Quantum algorithm for Petz recovery channels and pretty good measurements

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The Petz recovery channel plays an important role in quantum information science as an operation that approximately reverses the effect of a quantum channel. The pretty good measurement is a special case of the Petz recovery channel, and it allows for near-optimal state discrimination. A hurdle to the experimental realization of these vaunted theoretical tools is the lack of a systematic and efficient method to implement them. This paper sets out to rectify this lack: using the recently developed tools of quantum singular value transformation and oblivious amplitude amplification, we provide a quantum algorithm to implement the Petz recovery channel when given the ability to perform the channel that one wishes to reverse. Moreover, we prove that, in some sense, our quantum algorithm's usage of the channel implementation cannot be improved by more than a quadratic factor. Our quantum algorithm also provides a procedure to perform pretty good measurements when given multiple copies of the states that one is trying to distinguish.

Introduction—Pretty good measurements [1–5] and Petz recovery channels [6–10] are workhorses of quantum information theory: they are used ubiquitously to prove basic results in quantum communication and measurement [11]. Although important for attaining quantum channel capacities [12–16] and performing state discrimination [1–4, 9], these useful theoretical constructions are less common in experiment, for the simple reason that there has not been a systematic method for performing them efficiently in practice. Our goal here is to fill this gap.

The Petz recovery channel was introduced in the context of quantum sufficiency in [6, 7] and later rediscovered in [9] in the context of quantum error correction. It can be understood as a critical part of a quantum version of the Bayes theorem [17, Section IV]. To review it, let us begin with the classical case. A classical channel with input system X and output system Y over the alphabets $\mathcal{X}, \mathcal{Y}$ is a conditional probability distribution $\{p_{Y|X}(y|x)\}_{x \in \mathcal{X}, y \in \mathcal{Y}}$. We consider a probability distribution $p_X(x)$ over the alphabet $\mathcal{X}$ as the input to the channel. It then follows from the Bayes theorem that $p_X(x)p_{Y|X}(y|x) = p_Y(y)p_{X|Y}(x|y)$, where $p_Y(y) = \sum_x p_X(x)p_{Y|X}(y|x)$. Hence, for all $x \in \mathcal{X}, y \in \mathcal{Y}$, we define the "reversal channel" via the formula

$$p_{X|Y}(x|y) = \frac{p_X(x)p_{Y|X}(y|x)}{\sum_x p_X(x)p_{Y|X}(y|x)}. \tag{1}$$

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This channel acts on the output system Y. If the particular distribution $p_Y(y)$ defined above is "sent in" through this channel, then the input $p_X(x)$ is recovered perfectly: $p_X(x) = \sum_y p_{X|Y}(x|y)p_Y(y)$. The computation of the reversal channel $p_{X|Y}(x|y)$ requires a specification of the input probability distribution $p_X(x)$ and the forward channel $p_{Y|X}(y|x)$. The Petz recovery channel is a quantum generalization of the reversal channel above: it is a function of a quantum channel $\mathcal{N}$ and an input state σ to the channel, with the former generalizing $p_{Y|X}(y|x)$ and the latter $p_X(x)$. We discuss it in more detail in what follows.

The Petz recovery channel appears often in quantum information as a proof tool, showing that near-optimal recovery from undesired quantum operations is possible. Ref. [9] demonstrated how this recovery channel can be an effective means for reversing the effects of noise. Thereafter, [18] showed that the Petz recovery channel (therein called "transpose channel") is a universal recovery operation for approximate quantum error correction, which performs comparably to the best possible one in terms of worst-case fidelity (see also [19, 20]). The Petz recovery channel also goes by the name "pretty good recovery," as used in [21, 22], due to the result of [9]. Yet another application comes from the field of quantum communication: [16] showed explicitly how to use the Petz recovery channel in a decoder to achieve the coherent information rate of quantum communication. It has also found use in developing physically meaningful refinements of quantum entropy inequalities [23–27]. See [28–32] for further uses.

As an application of our results, our quantum algorithm can be used to implement the pretty good measurement (PGM) [1–5]. This measurement was used in [12, 15] as part of a coding scheme to approach the Holevo information rate for

classical communication over a quantum channel. It has also been instrumental in proving bounds for quantum algorithms. Ref. [33] showed that the PGM is an optimal measurement for solving the dihedral hidden subgroup problem and that it is helpful in proving a lower bound on the sample complexity of this problem. Similar techniques have been used for quantum probably-approximately-correct learning [34]. Ref. [35] showed how to implement the PGM for pure states, while our algorithm for Petz recovery channels is capable of performing the PGM in the general case.

We now begin the technical part of our paper, starting with an explicit description of the Petz recovery channel and the resources that we work with for its implementation.

Petz recovery channel—The Petz recovery channel is a function of a quantum state σ_A on a system A and a quantum channel $\mathcal{N}_{A \to B}$ taking system A to a system B. It is given explicitly as follows [10]:

$$\mathcal{P}^{\sigma,\mathcal{N}}_{B \to A}(\omega_B) := \sigma_A^{1/2} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma_A)^{-1/2} \omega_B \mathcal{N}(\sigma_A)^{-1/2} \right) \sigma_A^{1/2}, \quad (2)$$

where $\mathcal{N}^\dagger$ is the Hilbert–Schmidt adjoint [11] of the channel $\mathcal{N}$ and we have omitted the system labels of $\mathcal{N}_{A \to B}$ for brevity.

It is a composition of three completely positive (CP) maps:

$$(\cdot) \to [\mathcal{N}(\sigma_A)]^{-1/2} (\cdot) [\mathcal{N}(\sigma_A)]^{-1/2}, \quad (3)$$

$$(\cdot) \to \mathcal{N}^\dagger(\cdot), \quad (4)$$

$$(\cdot) \to \sigma_A^{1/2} (\cdot) \sigma_A^{1/2}. \quad (5)$$

None of these maps are trace preserving individually, but overall the map in (2) is trace preserving on the support of the state $\mathcal{N}(\sigma_A)$ [23]. We note here that the main idea behind our algorithm is to implement the Petz recovery channel as a composition of the three maps given in (3)–(5), while taking into account the fact that the overall map in (2) is trace-preserving in order to implement it deterministically with some desired accuracy.

Block-encoding—The Petz recovery channel depends on the state σ_A, and so our algorithm needs some form of access to it. In order to cover a wide range of scenarios, we employ the block-encoding formalism, which generalizes the most common input models for matrices used in quantum algorithms [36, 37].

Let $\|\cdot\|$ denote the spectral norm of a matrix (also known as the Schatten ∞-norm). For a complex matrix A and $\alpha \geq \|A\|$, the matrix A/α can be represented as the upper-left block of a unitary matrix:

$$U = \begin{bmatrix} A/\alpha & \cdot \\ \cdot & \cdot \end{bmatrix} \iff A = \alpha(\langle 0| \otimes I) U (|0\rangle \otimes I). \quad (6)$$

The unitary matrix U is said to be a *block-encoding* of A. Henceforth, we do not write identity operators explicitly, but we instead include system subscripts as a guide. If the linear map A/α acts on a qubits, then the unitary U can be thought of as a probabilistic implementation of this map: given an a-qubit input state $|\psi\rangle$, applying the unitary U to the state $|0\rangle|\psi\rangle$, measuring the first system, and post-selecting on the $|0\rangle$ outcome, the second system contains a state proportional to $A|\psi\rangle/\alpha$.

This generalizes the two most relevant input models in our case. If we are given copies of the quantum state σ_A, then we can implement an (approximate) block-encoding of σ_A by using density matrix exponentiation [38, 39] and "taking the logarithm" of the time evolution [37]. Alternatively, if we have access to a quantum circuit U^σ_{RA} that prepares a purification $|\psi^\sigma\rangle_{RA} := U^\sigma_{RA}|0\rangle_R|0\rangle_A$ of σ_A, such that $\text{Tr}_R[|\psi^\sigma\rangle\langle\psi^\sigma|_{RA}] = \sigma_A$, then we can directly implement an exact block-encoding of σ_A with only two uses of U^σ_{RA} as follows [36, 37]:

$$V^\sigma_{RAA'} := (U^\sigma_{RA})^\dagger (I_R \otimes \text{SWAP}_{AA'}) U^\sigma_{RA} = \begin{bmatrix} \sigma_A & \cdot \\ \cdot & \cdot \end{bmatrix}, \quad (7)$$

where system A' is isomorphic to system A.

