EEM015
Electromagnetic Field Theory

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V0.1

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Contents

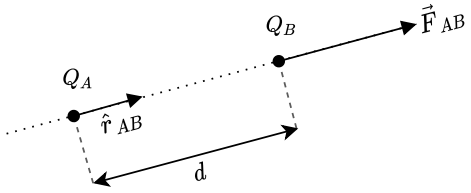
Part 1

Electrostatics

Contents

Electrostatics is the study of charged particles at rest and is fully described by Coulomb's Law and the associated electric field.

1.1 Coulomb's Law



Coulomb's Law states that a force $\vec{F}_{AB}$ acts upon point charge Q_B due to the proximity of point charge Q_A, and vice-versa. Simply:

$$\vec{F}_{AB} = \frac{1}{4\pi\epsilon_0}\frac{Q_A Q_B}{d^2}\hat{r}_{AB} \quad [\text{N}] \tag{1.1}$$

where $\epsilon_0 \approx 8.854 \cdot 10^{-12}$ $[\frac{C^2}{Nm^2}]$ or $[\frac{F}{m}]$ is the permittivity of vacuum.

Depending on the sign of the charges they will attract ($Q_A Q_B > 0$) or repel ($Q_A Q_B < 0$) each other or remain unaffected $Q_A Q_B = 0$.

More generally, having introduced a local coordinate system:

$$\vec{F}_{AB} = \frac{1}{4\pi\epsilon_0}\frac{Q_A Q_B}{|\vec{r}_B - \vec{r}_A|^2}\frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|} \tag{1.2}$$

This force is called the **Electrostatic Force**.

Notes on Coulomb's Law Coulomb's law is an empirical result, that is, it is the result of measurements.

It is an inverse-square law of the form

$$|F| = k_e\frac{|Q_A Q_B|}{r^2} \tag{1.3}$$

, where k_e is Coulomb's constant.

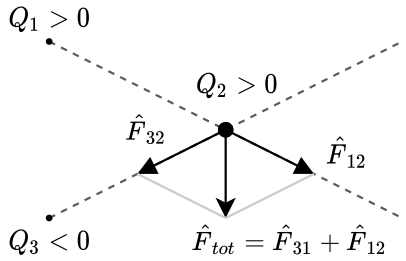
The net force $\vec{F}_{AB} + \vec{F}_{BA} = 0$, thus the coulomb force does not in any way move the charge constellation as a whole (eg. rotating or translating it).

1.2 The Superposition Principle

Coulomb's Law applies only to pairs of charges. In a system with a greater number of point charges, the force acted upon a charge is the superposition of forces from all other charges, as in the following example.

The total force $\vec{F}$ that acts on q located at $\vec{r}$ is:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n$$



resulting in the generalised Coulomb's law for multiple point charges (for a local coordinate system):

$$\vec{F} = \frac{q}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{Q_i}{|\vec{r} - \vec{r}_i|^2} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|} \tag{1.4}$$

Notes on the Superposition Principle There is no intrinsic proof that this principle holds, but has been observed through experimentation, just like Coulomb's Law itself.

1.3 The Electric Field

If we examine Coulomb's Law we find that, if Q_i and $\vec{r}_i$ are known and the **test charge** q (the point at which the force is evaluated) is ignored, the expression becomes a function of $\vec{r}$.

$$\vec{F} = q \underbrace{\frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{Q_i}{|\vec{r} - \vec{r}_i|^2} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|}}_{\vec{E}(\vec{r})} \tag{1.5}$$

This vector quantity, defined for all points $\vec{r}$ in space*, is the **electric field**. It is evaluated at the **field point** $\vec{r}$.

Definition: Electric Field

$$\vec{E}(\vec{r}) = \lim_{q \to 0} \frac{\vec{F}}{q} \quad \left[\frac{V}{m}\right] \tag{1.6}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{Q_i}{|\vec{r} - \vec{r}_i|^2} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|} \tag{1.7}$$

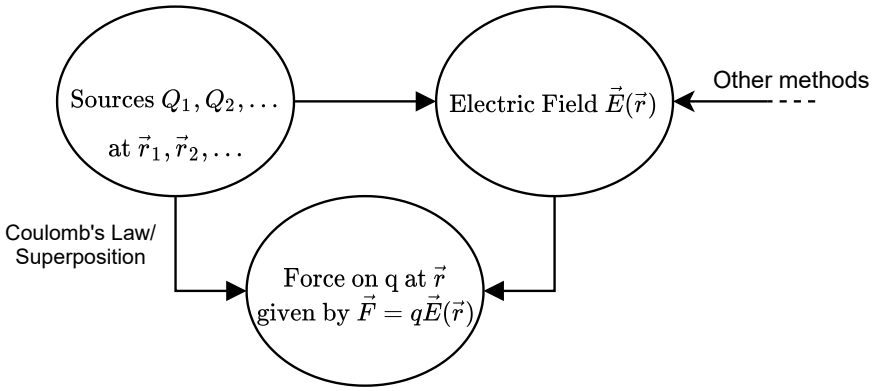
where $\vec{F}$ is the force that acts on q located at $\vec{r}$. $\vec{E}(\vec{r})$ is a field vector quantity: that is, it depends on a field and produces a field.

The limiting process implies that q tends to zero and therefore does not exert its own force on surrounding charges.

An implication of having derived the electric field from Coulomb's law is that it also follows the superposition principle. If $\vec{E}_1(\vec{r})$ is the electric field evaulated using a subset of point charges, and $\vec{E}_2(\vec{r})$ is the electric field evaluated using the remaining point charges, then $\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$

An argument for distinguishing the electric field is that it can be calculated in several ways, not exclusively with Coulomb's law. Having done that, it is easy to calculate the force exerted on a charge at a given field point using just the electric field. This is often easier than directly using Coulomb's law to solve for $\vec{F}$.

*$\vec{r}$ is defined in three components, assuming $\mathbb{R}^3$ space eg. x,y,z; ϕ, θ, r...



1.4 Flux through an open surface

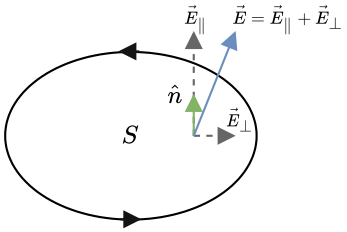
The (scalar) flux of a vector field E through an open surface S can be expressed:

$$\Phi_E = \int_S \vec{E} \cdot \underbrace{\hat{n} ds}_{d\vec{s}} \tag{1.8}$$

where $d\vec{s}$ is the vectorial surface integration element ie. a differential surface with its orientation perpendicular to the surface itself. The function results in a **scalar** quantity representing the "flow" of the field through the surface in its normal direction.

An open surface is oriented according to the **right-hand rule** such that, if the fingers of your right hand move in the direction of the surface *boundary*, your thumb will point in the normal direction.

The simplest example is a disk, illustrated below.



As $\vec{E}$ passes through the surface it may be expressed as having components parallel and tangential to the surface normal. It is the parallel component $\vec{E} \cdot \hat{n} = |\vec{E}||\hat{n}| \cos\alpha = |\vec{E}_\parallel|$ that contributes to the flux. The tangential component $\vec{E}_\perp$ by definition does not pass through the surface and thus does not contribute.

1.5 Continuum Charge Densities

Rather than deal with point charges when describing macroscopic objects, we can give a *charge density* as a function to describe the distribution of charges (protons and electrons).

A *volume charge density* is given by

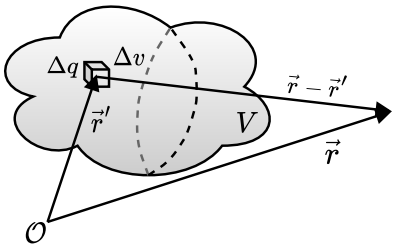
$$\rho_v(\vec{r}) = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} \qquad \left[\frac{C}{m^3}\right]$$

where Δq is the total charge enclosed by volume Δv.

Surface charge densities and *line charge densities* can also be defined similarly.

The total charge is given by integrating over the relevant integration domain. For instance:

$$Q = \int_V \rho_v(\vec{r})dv$$



Obs! Though the charge density is given as a limiting function $\lim_{\Delta v \to 0}$, if the scale becomes too small (e.g. smaller than an atom), then it will fail to be representative of the actual charge macroscopic distribution.

Source and field notation Here we introduce the prime ($'$) notation denoting source. $\vec{r'}$ is a vector to a source of the electric field in this case. The domain of $\vec{r'}$ is the volume charge density. Non-prime quantities are generally vectors to field points where the field, resulting from the source, is evaluated.

In the same way that the electric field was the superposition of the contributions of point charges in eq. 1.7, the electric field is the contributions of each infinitesimal charge Δq in a charge distribution.

A differential charge element can be thought of as the charge density summed over a small volume, and behaves as a point charge:

$$dq = \rho_v(\vec{r'})dv'$$

The reasoning is similar for other integration domains.

Integration Domain	$dq(\vec{r'})$
Volume	$dq = \rho_v(\vec{r'})dv'$
Surface	$dq = \rho_s(\vec{r'})ds'$
Line	$dq = \rho_l(\vec{r'})dl'$

By modifying eq. 1.7 we obtain an expression for the electric field resulting from a dq element.

$$d\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{dq(\vec{r'})}{|\vec{r} - \vec{r'}|^2}\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|}$$

The superposition of all these contributions gives the total electric field. That implies an integration over source domain $\vec{E}(\vec{r}) = \int d\vec{E}$. For a volume charge density, this is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_v(\vec{r'})}{|\vec{r} - \vec{r'}|^2}\frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|}dv' \tag{1.9}$$

1.6 Example - Superposition - Infinitely long straight line charge

Summary of video lecture

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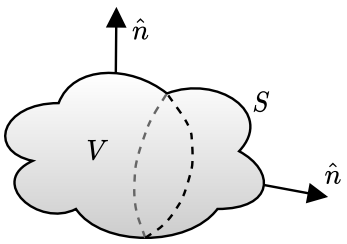
1. Set up appropriate cylindrical coordinate system.
2. Define $\vec{r}$, $\vec{r'}$, $\vec{r} - \vec{r'}$ and $|\vec{r} - \vec{r'}|$ in cylindrical coordinates.
3. Specify integration element $d\ell' = dz'$.

4. Express superposition integral $\vec{E}(\vec{r})$
5. Substitute $\zeta = z - z'$.
6. Recognise that $\hat{r}$-component is even function and $\hat{z}$-component is odd function, over even interval $\pm\infty$. Evaluating the component integrals separately shows that the former is non-zero and the latter is zero. The physical reason for this is due to the symmetry of the source in the $\hat{z}$ direction: any contribution in $\hat{z}$ from a source point at z_1 will be cancelled exactly by a source point at $-z_1$.
7. The anti-derivative of the $\hat{r}$-component is found in BETA.111. When evaluating at infinity, intuitively,
$$\lim_{\zeta\to\infty} \frac{\zeta}{\sqrt{r^2+\zeta^2}} = 1$$
$$\lim_{\zeta\to-\infty} \frac{\zeta}{\sqrt{r^2+\zeta^2}} = -1$$
8. This results in $\vec{E}(\vec{r}) = \hat{r}\frac{\rho_\ell}{2\pi\varepsilon_0 r}$. We note that the electric field depends inversely linearly on the $\hat{r}$-component only.

1.7 Gauss' Law

The surface integral of $\vec{E}$ over any closed surface S is equal to the total charge enclosed by the surface S, divided by ε_0.



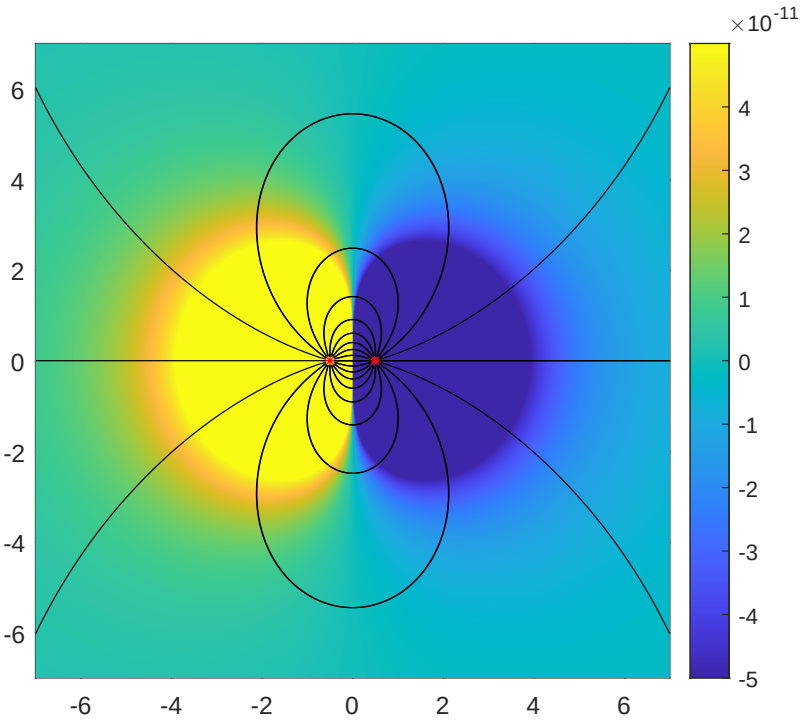
or, equivalently

The electric flux Φ_E through any closed surface S is equal to the total charge enclosed by the surface S, divided by ε_0.

$$(\Phi_E =) \oint_S \vec{E} \cdot d\vec{s} = \frac{Q_{\text{encl.}}}{\varepsilon_0} \tag{1.10}$$

Gauss' law is a *physical law* of electrostatic fields and must be fulfilled by all electrostatic fields.

