Jonathan Weisberg

Odds & Ends

Introducing Probability & Decision with a Visual Emphasis

V0.1 BETA

An Open Access Publication

Contents

Preface

This textbook is for introductory philosophy courses on probability and inductive logic. It is based on a typical such course I teach at the University of Toronto, where we offer "Probability & Inductive Logic" in the second year, alongside the usual deductive logic intro.

The book assumes no deductive logic. The early chapters introduce the little that's used. In fact almost no formal background is presumed, only very simple high school algebra.

Several well known predecessors inspired and shaped this book. Brian Skyrms' *Choice & Chance* and Ian Hacking's *An Introduction to Probability and Inductive Logic* were especially influential. Both texts are widely used with good reason—they are excellent. I've taught both myself many times, with great success. But this book blends my favourite aspects of each, organizing them in the sequence and style I prefer.

I hope this book also offers more universal benefits:

1. It is open access, hence free.
2. It's also open source, so other instructors can modify it to their liking.
3. It's available in both PDF and HTML. So it can be read comfortably on a range of devices, or printed.
4. It emphasizes visual explanations and techniques, to make the material more approachable.
5. It livens up the text with hyperlinks, images, and margin notes that highlight points of history and curiosity. I also hope to add some animations and interactive tools soon.

The book is divided into three main parts. The first explains the basics of logic and probability, the second covers basic decision theory, and the last explores the philosophical foundations of probability and statistics. This last, philosophical part focuses on the Bayesian and frequentist approaches.

A "cheat sheet" summarizing key definitions and formulas appears in Appendix A. Further appendices cover the axiomatic construction

of probability theory, Hume's problem of induction, and Goodman's new ridle of induction.

I usually get a mix of students in my course, with different ideological inclinations and varying levels of background. For some the technical material is easy, even review. For others, a healthy skepticism about scientific methods and discourses comes naturally. My goal is to get these students all more or less on the same page.

By the end of the course, students with little formal background have a bevy of tools for thinking about uncertainty. They can understand much more of the statistical and scientific discourse they encounter. And hopefully they have a greater appreciation for the value of formal methods. Students who already have strong formal tools and skills will, I hope, better understand their limitations. I want them to understand why these tools leave big questions open—not just philosophically, but also in very pressing, practical ways.

THE book was made with the `bookdown` package created by Yihui Xie. It's a wonderful tool, built on a bunch of other technologies I love, especially the R programming language and the pandoc conversion tool created by philosopher John MacFarlane. The book's visual style emulates the famous designs of Edward Tufte, thanks to more software created by Yihui Xie, J. J. Allaire, and many others who adapted Tufte's designs to HTML and PDF (via LaTeX).

If it weren't for these tools, I never would have written this book. It wouldn't have been possible to create one that does all the things this book is meant to do. I also owe inspiration to Kieran Healy's book *Data Visualization: A Practical Introduction*, which uses the same suite of tools. It gave me the idea to use those tools for an updated, open, and visually enhanced rendition of the classic material from Skyrms and Hacking.

Part I

1 *The Monty Hall Problem*

. . . in no other branch of mathematics is it so easy
for experts to blunder as in probability theory.
—Martin Gardner

IMAGINE you're on a game show. There are three doors, one with a prize behind it. You're allowed to pick any door, so you choose the first one at random, door A.

Now the rules of the game require the host to open one of the other doors and let you switch your choice if you want. Because the host doesn't want to give away the game, they always open an empty door.

In your case, the host opens door C: no prize, as expected. "Do you want to switch to door B?", the host asks.

Pause a moment to think about your answer before reading on.

WHAT did you decide? Did you conclude it doesn't matter whether you stick with door A or switch to door B?

If so, you're in good company. Most people find this answer sensible, including some professors of statistics and mathematics. They figure there are only two possibilities remaining, door A and door B, each with the same one-in-two chance of being the winner. So it doesn't matter which one you pick.

But the right answer is you should switch. Door B is now twice as likely to be the winner as door A. Why?

The reason is subtle. One way to think about it is that the host's choice of which door to open is a bit of a tell. Maybe they *had* to open door C, because the prize is behind door B and they didn't want to give that away. Of course, it could be behind door A instead, so maybe they just picked door C at random. But there was only a one-in-three chance the prize would be behind door A. Which means there's a two-in-three chance they didn't really have a choice, they had to open door C to avoid showing you the prize behind door B.

Here's another way to think about it. Imagine the game had a hundred doors instead of just three. And suppose again you start by picking the first door at random. Then the host opens *all the other doors*

The Monty Hall problem is named after the creator and host of the game show *Let's Make a Deal*.

Marilyn vos Savant made the Monty Hall problem famous when she solved it correctly in her "Ask Marilyn" column for *Parade* magazine. Read more about it in *The New York Times*.

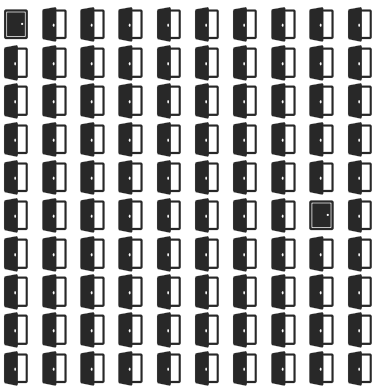


Figure 1.1: The hundred-door version of the Monty Hall problem, suggested by Marilyn vos Savant

but one, door 59 let's say. You have to ask yourself: why did they pick door 59 to leave closed?? Almost certainly because that's where the prize is hidden! Maybe you got really lucky and picked right with the first door at the beginning. But it's way more likely you didn't, and the host had to keep door 59 closed to avoid giving away the game.

1.1 *Diagramming the Solution*

A picture helps clarify things. At first the prize could be behind any of the three doors, with equal probability each way. So we draw a tree with three branches, each labeled with a probability of 1/3. Figure 1.2 shows the result.

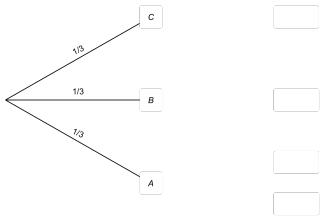


Figure 1.2: First stage of a tree diagram for the Monty Hall problem

Now, which door the host opens may depend on where the prize is, i.e. which branch we're on. If it's behind door C, they won't show you by opening that door. They would have to open door B in this case.

Likewise, if the prize is behind door C, then opening door B is their only option.

Only if the prize is behind door A do they have a choice: open either door B or door C. In that case it's a tossup which door they'll open, so each of those possibilities has a 1/2 chance. Check out Figure 1.3.

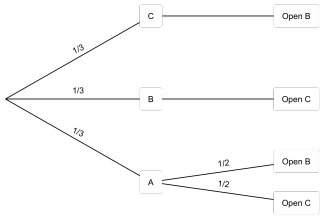


Figure 1.3: Second stage

Now imagine playing the game over and over. A third of the time things will follow the top path; a third of the time they'll follow the middle one; and the remaining third they'll follow one of the two bottom paths.

When things follow the bottom branches, half of those times the host will open door B, and half the time they'll open door C. So one in every six plays will follow the *A-and-Open-B* path. And one in every six plays will follow the *A-and-Open-C* path. See Figure 1.4.

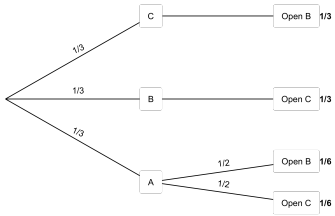


Figure 1.4: Third and final stage

Now we can understand what happens when the host opens door C. Usually it's because the prize is behind door B. Sometimes they open door C because the prize is behind door A instead. But that's only a sixth of the time, compared to a third of the time where they open door C because the prize is behind door B.

So when you see the host open door C, you should think it's more likely you're on the middle branch, with the prize behind door B. Switch!

1.2 *Lessons Learned*

Tree diagrams are a handy tool for solving probability problems. They also illustrate some central concepts of probability.

Probabilities are numbers assigned to possibilities. In the Monty Hall problem, there are three possibilities for where the prize is: door A, door B, and door C. Each of these possibilities has the same proba-

bility: 1/3.

SOME possibilities are ***mutually exclusive***, meaning only one of them can obtain. The prize can't be behind door A and door B, for example. Here are more examples of mutually exclusive possibilities:

- A coin can land heads or tails, but it can't do both on the same toss.
- A card drawn from a standard deck could be either an ace or a queen, but it can't be both.
- The temperature at noon tomorrow could be 20 degrees, or it could be 25 degrees, but it can't be both.

When possibilities are mutually exclusive, their probabilities add up. For example, the initial probability the prize will be behind either door A or door B is $1/3 + 1/3 = 2/3$. And the probability a card drawn from a standard deck will be either an ace or a queen is $4/52 + 4/52 = 8/52 = 2/13$.

ANOTHER key concept is possibilities that are ***exhaustive***. In the Monty Hall problem, the prize has to be behind one of the three doors, so A, B, and C "exhaust" all the possibilities. Here are more examples of exhaustive possibilities:

- A card drawn from a standard deck must be either red or black.
- The temperature at noon tomorrow must be either above zero, below zero, or zero.

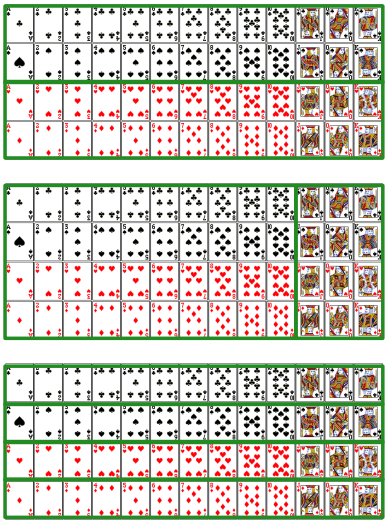


Figure 1.5: Three partitions for a card drawn from a standard deck

IN our tree diagrams, each branch-point always uses a set of possibilities that is *both* exclusive *and* exhaustive. The first split on the three doors covers all the possibilities for where the prize might be, and only one of those possibilities can be the actual location of the prize. Likewise for the second stage of the diagram. On the bottom branch for example, the host must open either door B or door C given the rules, but he will only open one or the other.

When a set of possibilities is both exclusive and exhaustive, it's called a ***partition***. A partition "carves up" the space of possibilities into distinct, non-overlapping units.

There can be more than one way to partition the space of possibilities. For example, a randomly drawn playing card could be black or red; it could be a face card or not; and it could be any of the four suits (♡, ♢, ♣, ♠).

WHEN possibilities form a partition, their probabilities must add up to 1. Initially, the probability the prize will be behind one of the three

doors is $1/3 + 1/3 + 1/3 = 1$. And the probability that a card drawn from a standard deck at random will be either red or black is $1/2 + 1/2 = 1$.

In a way, the fundamental principle of probability is that probabilities over a partition must add up to 1.

TREE diagrams follow a few simple rules based on these concepts. The parts of a tree are called *nodes*, *branches*, and *leaves*: see Figure 1.6.

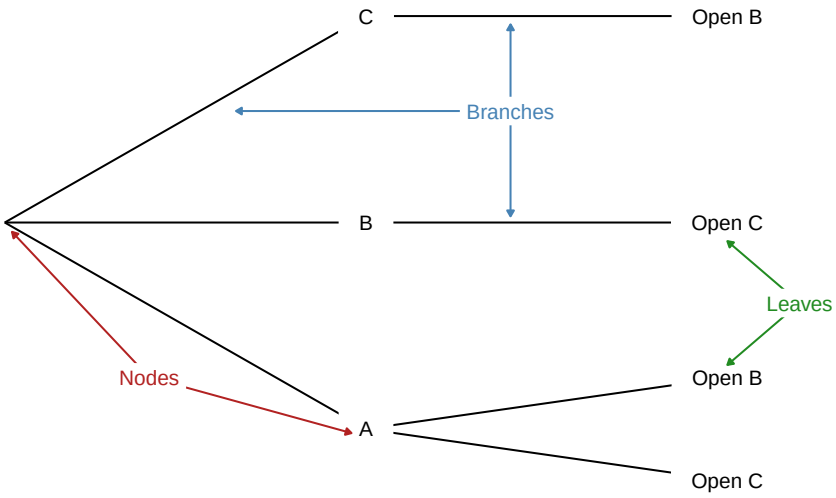


Figure 1.6: The parts of a tree diagram: nodes, branches, and leaves

The rules for a tree are as follows:

Rule 1. Each node must use a partition. The branches coming out of it must be mutually exclusive possibilities, and they must cover all the possibilities.