Assumptions—The resources that we use for implementing the Petz recovery channel are as follows: 1) Quantum circuits U^{σ_A} and $U^{\mathcal{N}(\sigma_A)}$ that are (approximate) block-encodings of σ_A and $\mathcal{N}(\sigma_A)$, respectively, and 2) a quantum circuit $U^{\mathcal{N}}_{E'A \to EB}$ that implements the channel $\mathcal{N}$, in the sense that $U^{\mathcal{N}}_{E'A \to EB}|0\rangle_{E'} =: V^{\mathcal{N}}_{A \to EB}$, where $V^{\mathcal{N}}_{A \to EB}$ is an isometric extension of $\mathcal{N}$ satisfying $\text{Tr}_E[V^{\mathcal{N}}_{A \to EB}(\omega_A)(V^{\mathcal{N}}_{A \to EB})^\dagger] = \mathcal{N}(\omega_A)$, for every input density operator ω_A.

We note that, given an efficient description of the channel $\mathcal{N}$ in terms of its Kraus operators, the unitary $U^{\mathcal{N}}_{E'A \to EB}$ can be efficiently implemented on a quantum computer [40]. Also, given copies or "purified access" to σ_A, we can achieve the corresponding access to $\mathcal{N}(\sigma_A)$ after applying $U^{\mathcal{N}}_{E'A \to EB}$, which then results in an efficient block-encoding for $\mathcal{N}(\sigma_A)$.

Rewriting the Petz recovery channel—Eq. (4) calls for the application of the adjoint $\mathcal{N}^\dagger$ of the channel $\mathcal{N}$. We now explain how this can be accomplished using $U^{\mathcal{N}}_{E'A \to EB}$. The action of the adjoint on an arbitrary operator ω_B is given by $\mathcal{N}^\dagger(\omega_B) = \langle 0|_{E'} U^{\mathcal{N}\dagger}(I_E \otimes \omega_B)U^{\mathcal{N}}|0\rangle_{E'}$ [11]. Let $\Gamma_{E\tilde{E}} := |\Gamma\rangle\langle\Gamma|_{E\tilde{E}}$ denote an operator proportional to the maximally entangled state on E and a reference system $\tilde{E}$, where $|\Gamma\rangle_{E\tilde{E}} := \sum_{i=0}^{d_E-1} |i\rangle_E|i\rangle_{\tilde{E}}$ and d_E is the dimension of system E. Then extending the identity operator with $\Gamma_{E\tilde{E}}$, we rewrite the previous identity as

$$\mathcal{N}^\dagger(\omega_B) = \text{Tr}_{\tilde{E}}[\langle 0|_{E'}(U^{\mathcal{N}}_{E'A \to EB})^\dagger (\Gamma_{E\tilde{E}} \otimes \omega_B)U^{\mathcal{N}}_{E'A \to EB}|0\rangle_{E'}]. \quad (8)$$

Now the interpretation of the adjoint map as a probabilistic quantum operation is clear: the adjoint map $\mathcal{N}^\dagger$ acting on the operator ω_B can be applied by tensoring in the maximally entangled state $\Gamma_{E\tilde{E}}/d_E$, performing the inverse of the unitary $U^{\mathcal{N}}$, measuring the system E', accepting if the all-zeros outcome occurs, and finally, ignoring the system $\tilde{E}$ (which corresponds to tracing it out).

Thus, our plan is to implement the linear extension of the adjoint map, as given in (8). Sandwiching this between the other two maps in (3) and (5) comprising the Petz recovery channel, we obtain the following isometric extension of the Petz recovery channel:

$$V^{\mathcal{P}}_{B \to \tilde{E}A} := (\langle 0|_{E'} \otimes I_{\tilde{E}} \otimes \sigma_A^{\frac{1}{2}})(U^{\mathcal{N}}_{E'A \to EB})^\dagger (|\Gamma\rangle_{E\tilde{E}} \otimes [\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}). \quad (9)$$

Tracing over $\tilde{E}$ then implements the Petz recovery channel $\mathcal{P}^{\sigma,\mathcal{N}}_{B \to A}(\omega_B)$. Note that in the rewriting above, the implementation of the adjoint map discussed in the preceding paragraph

is no longer contiguous. It proceeds in two phases: the application of the unitary $(U^{\mathcal{N}}_{E'A \to EB})^{\dagger}$ before multiplication by $\sigma_A^{1/2}$ (which applies (5)); and the measurement and post-selection after that step.

Quantum singular value transformation—Our implementation is based on quantum singular value transformation (QSVT) [37]. QSVT transforms the singular values of a block-encoded matrix and thus provides an efficient means of quantum matrix arithmetic. Often we need to rely on approximations, and so when doing so, we keep track of the error/precision δ, as well as the sub-normalization factor α: we say that U is an (α, δ)-block-encoding of A if $\|A - \alpha(\langle 0| \otimes I)U(|0\rangle \otimes I)\| \leq \delta$.

In what follows, we manipulate block-encodings U^{ρ} of density operators ρ. The power of QSVT is that it allows for transforming U^{ρ} to a block-encoding of $\tilde{f}(\rho)$, where $\tilde{f}$ is a function applied to the singular values of its argument. More precisely, $\tilde{f}$ denotes a polynomial approximation of some function f; in view of the maps given in (3) and (5) above, the particular functions of interest here are $f_1(x) := x^{-1/2}$ and $f_2(x) := x^{1/2}$.

The complexity of realizing the transformed block-encoding unitary $U^{\tilde{f}(\rho)}$ is stated in terms of the number of uses of U^{ρ} (which dominates the overall gate complexity), and it depends on the parameters of the functional approximation $\tilde{f}$. For a function f, let $\|f(x)\|_{\mathcal{J}} := \sup_{x \in \mathcal{J}} |f(x)|$. Using techniques from [41], for the two functions above, one can find polynomial approximations $\tilde{f}_1$, $\tilde{f}_2$ such that $\frac{\theta^{1/2}}{2} \|\tilde{f}_1(x) - x^{-1/2}\|_{[\theta, 1]} \leq \delta$, and $\frac{1}{2}\|\tilde{f}_2(x) - x^{1/2}\|_{[\theta, 1]} \leq \delta$ for $\theta, \delta \in (0, 1/2]$. If ρ has minimum singular value $\lambda_{\min}$, then it suffices to set $\theta \leq \lambda_{\min}$. Since $1/\lambda_{\min}$ behaves like a "condition number" for ρ, being proportional to the difficulty of transforming ρ, we denote it with the symbol κ and employ this notation later. Indeed, using the functional approximations from [41], QSVT achieves the desired transformations up to the errors indicated above, with $\mathcal{O}\left(\frac{1}{\theta} \log \frac{1}{\delta}\right)$ uses of U^{ρ}.

The quantum algorithm—We implement the isometric extension of the Petz recovery channel given in (9). This consists of applying the maps in (3), (4), and (5) sequentially, with the first and third steps employing QSVT. Eq. (9) also has a measurement component as the final step, arising from the implementation of the map in (4). By exploiting the trace-preserving property of the Petz recovery channel, we amplify the probability of success of this measurement (i.e., the projection onto $|0\rangle_{E'}$) using oblivious amplitude amplification [42], which is a special case of QSVT [37]. Overall, the implementation is precise up to ε error in diamond distance [43] (see [11] for a definition of diamond distance). Theorem 1 below states the guarantees of this technique.