Intuitively, for example, if we have two equal and oppositely charged particles enclosed within a surface then the field lines may look like the following plot (for a dipole).

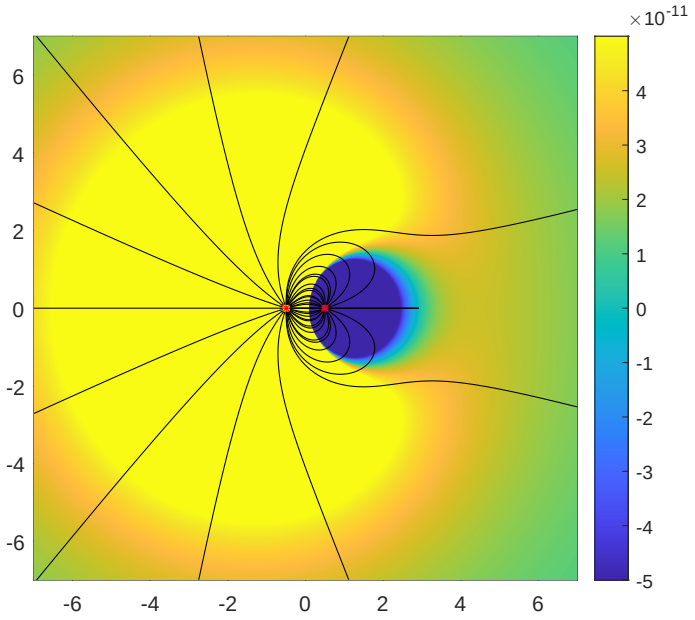


As we can see, for a surface enclosing this dipole, lines that start and end at a charge will either not cross the surface or cross it twice, cancelling out the flux. Lines that go off to infinity are in equal quantities coming in and going out, also cancelling.

Just as Gauss' law states, the total enclosed charge must be zero.

For an unbalanced dipole we see that most of the escaping field lines emanate from the stronger charge. The flux integral will thus be representative of the net enclosed charge.

Gauss' law is useful in problems where there is sufficient symmetry. That is, if a gauss-surface can be constructed such that the *normal component of the electric field $\vec{E} \cdot \hat{n}$ is constant over the closed surface.*



1.7.1 Using Gauss' law to evaluate a point charge

Using spherical coordinates, let S be a Gauss surface with radius R_0.

Using Gauss' law:

$$\oint_S \vec{E} \cdot d\vec{s} = \iint_S \frac{Q}{4\pi\varepsilon_0} \frac{1}{R_0^2} \hat{R} \cdot \hat{R} R_0^2 \sin\Theta d\Theta d\phi = \frac{Q}{\varepsilon_0}$$

1.7.2 Using Gauss' law to evaluate multiple point charges

Here we prove that Gauss' law applied to a Gauss surface S enclosing multiple point charges is valid.

This can then be extended to apply to any charge distribution.

If we consider that the total $\vec{E}$ field can be thought of as the superposition of contributions from individual charges, then the total field is

$$\vec{E} = \sum_i \vec{E}_i$$

, where $\vec{E}_i$ is the field contribution from charge Q_i.

$$\begin{aligned}
\oint_S \vec{E} \cdot d\vec{s} &= \oint_S (\sum_i \vec{E}_i) \cdot d\vec{s} \\
&= \sum_i \oint_S \vec{E}_i \cdot d\vec{s} \\
&= \text{Sum of Gauss' law for single charges} \\
&= \sum_i \frac{Q_i}{\varepsilon_0} = \frac{1}{\varepsilon_0} \sum_i Q_i \\
&= \frac{Q_{\text{encl.}}}{\varepsilon_0}
\end{aligned}$$

1.7.3 Using Gauss' law to evaluate an infinitely long line charge

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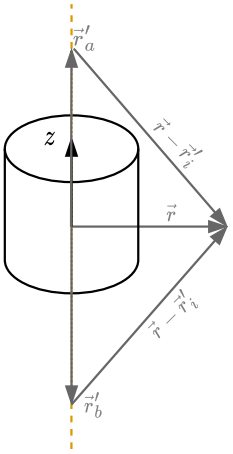
We first prove the inherent symmetry, using superposition. This eliminates dependency on $\hat{\phi}$ and $\hat{z}$, confirming that the field is irrotational. This is the first postulate. We use the second postulate of electrostatics (Gauss' law) to fully specify the $\hat{r}$-component.



1. Establish infinite line charge, with line charge density ρ_ℓ with field point in xy-plane.
2. Introduce pairwise points on line charge $\hat{z}z_a = -\hat{z}zb$ at equivalent distances from origo.
3. The local vector becomes $\vec{r} - \vec{r}' = \hat{r}(\phi)r - \hat{z}z_i$. Conclude that there is no ϕ-component, $E_\phi = 0$.
4. From superposition of the pairwise points $\vec{E} = d\vec{E}_a + d\vec{E}_b = ...$ conclude that $\hat{z}$-components cancel. $E_z = 0$.
5. Extend the above reasoning to all pairwise points along line charge.
6. Reason that a change in z has no effect since source is unchanged, while a change in ϕ will not affect the magnitude of $\vec{E}$, only the direction of $\hat{r}(\phi)$.
7. The resulting field expression $\vec{E} = \hat{r}(\phi)E_r(r)$ is conservative, which can be confirmed by taking the curl. Therefore one of the postulates of electrostatic fields is satisfied.
8. The other postulate (Gauss' law) states that the flux is proportional to the contained charge. $\oint_S \vec{E} \cdot d\vec{s} = \oint_S \vec{E}_r \hat{r}(\phi) \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$
9. Choose arbitrary cylindrical gauss surface with radius r and height L and integrate. $\oint_S \vec{E} \cdot d\vec{s} = E_r(r)2\pi rL = \frac{Q_{\text{encl}}}{\varepsilon_0}$
10. Since $Q_{\text{encl}} = \rho_\ell L$, we rearrange to get $\vec{E}(r) = \hat{r}E_r(r) = \frac{\rho_\ell}{\varepsilon_0 2\pi r}\hat{r}(\phi)$

1.7.4 Electric field from three important charge densities

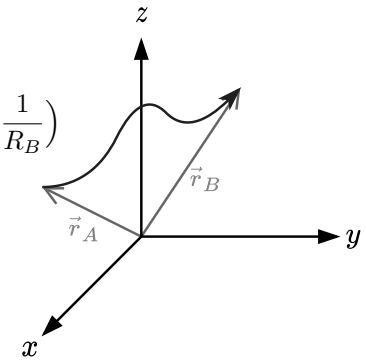
To summarize three important charge distribution constructs: the point charge, infinite line charge density and infinite plane charge density. The electric of each can be derived using Gauss' Law.

Point charge	$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\hat{R}(\Theta,\phi)$
Infinite Line Charge	$\vec{E}(r) = \frac{\rho_\ell}{\varepsilon_0 2\pi r}\hat{r}(\phi)$
Infinite Plane Charge	$\vec{E} = \frac{\rho_s}{2\varepsilon_0}(\pm\hat{z})$

1.8 Conservative Electric Field

The electric field is conservative (irrotational) as a consequence of Coulomb's Law. For a point charge placed at the origin of a spherical coordinate system, the line integral of the electric field is

$$\int_L \vec{E}\cdot d\vec{\ell} = \int_L \frac{Q}{4\pi\varepsilon_0}\frac{\hat{R}}{R^2}\cdot(\hat{R}dR + \underbrace{\hat{\Theta}Rd\Theta + \hat{\phi}R\sin\Theta d\phi}_{=0}) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_A} - \frac{1}{R_B}\right)$$



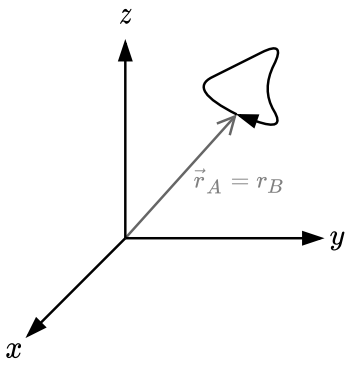
Clearly, only the $\hat{R}$ component will contribute to the line integral; any movement in orthogonal directions will not.

On a closed line A=B, and so the closed line integral evaluates to zero.

All possible closed line integrals of the electric field give

$$\boxed{\oint_L \vec{E} \cdot d\ell = 0} \tag{1.11}$$

making the electric field conservative, by definition (the value of the line integral is path-independent, ie. the closed line could take any shape). This is the **integral form of the conservative electric field requirement**[*].



The above reasoning was for a single point charge, but must also hold for a superposition of point charges, and therefore for any charge distribution.

1.9 Electrostatics - Postulates for Vacuum

Previously we introduced Gauss' law and the requirement of a conservative electric field. These are, in fact, the fundamental properties (postulates) of the electrostatic field in vacuum; everything else can be built atop these principles.

We have arrived at them with only a single assumption: that the empirical Coulomb's law is correct. We therefore also consider these postulates to be representations of laws of nature, confirmed to be correct by empirical observation.

Here we show how the two postulates give the curl and divergence of the electric field, which, according to the Helmholtz theorem, fully specifies the electric field.

1.9.1 Gauss' Law

Recall that Guass' law must be satisfied by any electrostatic field, for any closed surface S, and that that charge enclosed by a closed surface is equal to the volume integral of the volume charge density.

$$\boxed{\oint_S \vec{E} \cdot d\vec{s} = \frac{Q_{\text{encl.}}}{\varepsilon_0}} = \int_V \frac{\rho_v}{\varepsilon_0} dv$$

[*]We later show that this integral represents the voltage along the path. Thus this equation also expresses ***Kirchhoff's Voltage Law***: the sum of the voltage around any closed circuit is zero.

The divergence theorem (Gauss' theorem) (eq6.7.5) relates the surface integral of a vector field to the volume integral of the divergence

$$\boxed{\oint_S \vec{E} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{E}) dv} = \int_V \frac{\rho_v}{\varepsilon_0} dv$$

We now have a relationship between two volume integrals (and importantly, for the exact same volume) that we can combine in the same volume integral:

$$\int_V \left[(\nabla \cdot \vec{E}) - \frac{\rho_v}{\varepsilon_0} \right] dv = 0$$

This implies that the integrand is zero, and thus we have the **differential form of Gauss' law**:

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0} \tag{1.12}$$

Since the integral form of Gauss' law was valid for all closed surfaces, the differential form is valid for all volumes in space, and thus valid for every point in space.

To summarize both forms of Gauss' Law:

Integral	Differential
$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q_{\text{encl.}}}{\varepsilon_0}$	$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0}$

1.9.2 Conservative Electric Field

It was previously shown that the electric field is conservative:

$$\oint_L \vec{E} \cdot d\ell = 0$$

If we apply Stoke's theorem (eq:6.7.4) we can relate the closed line integral to an open surface integral of the curl of $\vec{E}$.

$$\oint_L \vec{E} \cdot d\vec{\ell} = \int_S (\nabla \times \vec{E}) \cdot d\vec{s} = 0$$

This implies that, for all possible open surfaces in space, the curl of the electric field must be zero at all points in space.

$$\nabla \times \vec{E} = \vec{0} \tag{1.13}$$

We now have the **conservative electric field requirement in differential form**.

Having derived that the electrostatic field is irrotational (curl-free) and nonsolenoidal (non-zero divergence), we have, in fact, fully specified the electrostatic field by Helmholtz' theorem. All other relationships in electrostatics can be derived from the two basic postulates. Further, it gives the insight that the electrostatic field is generated by a scalar source alone: that is, a charge or charge density. There is no vector source as in magnetics.

1.10 Derivation of Coulomb's Law using fundamental postulates of electrostatics

In ch. 1.7.3 we derived $\vec{E}$ for a line charge using the two postulates of electrostatics (among other arguments).

Here we demonstrate how even Coulomb's law itself may be easily derived.

Consider a single charge Q at origo in vacuum, with a spherical Gauss surface S of radius R enclosing Q. We reason that, since a point charge has no "preferred" direction, the electrical field must emanate radially (there is no other way to orient the field that it generates) and have the same intensity at all points on S.

Starting with Gauss' law and recognizing that we only have a radial component:

$$\oint_S \vec{E} \cdot d\vec{s} = \oint_S E_R \hat{R} \cdot \hat{R} ds \implies E_R 4\pi R^2 = \frac{Q\text{encl}}{\varepsilon_0}$$

We rearrange to get:

$$E_R = \frac{Q}{4\pi\varepsilon_0 R^2}$$
$$\vec{E} = \hat{R}E_R = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{R}$$

Which is exactly Coulomb's law. As a side note, $\vec{E}$ is *exactly* $\propto \frac{1}{R^2}$ at all points in space as a consequence of the fundamental postulate in eq. 1.12 (and the definition of divergence in spherical coordinates).