Rule 2. The probabilities at each node must add up to 1.

Rule 3. The probability on a branch is *conditional* on the branches leading up to it.

- For example, consider the bottom path in the Monty Hall problem. The probability the host will open door C is $1/2$ there because we're assuming the prize is behind door A.

Rule 4. The probability of a leaf is calculated by multiplying across the branches on the path leading to it. This number represents the probability that all possibilities on that path occur.

Notice, Rule 4 is how we got the final probabilities (the numbers in bold) we used to solve the Monty Hall problem.

Exercises

1. True or false: in the Monty Hall problem, it's essential to the puzzle that the host doesn't want to expose the prize. If they didn't care about giving away the location of the prize, there would be no reason to switch when they open door C.

2. In the version of the Monty Hall problem with a hundred doors, after the host opens every door except door 1 (your door) and door 59, the chance the prize is behind door 59 is:

   a. 1/100
   b. 1/99
   c. 1/2
   d. 99/100

3. Imagine three prisoners, A, B, and C, are condemned to die in the morning. But the king decides to pardon one of them first. He makes his choice at random and communicates it to the guard, who is sworn to secrecy. She can only tell the prisoners that one of them will be released at dawn, she can't say who.

   Prisoner A welcomes the news, as he now has a 1/3 chance of survival. Hoping to go even further, he says to the guard, "I know you can't tell me whether I am condemned or pardoned. But at least one other prisoner must still be condemned, so can you just name one who is?". The guard tells him that B is still condemned. "Ok", says A, "then it's either me or C who was pardoned. So my chance of survival has gone up to 1/2".

   Is prisoner A's reasoning correct? Use a probability tree to explain why/why not.

4. In a probability tree, each branch point should split into possibilities that are:

   a. Mutually exclusive.
   b. Exhaustive.
   c. Both mutually exclusive and exhaustive.
   d. None of the above.

5. Suppose you have two urns. The first has two black marbles and two white marbles. The second has three black marbles and one white marble. You are going to flip a fair coin to select one of the urns at random, and then draw one marble at random. What is the chance you will select a black marble?

Hint: draw a probability tree and ask yourself, "if I did this experiment over and over again, how often would I draw a black marble in the long run?"

a. 5/8
b. 3/8
c. 1/2
d. 1/4

2 *Logic*

I can win an argument on any topic, against any opponent. People know this, and steer clear of me at parties. Often, as a sign of their great respect, they don't even invite me.
—Dave Barry

Logic is the study of what follows from what. From the information that Tweety is a bird and all birds are animals, it follows that Tweety is an animal. But things aren't always so certain. Can Tweety fly? Most birds can fly, so probably. But Tweety might be a penguin.

Deductive logic is the branch of logic that studies what follows with certainty. *Inductive* logic deals with uncertainty, things that only follow with high probability.

This book is about inductive logic and probability. But we need a few concepts from deductive logic to get started.

2.1 *Validity & Soundness*

In deductive logic we study "valid" arguments. An argument is ***valid*** when the conclusion must be true if the premises are true. Take this example again:

Tweety is a bird.
All birds are animals.
Therefore, Tweety is an animal.

The first two lines are called the *premises* of the argument. The last line is called the *conclusion*. In this example, the conclusion must be true if the premises are. So the argument is valid.

Here's another example of a valid argument:

Tweety is taller than Kwazi.
Kwazi is taller than Peso.
Therefore, Tweety is taller than Peso.

The argument is valid because it's just not possible for the premises to be true and the conclusion false.

Here's an example of an *invalid* argument:

Tweety is a bird.
Most birds can fly.
Therefore, Tweety can fly.

It's not valid because validity requires the conclusion to follow *necessarily*. If there's any way for the the premises to be true yet the conclusion false, the argument doesn't count as valid. And like we said, Tweety might be a penguin.

Valid arguments are interesting because their logic is airtight. If the assumptions of the argument are correct, there's no way to go wrong accepting the conclusion. But what if the assumptions *aren't* correct? Validity isn't everything, we also want our arguments to build on true foundations.

We call an argument ***sound*** when it is valid *and* all the premises are true:

sound = valid + true premises.

For example, here's a sound argument:

The author of this book is human.
All humans are animals.
Therefore, the author of this book is an animal.

Sound arguments are important because their conclusions are always true. The premises of a sound argument are true by definition. And since it's valid by definition too, that guarantees the conclusion to be true as well.

Yet deductive logic studies validity, not soundness. Why?

Because logicians aren't in the business of determining when the premises of an argument are true. As a logician, I might have no idea who Tweety is, and thus no idea whether Tweety is a bird. I might not even know whether all birds fly, or just some, or even none. That's a job for an ornithologist.

A logician's job is to assess the *logic* of an argument, the connections between its assumptions and its conclusion. So a logician just takes the premises of an argument for granted and asks, how well do those assumptions support the conclusion? That's something you don't need to know any ornithology to study. Or biology, or medicine, or physics, or whatever topic a particular argument concerns.

Validity is a tricky, counterintuitive concept. It's very much a hypothetical notion: it's about whether the conclusion must be true *if* the

premises are true. So when we assess an argument's validity, we ignore what we know about the truth of its premises. We pretend they're true even if they aren't. We even have to ignore what we know about the conclusion.

Instead we suspend what we know about the topic, and just imagine the premises to be true. Then we ask: in this hypothetical scenario, is there any way the conclusion could be false? If there is, the argument is invalid. Otherwise, it's valid.

2.2 *Propositions*

ARGUMENTS are made out of statements, assertions that something is true. In logic we call these statements ***propositions***. And we use capital letters of the English alphabet to stand for them. For example, this argument:

> If Aegon is a tyrant, then Brandon is a wizard.
> Aegon is a tyrant.
> Therefore, Brandon is a wizard.

can be summarized like this:

> If A, then B.
> A.
> Therefore, B.

NOT all sentences are propositions. Some are questions, some are commands, some are expressions of worry. For example:

- What time is it?
- Pass the rooster sauce!
- Uh oh.

One way to distinguish propositions from other kinds of sentences is: propositions are capable of being true or false. It wouldn't make sense to respond to someone who asks you what time it is by saying, "what you just said is false!" And you wouldn't respond to someone's request to pass the sauce with "that's true!" Except maybe as a joke.

2.3 *Visualizing Propositions*

WE learned about mutually exclusive propositions in Section 1.2. Two propositions are mutually exclusive when one of them being true means the other must be false. For example:

- A: Confucius was born in the 6th Century A.D.

- *B*: Confucius was born in the 6th Century B.C.

There is no way for both of these propositions to be true, and we can visualize this relationship in a diagram (Figure 2.1).

Each circle represents a proposition. You can think of it as surrounding the possible situations where the proposition would be true. The circles don't overlap because there is no possible situation where both propositions in this example are true.

In contrast, these two propositions are not mutually exclusive:

- Confucius was born in Asia.
- Confucius was born in the 6th Century B.C.

When propositons are not mutually exclusive, we say they are ***compatible***. Compatible propositions overlap (Figure 2.2). The region where the circles overlap represents the possible scenarios where both propositions are true (the "*A* & *B* region").

These are called ***Euler diagrams***, after the mathematician Leonhard Euler (pronounced *oiler*).[1] You may have seen Venn diagrams before, which are very similar. But in an Euler diagram, the circles don't have to overlap.

SOMETIMES one circle will even contain another circle entirely. Take this example:

- Confucius was born in Asia.
- Confucius was born somewhere.

These propositions aren't just compatible. If the first is true, then the second *must* be true. Imagine an argument with the first proposition as the premise and the second proposition as the conclusion. The argument would be valid:

Confucius was born in Asia.
Therefore, Confucius was born.

In this case we say that the first proposition ***logically entails*** the second. In terms of an Euler diagram, he first circle is contained entirely in the second (Figure 2.3). Because there is no possible situation where the first proposition is true yet the second false.

What if an argument has multiple premises? For example:

Zhuangzi was born in the Chinese province of Anhui.
Zhuoru was born in the Chinese city of Beijing.
Therefore, both Zhuangzi and Zhuoru were born in China.

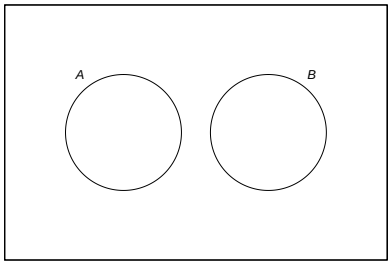


Figure 2.1: Mutually exclusive propositions

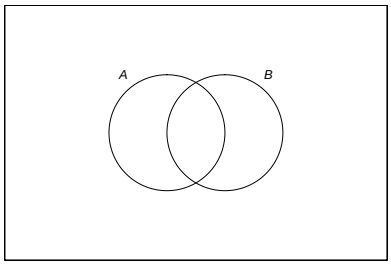


Figure 2.2: Compatible propositions

[1] Leonhard Euler lived from 1707 to 1783. You may have encountered some of his work before if you've worked with logarithms or taken calculus.

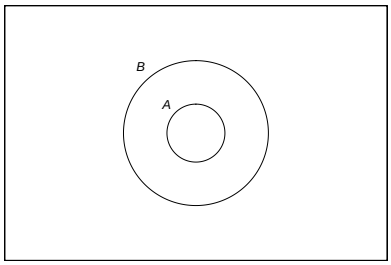


Figure 2.3: Logical entailment

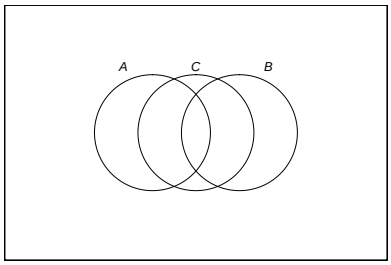


Figure 2.4: A valid argument with two premises

This argument is valid, and the diagram might look like Figure 2.4. Notice how the A & B region lies entirely within the C circle. This reflects the argument's validity: there is no way for the first two propositions to be true and the last one false.

In contrast, an invalid argument would have a diagram like Figure 2.5. This diagram allows for the possibility that A and B are both true yet C is false; part of the A & B region falls outside the C circle.

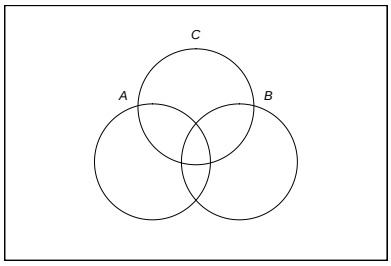


Figure 2.5: An invalid argument with two premises

2.4 *Strength*

INDUCTIVE logic studies arguments that aren't necessarily valid, but still "strong". A ***strong*** argument is one where the conclusion is highly probable, if the premises are true. For example:

> The sun has risen every day so far.
> Therefore, the sun will rise again tomorrow.

This argument isn't valid, because it's possible the conclusion is false even though the premise is true. Maybe the sun will explode in the night for some surprising reason. Or maybe the earth's rotation will be stopped by alien forces.

These possibilities aren't very likely, of course. So the argument is strong, even though it's not strictly valid. The premise gives us very good reason to believe the conclusion, just not a 100% guarantee.

In terms of an Euler diagram then, the premise circle isn't contained entirely within the conclusion circle (Figure 2.6). We have to leave some room for the possibility that the premise is true and the conclusion false. But we can still convey that this possibility has only a very slight chance of being true, by making it slim.

We could also label the *A-but-not-B* region with a small number, if we knew exactly how unlikely this possibility was.

