle$ is a *number*.

4. Clearly

$$||u\rangle|^2 \;\; = \;\; |1|^2 + |2+i|^2 + |-3i|^2 \;\; = \;\; 15$$

and

$$\langle u|u\rangle \;\; = \;\; \begin{bmatrix} 1 & 2+i & -3i \end{bmatrix} \begin{bmatrix} 1 \\ 2-i \\ 3i \end{bmatrix} \;\; = \;\; 15$$

As expected, they should be the same.

Inner products, projectors, Hermitians and unitaries

Problem:

Thus far we've seen projectors defined with unit-length vectors, as $|v\rangle\, v$, and the completeness relation for an orthonormal basis as $\sum_i |v_i\rangle\, v_i = I$. Does the completeness relation work for *non*-unit length vectors? If not, what is the tweak needed to make it work?

Solution:
Consider, for example, $|v_1\rangle = (1,1), |v_2\rangle = (-1,1)$. Both are not unit-length since

$$\begin{aligned} \langle v_1|v_1\rangle &= 2 \\ \langle v_2|v_2\rangle &= 2 \end{aligned}$$

The sum

$$|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq I$$

But it appears that scaling by the length (in this case, 2) will result in I. We can formalize this as follows.

Let $|v_1\rangle, \ldots, |v_n\rangle$ be an non-unit length but *orthogonal* basis and *ktu* be a vector. that we express as:

$$|u\rangle = \sum_i \alpha_i |v_i\rangle$$

where, as usual, each α_i is obtained by multiplying each side by $\langle v_i|$:

$$\langle v_i|u\rangle = \alpha_i \langle v_i|v_i\rangle$$

(all other inner products are zero). And so,

$$\alpha_i = \frac{\langle v_i|u\rangle}{\langle v_i|v_i\rangle} = \frac{\langle v_i|u\rangle}{|v_i|^2}$$

Now return to the expression of $|u\rangle$:

$$\begin{aligned} |u\rangle &= \sum_i \alpha_i |v_i\rangle \\ &= \sum_i \frac{\langle v_i|u\rangle}{|v_i|^2} |v_i\rangle \\ &= \sum_i \frac{1}{|v_i|^2} |v_i\rangle\langle v_i| \ |u\rangle \\ &= \left(\sum_i \frac{1}{|v_i|^2} |v_i\rangle\langle v_i| \right) |u\rangle \end{aligned}$$

Thus,

$$\sum_i \frac{1}{|v_i|^2} |v_i\rangle\langle v_i| = I$$

Which is the desired completeness relation. The real goal of this exercise is to get more comfortable with Dirac notation and "moving" the scalar to reformulate in terms of the outerproduct.

Problem:
This problem is intended to increase familiarity with Dirac symbols. Suppose $|0\rangle, |1\rangle, \ldots, |N-1\rangle$ are N basis vectors and that $|u\rangle, |v\rangle$ are two vectors in the same space expressed as

$$\begin{aligned} |u\rangle &= \alpha_0 |0\rangle + \ldots + \alpha_{N-1} |N-1\rangle \\ |v\rangle &= \beta_0 |0\rangle + \ldots + \beta_{N-1} |N-1\rangle \end{aligned}$$

Show that

$$\langle u|v\rangle = \sum_{i=0}^{N-1} \alpha_i^* \beta_i$$

Solution:

$$\begin{aligned} &\Big\langle \alpha_0 |0\rangle + \ldots + \alpha_{N-1} |N-1\rangle \Big| \beta_0 |0\rangle + \ldots + \beta_{N-1} |N-1\rangle \Big\rangle \\ &= \beta_0 \Big\langle \alpha_0 |0\rangle + \ldots + \alpha_{N-1} |N-1\rangle \Big| 0 \Big\rangle + \ldots + \beta_{N-1} \Big\langle \alpha_0 |0\rangle + \ldots + \alpha_{N-1} |N-1\rangle \Big| N-1 \Big\rangle \\ &= \beta_0 \left(\alpha_0^* \langle 0| + \ldots + \alpha_{N-1}^* \langle N-1| \right) |0\rangle + \ldots + \beta_{N-1} \left(\alpha_0^* \langle 0| + \ldots + \alpha_{N-1}^* \langle N-1| \right) |N-1\rangle \\ &= \beta_0 \alpha_0^* \langle 0|0\rangle + \ldots + \beta_{N-1} \alpha_{N-1}^* \langle N-1|N-1\rangle \\ &= \alpha_0^* \beta_0 + \ldots + \alpha_{N-1}^* \beta_{N-1} \end{aligned}$$