Theorem 1 *Let N_{σ}, $N_{\mathcal{N}(\sigma)}$ and $N_{\mathcal{N}}$ denote the number of elementary quantum gates needed to realize the unitaries U^{σ_A}, $U^{\mathcal{N}(\sigma_A)}$, and $U^{\mathcal{N}}_{E'A \to EB}$, respectively (noting that without loss of generality $N_{\mathcal{N}(\sigma)} \leq N_{\sigma} + 2N_{\mathcal{N}}$). Let κ_{σ} denote an upper bound on the reciprocal of the minimum non-zero eigenvalue of σ_A, and correspondingly, let $\kappa_{\mathcal{N}(\sigma)}$ denote the same for $\mathcal{N}(\sigma_A)$. There exists a quantum algorithm realizing the channel $\tilde{\mathcal{P}}^{\sigma_A, \mathcal{N}}_{B \to A}$, which is an approximate implementation of the ideal Petz re-*

covery channel in (2)*, in the sense that*

$$\left\|\tilde{\mathcal{P}}^{\sigma_A, \mathcal{N}}_{B \to A} - \mathcal{P}^{\sigma_A, \mathcal{N}}_{B \to A}\right\|_{\diamond} \leq \varepsilon, \tag{10}$$

with gate complexity (up to poly-logarithmic factors)

$$\widetilde{\mathcal{O}}\left(\sqrt{d_E \kappa_{\mathcal{N}(\sigma)}}\left(\kappa_{\mathcal{N}(\sigma)} N_{\mathcal{N}(\sigma)} + N_{\mathcal{N}} + N_{\sigma} \min\left(\kappa_{\sigma}, d_E \kappa_{\mathcal{N}(\sigma)}/\varepsilon^2\right)\right)\right). \tag{11}$$

In (11)*, d_E is the dimension of the system E, which is not smaller than the Kraus rank of the channel $\mathcal{N}(\cdot)$.*

In the Supplementary Material [1], we provide a modified algorithm that substitutes the dependence on d_E in (11) with the rank of the state $\widetilde{\mathcal{N}}(\sigma)$, where $\widetilde{\mathcal{N}}$ is a channel complementary to $\mathcal{N}$ [11]. For certain choices of $\mathcal{N}$ and σ, this provides a dramatic reduction in the running time.

We now break the algorithm down into its four steps and analyze each step individually (assuming without loss of generality that $\varepsilon = \mathcal{O}(1)$). We indicate the steps using the numbers **(1)**-**(4)**.

(1) To simulate the first step of the Petz recovery channel, as described by (3), we transform the block-encoding of $\mathcal{N}(\sigma_A)$ to a $\left(2\sqrt{\kappa_{\mathcal{N}(\sigma)}}, \frac{\mathcal{O}(\varepsilon)}{\sqrt{d_E}}\right)$-block-encoding $U^{\tilde{f}_1(\mathcal{N}(\sigma_A))}_{R'B}$ of $[\mathcal{N}(\sigma_A)]^{-1/2}$ using QSVT, which has gate complexity $\mathcal{O}\left(\kappa_{\mathcal{N}(\sigma)} N_{\mathcal{N}(\sigma)} \log \frac{d_E}{\varepsilon}\right)$. Then the following error bound holds

$$\left\|\tilde{f}_1(\mathcal{N}(\sigma_A)) - (\mathcal{N}(\sigma_A))^{-1/2}\right\| \leq \mathcal{O}(\varepsilon)/\sqrt{d_E}, \tag{12}$$

which suffices for our purposes, as shown later.

(2) Let $\tilde{E}$ be a system with dimension equal to that of E. The second step of the algorithm is simply to prepare the maximally entangled state $|\Phi\rangle_{E\tilde{E}} := |\Gamma\rangle_{E\tilde{E}}/\sqrt{d_E}$ alongside the state prepared above, and then apply the unitary $(U^{\mathcal{N}}_{AE' \to BE})^{\dagger}$. Note that $|\Phi\rangle_{E\tilde{E}}$ is a normalized quantum state, introducing an additional factor of $\frac{1}{d_E}$ in the output density operator, which resurfaces in the subnormalization factor of the overall unitary (see (15)). The maximally entangled state $|\Phi\rangle_{E\tilde{E}}$ is prepared by means of a unitary $U^{\Phi}_{E\tilde{E}}$ acting on the state $|0\rangle_{E\tilde{E}}$, so that $|\Phi\rangle_{E\tilde{E}} := U^{\Phi}_{E\tilde{E}}|0\rangle_{E\tilde{E}}$. Note that the unitary $U^{\Phi}_{E\tilde{E}}$ is easy to implement. For example, if systems E and $\tilde{E}$ consist of qubits, one can apply Hadamard gates on the qubits of E and CNOT gates between pairs of qubits of E and $\tilde{E}$. In this step, we have described the first half of the procedure for implementing a linear extension of (4); the final part, which consists of measurement and post-selection, is deferred to the fourth step.

(3) The third step of the algorithm is to apply an approximation of the map in (5) that conjugates the state by $\sigma_A^{1/2}$. Analogous to the first step, we transform the block-encoding of σ_A to a $\left(2, \frac{\mathcal{O}(\varepsilon)}{\sqrt{d_E \kappa_{\mathcal{N}(\sigma)}}}\right)$-block-encoding $U^{\tilde{f}_2(\sigma_A)}_{R''A}$ of $\tilde{f}_2(\sigma_A)$ using QSVT, which has gate complexity $\mathcal{O}\left(\kappa_{\sigma} N_{\sigma} \log\left(\frac{d_E \kappa_{\mathcal{N}(\sigma)}}{\varepsilon}\right)\right)$. Then the following error bound holds

$$\left\|\tilde{f}_2(\sigma_A) - \sigma_A^{1/2}\right\| \leq \mathcal{O}(\varepsilon)/\sqrt{d_E \kappa_{\mathcal{N}(\sigma)}}. \tag{13}$$

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[1] The Supplementary Material also includes Ref. [44].

We can now apply the unitary $U_{R''A}^{\tilde{f}_2(\sigma_A)}$ to the output of Step 2. In detail, letting ρ_A denote the output state of Step 2, we tensor in the state $|0\rangle\langle 0|_{R''}$ to the input state ρ_A and perform the unitary $U_{R''A}^{\tilde{f}_2(\sigma_A)}$.

Let us summarize the algorithm up to this point. We have described the addition of auxiliary systems as happening separately in each step. However, we are free to tensor them in to the input state ω_B at the start, enlarging the input state to $|0\rangle\langle 0|_{R''} \otimes |0\rangle\langle 0|_{E\tilde{E}} \otimes |0\rangle\langle 0|_{R'} \otimes \omega_B$. Then to this state, we apply the following product of unitaries:

$$\tilde{W} := U_{R''A}^{\tilde{f}_2(\sigma_A)}\left(U_{E'A\to EB}^{\mathcal{N}}\right)^{\dagger}\left(U_{E\tilde{E}}^{\Phi} \otimes U_{R'B}^{\tilde{f}_1(\mathcal{N}(\sigma_A))}\right), \quad (14)$$

where $U_{R''A}^{\tilde{f}_2(\sigma_A)}$ and $U_{R'B}^{\tilde{f}_1(\mathcal{N}(\sigma_A))}$ are implemented using QSVT. The unitary $\tilde{W}$ approximates the isometric extension in (9) and can be represented as the following block-encoding:

$$\tilde{W} = \begin{bmatrix} \frac{1}{4}\sqrt{\frac{1}{d_E \kappa_{\mathcal{N}(\sigma)}}}\tilde{V}_{B\to\tilde{E}A}^{\mathcal{P}} & \cdot \\ \cdot & \cdot \end{bmatrix}, \quad (15)$$

where the linear operator $\tilde{V}_{B\to\tilde{E}A}^{\mathcal{P}}$ is an approximate isometric extension of the Petz recovery channel and is defined through its action on a ket $|\psi\rangle_B$ as

$$\tilde{V}_{B\to\tilde{E}A}^{\mathcal{P}}|\psi\rangle_B := \tilde{f}_2(\sigma_A)\left(V_{A\to EB}^{\mathcal{N}}\right)^{\dagger}\tilde{f}_1(\mathcal{N}(\sigma_A))|\Gamma\rangle_{E\tilde{E}}|\psi\rangle_B. \quad (16)$$