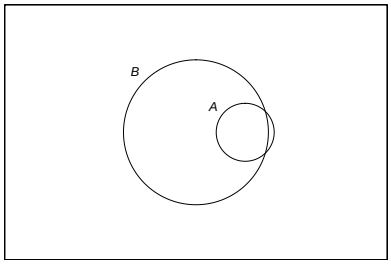


Figure 2.6: A strong argument with premise A and conclusion B

STRENGTH comes in degrees. An argument's premises can make the conclusion somewhat likely, very likely, almost certain, or perfectly certain. So arguments range from weak, to somewhat strong, to very strong, etc.

Strength differs from validity here, since validity is all-or-nothing. If there is any possible way for the premises to be true and the conclusion false, the argument is invalid—no matter how remote or bizarre that possibility is.

Notice though, valid arguments are strong by definition. Since it's impossible for a valid argument's conclusion to be false if the premises are true, the premises make the conclusion 100% probable. A valid argument is the strongest possible argument.

Figure 2.7: Pierre Simone Laplace (1749–1827) developed a formula for calculating the probability the sun will rise tomorrow. We'll learn how to do similar calculations in the coming chapters.

2.5 *Forms of Inductive Argument*

WHAT kinds of strong arguments are there, and how strong are they? That's what the rest of this book is about, in a way. But we can start by identifying some common forms of inductive argument right now.

GENERALIZING from observed instances is one extremely common form of argument:

> Every raven I have every seen has been black.
> Therefore, all ravens are black.

Arguments of this kind are usually stronger the more instances you observe. If you've only ever seen two ravens, this argument won't be very compelling. But if you've seen thousands, then it's much stronger.

It also helps to observe different kinds of instances. If you've only observed ravens in your city or town, then even the thousands you've seen won't count for much. Maybe the raven population in your area is unusual, and ravens on the other side of the world are all different colours.

GOING in the opposite direction, we can use what we know about a general population to draw conclusions about particular instances. We saw an example of this earlier:

> Most birds can fly.
> Tweety is a bird.
> Therefore, Tweety can fly.

Again, the strength of the inference depends on the details. If "most birds" means 99%, the argument is quite strong. If "most" only means 80%, then it's not so strong. (Usually "most" just means more than 50%.)

It also helps to know that Tweety is similar to the birds that can fly, and different from the ones that can't. If we know that Tweety is small and has feathers, that makes the argument stronger. If instead we know that Tweety is large, and coloured black and white, that makes the argument weaker. It suggests Tweety is a penguin.

INFERENCE to the best explanation is another common form of argument, quite different from the previous two. Here's an example:

> My car won't start and the gas gauge reads 'empty'.
> Therefore, my car is out of gas.

An empty tank would explain the sytmptoms described in the premise, so the premise makes the conclusion plausible. There could be other

possible explanations, of course. Maybe the engine and the gas gauge both just happened to break at the same time. But that would be quite a coincidence, so this explanation isn't as good.

What makes one explanation better than another? That turns out to be a very hard question, and there is no generally accepted answer. We'll come back to this issue later, once we have a better grip on the basics of probability.

Exercises

1. For each of the following arguments, say whether it is valid or invalid.

   a. All cats have whiskers.
   Simba has whiskers.
   Therefore, Simba is a cat.

   b. Ada Lovelace wrote the world's first computer program.
   Ada Lovelace was Lord Byron's daughter.
   Therefore, the first computer program was written by Lord Byron's daughter.

   c. All Canadian residents are Russian citizens.
   Vladimir Putin is a Canadian resident.
   Therefore, Vladimir Putin is a Russian citizen.

   d. Manitoba is located in either Saskatchewan, Ontario, or Quebec.
   Manitoba is not located in Saskatchewan.
   Manitoba is not located in Ontario.
   Therefore, Manitoba is located in Quebec.

   e. If snow is black then pigs can fly.
   Snow is not black.
   Therefore, pigs cannot fly.

   f. Either the moon is made of green cheese or pigs can fly.
   Pigs can't fly.
   Therefore the moon is made of green cheese.

2. For each pair of propositions, say whether they are mutually exclusive or compatible.

   a. Regarding the roll of an ordinary die:

      - The die will land on an even number.
      - The die will land either 4 or 5.

   b. Regarding the unemployment rate in your country tomorrow:

- The unemployment rate will be at least 5%.
- The unemployment rate will be exactly 5%.

c. Regarding a party tomorrow:

- Ani will be there and so will her sister PJ.
- PJ will not be there.

3. True or false? If A and B are mutually exclusive, then A logically entails that B is false.

4. True or false? It is possible for A to logically entail B even though the reverse does not hold (i.e. even though B does not logically entail A).

5. Create your own example of each of the three types of inductive argument described in this chapter:

a. Generalizing from Observed Instances
b. Inferring an Instance from a Generalization
c. Inference to the Best Explanation

3 *Truth Tables*

In this chapter we'll introduce the last few concepts we need from deductive logic, and we'll learn a useful technique in the process: truth tables.

3.1 Connectives

Complex propositions can be built up out of other, simpler propositions:

- Aegon is a tyrant **and** Brandon is a wizard.
- **Either** Aegon is a tyrant **or** Brandon is a wizard.
- **It's not true that** Aegon is a tyrant.

Notice, we call *it's not true that* a connective even though it doesn't actually connect two propositions together.

Here we've used two simple propositions to build up longer, more complex ones using the terms *and*, *either/or*, and *it's not true that*. Such terms are called ***connectives***.

The three connectives just listed are the only ones we'll need in this book. Each has a name and a shorthand symbol:

Name	English	Symbol	Example
conjunction	and	&	$A \,\&\, B$
disjunction	either/or	$\vee$	$A \vee B$
negation	it's not true that	$\sim$	$\sim A$

Here are some more examples of complex propositions:

- $F \,\&\, \sim G$: Florida is warm and Geneva is not.
- $\sim J \vee \sim K$: Either Jing won't come to the party or Kamal won't come.

Sometimes we also need parentheses, to avoid ambiguity. Consider an example from arithmetic:

$$4 \div 2 \times 2 = 1.$$

Is this equation true? That depends on what you mean. Does the division operation come first, or the multiplication? So we use parentheses to clarify: $4 \div (2 \times 2) = 1$, but $(4 \div 2) \times 2 = 4$.

In logic we use parentheses to prevent ambiguity similarly. Consider:

$$A \vee B \,\&\, C.$$

This proposition is ambiguous, it has two interpretations. In English we can distinguish them with a comma:

- Either Aegon is a tyrant or Brandon is a wizard, and Cerci is the queen.
- Either Aegon is a tyrant, or Brandon is a wizard and Cerci is the queen.

Notice how these statements make different claims. The first takes a definite stand on Cerci: she is the queen. It only leaves open the question whether Aegon is a tyrant or Brandon is a wizard. Whereas the second statement takes no definite stand on any of our three characters. Maybe Aegon is a tyrant, maybe not. Maybe Brandon is a wizard and Cerci is the queen, maybe not.

In logic we use parentheses to clarify which interpretation we mean:

- $(A \vee B) \,\&\, C$.
- $A \vee (B \,\&\, C)$.

Notice how the first statement is primarily an & statement. It uses & to combine the simpler statements C and $A \vee B$ together. Whereas the second statement is primarily a $\vee$ statement. It uses $\vee$ to combine A with $B \,\&\, C$.

We call the last connective used to build up the statement the ***main connective***.

- $(A \vee B) \,\&\, C$: main connective is &.
- $A \vee (B \,\&\, C)$: main connective is $\vee$.

Two more examples:

- $\sim(A \vee B)$: main connective is $\sim$.
- $\sim A \vee B$: main connective is $\vee$.

Technically, the last example should have parentheses to prevent ambiguity, like so: $(\sim A) \vee B$. But things get cluttered and hard to read if we add parentheses around every negation. So we have a special understanding for $\sim$ in order to keep things tidy.

This special understanding for $\sim$ mirrors the one for minus signs in arithmetic.

ℹ The negation symbol $\sim$ only applies to the proposition immediately following it.

So in the proposition $\sim A \vee B$, the $\sim$ only applies to A. And in $\sim(A \,\&\, B) \vee C$, it only applies to $A \,\&\, B$.

3.2 *Truth Tables*

The truth of a complex proposition built using our three connectives depends on the truth of its components. For example, $\sim A$ is false if A is true, and it's true if A is false:

Table 3.2: Truth table for $\sim$

A	$\sim A$
T	F
F	T

Slightly more complicated is the rule for &:

Table 3.3: Truth table for &

A	B	$A \,\&\, B$
T	T	T
T	F	F
F	T	F
F	F	F

There are four rows now because & combines two propositions A and B together to make the more complex proposition $A\&B$. Since each of those propositions could be either true or false, there are $2 \times 2 = 4$ possible situations to consider.

Notice that in only one of these situations is $A\&B$ true, namely the first row where both A and B are true.

The truth table for $\vee$ ("either/or") is a little more surprising:

Table 3.4: Truth table for $\vee$

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Now the complex proposition is always true, except in one case: the last row where A and B are both false. It makes sense that $A \vee B$ is false when both sides are false. But why is it true when both sides are true? Doesn't "Either A or B" mean that *just one* of these is true?

Sometimes it does have that meaning. But sometimes it means "Ei-

ther A or B, *or both*". Consider this exchange:

> X: What are you doing tomorrow night?
> Y: I'm either going to a friend's house or out to a club. I might even do both, if there's time.

Person Y isn't necessarily changing their mind here. They could just be clarifying: they're doing at least one of these things, possibly even both of them.

Although it's common to use "either/or" in English to mean *just* one or the other, in logic we use the more permissive reading. So $A \vee B$ means *either A, or B, or both*.

We can always convey the stricter way of meaning "either/or" with a more complex construction:

$$(A \vee B) \,\&\, \sim(A \,\&\, B).$$

That says:

> Either A or B is true, and it's not the case that both A and B are true.

Which is just a very explicit way of saying: either one or the other, but not both.

We can even verify that the complex construction captures the meaning we want using a truth table. We start with an empty table, where the header lists all the formulas we use to build up to the final, complex one we're interested in:

A	B	$A \vee B$	$A \& B$	$\sim(A \,\&\, B)$	$(A \vee B) \,\&\, \sim(A \,\&\, B)$

Then we fill in the possible truth values for the simplest propositions, A and B:

A	B	$A \vee B$	$A \& B$	$\sim(A \,\&\, B)$	$(A \vee B) \,\&\, \sim(A \,\&\, B)$
T	T				
T	F				
F	T				
F	F				

Next we consult the truth tables above for & and $\vee$ to fill in the columns at the next level of complexity:

A	B	$A \vee B$	$A\&B$	$\sim(A \,\&\, B)$	$(A \vee B) \,\&\, \sim(A \,\&\, B)$
T	T	T	T		
T	F	T	F		
F	T	T	F		
F	F	F	F		

Then move up to the next level of complexity. To fill in the column for $\sim(A \,\&\, B)$, we consult the column for $A \,\&\, B$ and apply the rules from the table for $\sim$:

A	B	$A \vee B$	$A\&B$	$\sim(A \,\&\, B)$	$(A \vee B) \,\&\, \sim(A \,\&\, B)$
T	T	T	T	F	
T	F	T	F	T	
F	T	T	F	T	
F	F	F	F	T	

Finally, we consult the columns for $A \vee B$ and $\sim(A \,\&\, B)$, and the table for &, to fill in the column for $(A \vee B) \,\&\, \sim(A\&B)$:

A	B	$A \vee B$	$A\&B$	$\sim(A \,\&\, B)$	$(A \vee B) \,\&\, \sim(A \,\&\, B)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Complex constructions like this are difficult at first, but don't worry. With practice they quickly become routine.

3.3 *Logical Truths & Contradictions*

Some propositions come out true in every row of the truth table. Consider $A \vee \sim A$ for example:

A	$\sim A$	$A \vee \sim A$
T	F	T
F	T	T

Such propositions are especially interesting because they *must* be true. Their truth is guaranteed, just as a matter of logic. So we call them ***logical truths***.

The other side of this coin is propositions that are false in every row

of the truth table, like $A \,\&\, {\sim}A$:

A	${\sim}A$	$A \,\&\, {\sim}A$
T	F	F
F	T	F

These propositions are called ***contradictions***.