Notice that, in the first line, for additional clarity, we have left the vector form in the left (conjugated) side of an inner-product:

$$\left\langle \boldsymbol{\alpha_0 \,|0\rangle + \ldots + \alpha_{N-1}\,|N-1\rangle} \,\middle|\, \beta_0\,|0\rangle + \ldots + \beta_{N-1}\,|N-1\rangle \right\rangle$$

Here, the intention is to allow anything to sit inside the inner-product, and to simplify it at the right time.

Problem:
Since inner products are computed often, let's do another example problem. Show that

$$\begin{aligned} |u\rangle &= \frac{i}{\sqrt{3}}|0\rangle - \frac{\sqrt{2}}{\sqrt{3}}|1\rangle \\ \left|u^\perp\right\rangle &= -\frac{\sqrt{2}}{\sqrt{3}}|0\rangle - \frac{i}{\sqrt{3}}|1\rangle \end{aligned}$$

are a 2D basis, and then compute the coefficients when expressing $|+\rangle$ in this basis.

Solution:
First, let's show they are orthogonal and of unit-length.

$$\begin{aligned} \left\langle u \middle| u^\perp \right\rangle &= \left\langle -\frac{i}{\sqrt{3}}\langle 0| - \frac{\sqrt{2}}{\sqrt{3}}\langle 1| \,\middle|\, -\frac{\sqrt{2}}{\sqrt{3}}|0\rangle + \frac{i}{\sqrt{3}}|1\rangle \right\rangle \\ &= \frac{i\sqrt{2}}{3}\langle 0|0\rangle - \frac{i\sqrt{2}}{3}\langle 1|1\rangle \\ &= 0 \end{aligned}$$

Unit-length:

$$\begin{aligned} \langle u|u\rangle &= \left\langle -\frac{i}{\sqrt{3}}\langle 0| - \frac{\sqrt{2}}{\sqrt{3}}\langle 1| \,\middle|\, \frac{i}{\sqrt{3}}|0\rangle - \frac{\sqrt{2}}{\sqrt{3}}|1\rangle \right\rangle \\ &= -\frac{i^2}{3} + \frac{2}{3} \\ &= 1 \end{aligned}$$

Next, the coefficients:

$$\langle u|+\rangle \;=\; \left\langle -\frac{i}{\sqrt{3}}\langle 0| - \frac{\sqrt{2}}{\sqrt{3}}\langle 1| \,\middle|\, \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right\rangle \;=\; -\frac{\sqrt{2}}{\sqrt{6}} - \frac{i}{\sqrt{6}}$$

and

$$\left\langle u^\perp \middle| + \right\rangle \;=\; \left\langle -\frac{\sqrt{2}}{\sqrt{3}}\langle 0| - \frac{i}{\sqrt{3}}\langle 1| \,\middle|\, \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right\rangle \;=\; -\frac{\sqrt{2}}{\sqrt{6}} - \frac{i}{\sqrt{6}}$$

Check:

$$|\langle u|+\rangle|^2 + \left|\left\langle u^\perp \middle| + \right\rangle\right|^2 \;=\; \frac{1+2}{6} + \frac{1+2}{6} \;=\; 1$$

(See Proposition 2.9). Thus,

$$|+\rangle \;=\; \langle u|+\rangle\,|u\rangle + \left\langle u^\perp \middle| + \right\rangle \left|u^\perp\right\rangle \;=\; \left(-\frac{\sqrt{2}}{\sqrt{6}} - \frac{i}{\sqrt{6}}\right)|u\rangle \;+\; \left(-\frac{\sqrt{2}}{\sqrt{6}} - \frac{i}{\sqrt{6}}\right)\left|u^\perp\right\rangle$$