If we have local coordinates (ie. Q not centered at origo) then we simply replace

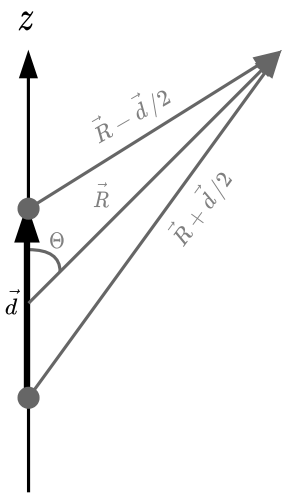
$$R \implies |\vec{r} - \vec{r'}|$$
$$\hat{R} \implies \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|}$$

to get the general equation in eq. 1.4.

$$\vec{F} = \frac{q}{4\pi\epsilon_0} \frac{Q}{|\vec{r} - \vec{r'}|^2} \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|} \tag{1.14}$$

1.11 Electric Dipole

The electric dipole consists of two charges $\pm q$ spaced a short distance d apart. The dipole is characterised by the assumption that $R >> d$. The two charges are oriented in the $\hat{z}$ direction with their midpoint at origo.



$\vec{E}$ is, as always, the superposition of the contributions from both point charges. In this case, it becomes:

$$\vec{E}_d = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{R} - \frac{\vec{d}}{2}}{\left|\vec{R} - \frac{\vec{d}}{2}\right|^3} - \frac{\vec{R} + \frac{\vec{d}}{2}}{\left|\vec{R} + \frac{\vec{d}}{2}\right|^3}\right]$$

In the above equation we see that, if d is small, then the exponential terms of d must be negligible. Our goal is to eliminate those terms to give an approximate expression that is more tractable.

1. We recognize that absolute value can be expressed as a dot product (ie. $\vec{A}\cdot\vec{A} = |A|^2$)

$$|\vec{R}\pm\vec{d}/2|^{-3} = [(\vec{R}\pm\vec{d}/2)\cdot(\vec{R}\pm\vec{d}/2)]^{-3/2} = \left(R^2 + \frac{d^2}{4} \pm \frac{\vec{R}\cdot\vec{d}}{R^2}\right)^{-3/2}$$

2. We remove the d^2 term and break out R

$$\implies R^{-3}\left(1 \pm \frac{\vec{R}\cdot\vec{d}}{R^2}\right)^{-3/2}$$

3. The term in parenthesis is now in a form that can be expanded using the binomial series* (BETA table 8.6.2).

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + ...$$

Clearly, any expansion beyond the second term will yield negligible d^2 terms.

$$\implies R^{-3}\left(1 \mp \frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}\right)$$

4. Substituting this back into the original $\vec{E}$ expression gives:

$$\vec{E}_d \approx \frac{Q}{4\pi\varepsilon_0 R^3}\left(3\frac{\vec{R}\cdot\vec{d}}{R^2}\vec{R} - \vec{d}\right)$$

OBS! Although it appears we are multiplying three vectors in the expression above, it is not a "triple" dot product! "Triple" scalar products do not exist! Think about it!

5. We define the *dipole moment* [Cm] as $\vec{p} = Q\vec{d}$. The expression thus becomes:

$$\vec{E}_d \approx \frac{1}{4\pi\varepsilon_0 R^3}\left(3\frac{\vec{R}\cdot\vec{p}}{R^2}\vec{R} - \vec{p}\right)$$

6. Conversion to spherical coordinates. We already have a 3D radial $\hat{R}$-component, but the dipole moment $\vec{p}$ is oriented in the $\hat{z}$ direction.

$$\vec{p} = \hat{z}p = (\hat{R}\cos\Theta - \hat{\Theta}\sin\Theta)p$$

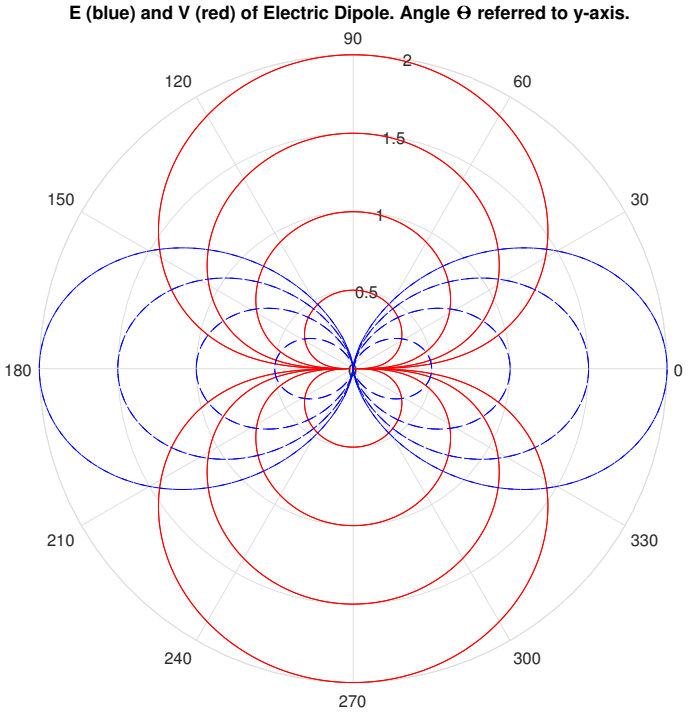
$$\vec{R}\cdot\vec{p} = R\hat{R}\cdot(\hat{R}\cos\Theta - \hat{\Theta}\sin\Theta)p = Rp\cos\Theta$$

Substituting the above expressions gives the final expression in spherical coordinates:

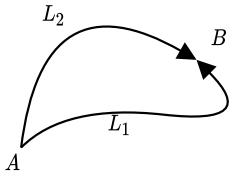
$$\begin{aligned}\vec{E}_d &\approx \frac{1}{4\pi\varepsilon_0 R^3}\left(3\frac{Rp\cos\Theta}{R^2}\vec{R} - (\hat{R}\cos\Theta - \hat{\Theta}\sin\Theta)p\right)\\ &\approx \frac{p}{4\pi\varepsilon_0 R^3}(\hat{R}2\cos\Theta - \hat{\Theta}\sin\Theta) \qquad (1.15)\end{aligned}$$

1.12 Electric Potential

*The binomial *series* is distinct from the binomial *expansion*. The latter is only defined for nonnegative integers and is a finite sum. The former is a generalized Taylor series, defined for arbitrary complex values and is an infinite series.



As previously established, the electrostatic field is conservative and thus $\oint_L \vec{E} \cdot d\vec{\ell} = 0$ and $\nabla \times \vec{E} = \vec{0}$.



The null-identity in eq. 6.24 implies that a curl-free vector field can always be expressed as the gradient of a scalar field, implying the existence of the ***electric potential***:

$$\boxed{\vec{E} = -\nabla V} \tag{1.16}$$

We now derive the electric potential more rigorously and interpret the physical meaning behind it. If we divide a closed line integral into two line integrals from A to B, and reverse the orientation of one of them, we obtain:

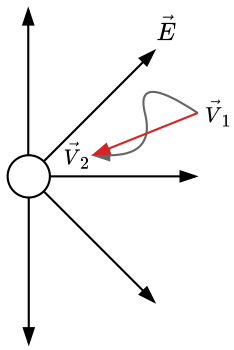
$$\oint_L \vec{E} \cdot d\vec{\ell} = \int_{L_1} \vec{E} \cdot d\vec{\ell} - \int_{L_2} \vec{E} \cdot d\vec{\ell} = 0$$

We thus have

$$\int_{L_1} \vec{E} \cdot d\vec{\ell} = \int_{L_2} \vec{E} \cdot d\vec{\ell}$$

and can reason that the line integral from A to B is path-independent; only the bounds matter.

Now recall Coulomb's law: If we wish to move a positive charge in a field generated by a positive charge, but *against* the field, we must perform positive *work** and consequently increase the potential energy of the system.



$$W = \int_A^B -\vec{F} \cdot d\vec{\ell} = \int_A^B -Q\vec{E} \cdot d\vec{\ell} \qquad [J] \tag{1.17}$$

$$W = W_B - W_A$$

This line integral from A to B gives us the difference in *potential energy* between points A and B. It is akin to doing work against gravity in lifting an object.

*Work can be expressed as the line integral of force: $W = \int_L \vec{F} \cdot d\ell$

If we move the charge in a closed loop, then the net work performed is zero (work is done *against* the electric field for a portion, then an equal amount of work is done *by* the field in the remaining portion). If we move the charge from point A to B, work W is performed and potential energy is added (in pos or neg quantities) to the system.

We know that the source of the work is the charge Q, and so we define the ***scalar electric potential*** as work per unit charge for convenience, using the work equation in (eq:1.17).

$$V = \frac{W}{Q} = -\int_A^B \vec{E} \cdot d\vec{\ell}$$

The scalar electric potential is the work performed in moving a unit charge between two points in space.

Through the $\vec{E} = -\nabla V$ relationship, we obtain the ***potential difference (voltage)***

$$\begin{aligned} V_{AB} &= -\int_A^B \vec{E} \cdot d\vec{\ell} = \int_A^B (\nabla V) \cdot d\vec{\ell} \\ &= \int_A^B dV = V_B - V_A \end{aligned}$$

where the space rate of increase of V along $\hat{\ell}$ is $(\nabla V) \cdot d\vec{\ell} = dV$ (from the definition of the gradient. See Cheng eq.2-88).

If we define one of the points (A in this case) to be at zero potential, then we have the absolute potential:

$$V(\vec{r}) = -\int_{L_{\text{ground}\to\vec{r}}} \vec{E} \cdot d\vec{\ell} \tag{1.18}$$

The zero-potential (ground) point is most often taken at infinity.

As a final note, ∇V is normal to planes of constant V (from the definition of the gradient): **therefore $\vec{E}$ is everywhere perpendicular to *equipotential* lines and surfaces.**

1.12.1 Electric potential due to point charges

The electric potential (potential difference) of a point at distance R from a point charge, referred to the potential at infinity (ie. ground):

$$\begin{aligned} V &= -\int_\infty^R \vec{E} \cdot d\vec{\ell} = -\int_\infty^R \frac{q}{4\pi\varepsilon_0 R^2} \hat{R} \cdot \hat{R} dR \\ &= -q\Big[\frac{1}{-R}\Big]_\infty^R = \frac{q}{4\pi\varepsilon_0 R} \qquad [V] \end{aligned}$$

Similarly, the potential difference between two points at finite distance r_1 and r_2, $r_1 < r_2$ is expressed:

$$V_{21} = V_2 - V_1 = \frac{q}{4\pi\varepsilon_0}\Big(\frac{1}{R_2} - \frac{1}{R_1}\Big)$$

We conclude that, indeed, the potential difference is only a function of distance R.

For a system of discrete charges, the potential at $\vec{R}$ (referenced to ∞) is a superposition of the potentials due to individual charges.

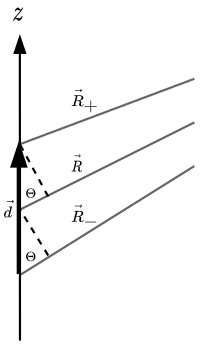
$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k}{|\vec{R} - \vec{R}'_k|}} \tag{1.19}$$

1.12.2 Electric potential due to a dipole

A dipole is simply a system of two opposing charges, therefore, using the equation for the potential due to discrete charges:

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{R} - \vec{d}/2|} - \frac{1}{|\vec{R} + \vec{d}/2|}\right)$$

We assume $d << R$, and therefore the angles between $\vec{d}$ and $\vec{R}$, $\vec{R}_+$ and $\vec{R}_-$ will be approximately the same. That is, we can approximate the three lines to the field point as being parallel.



The distances can then be expressed as

$$\frac{1}{R_+} \approx (R - d/2\cos\Theta)^{-1} = R^{-1}(1 - \frac{d}{2R}\cos\Theta)^{-1}$$
$$\frac{1}{R_-} \approx (R + d/2\cos\Theta)^{-1} = R^{-1}(1 + \frac{d}{2R}\cos\Theta)^{-1}$$

The term in parentheses can then be approximated using the binomial series, where terms $> d^2$ can be considered negligible. What remains is:

$$\frac{1}{R_+} \approx R^{-1}(1 + \frac{d}{2R}\cos\Theta)$$
$$\frac{1}{R_-} \approx R^{-1}(1 - \frac{d}{2R}\cos\Theta)$$

Thus

$$V \approx \frac{qd\cos\Theta}{4\pi\varepsilon_0 R^2}$$

or, equivalently, in terms of dipole moment $\vec{p}$:

$$\boxed{V \approx \frac{\vec{p}\cdot\hat{R}}{4\pi\varepsilon_0 R^2}} \tag{1.20}$$

The approximations thus far are the same as those made in the derivation of the electric field due to a dipole, though the process is significantly simpler.