After applying $\tilde{W}$ to the enlarged input state, we would like to measure the $R''E'R'$ systems and obtain the all-zeros state as the outcome (which corresponds to the top-left block of $\tilde{W}$). Receiving this outcome signals the successful implementation of the desired map $\tilde{V}_{B\to\tilde{E}A}^{\mathcal{P}}$, up to a sub-normalization factor of $4\sqrt{\kappa_{\mathcal{N}(\sigma)}d_E}$. To compare this to the ideal isometric extension in (9), we should account for the accumulated errors due to the approximate implementations of $\mathcal{N}(\sigma_A)^{-1/2}$ and $\sigma_A^{1/2}$ in $\tilde{W}$. It follows that

$$\left\|\tilde{V}_{B\to\tilde{E}A}^{\mathcal{P}} - V_{B\to\tilde{E}A}^{\mathcal{P}}\right\| \leq \mathcal{O}(\varepsilon), \quad (17)$$

where $\tilde{V}_{B\to\tilde{E}A}^{\mathcal{P}}$ is defined in (16) and $V_{B\to\tilde{E}A}^{\mathcal{P}}$ in (9). To see this, observe that the left-hand side of (17) can be bounded from above by the following quantity:

$$\left\|\sigma_A^{1/2} - \tilde{f}_2(\sigma_A)\right\|\left\|\left(V_{A\to EB}^{\mathcal{N}}\right)^{\dagger}\mathcal{N}(\sigma_A)^{-1/2}|\Gamma\rangle_{E\tilde{E}}\right\| +$$
$$\left\|\tilde{f}_2(\sigma_A)\left(V_{A\to EB}^{\mathcal{N}}\right)^{\dagger}\right\|\left\||\Gamma\rangle_{E\tilde{E}}\right\|\left\|\mathcal{N}(\sigma_A)^{-1/2} - \tilde{f}_1(\mathcal{N}(\sigma_A))\right\|, \quad (18)$$

which follows from applying the triangle inequality and sub-multiplicativity of the spectral norm. Noting that $|\Gamma\rangle_{E\tilde{E}}$ is the unnormalized maximally-entangled vector, we further bound the following terms:

$$\left\|\left(V_{A\to EB}^{\mathcal{N}}\right)^{\dagger}\mathcal{N}(\sigma_A)^{-1/2}|\Gamma\rangle_{E\tilde{E}}\right\| \leq \sqrt{d_E \kappa_{\mathcal{N}(\sigma_A)}}, \quad (19)$$

$$\left\|\tilde{f}_2(\sigma_A)\left(V_{A\to EB}^{\mathcal{N}}\right)^{\dagger}\right\|\left\||\Gamma\rangle_{E\tilde{E}}\right\| \leq 2\sqrt{d_E}. \quad (20)$$

The second bound follows because $\tilde{f}_2(\sigma_A)$ is a block-encoding with norm at most 2. Putting (18)–(20) together with the bounds in (12) and (13), we conclude an overall error between $V^{\mathcal{P}}$ and $\tilde{V}^{\mathcal{P}}$ of $\mathcal{O}(\varepsilon)$.

(4) Finally, we move on to the last step, which is a measurement of the $R''E'R'$ systems. Eq. (15) makes it clear that the probability p_{success} of measuring the all-zeros state, at this point, is approximately $\frac{1}{16d_E \kappa_{\mathcal{N}(\sigma)}}$. We would like to amplify this probability, and so we use oblivious amplitude amplification to implement an approximate projection onto this state. This too can be achieved using QSVT techniques [37] and requires a number of repetitions of $\tilde{W}$ that scales as $\mathcal{O}\left(1/\sqrt{p_{\text{success}}}\right)$, which in this case is $N_{\text{rep}} := \mathcal{O}\left(\sqrt{d_E \kappa_{\mathcal{N}(\sigma)}}\right)$. After applying (robust) oblivious amplitude amplification [45, Theorem 28], we obtain a unitary that is a $(1, \mathcal{O}(\varepsilon))$-block-encoding of the isometric extension $V_{B\to\tilde{E}A}^{\mathcal{P}}$, providing an $\mathcal{O}(\varepsilon)$-approximate implementation of the Petz recovery channel.

The complexity of our algorithm is given by N_{rep} times the complexity of implementing $\tilde{W}$. As we discussed previously, the cost of implementing the first step in $\tilde{W}$ is $\mathcal{O}\left(\kappa_{\mathcal{N}(\sigma)}N_{\mathcal{N}(\sigma)}\log\frac{d_E}{\varepsilon}\right)$. The complexity of implementing the second step is $\mathcal{O}(N_{\mathcal{N}} + \log d_E)$, where the logarithmic term is the cost of implementing $U_{E\tilde{E}}^{\Phi}$. Finally, the complexity of the third step is $\mathcal{O}\left(\kappa_{\sigma}N_{\sigma}\log\frac{d_E \kappa_{\mathcal{N}(\sigma)}}{\varepsilon}\right)$. An alternative for this last step is to consider choosing a threshold θ higher than $1/\kappa_{\sigma}$, and approximating the square root function by constant zero below the threshold. Indeed, then choosing $\theta \approx \varepsilon^2/(d_E \kappa_{\mathcal{N}(\sigma)})$ suffices, resulting in the alternative complexity $\mathcal{O}\left(\frac{d_E \kappa_{\mathcal{N}(\sigma)}}{\varepsilon^2}N_{\sigma}\log\frac{d_E \kappa_{\mathcal{N}(\sigma)}}{\varepsilon}\right)$ of the third step.

Lower bounds—Our algorithm uses the forward channel unitary $U_{E'A\to EB}^{\mathcal{N}}$ about $\mathcal{O}\left(\sqrt{d_E \kappa_{\mathcal{N}(\sigma)}}\right)$ times. We now prove that there is no generally applicable algorithm that uses $U_{E'A\to EB}^{\mathcal{N}}$ fewer than $\Omega\left(d_E^{\frac{1}{2}-\alpha}\kappa_{\mathcal{N}(\sigma)}^{\alpha}\right)$ times, for all $\alpha \in [0, \frac{1}{2}]$, thereby ruling out the possibility of large improvements on our algorithm that would simultaneously improve the dependence on both parameters d_E and $\kappa_{\mathcal{N}(\sigma)}$.

We consider solving the problem of unstructured search of $N \geq 2$ elements with only a single marked element. Let O be a search oracle that recognizes the single marked element. Let the input state σ_A be the maximally mixed state representing a uniformly random index $i \in [N]$. The forward channel $\mathcal{N}_{A\to B}$ applies the search oracle and outputs its output, which is equal to 1 if i is the marked element and is equal to 0 otherwise. Hence $\mathcal{N}_{A\to B}(\sigma_A) = \text{diag}(1 - \frac{1}{N}, \frac{1}{N})$ and $\kappa_{\mathcal{N}(\sigma)} = d_E = N$. Let $\mathcal{P}^{\mathcal{N},\sigma_A}$ be the Petz recovery channel defined from $\mathcal{N}$ and σ_A as specified above. Now applying the exact channel $\mathcal{P}^{\mathcal{N},\sigma_A}$ on the state $\omega_B = |1\rangle\langle 1|$ finds the marked element with certainty. Thus, for every constant $c < 1$, applying a c-approximate channel $\tilde{\mathcal{P}}^{\mathcal{N},\sigma_A}$ on ω_B still finds a marked element with probability at least $1 - c$. This requires $\Omega(\sqrt{N}) = \Omega\left(d_E^{\frac{1}{2}-\alpha}\kappa_{\mathcal{N}(\sigma)}^{\alpha}\right)$ uses of O, as the well known quantum search lower bound states [46].

Pretty good measurement—One can use our algorithm to implement the *pretty good measurement* [1–5], which is a special case of the Petz map. In this application, one is given

a set $\{\sigma_B^x\}_x$ of states and a probability distribution p_X. Let σ_{XB} denote the following classical–quantum state: $\sigma_{XB} := \sum_x p_X(x)|x\rangle\langle x|_X \otimes \sigma_B^x$. Let $\mathcal{N}_{XB\to B} := \mathrm{Tr}_X$ be the partial trace channel that discards system X.

We now plug these choices into (2). The adjoint map $(\mathcal{N}_{XB\to B})^\dagger$ appends the identity on system X. Let $\overline{\sigma}_B := \mathcal{N}_{XB\to B}(\sigma_{XB}) = \sum_x p_X(x)\sigma_B^x$. The resulting Petz recovery channel is as follows:

$$\mathcal{P}_{B\to XB}^{\sigma_{XB},\mathrm{Tr}_X}(\omega_B) := \sum_x [x]_X \otimes p_X(x)(\sigma_B^x)^{\frac{1}{2}}(\overline{\sigma}_B)^{-\frac{1}{2}}\omega_B(\overline{\sigma}_B)^{-\frac{1}{2}}(\sigma_B^x)^{\frac{1}{2}},$$

which is known as the "pretty good instrument" [23] and where $[x] \equiv |x\rangle\langle x|$. This is a generalization of the pretty good measurement that has a quantum output in addition to the usual classical measurement output; the PGM is obtained by discarding the quantum output.