Notice that the negation of a contradiction is a logical truth. For example, consider the truth table for ${\sim}(A \,\&\, {\sim}A)$:

A	${\sim}A$	$A \,\&\, {\sim}A$	${\sim}(A \,\&\, {\sim}A)$
T	F	F	T
F	T	F	T

3.4 *Mutually Exclusivity & Truth Tables*

TRUTH tables can be used to establish that two propositions are mutually exclusive. A very simple example is the propositions A and ${\sim}A$:

A	${\sim}A$
T	F
F	T

There is no row in the table where both propositions are true. And if two propositions can't both be true, they are mutually exclusive by definition.

A slightly more complex example is the propositions $A \vee B$ and ${\sim}A \,\&\, {\sim}B$:

A	B	${\sim}A$	${\sim}B$	$A \vee B$	${\sim}A \,\&\, {\sim}B$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	F	T

Again there's no row where $A \vee B$ and ${\sim}A \,\&\, {\sim}B$ are both true. So they are mutually exclusive.

3.5 *Entailment & Equivalence*

TRUTH tables can also be used to establish that an argument is valid. Here's a very simple example:

A & B.
Therefore, A.

Obviously it's not possible for the premise to be true and the conclusion false, so the argument is valid (if a bit silly). Accordingly, there is no line of the truth table where A & B comes out true, yet A comes out false:

A	B	A & B
T	T	T
T	F	F
F	T	F
F	F	F

The only line where A & B comes out true is the first one. And on that line A is true too. So the argument from A & B to A is valid.

One more example:

$A \vee B$.
$\sim A$.
Therefore, B.

This argument is valid because the first premise says that at least one of the two propositions A and B must be true, and the second line says it's not A. So it must be B that's true, as the conclusion asserts. And once again there is no line of the truth table where both $A \vee B$ and $\sim A$ are true, yet B is false:

A	B	$\sim A$	$A \vee B$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

The only line where both $A \vee B$ and $\sim A$ are true is the third row, and B is true on that row. So once again the truth table tells us this argument is valid.

In the previous chapter we introduced the concept of logical entailment. A logically entails B when it's impossible for A to be true and B false. When one proposition entails another, there is no line of the

truth table where the first proposition is true and the second is false.

SOMETIMES entailment goes in both directions: the first proposition entails the second *and the second entails the first*. For example, not only does $A \,\&\, B$ entail $B \,\&\, A$, but also $B \,\&\, A$ entails $A \,\&\, B$.

We say such propositions are ***logically equivalent***. In terms of truth tables, their columns match perfectly, they are identical copies of T's and F's.

A	B	$A \,\&\, B$	$B \,\&\, A$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

A more complex example is the propositions $\sim(A \vee B)$ and $\sim A \,\&\, \sim B$:

A	B	$\sim A$	$\sim B$	$A \vee B$	$\sim(A \vee B)$	$\sim A \,\&\, \sim B$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Here again the columns under these two propositions are identical.

3.6 *Summary*

CONNECTIVES can be used to build more complex propositions, like $A \,\&\, B$ or $A \vee \sim B$. We introduced three connectives:

- $\sim A$ means it's not true that A.
- $A \,\&\, B$ means both A and B are true.
- $A \vee B$ means either A is true, or B is true, *or both are true*.

In a complex proposition, the main connective is the last one used to build it up from simpler components. In $A \vee \sim B$ the main connective is the $\vee$.

An argument's validity can be established with a truth table, if there's no row where all the premises have a T and yet the conclusion has an F.

Truth tables can also be used to establish that two propositions are mutually exclusive, if there is no row of the table where both propositions have a T.

Logically equivalent propositions entail one another. When two propositions have identical columns in a truth table, they are logically equivalent.

Exercises

1. Using the following abbreviations:

$$A = \text{Asha loves Cerci},$$
$$B = \text{Balon loves Cerci},$$

translate each of the following into logicese (e.g. $\sim A \vee B$).

   a. Asha doesn't love Cerci.
   b. Asha loves Cerci and Balon loves Cerci.
   c. Asha loves Cerci but Balon does not.
   d. Neither Asha nor Balon loves Cerci.

2. For each pair of propositions, use a truth table to determine whether they are mutually exclusive.

   a. $A \,\&\, B$ and $A \,\&\, \sim B$.
   b. A and $\sim B$.
   c. $A \vee \sim A$ and $A \,\&\, \sim A$.

3. For each pair of propositions, use a truth table to determine whether they are logically equivalent.

   a. $\sim A \vee B$ and $\sim(A \,\&\, \sim B)$.
   b. A and $(A \,\&\, B) \vee (A \,\&\, \sim B)$.
   c. A and $(B \,\&\, A) \vee (B \,\&\, \sim A)$.

4. The proposition $A \vee (B \,\&\, C)$ features three simple propositions, so its truth table has 8 rows. Fill in the rest of the table:

A	B	C	$B \,\&\, C$	$A \vee (B \,\&\, C)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

5. Use a truth table to determine whether the propositions $A \vee (B \,\&\, C)$ and $(A \vee B) \,\&\, (A \vee C)$ are equivalent.

4 *The Gambler's Fallacy*

Applied statistics is hard.
—Andrew Gelman

My wife's family keeps having girls. My wife has two sisters and she and her sisters each have two daughters, with no other siblings or children. That's nine girls in a row!

So are they due for a boy next? Here are three possible answers.

Answer 1. Yes, the next baby is more likely to be a boy. Ten girls in a row would be a *really* unlikely outcome.

Answer 2. No, the next baby is actually more likely to be a girl. Girls run in the family! Something about this family clearly predispose them to have girls.

Answer 3. No, the next baby is equally likely to be a boy vs. girl. Each baby's sex is determined by a purely random event, similar to a coin flip. So it's equal odds every time. The nine girls so far is just a coincidence.

Which answer is correct?

4.1 Independence

It all hangs on whether the sex of each baby is "independent" of the others. Two events are ***independent*** when the outcome of one doesn't change the probability of the other.

A clear example of independence is ***sampling with replacement***. Suppose we have an urn with 50 black marbles and 50 white ones. You draw a ball at random, then put it back. Then you give the urn a good hard shake and draw at random again. The two draws are independent in this case. Each time you draw, the number of black vs. white marbles is the same, and they're all randomly mixed up.

Even if you were to draw ten white balls in a row, the eleventh draw would still be a 50-50 shot! It would just be a coincidence that you drew ten white balls in a row. Because there's always an even mix of black and white marbles, and you're always picking one at random.

But now imagine sampling ***without replacement***. The situation is

Note that girl and boy aren't the only two possibilities. The next child could also be intersex.

Boy and girl are the likely alternatives, since fewer than 2% of children born are intersex. (The exact percentage is unclear because the exact definition of "interesex" is controversial.)

But many intersex children have undergone risky or painful surgeries and treatments, sometimes without ever being told. So it's important to recognize intersex as a third category.

Fashion model Hanne Gaby Odiele is one example of a famous intersex person. She learned she was intersex just weeks before starting her modeling career.

the same, except now you set aside each ball you draw rather than put it back. Now the draws are ***dependent***. If you draw a black ball on the first draw, the odds of black on the next draw go down. There are only 49 black balls in the urn now, vs. 50 white.

4.2 *Fairness*

FLIPS of an ordinary coin are also independent. Even if you get ten heads in a row, the eleventh toss is still 50-50. If it's *really* an ordinary coin, the ten heads in a row was just a coincidence.

Coin flips aren't just independent, they're also *unbiased*: heads and tails are equally likely. A process is ***biased*** if some outcomes are more likely than others. For example, a loaded coin that comes up heads 3/4 of the time is biased.

So coin flips are unbiased *and* independent. We call such processes ***fair***.

Fair = Unbiased + Independent.

Another example of a fair process is drawing from our black/white urn with replacement. There are 50 black and 50 white marbles on every draw, so black and white have equal probability every time.

But drawing without replacement is not a fair process, because the draws are not independent. Removing a black ball makes the chance of black on the next draw go down.

4.3 *The Gambler's Fallacy*

GAMBLING often involves fair processes: fair dice, fair roulette wheels, fair decks of cards, etc. But people sometimes forget that fair processes are independent. If a roulette wheel comes up black nine times in a row, they figure it's "due" for red. Or if they get a bunch of bad hands in a row at poker, they figure they're due for a good one soon.

This way of thinking is called *the gambler's fallacy*. A fallacy is a mistake in reasoning. The mistake here is failing to fully account for independence. These gamblers know the process in question is fair, in fact that's a key part of their reasoning. They know it's unlikely that the roulette wheel will land on black ten times in a row because a fair wheel should land on black and red equally often. But then they overlook the fact that fair also means independent, and independent means the last nine spins tell us nothing about the tenth spin.

The gambler's fallacy is so seductive that it can be hard to find your way out of it. Here's one way to think about it that may help. Imagine the gambler's point of view at two different times: before the ten spins of the wheel, and after. Before, the gambler is contemplaing the likelihood of getting ten black spins in a row:

$$\_\_\_\_\_?$$

From that vantage point, the gambler is exactly right to think it's unlikely these ten spins will all land on black. But now imagine their point of view after observing (to their surprise) the first nine spins all landing black:

$$B\,B\,B\,B\,B\,B\,B\,B\,B\,\_\,?$$

Now how likely is it these ten spins will all land black? Well, just one more spin has to land black now to fulfill this unlikely prophecy. So it's not such a long shot anymore. In fact it's a 50-50 shot. Although it was very unlikely the first nine spins would turn out this way, now that they have, it's perfectly possible the tenth will turn out the same.

4.4 *Ignorance Is Not a Fallacy*

At this point you may have a nagging thought at the back of your mind. If we flipped a coin 100 times and it landed heads every time, wouldn't we conclude the next toss will probably land heads? How could that be a mistake?!

The answer: it's not a fallacy. It *would* be a fallacy if you knew the coin was fair. But if you didn't know that for sure, then landing 100 heads in a row would be enough to convince you it's not fair.

The gambler's fallacy only occurs when you know a process is fair, and then you fail to reason accordingly. If you don't know whether a process is fair, then you aren't making a logical error by reasoning according to a different assumption.

So, is the gambler's fallacy at work if my wife's family expects a boy next? As it turns out, the process that determines the sex of a child is pretty much fair.[1] So the correct answer to our opening question was Answer 3.

[1] The question isn't completely settled though, as far as I could tell from my (not very thorough) research.

Most people don't know about the relevant research, though. They may (like me) only know a bit from high school biology about how sex is usually determined at conception. But it's still possible for all they know that some people's eggs are more likely to select sperm cells with an X chromosome, for example.

So it's not necessarily a fallacy if my in-laws expect a boy next. It could just be a reasonable conclusion given the information available. A fallacy is an error in reasoning, not a lack of knowledge.

4.5 *The Hot Hand Fallacy*

Sometimes a basketball player hits a lot of baskets in a row and people figure they're on fire: they've got a "hot hand". But a famous study published in 1985 found that these streaks are just a coincidence. Each shot is still independent of the others. Is the hot hand an example of the gambler's fallacy?

Most people don't know about the famous 1985 study. Certainly nobody knew what the result of the study would be before it was conducted. So a lot of believers in the hot hand were in the unfortunate position of just not knowing a player's shots are independent. So the hot hand isn't the same as the gambler's fallacy.

Believers in the hot hand may be guilty of a different fallacy, though. That same study analyzed the reasoning that leads people to think a player's shots are dependent. Their conclusion: people tend to see patterns in sequences of misses and hits even when they're random. So there may be a second, different fallacy at work, the "hot hand fallacy".

Things might actually be even more complicated than that, though. Some recent studies found that the hot hand may actually be real after all! How could that be possible? What did the earlier studies miss?

It's still being researched, but one possibility is: defense. When a basketball player gets in the zone, the other team ups their defense. The hot player has to take harder shots. So one of the recent studies added a correction to account for increased difficulty. And another looked at baseball instead, where they did find evidence of streaks.