We can now easily derive the electric field from the potential

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial R}\hat{R} + \frac{\partial V}{R\,\partial\theta}\hat{\theta}\right)$$

$$\boxed{\vec{E} \approx \frac{p}{4\pi\varepsilon_0 R^3}\left(2\cos\theta\hat{R} + \sin\theta\hat{\theta}\right)} \tag{1.21}$$

1.12.3 Electric potential due to charge distribution

We simply integrate the contributions from each differential charge element $dq = \rho_v(\vec{r'})dv'$.

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho_v(\vec{r'})}{|\vec{r} - \vec{r'}|}dv' \tag{1.22}$$

The same reasoning is applied to surface and line charge densities.

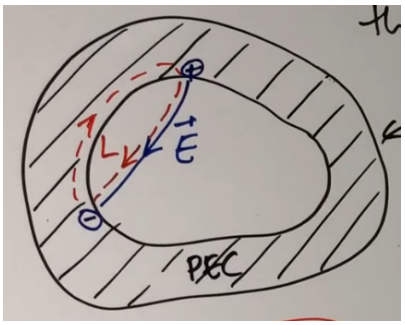
1.13 Metal Enclosures (Faraday shield)

Assume we have PEC volume with an enclosed cavity ie. a metal shell. We know that $\vec{E} = 0$ inside perfect electrical conductors (PECs).

1.13.1 Empty Cavity

This has the property that $\vec{E} = \vec{0}$ in the cavity when $\rho_v = 0$ in the cavity. Unaffected by external static charge densities.

Imagine that we have some unevenly distributed positive and negative charge on the inner surface. This would give rise to an electric field with field lines going from positive to negative charges.



If we consider the law of conservative electric fields in this scenario ($\oint_L \vec{E}\cdot d\vec{\ell} = 0$) for a path between two opposing charges, crossing the cavity and then passing completely within the metal shell:

$$\oint_L \vec{E}\cdot d\vec{\ell} = \int_{L_c} \vec{E}\cdot d\vec{\ell} + \underbrace{\int_{L_m} \vec{E}\cdot d\vec{\ell}}_{=0 \text{ since } \vec{E}=\vec{0}} = 0$$

So the path integral of the electric field inside the cavity is zero,
but we can show that as we integrate along the curve within the cavity, $\vec{E}\cdot d\vec{\ell} \geq 0$ ie. non-negative, as we are going from positive to negative charge. Thus, $\vec{E} = \vec{0}$ at every point along the curve, and hence every point within the cavity.

Thus, a static distribution of charges *outside* the metal shell will never produce any fields in the cavity.

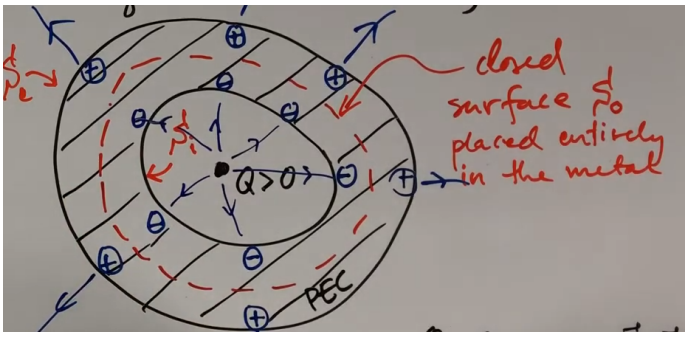
1.13.2 Charge within Cavity

Let there be a point charge $Q > 0$ in the cavity.

Surface charge densities are induced on the inner and outer surfaces of the enclosure, equal in total magnitude to enclosed Q. -Q on inner surface, +Q on outer surface.

If we use Gauss' Law and place a gauss surface S_0 within the metal enclosure itself then, since $\vec{E} = 0$ inside the metal, the flux integral is necessarily also zero:

$$\oint_{S_0} \vec{E}\cdot d\vec{s} = \frac{Q_{\text{encl}}}{\varepsilon_0} = 0$$



The enclosed charge must therefore be composed of the point charge Q as well as another charge distribution: an induced negative surface charge density on the inner wall of the enclosure, equal in magnitude to Q (since there is no E-field in the metal, there is no net charge in the volume of the metal ie. $\rho_v = \varepsilon_0 \nabla \cdot \vec{E} = 0$).

$$Q_{\text{encl}} = Q + \int_{S_i} \rho_s ds = 0$$
$$\hookrightarrow \int_{S_i} \rho_s ds = -Q$$

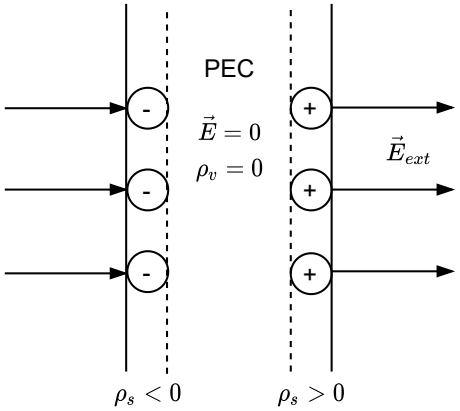
A negative surface charge density on the inner surface creates a charge imbalance in the metal, and so an equal positive charge density is induced on the outer surface of the enclosure. This is because the enclosure must contain zero *total* charge.

$$\int_{S_o} \rho_s ds = Q$$

The outer surface charge density distribution is not really dictated by the location of the charge within the cavity, only by external factors (eg. field lines must be perpendicular to PEC, external charges etc)

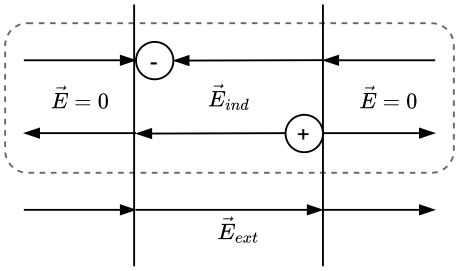
1.14 Perfect Electrical Conductor (PEC)

Characterised by unlimited number of freely flowing charge carriers. Copper is a good approximation in electrostatics. Consider a PEC plate in an external electric field.

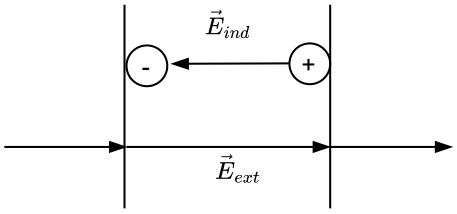


① $\vec{E} = \vec{0}$ **everywhere inside a PEC in an electrostatic configuration.** If $\vec{E} \neq \vec{0}$ then charges would experience a force and would be free to move and the situation would no longer be static. Hence, one can consider the static case an equilibrium after all charges have moved such that they fully cancel the overall E-field inside the PEC.

In terms of superposition of forces, consider a PEC with an external field and a positive and negative charge on opposite surfaces. There is a field induced by the charges themselves, which clearly cancel outside the PEC, but which sum within it. The internal induced field, however, is perfectly cancelled by the external field



This is motivated by considering the following: Initially, only the external field exists. In this case it carries positive charges to the right, and negative charges to the left. These charges accumulate on the surfaces, inducing their own electric field opposite to that of the external field, weakening it. The accumulation of charges on opposite surfaces continues until the induced electric field precisely cancels the external electric field, rendering the charges stationary.



② $\rho_v = 0$ inside a PEC since Gauss' Law (diff. form) states

$$\rho_v = \varepsilon_0 \nabla \cdot \vec{E} = \{\vec{E} = \vec{0}\} = \varepsilon_0 \nabla \cdot \vec{0} = 0$$

that is, if there is no electric field in the volume, then there is no net volume charge density. In other words, as many electrons as protons within the PEC volume.

③ $\rho_s \neq 0$. The previous property implies that the net charge must be entirely on the surfaces.

④ The PEC is an *equipotential* body since

$$\vec{E} = -\nabla V = \vec{0}$$
$$\hookrightarrow V = \text{const}$$

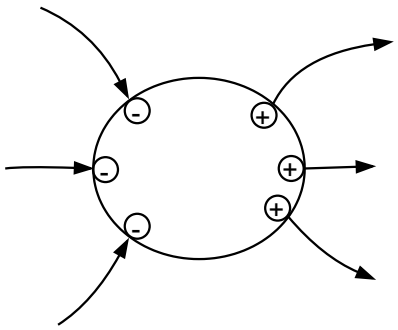
⑤ The electric field is perpendicular to the surface of the PEC.

$$\hat{n} \times \vec{E} = \vec{0}$$

ie. $\vec{E}$ is parallel to the surface normal $\hat{n}$.

(reminder: the cross product $\vec{A} \times \vec{B} = |AB \sin \Theta_{AB}| \hat{n}$ is indirectly a measure of how perpendicular two vectors are.)

Intuitively, this is because a non-perpendicular field would imply a field component parallel to the surface, which would exert a force on surface charges along the surface and hence the configuration would not be static or in equilibrium.



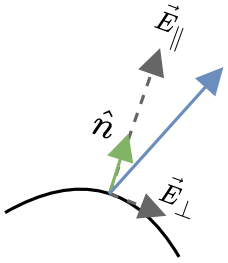
An alternative approach is: we have established that the PEC is an equipotential body (with an equipotential surface) since

$$\vec{E} = -\nabla V = \vec{0}$$

and, by definition, the gradient of a field is perpendicular to equipotential surfaces (see ch. 6.5.1).

$$\begin{aligned} \hat{n} \times \vec{E} &= \hat{n} \times (\vec{E}_\perp + \vec{E}_{||}) \\ &= \hat{n} \times \vec{E}_\perp + \underbrace{n \times \vec{E}_{||}}_{=0 \text{ since par.}} = 0 \end{aligned}$$

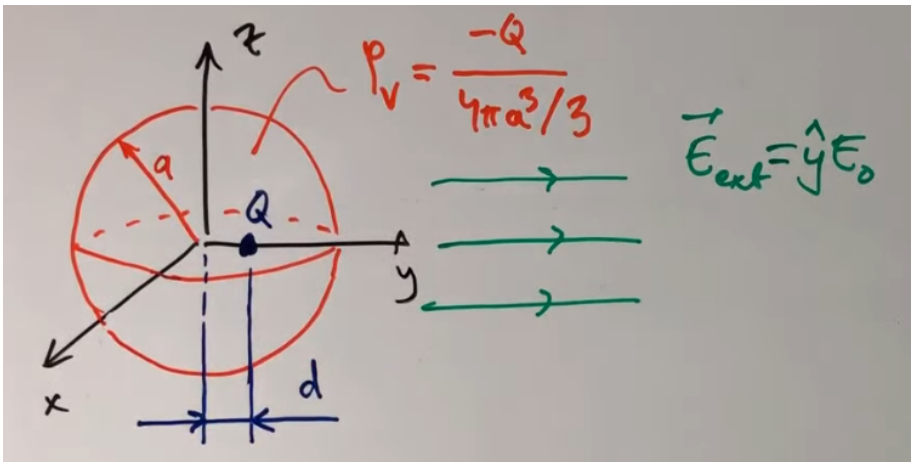
Therefore, $\hat{n} \times \vec{E}_\perp = 0$, so $\vec{E}_\perp$ must be zero.



1.15 Polarization of an Atom

In the presence of an external electric field, atomic nuclei are slightly displaced from their electron clouds, creating dipoles.

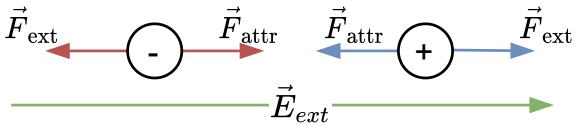
Consider an atom with proton of charge Q (considered as a point charge here) and a spherical electron cloud of radius a, with uniform charge density $\rho_v = \frac{-Q}{4\pi a^3/3}$.



The electric field due to *just* the electron cloud ($R < a$) is found readily using Gauss' Law:

$$E_R(R)\hat{R} = -\frac{QR}{4\pi\varepsilon_0 a^3}\hat{R}$$

An external electric field $\vec{E} = \hat{y}E_0$ is then applied, displacing the electron cloud from its nucleus by distance d.



Being of opposite sign, the electron cloud and the nucleus will attract each other with force:

$$F = \frac{Q^2 d}{4\pi\varepsilon_0 a^3}$$

The external field will exert a force on each, pulling them apart:

$$F = QE_0$$

Displacement will occur until an equilibrium is reached. This is expressed:

$$\frac{Q^2 d}{4\pi\varepsilon_0 a^3} = QE_0 \implies E_0 4\pi\varepsilon_0 a^3 = \underbrace{Qd}_{=p}$$

The atom has now formed a dipole, with associated dipole moment p. The dipole moment is proportional to the external field E_0, but also to the constant properties of the particular atom, such as the charge and size of electron cloud.

This can be abstracted as:

$$p = \alpha E_0$$

where α is the proportionality constant between the dipole moment of the atom and the external electric field: the **polarizability**. $\alpha = 4\pi\varepsilon_0 a^3 = 3\varepsilon_0 V_{\text{atom}}$, where V_{atom} is the volume of the atom/electron cloud.

We will explore this proportionality later, in the section on the permittivity of dielectric media.

