Math 551
Practice Problems for Final Exam

- ***Attention: Textbook, notes, calculators and other electronic devices are NOT allowed during exams.***

1) Find a basis for the column space col(A) of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 2 & 0 \\ 2 & -4 & 5 & 0 & 1 \\ -3 & 6 & 0 & 1 & 1 \end{bmatrix}.$$

Basis: $\left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

rank $A = 3$

Based on your finding, determine the rank of A.

2) Find a basis for the null space $\ker(A)$ of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Basis: $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

rank $A = 4 - 2 = 2$

Based on your finding, determine the rank of A.

3) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 2 \\ 1 & 4 & 5 \end{bmatrix}.$$

$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 7$

4)The eigenvalues of the matrix

$$A = \begin{bmatrix} -3 & 3 & 1 \\ -10 & 8 & 2 \\ 20 & -12 & -2 \end{bmatrix}$$

$E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\}$, dim $= 1$

$E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} \frac{3}{5} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{5} \\ 0 \\ 1 \end{pmatrix} \right\}$, dim $= 2$

are $\lambda_1 = -1$ and $\lambda_2 = 2$. Find the corresponding eigenspaces $E_{\lambda_1}(A)$ and $E_{\lambda_2}(A)$ and their dimensions. Based on your findings, determine whether A is diagonalizable.

Yes.

5) Consider the following subspace in $\mathbb{R}^4$:

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

$\left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \frac{1}{2\sqrt{65}} \begin{pmatrix} 3 \\ 11 \\ 7 \\ 9 \end{pmatrix} \right\}$

Find an orthonormal basis for W.

6) Consider the following set of vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

It's an orthogonal basis but not an orthonormal basis since $\left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{2} \neq 1$

Determine whether this set of vectors forms an orthogonal basis of $\mathbb{R}^3$. If it does, determine whether it also forms an orthonormal basis.

7) Determine a, b, c and d such that the following matrix is an orthogonal matrix

$$A = \begin{bmatrix} 2/3 & \sqrt{2}/2 & \sqrt{2}/6 \\ 2/3 & a & b \\ c & 0 & d \end{bmatrix}.$$

$\begin{cases} a = -\frac{\sqrt{2}}{2} \\ b = \frac{\sqrt{2}}{6} \\ c = \frac{1}{3} \\ d = -\frac{\sqrt{2}}{3} \end{cases}$ or $\begin{cases} a = -\frac{\sqrt{2}}{2} \\ b = \frac{\sqrt{2}}{6} \\ c = -\frac{1}{3} \\ d = \frac{\sqrt{2}}{3} \end{cases}$

8) Is the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -6 & 2 \\ -1 & -3 & 1 \end{bmatrix}$$

Yes. $A^{-1} = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 4 \\ 3 & -5 & 11 \end{pmatrix}$

invertible? If yes find its inverse A^{-1}. If no explain why.

9) In $\mathbb{R}^4$, consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -4 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 4 \\ -28 \\ 30 \\ 0 \end{bmatrix}.$$

No since $\{v_1, v_2, v_3, w\}$ is linearly indep.

Determine whether $\vec{w}$ belongs to the subspace $V = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_2\}$.

10) Find *all* solutions of the linear system

$$\begin{cases} x_1 + 2x_2 + x_3 + 12x_5 = -2 \\ x_1 + 2x_2 + 2x_3 - 2x_4 + 4x_5 = 1 \\ x_1 + 2x_2 + 5x_3 - 7x_4 - 18x_5 = 4 \end{cases}$$

$$\begin{pmatrix} 7 \\ 0 \\ -9 \\ -6 \\ 0 \end{pmatrix} + \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -16 \\ 0 \\ 4 \\ -2 \\ 1 \end{pmatrix} \right\}$$