The full story of the hot hand fallacy has yet to be told it seems.

But here's Selena Gomez and Nobel prize winner Richard Thaler telling a bit of the story in a clip from the 2015 movie *The Big Short*.

Exercises

1. Suppose you are going to draw a card at random from a standard deck. For each pair of propositions, say whether they are independent or not.

   a. A: the card is red.
      B: the card is an ace.

   b. A: the card is red.
      B: the card is a diamond.

   c. A: the card is an ace.
      B: the card is a spade.

   d. A: the card is a Queen.
      B: the card is a face card.

2. Drawing from an urn filled with 50 black and 50 white marbles with replacement is a _____ process.

   a. Independent
   b. Fair
   c. Unbiased
   d. All of the above
   e. None of the above

3. Drawing from an urn filled with 50 black and 50 white marbles *without* replacement is a _____ process.

   a. Independent
   b. Fair
   c. Unbiased
   d. All of the above
   e. None of the above

4. For each of the following examples, say whether it is an instance of the gambler's fallacy.

   a. You're playing cards with your friends using a standard, randomly shuffled deck of 52 cards. You're about half-way through the deck and no aces have been drawn yet. You conclude that an ace is due soon, and thus the probability the next card is an ace has gone up.

   b. You're holding a six-sided die, which you know to be fair. You're going to roll it 60 times. You figure about 10 of those rolls should be threes. But after 59 rolls, you've rolled a three only five times. You figure that the probability of a three on the last roll has gone up: it's higher than just 1/6.

   c. You know the lottery numbers in Ontario are selected using a fair process. So it's really unlikely that someone will win two weeks in a row. Your friend won last week, so you conclude their chances of winning this week too are even lower than usual.

   d. You're visiting a new country where corruption is common, so you aren't sure whether the lottery there is fair. You see on the news that the King's cousin won the lottery two weeks in a row. You conclude that their chances of winning next week are higher than normal, because the lottery is rigged in their favour.

5 Calculating Probabilities

Imagine you're going to flip a fair coin twice. You could get two heads, two tails, or one of each. How probable is each outcome?

It's tempting to say they're equally probable, $1/3$ each. But actually the first two are only $1/4$ likely, while the last is $1/2$ likely. Why?

There are actually four possible outcomes here, but we have to consider the order of events to see how. If you get one each of heads and tails, what order will they come in? You could get the head first and then the tail, or the reverse.

So there are four possible sequences: HH, TT, HT, and TH. And all four sequences are equally likely, a probability of $1/4$.

How do we know each sequence has $1/4$ probability though? And how does that tell us the probability is $1/2$ that you'll get one each of heads and tails? We need to introduce some mechanics of probability to settle these questions.

5.1 Multiplying Probabilities

We denote the probability of proposition A with $Pr(A)$. For example, $Pr(A) = 2/3$ means there's a $2/3$ chance A is true.

Now, our coin is fair, and by definition that means it always has a $1/2$ chance of landing heads and a $1/2$ chance of landing tails. For a single toss, we can use H for the proposition that it lands heads, and T for the proposition that it lands tails. We can then write $Pr(H) = 1/2$ and $Pr(T) = 1/2$.

For a sequence of two tosses, we can use H_1 for heads on the first toss, and H_2 for heads on the second toss. Similarly, T_1 and T_2 represent tails on the first and second tosses, respectively. The four possible sequences are then expressed by the complex propositions:

- $H_1 \,\&\, H_2$,
- $T_1 \,\&\, T_2$,
- $H_1 \,\&\, T_2$,
- $T_1 \,\&\, H_2$.

We want to calculate the probabilities of these propositions. For exam-

ple, we want to know what number $Pr(H_1 \,\&\, H_2)$ is equal to.

Because the coin is fair, we know $Pr(H_1) = 1/2$ and $Pr(H_2) = 1/2$. The probability of heads on any given toss is always $1/2$, no matter what came before. To get the probability of $H_1 \,\&\, H_2$ it's then natural to compute:

$$\begin{aligned} Pr(H_1 \,\&\, H_2) &= Pr(H_1) \times Pr(H_2) \\ &= 1/2 \times 1/2 \\ &= 1/4. \end{aligned}$$

And this is indeed correct, but *only because the coin is fair and thus the tosses are independent*. The following is a general rule of probability:

The Multiplication Rule If A and B are independent, then $Pr(A \,\&\, B) = Pr(A) \times Pr(B)$.

So, because our two coin tosses are independent, we can multiply to calculate $Pr(H_1 \,\&\, H_2) = 1/4$. And the same reasoning applies to all four possible sequences, so we have:

$$\begin{aligned} Pr(H_1 \,\&\, H_2) &= 1/4, \\ Pr(T_1 \,\&\, T_2) &= 1/4, \\ Pr(H_1 \,\&\, T_2) &= 1/4, \\ Pr(T_1 \,\&\, H_2) &= 1/4. \end{aligned}$$

THE Multiplication rule only applies to independent propositions. Otherwise it gives the wrong answer.

For example, the propositions H_1 and T_1 are definitely not independent. If the coin lands heads on the first toss (H_1), that drastically alters the chances of tails on the first toss (T_1). It changes that probability to zero! If you were to apply the Multiplication Rule though, you would get $Pr(H_1 \,\&\, T_1) = Pr(H_1) \times Pr(T_1) = 1/2 \times 1/2 = 1/4$, which is definitely wrong.

⊘ Only use the Multiplication Rule on independent propositions.

5.2 *Adding Probabilities*

WE observed that you can get one head and one tail two different ways. You can either get heads then tails ($H_1 \,\&\, T_2$), or you can get tails then heads ($T_1 \,\&\, H_2$). So the logical expression for "one of each" is:

$$(H_1 \,\&\, T_2) \vee (T_1 \,\&\, H_2).$$

This proposition is a disjunction: its main connective is $\vee$. How do we calculate the probability of a disjunction?

The Addition Rule If A and B are mutually exclusive, then $Pr(A \vee B) = Pr(A) + Pr(B)$.

In this case the two sides of our disjunction are mutually exclusive. They describe opposite orders of affairs. So we can apply the Addition Rule to calculate:

$$\begin{aligned} Pr((H_1 \,\&\, T_2) \vee (T_1 \,\&\, H_2)) &= Pr(H_1 \,\&\, T_2) + Pr(T_1 \,\&\, H_2) \\ &= 1/4 + 1/4 \\ &= 1/2. \end{aligned}$$

THIS completes the solution to our opening problem. We've now computed the three probabilities we wanted:

- $Pr(\text{2 heads}) = Pr(H_1 \,\&\, H_2) = 1/2 \times 1/2 = 1/4$,
- $Pr(\text{2 tails}) = Pr(T_1 \,\&\, T_2) = 1/2 \times 1/2 = 1/4$,
- $Pr(\text{One of each}) = Pr((H_1 \,\&\, T_2) \vee (T_1 \,\&\, H_2)) = 1/4 + 1/4 = 1/2$.

In the process we introduced two central rules of probability, one for & and one for $\vee$. The multiplication rule for & only applies when the propositions are independent. The addition rule for $\vee$ only applies when the propositions are mutually exclusive.

WHY does the addition rule for $\vee$ sentences only apply when the propositions are mutually exclusive? Well imagine the weather forecast says there's a 90% chance of rain in the morning, and there's also a 90% chance of rain in the afternoon. What's the chance it'll rain at some point during the day, either in the morning or the afternoon? If we calculate $Pr(M \vee A) = Pr(M) + Pr(A)$, we get $90\% + 90\% = 180\%$, which doesn't make any sense. There can't be a 180% chance of rain tomorrow.

The problem is that M and A are not mutually exclusive. It could rain all day, both morning and afternoon. We'll see the correct way to handle this kind of situation in Chapter 7. In the meantime just be careful:

⊘ Only use the Addition Rule on mutually exclusive propositions.

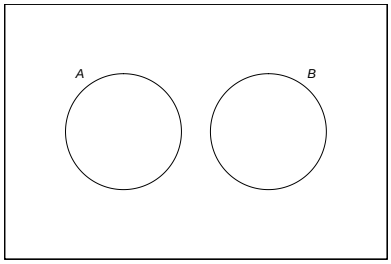


Figure 5.1: Mutually exclusive propositions don't overlap

5.3 *Exclusivity vs. Independence*

EXCLUSIVITY and independence can be hard to keep straight at first. One way to keep track of the difference is to remember that mutually exclusive propositions don't overlap, but independent propositions usually do. Independence means the truth of one proposition

doesn't affect the chances of the other. So if you find out that A is true, B still has the same chance of being true. Which means there have to be some B possibilities within the A circle (unless the probability of A was zero to start with).

So independence and exclusivity are very different. Generally speaking, exclusive propositions are not independent, and independent propositions are not exclusive.

There is one exception. If $Pr(A) = 0$, then A and B can be both independent and mutually exclusive. If they're mutually exclusive, the probability of A just stays 0 after B we learn B. But otherwise, independence and mutual exclusivity are incompatible with one another.

Another marker that may help you keep these two concepts straight: exclusivity is a concept of deductive logic. It's about whether it's *possible* for both propositions to be true (even if that possibility is very unlikely). But independence is a concept of inductive logic. It's about whether one proposition being true changes the *probability* of the other being true.

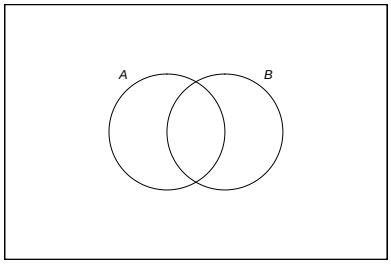


Figure 5.2: Independent propositions do overlap (unless one of them has zero probability).

5.4 *Tautologies, Contradictions, and Equivalent Propositions*

A tautology is a proposition that must be true, so its probability is always 1.

The Tautology Rule $Pr(T) = 1$ for every tautology T.

For example, $A \vee {\sim}A$ is a tautology, so $Pr(A \vee {\sim}A) = 1$. In terms of an Euler diagram, the A and ${\sim}A$ regions together take up the whole diagram. To put it a bit colourfully, $Pr(A \vee {\sim}A) = \blacksquare + \blacksquare = 1$.

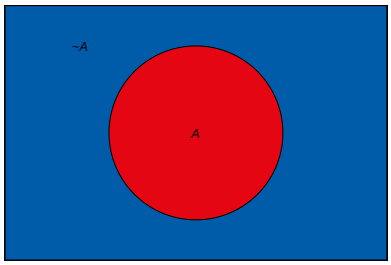


Figure 5.3: The Tautology Rule. Every point falls in either the A region or the ${\sim}A$ region, so $Pr(A \vee {\sim}A) = 1$.

The flipside of a tautology is a contradiction, a proposition that can't possibly be true. So it has probability 0.

The Contradiction Rule $Pr(C) = 0$ for every contradiction C.

For example, $A \,\&\, {\sim}A$ is a contradiction, so $Pr(A \,\&\, {\sim}A) = 0$. In terms of our Euler diagram, there is no region where A and ${\sim}A$ overlap. So the portion the diagram devoted to $A \,\&\, {\sim}A$ is nil, zero.

Equivalent propositions are true under exactly the same circumstances (and false under exactly the same circumstances). So they have the same chance of being true (ditto false).

The Equivalence Rule $Pr(A) = Pr(B)$ if A and B are logically equivalent.

For example, $A \vee B$ is logically equivalent to $B \vee A$, so $Pr(A \vee B) = Pr(B \vee A)$.

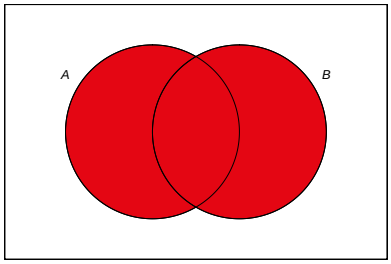


Figure 5.4: The Equivalence Rule. The $A \vee B$ region is identical to the $B \vee A$ region, so they have the same probability.

In terms of an Euler diagram, the $A \vee B$ region is exactly the same as the $B \vee A$ region: the red region. So both propositions take up the same amount of space in the diagram.