1.16 Polarization of macroscopic objects (insulators)

We can extend the previous section on atoms to entire macroscopic objects. Objects that can be polarized but in which electrons do not entirely leave the nuclei are *insulators*.*

1.16.1 The Polarization Vector & Bound Charge Densities

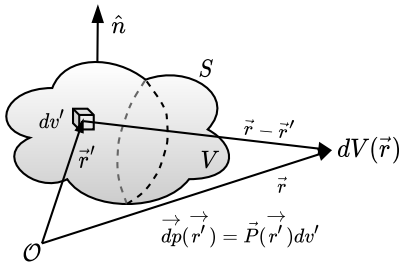
Just as discrete charges make up a volume charge density, individual molecular dipoles make up a *polarization vector*:

$$\vec{P}(\vec{r'}) = \lim_{\Delta v \to 0} \frac{\sum_{i=1}^{N} \vec{p}_i}{\Delta v} \qquad \left[\frac{C}{m^2}\right] \tag{1.23}$$

that is, dipole moment per unit volume.

Of course, despite the limiting function, Δv must be large enough to contain "enough" dipoles to be representative.

We know the potential generated by a single dipole $V(\vec{r'}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{R}}{R^2}$ (eq. 1.20), and the total electric potential of the object is given by the superposition of contributions. An infinitesimal contribution dV is thus:



$$dV(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{dp}(\vec{r'})}{|\vec{r}-\vec{r'}|^2}\cdot\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|}$$

(OBS! Note the scalar product above.)

This is the contribution of a small volume dv' with a total dipole moment $\vec{dp}(\vec{r'}) = \vec{P}(\vec{r'})dv'$.

The total electric potential of the object is the volume integration of such infinitesimal contributions:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\vec{P}(\vec{r'})\cdot(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}dv' \tag{1.24}$$

We now rewrite the above expression in such a way that provides an interpretation of *bound charges*.

BETA pg.249, operation 6 gives the identity

$$\nabla'\cdot(\alpha\vec{\beta}) = \underbrace{(\nabla'\alpha)\cdot\vec{\beta}}_{\text{Integrand above}} + \alpha(\nabla'\cdot\vec{\beta}) \tag{1.25}$$

where the prime on ∇' means we derive w.r.t. the source coordinate, and where α is a scalar field and β a vector field. If we assign $\alpha = 1/(|\vec{r}-\vec{r'}|)$ and $\vec{\beta} = \vec{P}$, and recognise that $\nabla'\alpha = \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}$ we see that the integrand of eq. 1.24 is simply the first term on the right of the identity.

This allows us to express the integrand with the other two terms:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \left[\nabla'\cdot\left(\frac{\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|}\right) - \frac{\nabla'\cdot\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|}\right]dv' \tag{1.26}$$

*As an aside, if the external E-field is intense enough, exceeding the *dielectric strength* of the material, electrons will be ripped from atoms anyway, creating positively charged ions and causing collisions and permanent dislocation and damage in the material. As an example, the dielectric strength of air is 3 kV/mm. Once the electric field exceeds this value, massive ionization and corona discharge occurs. Oil has a higher dielectric strength at 15 kV/mm, which is one reason why high voltage equipment is kept submerged in oil.

The first integral can be rewritten using the divergence theorem (Gauss' Theorem), which states that the volume integral of divergence of a vector field equals the flux integral of the vector field. The second integral is left as it is.

Thus:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\vec{P}(\vec{r'})\cdot\hat{n}}{|\vec{r}-\vec{r'}|}ds' + \frac{1}{4\pi\varepsilon_0}\int_V \frac{-\nabla'\cdot\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|}dv'$$

We have now arrived at a sum of two terms that are similar to the expressions of the electric potential due to a surface charge density and a volume charge density (eq. 1.22), the difference being the numerator of the integrands (though they have the same units).

We can define an *equivalent polarization surface charge density* $\rho_{ps} = \hat{n}\cdot\vec{P}$ and a *polarization volume charge density* $\rho_{pv} = -\nabla\cdot\vec{P}$.

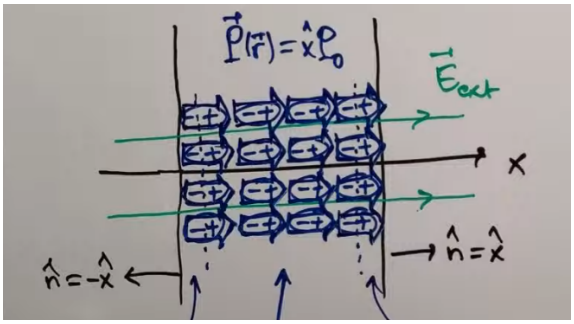
$$\begin{array}{ll} \rho_{ps} = \hat{n}\cdot\vec{P} & \left[\frac{C}{m^2}\right] \\ \rho_{pv} = -\nabla\cdot\vec{P} & \left[\frac{C}{m^3}\right] \end{array}$$

Since these polarization charge densities are the result of bound charges (ie. they cannot redistribute themselves), we can call them *bound charge densities.*

They cannot be directly compared with charge densities in conductors since the origin is completely different. We cannot add or remove to this type of charge density. The overall charge in the material is neutral.

1.16.2 Polarization of an insulator plate

Now, consider a plate of insulator material exposed to an external electric field.

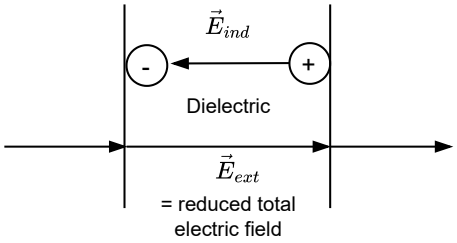


The external field causes electric dipoles to form due to the displacement of electrons relative to their nuclei. The polarization vector is thus parallel to the electric field $\vec{P} = \hat{x}P_0$.

Intuitively, we observe that each surface has accumulated a net positive or negative charge. This explains the existence of bound surface charge densities with a magnitude of $\rho_{ps} = \hat{n}\cdot\vec{P} = -\hat{x}\cdot(\hat{x}P_0) = -P_0$ and $+P_0$, for the left and right surfaces, resp.

In the volume of the plate, the polarization vector is constant and the divergence is therefore zero. Therefore we have no net bound volume charge density, $\rho_{pv} = -\nabla\cdot(\hat{x}P_0) = 0$. This can also be intuitively explained by observing that the number of positive charges precisely cancels the number of negative charges, giving a net charge of zero.

The bound surface charge densities will give an induced electric field that opposes the external electric field. Unlike in a PEC, this induced electric field does not cancel the external electric field, but only reduce it.



1.17 Gauss' Law for Dielectric Media

Gauss' Law, as previously presented, though general and valid in every case, requires a modification in order to explicitly take into account the material-dependent effects of polarization.

We start with the basic law, which states

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0}$$

But we now know what there are two types of volume charge density within a material: one due to free charges and one due to polarization $\rho_v = \rho_{\text{fv}} + \rho_{\text{pv}}$, where ρ_{fv} is the *free* volume charge density. A material does not strictly have to be a conductor to contain a free charge volume density, it is simply that volume charge density that can not be attributed to the polarization; the rest, if you will.

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho_v}{\varepsilon_0} \\ &= \frac{\rho_{\text{fv}} + \rho_{\text{pv}}}{\varepsilon_0} \\ &= \frac{-\nabla \cdot \vec{P} + \rho_{\text{fv}}}{\varepsilon_0} \\ \implies \nabla \cdot \underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{\triangleq \vec{D}} &= \rho_{\text{fv}} \end{aligned}$$

$$\boxed{\nabla \cdot \vec{D} = \rho_{\text{fv}} \qquad \left[\frac{C}{m^3}\right]} \tag{1.27}$$

Thus, we define a new vector field quantity $\vec{D}$, the ***electric flux density*** (or ***electric displacement***), which includes material polarization properties. The name presumably comes from the units.

It has the *constitutive relationship*

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \qquad \left[\frac{C}{m^2}\right]} \tag{1.28}$$

Note: It is important to note that nothing has fundamentally changed with the introduction of this new vector field (Guass' Law remains unchanged), but it does make dealing with dielectric media much more convenient since ρ_{pv} is typically both difficult to describe and to control, whereas ρ_{fv} is more controllable in practice. By incorporating the polarization effects into the electric flux density we, in a sense, circumvent the problem of having to define ρ_{pv}. If we are in vacuum, then $\vec{P} = \vec{0}$ and the law reduces to the basic version.

By having only ρ_{fv} as the source of the D-field, it becomes easier to calculate. The E-field can then be derived via the constitutive relationship.

We can express Gauss' Law for dielectric media in integral form as well just as we did for the basic expression, using the divergence theorem:

$$\int_V (\nabla \cdot \vec{D}) dv \xrightarrow[Theorem]{Gauss'} \oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_{\text{fv}} dv \tag{1.29}$$

$$\implies \boxed{\oint_S \vec{D} \cdot d\vec{s} = Q_{\text{f,encl}}}$$

OBS! In literature the subscript f is not included, as it should be understood that the electric flux density only depends the *free* volume charge density. Abuse of notation be damned.

In summary:

Gauss' Law for Dielectric Media

Integral	Differential
$\oint_S \vec{D} \cdot d\vec{s} = Q_{\text{encl}}$	$\nabla \cdot \vec{D} = \rho_{\text{v}}$

1.18 Permittivity of Dielectric Media

When describing polarization of a single atom, we discovered that the electric dipole moment is proportional to the external electric field:

$$\vec{p} = \alpha \vec{E}$$

where α is a constant that depends on properties of the particular atom: the *polarizability*.

For *linear* and *isotropic* media, we similarly have a proportional relationship between the polarization vector and the external electric field.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \tag{1.30}$$

where χ_e is a dimensionless quantity called **electric susceptibility**, a proportionality constant.

Because the relationship has been defined in this way, rewriting the electric flux density results in:

$$\begin{aligned} \vec{D} &= \varepsilon_0 \vec{E} + \vec{P} \\ &= \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} \\ &= \varepsilon_0 \underbrace{(1 + \chi_e)}_{\varepsilon_r} \vec{E} \end{aligned}$$

$$\boxed{\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}} \tag{1.31}$$

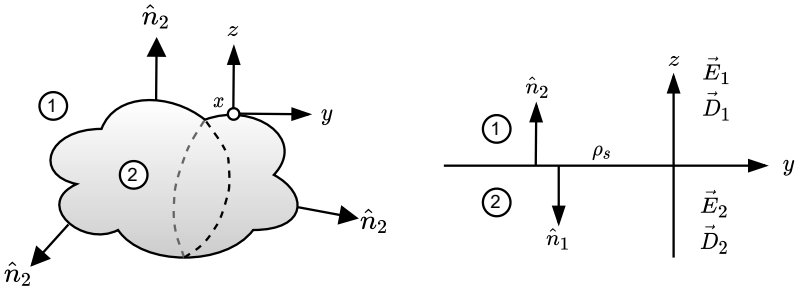
where ε_r is a dimensionless constant known as the **relative permittivity**.

$\varepsilon \left[\frac{F}{m}\right]$ is the **absolute permittivity**.

1.19 Boundary Conditions

If we have two different media, the *boundary conditions* are the conditions that the electrostatic fields must satisfy on the interface.

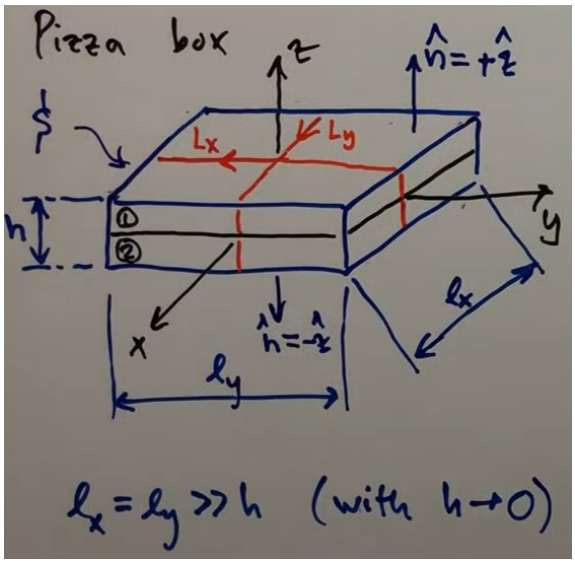
Consider such a situation, perhaps a dielectric object surrounded by air. If we set up a local coordinate system such that $\hat{z}$ is in the direction of the surface normal (the left diagram) and then zoom in on this region to the point that the surface is approximately flat (the right diagram).



Over such a small portion of the surface, the fields can be considered to be approximately constant.

As when deriving Coulomb's law using the two fundamental postulates of electrostatics, we can again do so here. Both must be fulfilled, for the condition to be valid. We start with Gauss' Law.

We introduce a gauss surface in the shape of a "pizza box"; a cuboid closed surface whose sides are of equal length and which has a small height that tends to zero. The surface is divided into two halves by the material interface plane.



As $h \to 0$, the charge enclosed by the box (which always has the interface in the xy-plane) will become simply that on the dividing surface ie. $\rho_s \ell_x \ell_y$.