We check the necessary assumptions for our technique against what is potentially available for experiments. The isometric extension of the channel $\mathrm{Tr}_X(\cdot)$ is simply the identity. If we have copies of σ_{XB} then our algorithm is applicable, but it is more efficient in the case when we can prepare a purification of σ_{XB}. Applying Theorem 1, we arrive at a quantum algorithm implementing the pretty good instrument with per-formance guarantees as in (10) and (11), where

$$d_E = |X|, \quad \kappa_{\mathcal{N}(\sigma)} = \kappa_{\overline{\sigma}}, \quad \kappa_\sigma = \min_x p_X(x)\kappa_{\sigma_B^x}. \quad (21)$$

Conclusion—We have developed a quantum algorithm for implementing the Petz recovery channel and the pretty good measurement. This solves an important open problem in quantum computation, and more generally, it opens up a new research paradigm for realizing fully quantum Bayesian inference on quantum computers.

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SUPPLEMENTARY MATERIAL

I. INTRODUCTION

In this supplementary material, we show how the running time of the algorithm proposed in the main text can be improved substantially in some cases. That is, we remove the dependence of the algorithm on the dimension d_E of the environment system E, and we replace this parameter with the dimension of the projection $\Pi_E^{\widetilde{\mathcal{N}}(\sigma)}$ onto the support of the state $\widetilde{\mathcal{N}}_{A\rightarrow E}(\sigma_A)$, where $\widetilde{\mathcal{N}}_{A\rightarrow E}$ is the channel complementary to the original channel $\mathcal{N}_{A\rightarrow B}$ and defined from the unitary extension $U_{E'A\rightarrow EB}^{\mathcal{N}}$. In particular, the complementary channel is defined by

$$\widetilde{\mathcal{N}}_{A\rightarrow E}(\omega_A) = \mathrm{Tr}_B[U_{E'A\rightarrow EB}^{\mathcal{N}}(|0\rangle\langle0|_{E'}\otimes\omega_A)(U_{E'A\rightarrow EB}^{\mathcal{N}})^{\dagger}], \quad (22)$$

for every input state ω_A.

For some examples of $U_{E'A\rightarrow EB}^{\mathcal{N}}$ and σ_A, the dimension of the subspace onto which $\Pi_E^{\widetilde{\mathcal{N}}(\sigma)}$ projects is substantially smaller than d_E, i.e.,

$$\mathrm{Tr}[\Pi_E^{\widetilde{\mathcal{N}}(\sigma)}] \ll d_E, \quad (23)$$

and so this replacement can lead to a substantial improvement of the algorithm's running time in some cases. For example, suppose that $\sigma = |0\rangle\langle0|$ and $\mathcal{N}$ is the qudit erasure channel, defined as

$$\rho \rightarrow (1-p)\rho + p|e\rangle\langle e|, \quad (24)$$

where $p \in [0,1]$ is the erasure probability and $|e\rangle\langle e|$ is an erasure symbol (i.e., a rank-one density operator orthogonal to every d-dimensional input state ρ). The complementary channel in this case is given by[2]

$$\rho \rightarrow p\rho + (1-p)|e\rangle\langle e|, \quad (25)$$

and we thus find that $\widetilde{\mathcal{N}}(\sigma) = p|0\rangle\langle0| + (1-p)|e\rangle\langle e|$. This density operator has rank two, which is substantially smaller than $d_E = d+1$, with d_E being the dimension of the environment, i.e., the parameter included in the running time of the quantum algorithm given in the main text.

Notation. Let us remind readers that, as in the main text, we define $|\Gamma\rangle_{E\tilde{E}} := \sum_{i=1}^{d_E}|i\rangle_E|i\rangle_{\tilde{E}}$ which is not a true quantum state vector but is instead only proportional to one. Later we introduce a subspace F of E and use the notation $|\Gamma\rangle_{F\tilde{F}}$ to denote the maximally entangled vector $\sum_{i=1}^{d_F}|i\rangle_F|i\rangle_{\tilde{F}}$, where $\tilde{F} \subseteq \tilde{E}$ is some (sub)system isomorphic to F. We also employ the following definition:

$$\tilde{\mathcal{O}}(f(a,b,c)) := \mathcal{O}(f(a,b,c) \cdot \mathrm{polylog}(f(a,b,c))). \quad (26)$$

---

[2] An isometry for the erasure channel can be implemented as follows: first embed the input $\rho \in \mathbb{C}^{d\times d}$ into $\mathbb{C}^{(d+1)\times(d+1)}$ by extending the input space $\mathbb{C}^d$ with $|e\rangle$ resulting in $\tilde{\rho} \in \mathbb{C}^{(d+1)\times(d+1)}$, and append the state $|e\rangle\langle e|$ to it, resulting in $\tilde{\rho}\otimes|e\rangle\langle e|$. Finally, apply the partial SWAP operator $\sum_{j\in[d+1]}|jj\rangle\langle jj| + \sum_{j<k\in[d+1]}\sqrt{1-p}(|jk\rangle\langle jk|+|kj\rangle\langle kj|) + i\sqrt{p}(|jk\rangle\langle kj|+|kj\rangle\langle jk|)$ to $\tilde{\rho}\otimes|e\rangle\langle e|$.

II. OVERVIEW OF THE MODIFIED ALGORITHM

Recall from the main text that our algorithm implements an approximation of the following isometric extension of the Petz recovery channel:

$$V_{B\rightarrow\tilde{E}A}^{\mathcal{P}} := (\langle0|_{E'}\otimes I_{\tilde{E}}\otimes\sigma_A^{\frac{1}{2}})(U_{E'A\rightarrow EB}^{\mathcal{N}})^{\dagger}(|\Gamma\rangle_{E\tilde{E}}\otimes[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}). \quad (27)$$

Writing the isometric extension in this way allowed us to break down its implementation into the following basic steps:

1. Apply an approximation of $[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}$ to the system B using QSVT.

2. Append a maximally entangled state $|\Phi\rangle_{E\tilde{E}}$ proportional to $|\Gamma\rangle\langle\Gamma|_{E\tilde{E}}$. (Note that the reduced density operator of $|\Phi\rangle_{E\tilde{E}}$ is proportional to $I_E = \Pi_E$.)

3. Apply $(U_{E'A\rightarrow EB}^{\mathcal{N}})^{\dagger}$ to systems EB.

4. Apply an approximation of $\sigma_A^{\frac{1}{2}}$ to the system A using QSVT.

5. The above steps need to be repeated a number of times (detailed in the main text), by performing (robust) oblivious amplitude amplification, in order to boost the success probability of projecting onto $|0\rangle\langle0|_{E'}$ such that it is nearly equal to one.

This implementation brings the dimension factor d_E into the running time due to the usage of the state $|\Phi\rangle_{E\tilde{E}}$, and it is our goal to remove this factor. In order to do so, we modify Step 2 of the algorithm: instead of appending a state $|\Phi\rangle_{E\tilde{E}}$ whose reduced density operator on the system E is proportional to $\Pi_E = I_E$, we append a state $|\widetilde{\Phi}\rangle_{E\tilde{E}G}$, such that the reduced density operator of $(I_{E\tilde{E}}\otimes\langle0|_G|\widetilde{\Phi}\rangle)_{E\tilde{E}G}$ on the system E is proportional to $\Pi_E^{\widetilde{\mathcal{N}}(\sigma)}$, where $\Pi_E^{\widetilde{\mathcal{N}}(\sigma)}$ is a projector onto the subspace $F := \mathrm{supp}(\widetilde{\mathcal{N}}(\sigma)) \subseteq E$ having dimension d_F.