5.5 The Language of Events

In math and statistics books you'll often see a lot of the same concepts from this chapter introduced in different language. Instead of propositions, they'll discuss *events*, which are sets of possible outcomes.

For example, the roll of a six-sided die has six possible outcomes: $1, 2, 3, 4, 5, 6$. And the event of the die landing on an even number is the set $\{2, 4, 6\}$.

In this way of doing things, rather than consider the probability that a proposition A is true, we consider the probability that event E occurs. Instead of considering a conjunction of propositions like $A \,\&\, B$, we consider the *intersection* of two events, $E \cap F$. And so on.

If you're used to seeing probability presented this way, there's an easy way to translate into logic-ese. For any event E, there's the corresponding proposition that event E occurs. And you can translate the usual set operations into logic as follows:

Table 5.1: Translating between events and propositions

Events	Propositions
E^c	$\sim A$
$E \cap F$	$A \,\&\, B$
$E \cup F$	$A \vee B$

We won't use the language of events in this book. I'm just mentioning it in case you've come across it before and you're wondering how it connects. If you've never seen it before, you can safely ignore this section.

5.6 Summary

In this chapter we learned how to represent probabilities of propositions using the $Pr(\ldots)$ operator. We also learned some fundamental rules of probability.

There were three rules corresponding to the concepts of tautology, contradiction, and equivalence.

- $Pr(T) = 1$ for every tautology T.
- $Pr(C) = 0$ for every contradiction C.
- $Pr(A) = Pr(B)$ if A and B are logically equivalent.

And there were two rules corresponding to the connectives & and $\vee$.

- $Pr(A \vee B) = Pr(A) + Pr(B)$, if A and B are mutually exclusive.
- $Pr(A \,\&\, B) = Pr(A) \times Pr(B)$, if A and B are independent.

The restrictions on these two rules are essential. If you ignore them, you will get wrong answers.

Exercises

1. What is the probability of each of the following propositions?

   a. $A \,\&\, (B \,\&\, \sim A)$
   b. $\sim(A \,\&\, \sim A)$

2. Give an example of each of the following:

   a. Two statements that are mutually exclusive.
   b. Two statements that are independent.

3. For each of the following, say whether it is true or false.

   a. If propositions are independent, then they must be mutually exclusive.
   b. Independent propositions usually aren't mutually exclusive.
   c. If propositions are mutually exclusive, then they must be independent.
   d. Mutually exclusive propositions usually aren't independent.

4. Assume $Pr(A \,\&\, B) = 1/3$ and $Pr(A \,\&\, \sim B) = 1/5$. Answer each of the following:

   a. What is $Pr((A \,\&\, B) \vee (A \,\&\, \sim B))$?
   b. What is $Pr(A)$?
   c. Are $(A \,\&\, B)$ and $(A \,\&\, \sim B)$ independent?

5. Suppose A and B are independent, and A and C are mutually exclusive. Assume $Pr(A) = 1/3, Pr(B) = 1/6, Pr(C) = 1/9$, and answer each of the following:

   a. What is $Pr(A \,\&\, C)$?
   b. What is $Pr((A \,\&\, B) \vee C)$?
   c. Must $Pr(A \,\&\, B) = 0$?

6. Consider the following argument:

   > If a coin is fair, then the probability of getting at least one heads in a sequence of four tosses is quite high: above 90%.

Therefore, if a fair coin has landed tails three times in a row, the next toss will probably land heads.

Answer each of the following questions.

a. Is the premise of this argument true?
b. Is the argument valid?
c. Is the argument sound?

7. The Addition Rule can be extended to three propositions. If A, B, and C are all mutually exclusive with one another, then

$$Pr(A \vee B \vee C) = Pr(A) + Pr(B) + Pr(C).$$

Explain why this rule is correct. Would the same idea extend to four mutually exclusive propositions? To five?

(Hint: there's more than one way to do this. You can use an Euler diagram. Or you can derive the new rule from the original one, by thinking of $A \vee B \vee C$ as a disjunction of $A \vee B$ and C.)

8. You have a biased coin, where each toss has a 3/5 chance of landing heads. But each toss is independent of the others. Suppose you're going to flip the coin 1,000 times. The first 998 tosses all land tails. What is the probability at least one of the last two flips will be tails?

6 Conditional Probability

The chances of crashing your car are pretty low, but they're considerably higher if you're drunk. Probabilities change depending on the conditions.

We symbolize this idea by writing $Pr(A \mid B)$, the probability that A is true *given* that B is true. And we call this kind of probability ***conditional probability***.

For example, to say the probability of A given B is 30%, we write:

$$Pr(A \mid B) = .3.$$

But how do we calculate conditional probabilities?

To say B increases the chance of A we write $Pr(A \mid B) > Pr(A)$. And to say B doubles the chance of A write $Pr(A \mid B) = 2 \times Pr(A)$.

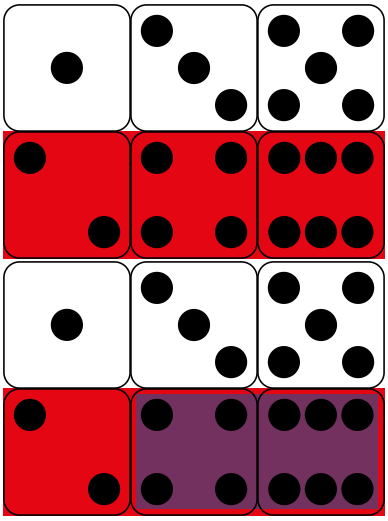


Figure 6.1: Conditional probability in a fair die roll

6.1 Calculating Conditional Probability

Suppose I roll a fair, six-sided die behind a screen. You can't see the result, but I tell you it's an even number. What's the probability it's also a "high" number: either a 4, 5, or 6?

Maybe you figured the correct answer: 2/3. But why is that correct? Because, out of the three even numbers (2, 4, and 6), two of them are high (4 and 6). And since the die is fair, we expect it to land on a high number 2/3 of the times it lands on an even number.

This hints at a formula for $Pr(A \mid B)$.

Conditional Probability

$$Pr(A \mid B) = \frac{Pr(A \,\&\, B)}{Pr(B)}.$$

In the die-roll example, we considered how many of the B possibilities were also A possibilities. Which means we divided $Pr(A \,\&\, B)$ by $Pr(B)$.

In fact, this formula is our official definition for the concept of conditional probability. When we write the sequence of symbols $Pr(A \mid B)$, it's really just shorthand for the fraction $Pr(A \,\&\, B)/Pr(B)$.

In terms of an Euler diagram (Figure 6.2), the definition of conditional probability compares the size of the purple $A \,\&\, B$ region to the

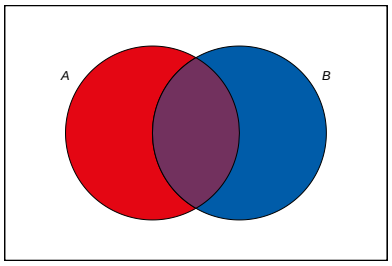


Figure 6.2: Conditional probability is the size of the $A \,\&\, B$ region compared to the entire B region.

size of the whole B region, purple and blue together. If you don't mind getting a little colourful with your algebra:

$$Pr(A \mid B) = \frac{\blacksquare}{\blacksquare + \blacksquare}.$$

So the definition works because, informally speaking, $Pr(A \,\&\, B)/Pr(B)$ is the proportion of the B outcomes that are also A outcomes.[1]

[1] The comedian Steven Wright once quipped that "black holes are where God divided by zero."

Dividing by zero is a common pitfall with conditional probability. Notice how the definition of $Pr(A \mid B)$ depends on $Pr(B)$ being larger than zero. If $Pr(B) = 0$, then the formula

$$Pr(A \mid B) = \frac{Pr(A \,\&\, B)}{Pr(B)}$$

doesn't even make any sense. There is no number that results from the division on the right hand side.[2]

In such cases we say that $Pr(A \mid B)$ is *undefined*. It's not zero, or some special number. It just isn't a number.

[2] There are alternative mathematical systems of probability, where conditional probability is defined differently to avoid this problem. But in this book we'll stick to the standard system. Where there's just no such thing as "the probability of A given B" when B has zero probability.

6.2 *Conditional Probability & Trees*

We already encountered conditional probabilities informally, when we used a tree diagram to solve the Monty Hall problem.

In a tree diagram, each branch represents a possible outcome. The number placed on that branch represents the chance of that outcome occurring. But that number is based on the assumption that all branches leading up to it occur. So the probability on that branch is conditional on all previous branches.

For example, suppose there are two urns of coloured marbles:

- Urn X contains 3 black marbles, 1 white.
- Urn Y contains 1 black marble, 3 white.

I flip a fair coin to decide which urn to draw from, heads for Urn B and tails for Urn W. Then I draw one marble at random.

The probability of drawing a black marble on the top path is $3/4$ because we are assuming the coin landed heads, and thus I'm drawing from Urn X. If the coin lands tails instead, and I draw from Urn Y, then the chance of a black marble is instead $1/4$. So these quantities are conditional probabilities:

$$Pr(B \mid H) = 3/4,$$
$$Pr(B \mid T) = 1/4.$$

Notice, though, the first branch in a tree diagram is different. In the H-vs.-T branch, the probabilities are *un*conditional, since there are no previous branches for them to be conditional on.

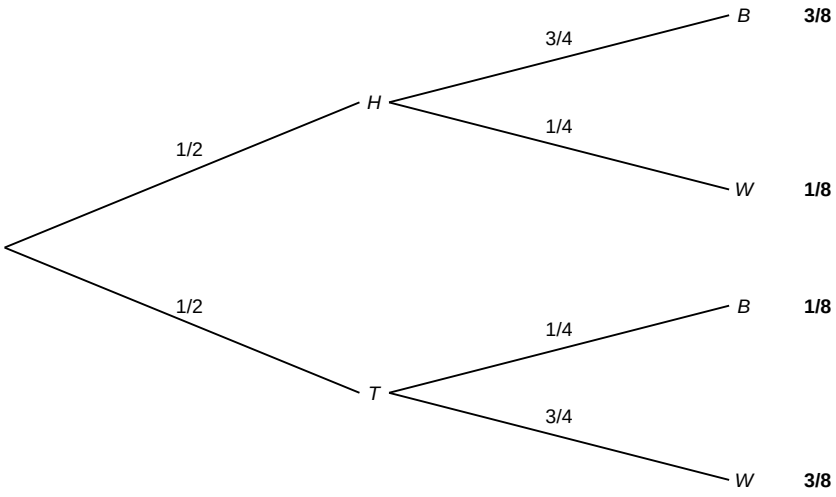


Figure 6.3: Tree diagram for an urn problem

6.3 More Examples

IMAGINE an urn contains marbles of three different colours: 20 are red, 30 are blue, and 40 are green. I draw a marble at random. What is $Pr(R \mid \sim B)$, the probability it's red given that it's not blue?

$$\begin{aligned} Pr(R \mid \sim B) &= \frac{Pr(R \,\&\, \sim B)}{Pr(\sim B)} \\ &= \frac{Pr(R)}{Pr(\sim B)} \\ &= \frac{20/90}{60/90} \\ &= 1/3. \end{aligned}$$

This calculation relies on the fact that $R \,\&\, \sim B$ is logically equivalent to R. A red marble is automatically not blue, so R is true under exactly the same circumstances as $R \,\&\, \sim B$. The Equivalence Rule thus tells us $Pr(R \,\&\, \sim B) = Pr(R)$.

SUPPOSE a university has 10,000 students, and each student is studying under one of four broad headings: Humanities, Social Sciences, STEM, or Professional. Within each of these categories, the number of students with an average grade of A, B, C, or D is as follows:

	Humanities	Social Sciences	STEM	Professional
A	200	600	400	900
B	500	800	1600	900
C	250	400	1500	750
D	50	200	500	450

What is the probability a randomly selected student will have an A average, given that they are studying either Humanities or Social Sciences?

$$\begin{aligned} Pr(A \mid H \vee S) &= \frac{Pr(A \,\&\, (H \vee S))}{Pr(H \vee S)} \\ &= \frac{800/10,000}{3,000/10,000} \end{aligned}$$

Humanities or Social Sciences given that they have an A average?

$$\begin{aligned} Pr(H \vee S \mid A) &= \frac{Pr((H \vee S) \,\&\, A)}{Pr(A)} \\ &= \frac{800/10,000}{2,100/10,000} \\ &= 8/21. \end{aligned}$$

Notice how we get a different number now.