Top surface:

$$\oint_{S,\text{top}} \vec{D} \cdot d\vec{s} = D_{(1,z)} \ell_x \ell_y$$

Bottom surface:

$$\oint_{S,\text{bottom}} \vec{D} \cdot d\vec{s} = -D_{(2,z)} \ell_x \ell_y$$

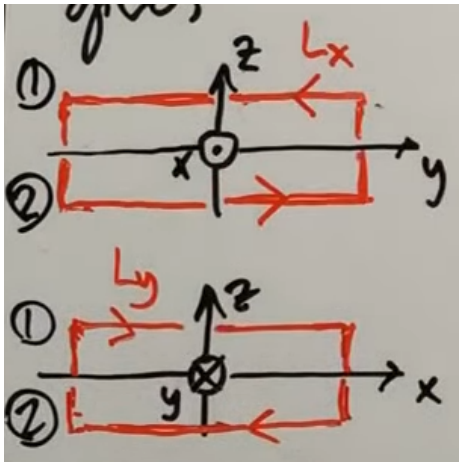
Another consequence of $h \to 0$ is that the flux through the side walls becomes negligible, and so will be skipped.

The result is thus:

$$\begin{aligned} & D_{(1,z)}\,\ell_x\ell_y - D_{(2,z)}\,\ell_x\ell_y \approx \rho_s\ell_x\ell_y \\ \implies & D_{(1,z)} - D_{(2,z)} = \rho_s \\ \implies & \hat{z}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s \\ \implies & \boxed{\hat{n}_2\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s} \end{aligned}$$

making it independent of our local coordinate system.

The other postulate of electrostatics states that the electric field is conservative ie. $\oint_L \vec{E}\cdot d\vec{\ell} = 0$. The two closed contours we shall consider are on the intersections of the pizza box and the xz and yz planes.



For L_x, in which the contour is in the plane perpendicular to x (and right-hand oriented), we integrate along the horizontal curves and skip the vertical curves (since they are of negligible length).

$$\begin{aligned} \oint_{L_x}\vec{E}\cdot d\vec{\ell} &\approx -E_{(1,y)}\ell_y + E_{(2,y)}\ell_y = 0 \\ &\implies -E_{(1,y)} + E_{(2,y)} = 0 \end{aligned}$$

Contour L_y we evaluate in a similar fashion:

$$\begin{aligned} \oint_{L_y}\vec{E}\cdot d\vec{\ell} &\approx E_{(1,x)}\ell_x - E_{(2,x)}\ell_x = 0 \\ &\implies E_{(1,x)} - E_{(2,x)} = 0 \end{aligned}$$

This is infact equal to $\hat{n}_2\times(\vec{E}_1 - \vec{E}_2) = \vec{0}$, since

$$\begin{aligned} \hat{n}_2\times(\vec{E}_1 - \vec{E}_2) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ E_{1,x}-E_{2,x} & E_{1,y}+E_{2,y} & E_{1,z}+E_{2,z} \end{vmatrix} \\ &= \hat{x}(\underbrace{-E_{1,y}+E_{2,y}}_{=0}) + \hat{y}(\underbrace{E_{1,x}-E_{2,x}}_{=0}) \end{aligned}$$

$$\boxed{\hat{n}_2\times(\vec{E}_1 - \vec{E}_2) = \vec{0}}$$

1.19.1 Two Important Boundary Condition Examples

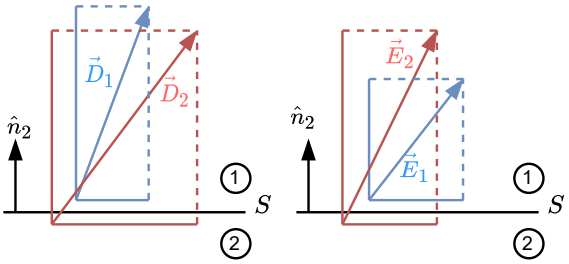
Two special cases for the boundary conditions are examined: the case where we have two dielectric media but with $\rho_s = 0$ on the interface, and the case where one of the regions is a perfect electric conductor.

①: If we have two different dielectric media with $\rho_s = 0$ on the interface, then we have the boundary condition

$$\hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = 0$$

and therefore

$$D_{\perp 1} = D_{\perp 2}$$



The normal components of $\vec{D}$ must be continuous on the interface (it is *dis*continuous for $\rho_s \neq 0$).

The second boundary condition

$$\hat{n}_2 \times (\vec{E}_1 - \vec{E}_2) = \vec{0}$$

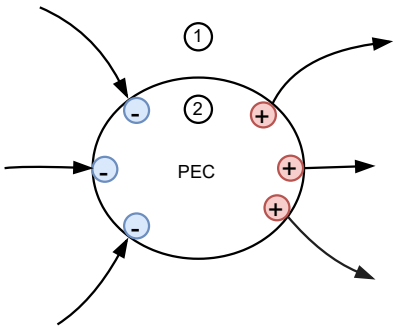
is unaffected and states that the surface-parallel component of $\vec{E}$ is always continuous.

$$E_{||1} = E_{||2}$$

The two boundary conditions are, of course, related via the constitutive relations $\vec{D}_1 = \varepsilon_1 \vec{E}_1$ and $\vec{D}_2 = \varepsilon_2 \vec{E}_2$.

②: If we have a dielectric and a PEC in region 2, then, as we know, $\vec{E}_2 = \vec{0}$ and $\vec{D}_2 = \vec{0}$, and consequently the boundary conditions are

$$\hat{n}_2 \cdot \vec{D}_2 = \rho_s$$
$$\hat{n}_2 \times \vec{E}_2 = \vec{0}$$



This directly implies that the surface parallel components of both fields *do not exist*. Field lines for $\vec{E}$ and $\vec{D}$ are normal to the PEC everywhere on the interface, just as we argued before in a previous chapter.

1.20 Electrostatics - Postulates for Dielectric Media

We now have two postulates and one constitutive relation, as well as two boundary conditions which we summarize here.

Gauss' Law (diff. form, integral form and associated boundary condition)

$$\nabla \cdot \vec{D} = \rho_v \Leftrightarrow \oint_S \vec{D} \cdot d\vec{s} = Q_{\text{encl}} \implies \hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s$$

Conservative electric field (diff. form, integral form and associated boundary condition)

$$\nabla \times \vec{E} = \vec{0} \Leftrightarrow \oint_L \vec{E} \cdot d\vec{\ell} = 0 \implies \hat{n}_2 \times (\vec{E}_1 - \vec{E}_2) = \vec{0}$$

Constitutive relation between electric field $\vec{E}$ and electric flux density $\vec{D}$

$$\vec{D} = \varepsilon \vec{E}$$

We can now formulate a differential equation for the electrostatic potential at every point in space using the above

$\nabla \times \vec{E} = \vec{0}$	Conservative electric field implies...
$\vec{E} = -\nabla V$	Electric field is negative gradient of potential
$\vec{D} = -\varepsilon \nabla V$	Expressed via constitutive relation
$\boxed{\nabla \cdot (-\varepsilon \nabla V) = \rho_v}$	Substituted into Gauss' Law (diff. form)

Once this single differential equation is satisfied, along with the boundary conditions, then the E-field is fully specified in the static case. *This solution is unique.*

In the special case that ε = const., we can formulate the differential equation for the electric potential as **Poisson's equation**:

$$\nabla \cdot (\nabla V) = \boxed{\nabla^2 V - \frac{\rho_v}{\varepsilon}} \tag{1.32}$$

Further, in the special case that $\rho_v = 0$, Poisson's equation reduces to **Laplace's equation**:

$$\nabla \cdot (\nabla V) = \nabla^2 V = 0 \tag{1.33}$$

where the $\nabla \cdot \nabla$ operator ("the divergence of the gradient of") is known as the **Laplacian operator**, and is also written ∇^2. It is a sum of second order spatial derivatives:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.34}$$

1.21 Capacitance

Given a metal object of total charge Q and potential V_0, the electric potential is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_s(\vec{r'})}{|\vec{r} - \vec{r'}|} ds'$$

where $Q = \int_S \rho_s \, ds$.

If we were to add charge such that $\widetilde{Q} = kQ$, then the surface charge density would also increase. However, due to the electrostatic situation, the distribution would not change and the body would remain equipotential. The charges would arrange themselves on the surface in the same manner as before but with a higher density. Thus $\widetilde{\rho_s}(\vec{r}) = k\rho_s(\vec{r})$.

Naturally this changes the voltage of the object

$$\widetilde{V}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\widetilde{\rho_s}(\vec{r'})}{|\vec{r}-\vec{r'}|}ds' = k\frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_s(\vec{r'})}{|\vec{r}-\vec{r'}|}ds' = kV(\vec{r})$$

Thus, the total charge Q is proportional to the potential V_0. The previous potential expression, evalutated anywhere on the surface, gives $\widetilde{V_0} = kV_0$.

This proportionality can also be observed in a different way: $k\vec{E} = -\nabla\widetilde{V_0}$, and on the surface boundary of the metal object $\hat{n}_2 \cdot k\vec{E} = k\frac{\rho_s}{\varepsilon_0} \rightarrow \widetilde{Q} = kQ$. The ratio Q/V_0 remains constant, and is therefore obviously independent of Q and V_0.

We can simply express this relationship explicitly using a proportionality constant C:

$$Q = CV_0$$

where the proportionality constant

$$C = \frac{Q}{V_0} \qquad [C/V] = [F] \tag{1.35}$$

is the **capacitance**: the total amount of charge in an object of given material for a given potential. The higher the charge per given potential, the greater the capacitance.

The capacitance is *independent* of Q and V_0, and is instead a function of ρ and therefore a function of geometrical shape and dielectric properties.

1.22 Energy - Assemble Point Charges

Consider a point charge Q_1 at origo that produces an electric field. Then consider that we want to move another point charge Q_2 from ∞ to a proximal position $\vec{r}_2$.

This requires work and thus a force to counteract that of the electric field acting on Q_2. We can consider the force moving Q_2 against the field to be the "mechanical" force.

$$\begin{aligned} W = \int_{L_{\infty\rightarrow\vec{r}_2}} \vec{F}_{\text{mech}} \cdot d\vec{\ell} &= \int_{L_{\infty\rightarrow\vec{r}_2}} (-\vec{F}_{\text{el}}) \cdot d\vec{\ell} \\ &= \int_{L_{\infty\rightarrow\vec{r}_2}} (-Q_2\vec{E}) \cdot d\vec{\ell} \\ &= Q_2 V(\vec{r}_2) \\ &= Q_2 V_2 \end{aligned}$$

We knew from previous chapters that the potential can be expressed as *work per unit charge*, and so it is no surprise that it appears when considering static energy *(Remember: work = energy)*.

1.23 Energy - Continuum Charge Density in Vacuum

1.24 Energy - Dielectric Media

1.25 Torque and Net Force on Electric Dipole

1.25.1 Work to Turn Electric Dipole

1.26 Summary of Electrostatics

Part 2

Steady Currents

Part 3

Magnetostatics

Part 4

Quasi-statics

Part 5

Electrodynamics

5.1 Rectangular Waveguides

waveguides r harddd :D

Computing the $\vec{E}(t, x, y, z)$ field in time involves evolving a Maxwell's equations in time.

Poisson's equation:

$$\underbrace{\nabla^2 V}_{\nabla\cdot\nabla V} = -\frac{\rho_{\text{free}}}{\varepsilon}$$

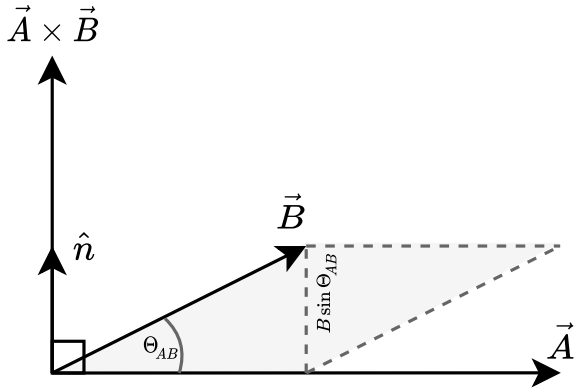
if ε constant.

Part 6

Vector Calculus

6.1 Vector Algebra

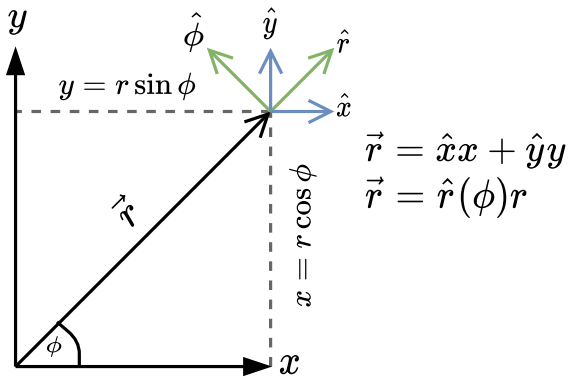
- Scalar/Dot product
  Produces a scalar from two vectors $\vec{A} \cdot \vec{B} = |A||B| \cos \phi_{AB}$.
  Can be interpreted as *product of magnitude of A and projection of B on A.*
  0 if vectors are orthogonal.
- Vector/Cross product
  Produces a vector, perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.
  $\vec{A} \times \vec{B} = |AB \sin \Theta_{AB}| \hat{n} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$, where the rightmost expression is a determinant.
  Magnitude of cross-product is equal to the area of the parallelogram containing $\vec{A}$ and $\vec{B}$.
  0 when parallel.