On a high level, the modified algorithm applies an approximation to the following map (cf., Eq. (27))

$$X_{B\rightarrow\tilde{E}A}^{\mathcal{P}} := (\langle0|_{E'}\otimes I_{\tilde{E}}\otimes\sigma_A^{\frac{1}{2}})(U_{E'A\rightarrow EB}^{\mathcal{N}})^{\dagger}(|\Gamma\rangle_{F\tilde{F}}\otimes[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}). \quad (28)$$

Now we show that $X_{B\rightarrow\tilde{E}A}^{\mathcal{P}} = V_{B\rightarrow\tilde{E}A}^{\mathcal{P}}$, proving the correctness of the modified algorithm. Indeed, without loss of generality we can assume $|\Gamma\rangle_{F\tilde{F}} = (\Pi_E^{\widetilde{\mathcal{N}}(\sigma)}\otimes I_{\tilde{E}})|\Gamma\rangle_{E\tilde{E}}$, and so we have that

$$V_{B\rightarrow\tilde{E}A}^{\mathcal{P}} - X_{B\rightarrow\tilde{E}A}^{\mathcal{P}}$$

$$= (\langle0|_{E'}\otimes I_{\tilde{E}}\otimes\sigma_A^{\frac{1}{2}})(U_{E'A\rightarrow EB}^{\mathcal{N}})^{\dagger}\times$$

$$\qquad (((I_E - \Pi_E^{\widetilde{\mathcal{N}}(\sigma)})\otimes I_{\tilde{E}})|\Gamma\rangle_{E\tilde{E}}\otimes[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}) \quad (29)$$

$$= (\underbrace{(\langle0|_{E'}\otimes\sigma_A^{\frac{1}{2}})(U_{E'A\rightarrow EB}^{\mathcal{N}})^{\dagger}(I_E - \Pi_E^{\widetilde{\mathcal{N}}(\sigma)})}_{M}\otimes I_{\tilde{E}})\times$$

$$\qquad (|\Gamma\rangle_{E\tilde{E}}\otimes[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}). \quad (30)$$

We conclude by showing that in fact $M = 0$ by arguing that $\text{Tr}[M^\dagger M] = 0$. To this end, observe that

$$\text{Tr}_B[M^\dagger M] = (I_E - \Pi_E^{\widetilde{\mathcal{N}}(\sigma)})\widetilde{\mathcal{N}}(\sigma_A)(I_E - \Pi_E^{\widetilde{\mathcal{N}}(\sigma)}) = 0, \quad (31)$$

since the image of $\Pi_E^{\widetilde{\mathcal{N}}(\sigma)}$ is equal to the support of $\widetilde{\mathcal{N}}(\sigma_A)$, by definition.

III. DETAILED ANALYSIS OF THE MODIFIED ALGORITHM

We now state the modified algorithm more rigorously. To apply the Petz recovery channel $\mathcal{P}^{\mathcal{N},\sigma_A}$ to an input state ω_B, consider the steps given in Algorithm 1 below.

Algorithm 1 Petz recovery channel associated with the channel $\mathcal{N}_{A \to B}$ and the state σ_A.

Input: $U^{\mathcal{N}}_{AE' \to BE}$; U^σ_{RA}, a unitary that prepares a purification of the state σ_A; $U^{\mathcal{N}(\sigma_A)}$, an approximate block-encoding of $\mathcal{N}(\sigma_A)$.

Classical pre-processing:

1: Compute the circuit $U^{\tilde{f}_1(\mathcal{N}(\sigma_A))}_{R'B}$ that enacts a $\left(2\sqrt{\kappa_{\mathcal{N}(\sigma)}}, \frac{\tilde{\mathcal{O}}(\varepsilon)}{\sqrt{d_F}}\right)$-block-encoding of $[\mathcal{N}(\sigma_A)]^{-1/2}$, by applying QSVT to a block-encoding of $\mathcal{N}(\sigma_A)$. The relevant block $\tilde{f}_1(\mathcal{N}(\sigma_A))$ satisfies

$$\left\| \mathcal{N}(\sigma_A)^{-\frac{1}{2}} - \tilde{f}_1(\mathcal{N}(\sigma_A)) \right\|_\infty \leq \frac{\tilde{\mathcal{O}}(\varepsilon)}{\sqrt{d_F}}. \quad (32)$$

2: Using Lemma 4 below, compute the circuit $U^\Pi_{E\tilde{E}G}$ that prepares a subnormalized state $|\widetilde{\Phi}\rangle_{E\tilde{E}G}$, which approximately purifies $\Pi_E^{\widetilde{\mathcal{N}}(\sigma)}|\Gamma\rangle_{E\tilde{E}}$ up to a prefactor of $2\sqrt{\kappa_{\widetilde{\mathcal{N}}(\sigma)}}$. That is,

$$\left\| \Pi_E^{\widetilde{\mathcal{N}}(\sigma)}|\Gamma\rangle_{E\tilde{E}} - 2\sqrt{\kappa_{\widetilde{\mathcal{N}}(\sigma)}}(\langle 0|_G \otimes I_{E\tilde{E}})|\widetilde{\Phi}\rangle_{E\tilde{E}G} \right\|_2 \leq \tilde{\mathcal{O}}(\varepsilon). \quad (33)$$

3: Compute the circuit $U^{\tilde{f}_2(\sigma_A)}_{R''A}$ that enacts a $\left(2, \frac{\tilde{\mathcal{O}}(\varepsilon)}{\sqrt{d_F \kappa_{\mathcal{N}(\sigma)}}}\right)$-block-encoding of $\sigma_A^{-\frac{1}{2}}$, by applying QSVT to a block-encoding of σ_A. The relevant block $\tilde{f}_2(\sigma_A)$ satisfies

$$\left\| \sigma_A^{1/2} - \tilde{f}_2(\sigma_A) \right\|_\infty \leq \frac{\tilde{\mathcal{O}}(\varepsilon)}{\sqrt{d_F \kappa_{\mathcal{N}(\sigma)}}}. \quad (34)$$

Quantum steps:

4: Perform $O(\sqrt{\kappa_{\mathcal{N}(\sigma)} \cdot \kappa_{\widetilde{\mathcal{N}}(\sigma)}})$ rounds of oblivious amplitude amplification on the unitary formed by composing the unitaries prepared in the first three steps

$$U^{\tilde{f}_2(\sigma_A)}_{R''A}\left(U^{\mathcal{N}}_{AE' \to BE}\right)^\dagger\left(U^\Pi_{E\tilde{E}G} \otimes U^{\tilde{f}_1(\mathcal{N}(\sigma_A))}_{R'B}\right) \quad (35)$$

and act with the resultant unitary on the input state of system B.

Result: The above map acts on an arbitrary input state as an approximation to (27), which is an isometric extension of the Petz recovery channel $\mathcal{P}^{\sigma,\mathcal{N}}_{B \to A}$.

A. Complexity

1. Implementing Step 2

We now explain how to prepare the state $|\widetilde{\Phi}\rangle_{E\tilde{E}G}$ that features in our new Step 2. We require the following definitions and adaptation of a lemma from [41, 44]:

Definition 2 (Subnormalized density operator) *A subnormalized density operator ϱ is a positive semi-definite matrix of trace at most one (i.e., $\text{Tr}[\varrho] \leq 1$). In particular, it need not be a valid density operator, which has trace equal to one.*

Definition 3 (Purification) *A* purification *of a subnormalized density operator ϱ is a pure state consisting of three registers, such that tracing out the third register and projecting onto the subspace for which the second register is $|0\rangle$ yields ϱ.*

Lemma 4 (Implementing a purification of a projector)
Let $q \in (0,1)$, Π be a projector, and ϱ a subnormalized density operator acting on the system E. Suppose that $q\Pi \preceq \varrho$ and $(I - \Pi)\varrho(I - \Pi) = 0$, and suppose that we have access to an a-qubit unitary U_ϱ preparing a purification of a subnormalized density operator ϱ. Then we can a prepare a pure state $|\widetilde{\Phi}\rangle_{E\tilde{E}G}$ satisfying

$$\left\| (\langle 0|_G \otimes I_{E\tilde{E}})|\widetilde{\Phi}\rangle_{E\tilde{E}G} - \frac{\sqrt{q}}{2}(\Pi_E \otimes U_{\varphi,\tilde{E}})|\Gamma\rangle_{E\tilde{E}} \right\|_2 \leq \varepsilon, \quad (36)$$

where $U_{\varphi,\tilde{E}}$ is a unitary acting on the ancillary system $\tilde{E}$ (which is traced over in our applications). We can do so with $\mathcal{O}((1/q)\,\text{polylog}(1/q, 1/\varepsilon))$ queries to $U_{\tilde{\varrho}}$ and its inverse.