6.4 Order Matters

In general, the probability of A given B will be different from the probability of B given A. These are different concepts.

For example, university students are usually young, but young people aren't usually university students. Most aren't even old enough to be in university. So the probability someone is young given they are in university is high. But the probability someone is in university given that they are young is low. So $Pr(Y \mid U) \neq Pr(U \mid Y)$.

Once in a while we do find cases where $Pr(A \mid B) = Pr(B \mid A)$. For example, suppose we throw a dart at random at a circular board, divided into four quadrants. The chance the dart will land on the left half given that it lands on the top half is the same as the chance it lands on the top half given it lands on the left. Both probabilities are $1/2$.

But this kind of thing is the exception rather than the rule. Usually, $Pr(A \mid B)$ will be a different number from $Pr(B \mid A)$. So it's important to remember how order matters.

⊘ When we write $Pr(A \mid B)$, we are discussing the probability of A. But we are discussing it under the assumption that B is true.

6.5 Declaring Independence

We explained independence informally back in Chapter 4: A and B are independent if the truth of one doesn't change the probability of the other. Now that we've formally defined conditional probability, we can formally define independence too.

Independence A is independent of B if $Pr(A \mid B) = Pr(A)$ and $Pr(A) > 0$.

In other words, they're independent if A's probability is the same after B is given as it was before (and not just for the silly reason that there was no chance of A being true to begin with).

Now we can establish three useful facts about independence.

THE first is summed up in the mantra "independence means multiply". This actually has two parts.

We already learned the first part with the Multiplication Rule: if A is independent of B, then $Pr(A \,\&\, B) = Pr(A)Pr(B)$. Except now we can see why this rule holds, using the definition of conditional probability and some algebra:

$$\begin{aligned} Pr(A \mid B) &= \frac{Pr(A \,\&\, B)}{Pr(B)} && \text{by definition} \\ Pr(A \mid B)Pr(B) &= Pr(A \,\&\, B) && \text{by algebra} \\ Pr(A)Pr(B) &= Pr(A \,\&\, B) && \text{by independence.} \end{aligned}$$

The second part of the "independence means multiply" mantra is new though. It basically says that the reverse also holds. As long as $Pr(A) > 0$ and $Pr(B) > 0$, if $Pr(A \,\&\, B) = Pr(A)Pr(B)$, then A is independent of B.

Bottom line: as long as there are no zeros to worry about, independence is the same thing as $Pr(A \,\&\, B) = Pr(A)Pr(B)$.

SECOND, independence is symmetric. If A is independent of B then B is independent of A. Informally speaking, if B makes no difference to A's probability, then A makes no difference to B's probability.

This is why we often say "A and B are independent", without specifying which is independent of which. Since independence goes both ways, they're automatically independent of each other.

THIRD, independence extends to negations. If A is independent of B, then it's also independent of $\sim B$ (as long as $Pr(\sim B) > 0$, so that $Pr(A \mid \sim B)$ is well-defined).

Notice, this also means that if A is independent of B, then $\sim A$ is independent of $\sim B$ (as long as $Pr(\sim A) > 0$).

So far our definition of independence only applies to two propositions. We can extend it to three as follows:

Three-way Independence A, B, and C are independent if

i. A is independent of B, A is independent of C, and B is independent of C, and
ii. $Pr(A \,\&\, B \,\&\, C) = Pr(A)Pr(B)Pr(C)$.

In other words, a trio of propositions is independent if each pair of them is independent, and the multiplication rule applies to their conjunction. The same idea can be extended to define independence for four propositions, five, etc.

Exercises

1. Answer each of the following:

   a. On a fair die with six sides, what is the probability of rolling a low number (1, 2, or 3) given that you roll an even number.
   b. On a fair die with eight sides, what is the probability of rolling an even number given that you roll a high number (5, 6, 7, or 8)?

2. Suppose $Pr(B) = 4/10$, $Pr(A) = 7/10$, and $Pr(B \,\&\, A) = 2/10$. What are each of the following probabilities?

   a. $Pr(A \mid B)$
   b. $Pr(B \mid A)$

3. Five percent of tablets made by the company Ixian have factory defects. Ten percent of the tablets made by their competitor company Guild do. A computer store buys 40% of its tablets from Ixian, and 60% from Guild.

   Draw a probability tree to answer the following questions.

   a. What is the probability a randomly selected tablet in the store is made by Ixian and has a factory defect?
   b. What is the probability a randomly selected tablet in the store has a factory defect?
   c. What is the probability a tablet from this store is made by Ixian, given that it has a factory defect?

   This exercise and the next one are based on very similar exercises from Ian Hacking's wonderful book, *An Introduction to Probability and Inductive Logic*.

4. In the city of Elizabeth, the neighbourhood of Southside has lots of chemical plants. 2% of Elizabeth's children live in Southside, and 14% of those children have been exposed to toxic levels of lead. Elsewhere in the city, only 1% of the children have toxic levels of exposure.

   Draw a probability tree to answer the following questions.

   a. What is the probability that a randomly chosen child from Elizabeth lives in Southside and has toxic levels of lead exposure?
   b. What is the probability that a randomly chosen child from Elizabeth has toxic levels of lead exposure?
   c. What is the probability that a randomly chosen child from Elizabeth who has toxic levels of lead exposure lives in Southside?

5. Imagine 100 prisoners are sentenced to death. 70 of them are housed in cell block A, the other 30 are in cell block B. Of the prisoners in cell block A, 9 are innocent. Only 1 prisoner in cell block B is innocent.

   The law requires that one prisoner be pardoned. The lucky prisoner will be selected by flipping a fair coin to choose either cell block A or B. Then a fair lottery will be used to select a random prisoner from the chosen cell block.

   What is the probability the pardoned prisoner comes from cell block A if she is innocent? Answer each of the following to find out.

   I = The pardoned prisoner is innocent.
   A = The pardoned prisoner comes from cell block A.

   a. What is $Pr(I \mid A)$?
   b. What is $Pr(A \,\&\, I)$?
   c. What is $Pr(I \mid B)$?
   d. What is $Pr(B \,\&\, I)$?
   e. What is $Pr(I)$?
   f. What is $Pr(A \mid I)$?
   g. Draw a probability tree to visualize and verify your calculations.

6. Suppose A, B, and C are independent, and they each have the same probability: $1/3$. What is $Pr(A \,\&\, B \mid C)$?

7. If A and B are mutually exclusive, what is $Pr(A \mid B)$? Justify your answer using the definition of conditional probability.

8. Which of the following situations is impossible? Justify your answer.

   a. $Pr(A) = 1/2, Pr(A \mid B) = 1/2, Pr(B \mid A) = 1/2$.
   b. $Pr(A) = 1/2, Pr(A \mid B) = 1, Pr(A \mid \sim B) = 1$.

9. Is the following statement true or false: if A and B are mutually exclusive, then $Pr(A \vee B \mid C) = Pr(A \mid C) + Pr(B \mid C)$. Justify your answer.

10. Justify the second part of the "independence means multiply" mantra: if $Pr(A) > 0$, $Pr(B) > 0$, and $Pr(A \,\&\, B) = Pr(A)Pr(B)$, then A is independent of B.

    Hint: start by supposing $Pr(A) > 0$, $Pr(B) > 0$, and $Pr(A \,\&\, B) = Pr(A)Pr(B)$. Then apply some algebra and the definition of conditional probability.

11. Justify the claim that independence is symmetric: if A is independent of B, then B is independent of A.

    Hint: start by supposing that A is independent of B. Then write out $Pr(A \mid B)$ and apply the definition of conditional probability.

7 Calculating Probabilities, Part II

We learned rules for $\vee$ and for & back in Chapter 5:

The Addition Rule $Pr(A \vee B) = Pr(A) + Pr(B)$ if A and B are mutually exclusive.

The Multiplication Rule $Pr(A\&B) = Pr(A) \times Pr(B)$ if A and B are independent.

In this chapter we'll learn new, more powerful rules for $\vee$ and &. But we'll start with negation, a rule for calculating $Pr(\sim A)$.

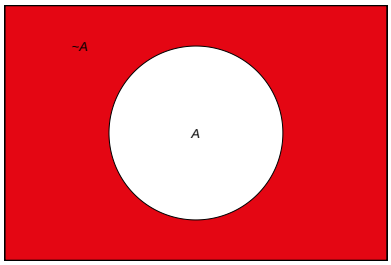


Figure 7.1: The Negation Rule. $Pr(\sim A) = 1 - Pr(A)$.

7.1 The Negation Rule

If there's a 70% chance of rain, then there's a 30% chance it won't rain. In symbols, if $Pr(R) = .7$ then $Pr(\sim R) = .3$. So the rule for $Pr(\sim A)$ is:

The Negation Rule $Pr(\sim A) = 1 - Pr(A)$.

In terms of an Euler diagram, the probability of $\sim A$ is the size of the red region. So $Pr(\sim A)$ is $1 - Pr(A)$.

It's important to notice that this rule can be flipped around, to calculate the probability of a positive statement:

$$Pr(A) = 1 - Pr(\sim A).$$

Sometimes we really want to know the probability of A, $Pr(A)$, but it turns out to be much easier to calculate $Pr(\sim A)$ first. Then we use this flipped version of the negation rule to get what we're after.

7.2 The General Addition Rule

The Addition Rule for calculating $Pr(A \vee B)$ depends on A and B being mutually exclusive. What if they're not? Then we can use:

The General Addition Rule $Pr(A \vee B) = Pr(A) + Pr(B) - Pr(A \,\&\, B)$.

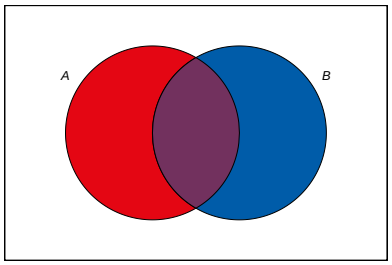


Figure 7.2: The General Addition Rule in an Euler diagram.

This rule always applies, whether A and B are mutually exclusive or not.

To understand the rule, consider an Euler diagram where A and B are not mutually exclusive. In terms of colour, the size of the $A \vee B$-region is:

$$\blacksquare + \blacksquare + \blacksquare.$$

Which is the same as:

$$(\blacksquare + \blacksquare) + (\blacksquare + \blacksquare) - \blacksquare.$$

In algebraic terms this is:

$$Pr(A) + Pr(B) - Pr(A \,\&\, B).$$

To think of it another way, when we add $Pr(A) + Pr(B)$ to get the size of the $A \vee B$ region, we double-count the $A \,\&\, B$ region. So we have to subtract out $Pr(A \,\&\, B)$ at the end.

WHAT if there is no $A \,\&\, B$ region? Then $Pr(A \,\&\, B) = 0$, so subtracting it at the end has no effect. Then we just have the old Addition Rule:

$$\begin{aligned} Pr(A \vee B) &= Pr(A) + Pr(B) - Pr(A \,\&\, B) \\ &= Pr(A) + Pr(B) - 0 \\ &= Pr(A) + Pr(B). \end{aligned}$$

And this makes sense. If there is no $A \,\&\, B$ region, that means A and B are mutually exclusive. So the old Addition Rule applies.

That's why we call the new rule the *General* Addition Rule. It applies in general, even when A and B are not mutually exclusive. And in the special case where they are mutually exclusive, it gives the same result as the Addition Rule we already learned.

A tree diagram also works to explain the General Addition Rule. Consider Figure 7.3, where we start with branches for A and $\sim A$, then subdivide into branches for B and $\sim B$.