- Scalar triple product $\vec{A} \cdot (\vec{B} \times \vec{C})$
- Vector Triple
-

6.2 2D Position Vector (ortsvektor)

A position vector (ortsvektor) is simply a vector from origo to a point in space.



A position vector $\vec{r}$ in $\mathbb{R}^2$ can be expressed in Cartesian and Polar coordinates. Starting with straightforward Cartesian coordinates, we can derive a new basis vector in the direction of $\vec{r}$:

$$\vec{r} = \hat{x}x + \hat{y}y = \hat{x}r\cos\phi + \hat{y}r\sin\phi = \underbrace{(\hat{x}\cos\phi + \hat{y}\sin\phi)}_{\hat{r}}r = \hat{r}(\phi)r$$

The new basis vector $\hat{r}($

The position vector does not have a $\hat{\phi}$-component, as it simply points radially outwards, but the field in general will. Since $\hat{\phi}$ is orthogonal to $\hat{r}$, it is:

$$\hat{\phi}(\phi) = \hat{r}(\phi + \frac{\pi}{2}) = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

We can summarize using a *transformation matrix*:

$$\begin{bmatrix}\hat{r}\\ \hat{\phi}\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}}_{=\mathbb{A}}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} \tag{6.1}$$

The inverse is obtained by writing r in terms of x and y (ie. inverting the transformation matrix).

$$\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}}_{=\mathbb{A}^{-1}}\begin{bmatrix}\hat{r}\\ \hat{\phi}\end{bmatrix} \tag{6.2}$$

OBS! These transformation matricies apply to the vector component magnitudes as well:.

$$\begin{bmatrix}F_r\\ F_\phi\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}F_x\\ F_y\end{bmatrix} \tag{6.3}$$

$$\begin{bmatrix}F_x\\ F_y\end{bmatrix} = \begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}F_r\\ F_\phi\end{bmatrix} \tag{6.4}$$

A way to prove this is to express an arbitrary vector field $\vec{U}$ at position $\vec{r}$. The field at that point is given by $\vec{u}$.

$$\vec{u} = \hat{x}u_x + \hat{y}u_y = \hat{r}u_r + \hat{\phi}u_\phi$$

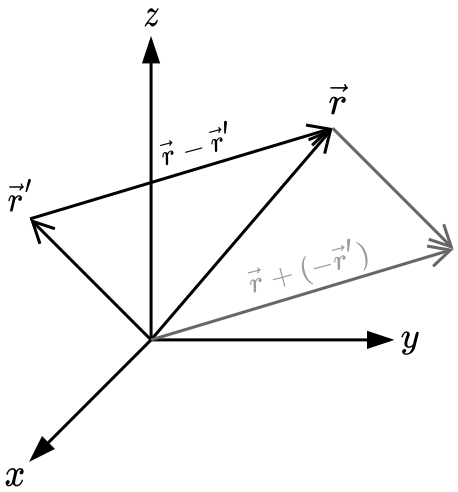
$$\begin{aligned}\vec{u}\cdot\hat{x} = u_x &= \hat{r}\cdot\hat{x}u_r + \hat{\phi}\cdot\hat{x}u_\phi\\ &= \{\vec{A}\cdot\vec{B} = |A||B|\cos_{AB}\} = ...\\ &= u_r\cos\phi - u_\phi\sin\phi\end{aligned}$$

$$\begin{aligned}\vec{u}\cdot\hat{y} = u_y &= \hat{r}\cdot\hat{y}u_r + \hat{\phi}\cdot\hat{y}u_\phi\\ &= \{\vec{A}\cdot\vec{B} = |A||B|\cos_{AB}\} = ...\\ &= u_r\sin\phi + u_\phi\cos\phi\end{aligned}$$

This somewhat involved expression is reasonable as ϕ will change depending on both $\vec{r}$ and $\vec{u}$.

6.3 3D Position Vector (ortsvektor)

TODO



6.4 Coordinate Systems

A point in $\mathbb{R}^3$ is the intersection of three planes. Each plane is given by a basis vector. The basis vectors (or dimensions) are ordered such that the vector product of two sequential basis vectors (in cyclic sequence) gives the following one in the sequence.

$$\hat{u}_1 \times \vec{u}_2 = \vec{u}_3$$
$$\hat{u}_2 \times \vec{u}_3 = \vec{u}_1$$
$$\hat{u}_3 \times \vec{u}_1 = \vec{u}_2$$

Since the vector product is right-hand oriented, so too will the resulting ON-coordinate system.

In any ON-system, we must define the derivative as we travel a length in a particular direction: the *displacement vector** $d\vec{r}$. In cartesian coordinates each basis vector is itself a length, so this is trivially:

$$\begin{aligned} d\vec{r} &= \hat{u}_1 dr_1 + \hat{u}_2 dr_2 + \hat{u}_3 dr_3 \\ &= \hat{x}dx + \hat{y}dy + \hat{z}dz \end{aligned}$$

In, for example, cylindrical coordinates $\hat{r}$ and $\hat{z}$ are lengths, but ϕ is an angle. This requires a conversion factor to convert $d\phi$ to dr_2, that is $dr_2 = rd\phi$ (by definition of the radian).

$$d\vec{r} = \hat{r}dr + \hat{\phi}rd\phi + \hat{z}dz$$

In other words, a displacement in $\hat{\phi}$ requires a scaling factor of r.

This scaling factor is called a *metric coefficient* (denoted h_i, for each ordered dimension). For Cartesian coordinates they are

$$h_1 = 1, h_2 = 1, h_3 = 1$$

For cylindrical coordinates they are

$$h_1 = 1, h_2 = r, h_3 = 1$$

For spherical coordinates they are

$$h_1 = 1, h_2 = r, h_3 = r\sin\Theta$$

*BETA terminology

6.4.1 Cartesian Coordinates

Position Vector

$$\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z$$

A point is defined

$$P(x, y, z)$$

Unit vectors

$$\hat{x}, \quad \hat{y}, \quad \hat{z}$$

$\hat{x}, \hat{y}, \hat{z}$ have constant direction and are fully independent of each other.
Basis vectors

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.5}$$

6.4.2 Cylindrical Coordinates

Position Vector

$$\vec{r} = \hat{r}(\phi)r + \hat{z}z$$

A point is defined

$$P(r, \phi, z)$$

Unit vectors

$$\hat{r}(\phi), \quad \hat{\phi}(\phi), \quad \hat{z}$$

$\hat{z}$ has constant direction and is independent.
Basis vectors

$$\hat{r} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \qquad \hat{\phi} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \qquad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.6}$$

Basis vector relationship (cartesian → cylindrical)*

$$\begin{bmatrix} \hat{r} \\ \hat{\phi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{6.7}$$

Basis vector relationship (cylindrical → cartesian)

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\phi} \\ \hat{z} \end{bmatrix} \tag{6.8}$$

check the above

Coordinate transformations

$$\begin{cases} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{cases} \tag{6.9}$$

$$\begin{cases} r &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1}(\frac{y}{x}) \\ z &= z \end{cases} \tag{6.10}$$

*Remember how to correctly evaluate matrix products (in this case, a dot product). Sum of elementwise products between row of A and column of B.

6.4.3 Spherical Coordinates

Position vector

$$\vec{R} = \hat{R}(\Theta, \phi)R$$

Coordinate

$$P(R, \Theta, \phi)$$

Unit vectors

$$\hat{R}(\Theta, \phi), \quad \hat{\Theta}(\Theta, \phi), \quad \hat{\phi}(\phi)$$

Basis vectors

$$\hat{R} = \begin{bmatrix} \sin\Theta\cos\phi \\ \sin\Theta\sin\phi \\ \cos\Theta \end{bmatrix} \qquad \hat{\Theta} = \begin{bmatrix} \cos\Theta\cos\phi \\ \cos\Theta\sin\phi \\ -\sin\Theta \end{bmatrix} \qquad \hat{\phi} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \tag{6.11}$$

Add spherical -¿ cartesian transformation Coordinate transformations

$$\begin{cases} x &= R\sin\Theta\cos\phi \\ y &= R\sin\Theta\sin\phi \\ z &= R\cos\Theta \end{cases} \tag{6.12}$$

6.5 Nabla Operators

There exist three nabla operators, which describe the derivative of fields.

- ∇V: the *gradient* of a scalar field – the space rate of change of a *scalar field*. Results in a vector field, each point in the direction of maximum increase.
- $\nabla \cdot \vec{A}$: the *divergence* of a vector field – . Results in a scalar field.
- $\nabla \times \vec{A}$: the *curl* of a vector field – . Results in a vector field, each point normal to the plane of maximum curl.

The two derivatives, curl and divergence, are needed to fully specify a vector field, by the Helmholtz theorem.

6.5.1 Gradient of scalar field ∇V

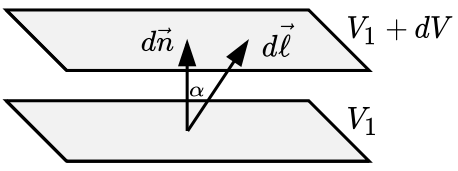
$$\nabla V \triangleq \hat{n}\frac{dV}{dn} \tag{6.13}$$

The gradient of a scalar field is the vector field whose magnitude and direction represent the maximum space rate of increase of the scalar field.

Intuition Let V_1 and $V_1 + dV$ be two arbitrary equipotential surfaces. The maximum space rate of change occurs in the direction that results in the shortest path, that is the surface normal $\hat{n}$. For the same change dV in V, the space rate of change in some other direction $d\vec{\ell} \neq d\vec{n}$ (not in the direction of maximum change) is obviously not as great.

The directional derivative along $d\vec{\ell}$ is:

$$\begin{aligned} \frac{dV}{d\ell} &= \frac{dV}{dn}\frac{dn}{d\ell} = \frac{dV}{dn}\cos\alpha = \frac{dV}{dn}\hat{n}\cdot\hat{\ell} \\ &= (\nabla V)\cdot\hat{\ell} \\ dV &= (\nabla V)\cdot d\vec{\ell} \end{aligned} \tag{6.14}$$



Therefore, the space rate increase of V in $\hat{\ell}$ is the projection (the component) of the gradient of V in that direction.

In cartesian coordinates:

$$\nabla V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z} \tag{6.15}$$

In general for curvilinear coordinate systems:

$$\nabla = \hat{u}_1\frac{\partial}{h_1\partial u_1} + \hat{u}_2\frac{\partial}{h_2\partial u_2} + \hat{u}_3\frac{\partial}{h_3\partial u_3} \tag{6.16}$$

6.5.2 Divergence of a vector field

The divergence is defined at a point as the **net outward flux of a vector field $\vec{E}$ per unit volume Δv, as the volume about the point tends to zero.**

In general, this is

$$\nabla \cdot \vec{A} \triangleq \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v} \tag{6.17}$$

In cartesian coordinates:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{6.18}$$

6.6 Commonly used Vector Integrals

$$\int_V \vec{F} dv \qquad \vec{\mathbb{R}}^3 \to \mathbb{R} \qquad \text{Simple volume integral. Shorthand for triple integral.} \tag{6.19}$$

$$\int_L V d\vec{\ell} \qquad \vec{\mathbb{R}} \to \mathbb{R} \qquad \text{Line integral over scalar field.} \tag{6.20}$$

$$\int_L \vec{F} \cdot d\vec{\ell} \qquad \vec{\mathbb{R}}^3 \to \mathbb{R} \qquad \text{Line integral over vector field.} \tag{6.21}$$

$$\int_S \vec{A} \cdot d\vec{s} \qquad \vec{\mathbb{R}}^3 \to \mathbb{R} \qquad \text{"Flux integral". Flux of } \vec{A} \text{ through surface S.} \tag{6.22}$$

The function notation is fucky. I want to indicate when a vector field as function of 3 coordinates results in a single non-field scalar.

6.7 Fundamental Theorems of Vector Calculus

The four fundamental theorems of vector calculus are generalizations of the fundamental theorem of calculus which, to repeat, equates the integral of the derivative $G'(x)$ to the values of $G(x)$ and the interval boundaries. *

$$\int_a^b G'(x)dx = G(b) - G(a) \tag{6.23}$$

In a similar fashion, the fundamental theorems of vector calculus state that the integral of some type of derivative over some object is equal to the values of a function along the boundary of that object.

The four theorems are:

*https://mathinsight.org/fundamental_theorems_vector_calculus_summary

1. The gradient theorem for line integrals
2. Green's Theorem
3. Stoke's Theorem
4. The divergence Theorem

Also useful are a number of vector *null identities.*

6.7.1 Null-Identities

1.
$$\nabla \times (\nabla V) \equiv 0 \tag{6.24}$$

The curl of the gradient of any scalar field is zero. (The existence of V and ∇V is implied here.) Consequentially: *If a vector field is curl-free (conservative), then it can be expressed as the gradient of a scalar field.* That is, if $\nabla \times \vec{E} = 0$ then $\vec{E} = -\nabla V^*$.