Proof. This lemma follows from Lemma 6.4.11 of [41]. From there, we conclude that it is possible to implement a unitary W_{EG} that is a $(1, \tilde{\mathcal{O}}_{\nu,q}(a), 0)$-block-encoding of an operator V_E satisfying

$$\left\| \left(V - \frac{\sqrt{q}}{2\sqrt{\varrho}} \right)\Pi \right\|_\infty \leq \nu \quad (37)$$

with $\mathcal{O}((1/q)\,\text{polylog}(1/q, 1/\varepsilon))$ queries to U_ϱ and its inverse. By the assumptions of the lemma (i.e., $q\Pi \preceq \varrho$ and $(I-\Pi)\varrho(I-\Pi) = 0$), the operator ϱ_E is supported on the image of Π_E, which implies that $\Pi\varrho\Pi = \varrho$ and $[\Pi, \varrho] = 0$.

We now apply the inequality in (37) to arrive at the claim in (36). Let $|\psi_\varrho\rangle_{E\tilde{E}} := (\sqrt{\varrho}_E \otimes I_{\tilde{E}})|\Gamma\rangle_{E\tilde{E}}$ denote the canonical purification of ϱ_E. Then, omitting system labels for conciseness, we find that

$$\left\| \left(V - \frac{\sqrt{q}}{2\sqrt{\varrho}} \right)\Pi \right\|_\infty \leq \nu \quad (38)$$

$$\Rightarrow \quad \left\| V\Pi \otimes I - \frac{\sqrt{q}}{2\sqrt{\varrho}}\Pi \otimes I \right\|_\infty \leq \nu \quad (39)$$

$$\Rightarrow \quad \left\| V \otimes I|\psi_\varrho\rangle - \frac{\sqrt{q}}{2\sqrt{\varrho}}\Pi \otimes I|\psi_\varrho\rangle \right\|_2 \leq \nu \quad (40)$$

$$\Rightarrow \quad \left\| V \otimes I|\psi_\varrho\rangle - \frac{\sqrt{q}}{2}\Pi \otimes I|\Gamma\rangle \right\|_2 \leq \nu. \quad (41)$$

The above steps follow because $\|A\|_\infty = \|A \otimes I\|_\infty$, from the definition of the spectral norm, and from the assumption that Π commutes with ϱ.

Now let $|\varphi_\varrho\rangle$ be the actual purification of ϱ prepared by the unitary U_ϱ, which is related to the canonical purification $|\psi_\varrho\rangle$ by a unitary U_φ acting on the reference system:

$$|\varphi_\varrho\rangle = I \otimes U_\varphi |\psi_\varrho\rangle. \tag{42}$$

This follows by the well known fact that every two purifications of a density operator are related to each other by a unitary acting on the purifying system [11]. Continuing by multiplying each term in (41) by $I \otimes U_\varphi$ (due to unitary invariance of the norm), we conclude that

$$\left\| V \otimes I |\varphi_\varrho\rangle - \frac{\sqrt{q}}{2}\Pi \otimes U_\varphi |\Gamma\rangle \right\|_2 \le \nu. \tag{43}$$

Thus, we can prepare the unnormalized state $(V_E \otimes I_{\tilde E})|\varphi_\varrho\rangle_{E\tilde E}$ by acting on $|\varphi_\varrho\rangle_{E\tilde E} \otimes |0\rangle_G$ with W_{EG} and post-selecting on the $|0\rangle_G$ subspace. Therefore the first term in (43) is equal to $(\langle 0|_G \otimes I_{E\tilde E})|\widetilde\Phi\rangle_{E\tilde E G}$. The second term in (43) is a purification of the desired projector, which follows because

$$\mathrm{Tr}_{\tilde E}[(\Pi_E \otimes U_{\varphi,\tilde E})|\Gamma\rangle\langle\Gamma|_{E\tilde E}(\Pi_E \otimes (U_{\varphi,\tilde E})^\dagger)] = \Pi_E. \tag{44}$$

(We note that Π projects onto some subspace of the system E, not necessarily the whole system.) Thus, (43) implies that we can prepare a state $|\widetilde\Phi\rangle_{E\tilde E G}$ which, after postselection on $|0\rangle_G$, well-approximates the desired purification $(\Pi \otimes I)|\Gamma\rangle$ up to a scale factor and a unitary $U_{\varphi,\tilde E}$ acting on a reference system. Furthermore, the unitary

$$U^\Pi := W_{EG} U_{\varrho,E\tilde E}, \tag{45}$$

acting on the initial state $|0\rangle_{E\tilde E G}$, prepares this state, where $U_{\varrho,E\tilde E}$ is the unitary defined in the lemma statement. ∎

We now explain how to use Lemma 4 to implement Step 2. First, we need to slightly modify our access assumptions: instead of our previous assumption of access to the block-encoding U^{σ_A} directly, we assume we have access to a unitary U_{RA}^σ that prepares a purification of the state σ_A. (As mentioned in the main text, with two uses of U_{RA}^σ, we can prepare U^{σ_A}.) The unitary U_{RA}^σ takes the role of the unitary U_ϱ in Lemma 4. Accordingly, we keep track of the number of uses of U_{RA}^σ and denote its gate complexity as N'_σ.[3]

Now, it is easy to see that the unitary

$$U_{\tilde\rho} = U_{A\to BE}^{\mathcal N} \circ U_{RA}^\sigma \tag{46}$$

acting on $|0\rangle\langle 0|$ prepares a state $|\psi\rangle\langle\psi|_{RBE}$ satisfying

$$\mathrm{Tr}_{RB}(|\psi\rangle\langle\psi|_{RBE}) = \widetilde{\mathcal N}_{A\to E}(\sigma_A) =: \varrho_E. \tag{47}$$

Thus, in Lemma 4, we set

$$\varrho \leftarrow \varrho_E. \tag{48}$$

Our goal is to implement a unitary that prepares a block-encoded projector onto $\mathrm{supp}(\varrho_E)$, and so we also set

$$\Pi \leftarrow \Pi_E^{\widetilde{\mathcal N}(\sigma)}. \tag{49}$$

That is, considering (43), the state we can prepare is close to a purification of the projector $\Pi_E^{\widetilde{\mathcal N}(\sigma)}$. Also, by the condition of Lemma 4 that $q\Pi \le \varrho$, we see that it also suffices to set

$$q \leftarrow \lambda_{\min}(\widetilde{\mathcal N}(\sigma)). \tag{50}$$

Putting together Lemma 4 with (46), we see that with a gate complexity of

$$\mathcal O\big(\kappa_{\widetilde{\mathcal N}(\sigma)}(N_{\mathcal N} + N'_\sigma)\mathrm{polylog}(\kappa_{\widetilde{\mathcal N}(\sigma)}, 1/\varepsilon)\big), \tag{51}$$

we can prepare the desired state $|\widetilde\Phi\rangle_{E\tilde E G}$.

2. Detailed Complexity Analysis

We now analyze the complexity of the entire modified algorithm:

- Step 1 has gate complexity

$$\mathcal O(\kappa_{\mathcal N(\sigma)} N_{\mathcal N(\sigma)}\mathrm{polylog}(d_F, \kappa_{\mathcal N(\sigma)}, 1/\varepsilon)).$$

- Step 2 has gate complexity

$$\mathcal O\big(\kappa_{\widetilde{\mathcal N}(\sigma)}\mathrm{polylog}\big(\kappa_{\widetilde{\mathcal N}(\sigma)}, 1/\varepsilon\big)(N_{\mathcal N} + N'_\sigma)\big).$$

The gate complexity to implement $\left(U_{AE'\to BE}^{\mathcal N}\right)^\dagger$ is, by definition, $N_{\mathcal N}$.

- Step 3 has gate complexity

$$\mathcal O(\kappa_\sigma N'_\sigma\mathrm{polylog}(d_F, \kappa_{\mathcal N(\sigma)}, 1/\varepsilon)).$$

Finally, for the actual quantum computational implementation, oblivious amplitude amplification multiplies the overall gate complexity by the number of rounds of amplification $\mathcal O\big(\sqrt{\kappa_{\mathcal N(\sigma)}\kappa_{\widetilde{\mathcal N}(\sigma)}}\big)$ (which is just the product of the accumulated subnormalization factors). The overall gate complexity is thus

$$\widetilde{\mathcal O}\Big(\sqrt{\kappa_{\mathcal N(\sigma)}\kappa_{\widetilde{\mathcal N}(\sigma)}}\big(\kappa_{\mathcal N(\sigma)}N_{\mathcal N(\sigma)} + (\kappa_\sigma + \kappa_{\widetilde{\mathcal N}(\sigma)})N'_\sigma + \kappa_{\widetilde{\mathcal N}(\sigma)}N_{\mathcal N}\big)\Big). \tag{52}$$

where the symbol $\sim$ hides $\mathrm{polylog}(d_F, \kappa_{\widetilde{\mathcal N}(\sigma)}, \kappa_{\mathcal N(\sigma)}, 1/\varepsilon)$ factors, as indicated in (26).

The error analysis proceeds similarly to that in the main text. The overall unitary resulting from Steps 1, 2, and 3 is as follows:

$$U_{R''A}^{\tilde f_2(\sigma_A)}(U_{E'A\to EB}^{\mathcal N})^\dagger U_{F\tilde F G}^{\overline\Pi} \otimes U_{R'B}^{\tilde f_1(\mathcal N(\sigma_A))} =$$

---

[3] Here the prime $'$ reflects that it is closely related to, and at most twice as large as, N_σ.

$$\left[\begin{array}{cc} \frac{1}{4\sqrt{\kappa_{\mathcal{N}(\sigma)}}\cdot 2\sqrt{\kappa_{\widetilde{\mathcal{N}}(\sigma)}}} \tilde{X}^{\mathcal{P}}_{B\to\tilde{E}A} & \cdot \\ & \cdot \\ \cdot & \end{array} \right] \qquad (53)$$

where the relevant block is the $|0\rangle_{E'G}$ subspace and

$$\tilde{X}^{\mathcal{P}}_{B\to\tilde{E}A} := \tilde{f}_2(\sigma_A)\big(U^{\mathcal{N}}_{E'A\to EB}\big)^\dagger \tilde{f}_1(\mathcal{N}(\sigma_A))2\sqrt{\kappa_{\widetilde{\mathcal{N}}(\sigma)}}\times$$
$$(\langle 0|_G \otimes I_{E\tilde{E}})|\widetilde{\Phi}\rangle_{E\tilde{E}G} \otimes I_B \quad (54)$$

(cf., Eqs. (15) and (16) in the main text). The oblivious amplitude amplification in Step 4 boosts the probability of success,

which is currently $\tilde{\mathcal{O}}\big(\sqrt{\kappa_{\mathcal{N}(\sigma)}}\sqrt{\kappa_{\widetilde{\mathcal{N}}(\sigma)}}\big)$, to ≈ 1, thus implementing the map $\tilde{X}^{\mathcal{P}}_{B\to\tilde{E}A}$ with the boosted probability. However, this is an approximation of the ideal map (cf., Eq. (27)) given by

$$X^{\mathcal{P}}_{B\to\tilde{E}A} = \sigma_A^{\frac{1}{2}}(U^{\mathcal{N}}_{E'A\to EB})^\dagger[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}|\Gamma\rangle_{F\tilde{F}} \otimes I_B. \quad (55)$$

(Note that any map which differs from (55) by a unitary on the $\tilde{F}$ system works just as well; hence, without loss of generality, we have set this unitary to the identity.)

We now compute the ∞-norm distance between these two operators, and we omit the identity on system B to save space:

$$\left\| X^{\mathcal{P}}_{B\to\tilde{E}A} - \tilde{X}^{\mathcal{P}}_{B\to\tilde{E}A} \right\|_\infty$$
$$= \left\| \sigma_A^{\frac{1}{2}}(U^{\mathcal{N}}_{E'A\to EB})^\dagger[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}|\Gamma\rangle_{F\tilde{F}} - \tilde{f}_2(\sigma_A)\big(U^{\mathcal{N}}_{E'A\to EB}\big)^\dagger \tilde{f}_1(\mathcal{N}(\sigma_A))2\sqrt{\kappa_{\widetilde{\mathcal{N}}(\sigma)}}(\langle 0|_G \otimes I_{E\tilde{E}})|\widetilde{\Phi}\rangle_{F\tilde{F}G} \right\|_\infty \quad (56)$$
$$\leq \left\| \sigma_A^{1/2} - \tilde{f}_2(\sigma_A) \right\|_\infty \left\| (U^{\mathcal{N}}_{E'A\to EB})^\dagger[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}(|\Gamma\rangle_{F\tilde{F}}) \right\|_2$$
$$+ \left\| \tilde{f}_2(\sigma_A)\big(U^{\mathcal{N}}_{E'A\to EB}\big)^\dagger \right\|_\infty \left\| [\mathcal{N}(\sigma_A)]^{-\frac{1}{2}} - \tilde{f}_1(\mathcal{N}(\sigma_A)) \right\|_\infty \left\| |\Gamma\rangle_{F\tilde{F}} \right\|_2$$
$$+ \left\| \tilde{f}_2(\sigma_A)\big(U^{\mathcal{N}}_{E'A\to EB}\big)^\dagger \tilde{f}_1(\mathcal{N}(\sigma_A)) \right\|_\infty \left\| |\Gamma\rangle_{F\tilde{F}} - 2\sqrt{\kappa_{\widetilde{\mathcal{N}}(\sigma)}}(\langle 0|_G \otimes I_{E\tilde{E}})|\widetilde{\Phi}\rangle_{F\tilde{F}G} \right\|_2 \quad (57)$$

Then, noting that

$$\left\| (U^{\mathcal{N}}_{E'A\to EB})^\dagger[\mathcal{N}(\sigma_A)]^{-\frac{1}{2}}(|\Gamma\rangle_{F\tilde{F}}) \right\|_2 \leq \sqrt{d_F \kappa_{\mathcal{N}(\sigma)}}, \quad (58)$$

$$\left\| \tilde{f}_2(\sigma_A)\big(U^{\mathcal{N}}_{E'A\to EB}\big)^\dagger \right\|_\infty \left\| |\Gamma\rangle_{F\tilde{F}} \right\|_2 \leq 2\sqrt{d_F}, \quad (59)$$

$$\left\| \tilde{f}_2(\sigma_A)\big(U^{\mathcal{N}}_{E'A\to EB}\big)^\dagger \tilde{f}_1(\mathcal{N}(\sigma_A)) \right\|_\infty \leq 4, \quad (60)$$

we see that if we choose

$$\left\| \mathcal{N}(\sigma_A)^{-\frac{1}{2}} - \tilde{f}_1(\mathcal{N}(\sigma_A)) \right\|_\infty \leq \frac{\tilde{\mathcal{O}}(\varepsilon)}{\sqrt{d_F}} \quad \text{(Step 1 error),} \quad (61)$$

$$\left\| |\Gamma\rangle_{F\tilde{F}} - 2\sqrt{\kappa_{\widetilde{\mathcal{N}}(\sigma)}}(\langle 0|_G \otimes I_{E\tilde{E}})|\widetilde{\Phi}\rangle_{F\tilde{F}G} \right\|_2 \leq \tilde{\mathcal{O}}(\varepsilon)$$

$$\text{(Step 2 error),} \quad (62)$$

$$\left\| \sigma_A^{1/2} - \tilde{f}_2(\sigma_A) \right\|_\infty \leq \frac{\tilde{\mathcal{O}}(\varepsilon)}{\sqrt{d_F \kappa_{\mathcal{N}(\sigma)}}} \quad \text{(Step 3 error),} \quad (63)$$

then we obtain a final approximation error of

$$\left\| X^{\mathcal{P}}_{B\to\tilde{E}A} - \tilde{X}^{\mathcal{P}}_{B\to\tilde{E}A} \right\|_\infty \leq \tilde{\mathcal{O}}(\varepsilon), \quad (64)$$

which means the actual error in the relevant block of Eq. (53) is $\frac{\varepsilon}{\sqrt{\kappa_{\mathcal{N}(\sigma)}\kappa_{\widetilde{\mathcal{N}}(\sigma)}}}$. The $\sqrt{\kappa_{\mathcal{N}(\sigma)}\kappa_{\widetilde{\mathcal{N}}(\sigma)}}$ rounds of oblivious amplitude amplification then magnify the overall error to $\tilde{\mathcal{O}}(\varepsilon)$.