The derivation is simple: for a general vector field, given as the gradient of a scalar field $\vec{A} = \nabla V$, Stoke's law (eq. 6.32) states that

$$\int_S \nabla \times (\nabla V) \cdot d\vec{s} = \oint_L \nabla V \cdot d\vec{\ell}$$

However, the space rate of increase of V in the $\hat{\ell}$ direction is given by

$$dV = (\nabla V) \cdot d\vec{\ell}$$

Therefore

$$\int_S \nabla \times (\nabla V) \cdot d\vec{s} = \oint_L dV = 0$$

This is explained by the *gradient theorem for line integrals* (eq. 6.26), and by taking the limits at the same point.

Then, if we let a vector field be $\vec{E}$, and $\nabla \times \vec{E} = 0$ (is conservative), the scalar field V can be defined as

$$\vec{E} = -\nabla V$$

2.
$$\nabla \cdot (\nabla \times \vec{F}) \equiv 0 \tag{6.25}$$

The divergence of the curl of any vector field is zero.

This culminates in the *Helmholtz Theorem* which states that a vector field is uniquely defined (within an additive constant) by its divergence and curl.

6.7.2 The Gradient Theorem for line integrals

If vector field $\vec{\nabla} f$ is conservative ($\nabla \times (\nabla f) = 0$), then the gradient theorem for line integrals *relates the line integral of the gradient of a function to the values of the function at its endpoints.*

$$\int_L \nabla f \cdot d\vec{\ell} = f(\vec{r}_b) - f(\vec{r}_a) \tag{6.26}$$

where $\vec{r}_a$ and $\vec{r}_b$ are the endpoints of L.

This implies

*Minus sign unimportant in identity, included due to relevance to electric field and potential.

1. that the value does not depend on the *path* of L, just the endpoints.
2. that the closed loop integral is zero.

An EMFT example of this is the *electric potential difference V*

$$V(\vec{r}_b) - V(\vec{r}_a) = -\int_{L_{\vec{r}_a \to \vec{r}_b}} \vec{E} \cdot d\vec{\ell} = \int_{L_{\vec{r}_a \to \vec{r}_b}} \nabla V \cdot d\vec{\ell}$$

6.7.3 Green's Theorem

Green's theorem relates a closed line integral around a simple closed curve L to a double integral over the plane region S bounded by L. It is a special 2-D case of *Stoke's theorem* in the xy-plane..

In terms of 3d-*curl*, it is:

$$\oint_L \vec{F} \cdot d\vec{\ell} = \int_S (\nabla \times \vec{F}) \cdot d\vec{s} \implies \int_S (\nabla \times \vec{F}) \cdot \hat{n} ds$$

Since we are exclusively in the xy-plane, the normal is always in the $\hat{z}$ direction, and thus the integrand will be the $\hat{z}$-component of the curl if $\vec{F}$.

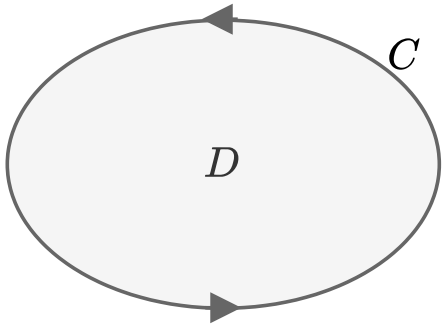
Thus:

$$\oint_L \vec{F} \cdot d\vec{l} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy \tag{6.27}$$

,where the path of integration along L is anticlockwise and $d\vec{l} = (dx, dy)$, an infinitesimal length along the curve. $\vec{F}$ must be continuously differentiable on S.

Component-wise ($\mathbb{R}^2$), this is

$$\oint_L (F_1 dx + F_2 dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \tag{6.28}$$



The integrand of the double integral can be regarded as the *2D-curl** at each point on S, or the "microscopic circulation".

*It is, in fact, simply the $\hat{z}$ component of the proper 3D-curl (perpendicular to the plane of interest). The result is scalar.

Intuition Green's Theorem equates the circulation around a planar region to the total "microscopic circulation" in S. Consider dividing S into small squares and evaluating the circulation around the four edges. Each set of neighbours will share one edge which, when summed, will cancel out. As the area of each square tends to zero, all contributions will be cancelled out except for those at the surface boundary – equal to the closed line integral.

$$\oint_L \vec{F} \cdot d\vec{l} = \iint_S \text{"2D-curl"} dA \tag{6.29}$$

Area calculation

Interestingly, Green's Theorem can be used to calculate surface area by line integral. Area is simply

$$A = \iint_D dA \tag{6.30}$$

If P and Q are selected such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then the area is given by

$$A = \oint_C (P dx + Q dy) \tag{6.31}$$

one such choice is $\vec{F} = (-y/2, x/2)$.

Example - Area using Green's Theorem Disk S defined by $x^2 + y^2 \leq r^2$

Parameterize

$$\begin{aligned} \frac{x^2}{r^2} + \frac{y^2}{r^2} &= 1 \\ \frac{(r\cos t)^2}{r^2} + \frac{(r\sin t)^2}{r^2} &= 1 \\ \cos^2 t + \sin^2 t &= 1 \\ x = r\cos t \qquad & y = r\sin t \end{aligned}$$

Boundary L defined parametrically in counterclockwise orientation. Position vector $\vec{l}$:

$$\vec{l}(t) = (r\cos t, r\sin t), \quad 0 \leq t \leq 2\pi$$

Then, a line element is

$$\begin{aligned} \frac{d\vec{l}(t)}{dt} &= (\underbrace{-r\sin t}_{dx}, \underbrace{r\cos t}_{dy}) \\ d\vec{l}(t) &= (\underbrace{-r\sin(t)dt}_{dx}, \underbrace{r\cos(t)dt}_{dy}) \end{aligned}$$

By Green's theorem

$$\begin{aligned} \iint_S dA &= \frac{1}{2}\oint_L (x dy - y dx) \\ &= \frac{1}{2}\oint_0^{2\pi} (r\cos t(r\cos(t)dt) - r\sin t(-r\sin(t)dt)) \\ &= \frac{1}{2}\oint_0^{2\pi} [r\cos t(r\cos t) - r\sin t(-r\sin t)]dt \\ &= \frac{1}{2}\oint_0^{2\pi} r^2(\cos^2 t + \sin^2 t)dt = \frac{r^2}{2}\oint_0^{2\pi} dt = \pi r^2 \qquad \square \end{aligned}$$

This is also easily calculated by converting the problem to polar coordinates, in the form:

$$\begin{aligned}\iint_S dA &= \oint_L \vec{F} \cdot d\vec{l} \\ &= \frac{1}{2} \oint_0^{2\pi} (-r \sin\phi \hat{x} + r \cos\phi \hat{y}) \cdot \hat{\phi} r d\phi\end{aligned}$$

, one then only needs to transform the basis vectors and the rest will make itself apparent.

6.7.4 Stoke's Theorem

The closed line integral of a vector field of a contour bounding open surface S is equal to the flux (surface integral) of the curl of the vector field.

$$\oint_L \vec{A} \cdot d\vec{\ell} = \int_S (\nabla \times \vec{A}) \cdot d\vec{s} \tag{6.32}$$

Intuition Stoke's theorem is a generalisation of Green's theorem (where Green's is a special case in just the xy-plane, using the $\hat{z}$-component of the curl). Imagine taking the plane surface of Green's theorem and "inflating" it to a 3D surface, but with the same bounding contour L. The integrand of the open surface integral is now the 3D *curl* instead of the 2D "circulation".

6.7.5 The Divergence Theorem (Gauss' Theorem)

Divergence, briefly: The flux of a vector is analogous to the flow of an incompressible liquid. For a volume *which is enclosed by a closed surface*, the inward and outward fluxes through the surface are balanced unless the volume contains a sink or source, which will of course result in a net in inward or outward flux. This net "flow", or *divergence*, is a measure of the enclosed sink or source, depending on direction.

Divergence: The divergence of a vector field $\vec{A}$ at a point is defined as the net outward flux of $\vec{A}$ per unit volume, as the volume about the point tends to zero.

$$\nabla \cdot \vec{A} \triangleq \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v} \tag{6.33}$$

The result is a *scalar field* giving the quantity of the vector field's source at each point.

The *divergence theorem* is the relationship between divergence within a volume and the outward flux through a closed surface.

Divergence Theorem (Gauss' Theorem): The volume integral of the divergence of a vector field equals the total outward flux of the vector field through a closed bounding surface.

$$\int_V (\nabla \cdot \vec{A}) dv = \oint_S \vec{A} \cdot d\vec{s} \tag{6.34}$$

Intuition Divide a volume V into N differential volumes Δv. The divergence at point j is the outward flux through the surface that bounds Δv_j (as per eq. 6.33). However, wherever Δv_j interfaces with its neighbours, the fluxes will be equal and opposite, cancelling out. This occurs everywhere within the volume, except at the bounding surface, where there will be a an outward flux.

Thus, the sum of all divergence within the volume equals the flux at just the bounding surface.

6.7.6 The Helmholtz Theorem

A vector field (vector point function) is uniquely defined(within an additive constant) if its divergence and curl are specified everywhere.

In other words, a specification of the sources of a vector field is sufficient to specify the vector field itself.

The Helmholtz theorem is written:

$$\vec{B} = -\nabla V + \nabla \times \vec{A} \tag{6.35}$$

Any vector field can be decomposed into two terms: the gradient of a scalar field and the curl of a vector field.

Taking the divergence and curl of $\vec{B}$:

$$\nabla \cdot \vec{B} = \nabla \cdot (-\nabla V) = \rho$$

$$\boxed{\nabla \cdot \vec{B} = \rho}$$

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \vec{J}$$

$$\boxed{\nabla \times \vec{B} = \vec{J}}$$

Taking the divergence removes the right term and taking the curl removes the left term, due to the null identities. *rho* is in general a scalar density function and $\vec{J}$ can be regarded as the strength of the vector source.

This allows us to classify fields into four varieties, depending on whether ρ and $\vec{J}$ are non-zero or not.

$\nabla \cdot \vec{B} = \rho$: the field is non-solenoidal.
$\nabla \times \vec{B} = \vec{J}$: the field is rotational

Example: the electrostatic field is non-solenoidal and irrotational since $\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0}$ and $\nabla \times \vec{E} = \vec{0}$.

Part 7

Appendix

7.1 Maxwell's Equations

Ampére's Law

$$\nabla \times \vec{H} = \vec{J}_v + \frac{\partial \vec{D}}{\partial t} \tag{7.1}$$

$$\vec{D} = \epsilon \vec{E} \tag{7.2}$$

Faraday's Law

$$\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \tag{7.3}$$

$$\vec{B} = \mu \vec{H} \tag{7.4}$$

Gauss' Law

7.2 Quantities

Symbol	Quantity	Units (Derived and SI)	Relationship
E	Electric Field Intensity	$\frac{V}{m}$	$E = qF$
H	Magnetic Field Strength	$\frac{A}{m}$	
Φ_B	Magnetic Flux	$\text{Wb} = V \cdot s$	
B	Magnetic Flux Density	$T = \frac{\text{Wb}}{m^2} = \frac{V \cdot s}{m^2}$	$\frac{\phi}{A}$
L	Inductance	$H = \frac{\text{Wb}}{A}$	
F_M	Magnetomotive Force (mmf)	$A[\text{turns}]$	$NI = \Re \cdot \phi_B = H\ell$
$\Re$	Reluctance	$H^{-1} = \frac{A}{\text{Wb}}$	$\frac{F_M}{\phi_B} = \frac{NI}{\phi_B} = \frac{H\ell}{BA} = \frac{\ell}{\mu A}$
μ_0	Permeability	$\frac{H}{m} = \frac{\text{Wb}}{Am} = \frac{V \cdot s}{Am}$	$\frac{B}{H}$
μ_r	Relative Permeability	*Ratio*	$\frac{B_{\text{material}}}{B_{\text{vacuum}}}$
Q	Charge	C	$I \cdot t$
I	Current	A	
C	Capacitance	$F = \frac{Q}{V}$	
E	Energy	$J = W \cdot t$	
ε_0	Permittivity	$\frac{C}{V \cdot m}$	$\frac{D}{E}$
ε_r	Relative Permittivity	*Ratio*	$\frac{D_{\text{material}}}{D_{\text{vacuum}}}$

Table 7.1: Electromagnetic quantities and units

Electrical Circuit	Magnetic Circuit
e.m.f E (V)	mmf F_M (A)
current I (A)	flux ϕ_B (Wb)
resistance R (Ω)	reluctance $\Re$ H^{-1}
$I = \frac{E}{R}$	$\phi_B = \frac{F_M}{\Re}$
$R = \frac{\rho \ell}{A}$	$\Re = \frac{\ell}{\mu A}$

Table 7.2: Comparison between electrical and magnetic quantities

7.3 Links

https://mathinsight.org/

Intuitive understanding of advanced vector calculus concepts such as Green's Theorem, conservative fields etc...

https://tutorial.math.lamar.edu/classes/calciii/calciii.aspx

Multivariable vector calculus.

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