There are three leaves where $A \vee B$ is true, marked with emoji. If we add $Pr(A) + Pr(B)$, we're adding the two leaves where A is true (😎 and 😍) to the two leaves where B is true (😎 and 🤓). So we've double-counted the $A \,\&\, B$ leaf (😎). To get $Pr(A \vee B)$ then, we have to subtract one of those $A \,\&\, B$ leaves (😎).

THERE is a catch to the General Addition Rule. You need to know $Pr(A \,\&\, B)$ in order to apply it. Sometimes that information is given to us. But when it's not, we have to figure it out somehow. If A and B are mutually exclusive, then it's easy: $Pr(A \,\&\, B) = 0$. Or, if they're independent, then we can calculate $Pr(A \,\&\, B) = Pr(A) \times Pr(B)$. But in other cases we have to turn elsewhere.

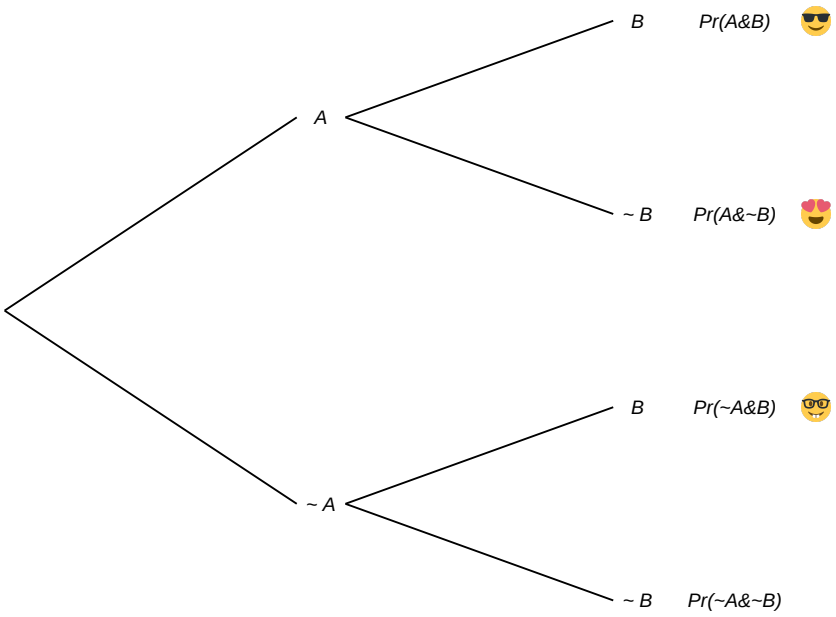


Figure 7.3: Tree diagram with the three $A \vee B$ leaves marked

7.3 *The General Multiplication Rule*

How can we calculate $Pr(A \,\&\, B)$ in general?

The General Multiplication Rule $Pr(A \,\&\, B) = Pr(A \mid B)Pr(B)$.

The intuitive idea is, if you want to know how likely it is A and B will both turn out to be true, first ask yourself how likely A is to be true *if* B is true. Then weight the answer according to B's chances of being true.

Notice, if A and B are independent, then this rule just collapses into the familiar Multiplication Rule we already learned. If they're independent, then $Pr(A \mid B) = Pr(A)$ by definition. So substituting into the General Multiplication Rule gives:

$$\begin{aligned} Pr(A \,\&\, B) &= Pr(A \mid B)Pr(B) \\ &= Pr(A)Pr(B). \end{aligned}$$

Which is precisely the Multiplication Rule.

So we now have two rules for &. The first one only applies when the two sides of the & are independent. The second applies whether they're independent or not. The second rule ends up being the same as the first one when they are independent.

A tree diagram helps us understand this rule too. Recall this problem from Chapter 6, with two urns of coloured marbles:

- Urn X contains 3 black marbles, 1 white.
- Urn Y contains 1 black marble, 3 white.

I flip a fair coin to decide which urn to draw from, heads for Urn X and tails for Urn Y. Then I draw one marble at random.

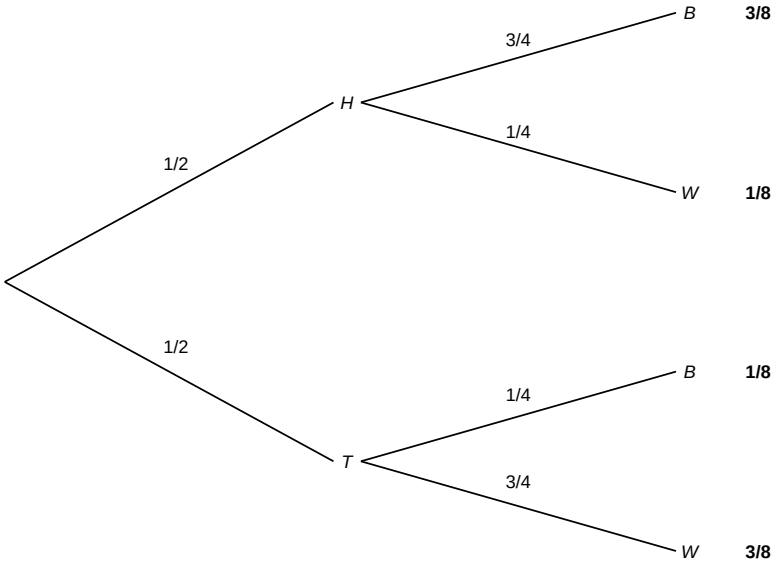


Figure 7.4: Tree diagram for an urn problem

Now suppose we want to know the probability the coin will land tails and the marble drawn will be white, $Pr(T \,\&\, W)$. The General Multiplication Rule tells us the answer is:

$$\begin{aligned} Pr(T \,\&\, W) &= Pr(W \,\&\, T) \\ &= Pr(W \mid T)Pr(T) \\ &= 3/4 \times 1/2 \\ &= 3/8. \end{aligned}$$

In the tree diagram, this corresponds to following the bottom-most path, multiplying the probabilities as we go. And this makes sense: half the time the coin will land tails, and on 3/4 of those occasions the marble drawn will be white. So, if we were to repeat the experiment again and again, we would get tails followed by a white marble in 3 out of every 8 trials.

⊘ Black hole warning: notice that the General Multiplication Rule depends on $Pr(A \mid B)$ being well-defined. So it only applies when $Pr(B) > 0$.

7.4 *Laplace's Urn Puzzle*

THE same urn scenario was used by 18th Century mathematician Laplace in one of his favourite puzzles. He asked what happens if we do *two* draws, with replacement. What's the probability both draws will come up black?

It's tempting to say $1/4$. The probability of drawing a black marble on each draw is $1/2$. So it *seems* the probability of two blacks is just $1/2 \times 1/2 = 1/4$.

But the correct answer is actually $5/16$. Why? Let's use a probability tree again.

Depending on how the coin lands, you could end up drawing either from Urn X or from Urn Y, with equal probability.

If you end up drawing from Urn X, the probability of a black marble on any given draw is $3/4$. Because the draws are independent (we're drawing with replacement), the probability they'll both come up black is $3/4 \times 3/4 = 9/16$.

If instead you end up drawing from Urn Y, the probability of a black marble on any given draw is $1/4$. The draws are still independent though, so the chance of both being black in this case is $1/4 \times 1/4 = 1/16$.

So the probability of drawing two black marbles from Urn X is:

$$\begin{aligned} Pr(X \,\&\, BB) &= Pr(X)Pr(BB \mid X) \\ &= 1/2 \times 9/16 \\ &= 9/32. \end{aligned}$$

And the probability of drawing two black marbles from Urn Y is:

$$\begin{aligned} Pr(Y \,\&\, BB) &= Pr(Y)Pr(BB \mid Y) \\ &= 1/2 \times 1/16 \\ &= 1/32. \end{aligned}$$

Now we can apply the Addition Rule to calcualte $Pr(BB)$:

$$\begin{aligned} Pr(BB) &= Pr(X \,\&\, BB) + Pr(Y \,\&\, BB) \\ &= 9/32 + 1/32 \\ &= 5/16. \end{aligned}$$

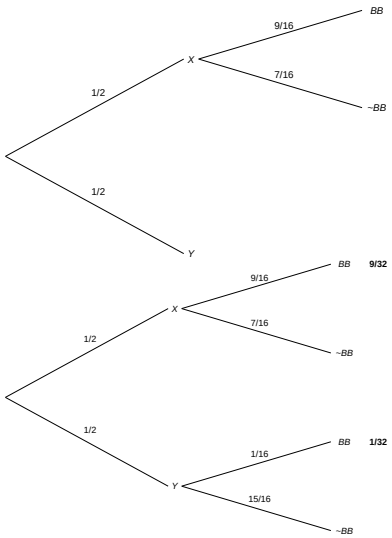


Figure 7.5: Building a probability tree to solve Laplace's urn puzzle

7.5 *The Law of Total Probability*

THIS kind of calculation comes up a lot. Since it would be tedious to figure it out from scratch every time, we make a general rule instead:

The Law of Total Probability $Pr(A) = Pr(A \mid B)Pr(B) + Pr(A \mid \sim B)Pr(\sim B)$.

There's an intuitive idea at work here. To figure out how likely A is, consider how likely it would be if B were true, and how likely it would be if B were false. Then weight each of those hypothetical possibilities according to their probabilities.

We can also use an Euler diagram. The size of the A region is the sum of the $A \,\&\, B$ region and the $A \,\&\, \sim B$ region: ■ + ■. And each of those regions can be calculated using the General Multiplication Rule. For example, $Pr(A \,\&\, B) = Pr(A \mid B)Pr(B)$. So in algebraic terms we have:

$$\begin{aligned} Pr(A) &= \blacksquare + \blacksquare \\ &= Pr(A \,\&\, B) + Pr(A \,\&\, \sim B) \\ &= Pr(A \mid B)Pr(B) + Pr(A \mid \sim B)Pr(\sim B). \end{aligned}$$

Which is precisely the Law of Total Probability.

We can also use a tree diagram to illustrate the same reasoning. There are two leaves where A is true, marked 😎 and 😍. To get the probability of each leaf we multiply across the branches (that's the General Multiplication Rule). And then to get the total probability for A, we add up the two leaves: $Pr(A) =$ 😎 + 😍. Once again the result is the Law of Total Probability:

$$Pr(A) = Pr(A \mid B)Pr(B) + Pr(A \mid \sim B)Pr(\sim B).$$

⊘ Black hole warning: notice that the Law of Total Probability depends on $Pr(A \mid B)$ and $Pr(A \mid \sim B)$ both being well-defined. So it only applies when $Pr(B) > 0$ and $Pr(\sim B) > 0$.

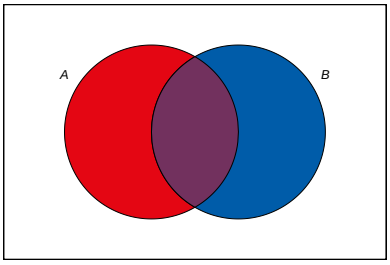


Figure 7.6: The Law of Total Probability calculates the size of the A region by summing its two part.

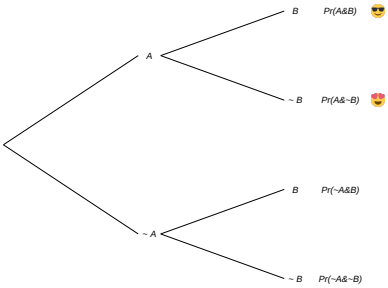


Figure 7.7: The Law of Total Probability in a tree diagram

7.6 *Example*

Every day Professor X either drives her car to campus or takes the bus. Mostly she drives, but one time in four she takes the bus. When she drives, she's on time 80% of the time. When she takes the bus, she's on-time 90% of the time.

What is the probability she'll be on time for class tomorrow?

First let's solve this by just applying the Law of Total Probability directly:

$$\begin{aligned} Pr(O) &= Pr(O \mid B)Pr(B) + Pr(O \mid D)Pr(D) \\ &= (9/10)(1/4) + (8/10)(3/4) \\ &= 33/40. \end{aligned}$$

Now let's solve it slightly differently, thinking the problem through from more basic principles.

There are two, mutually exclusive cases where Professor X is on time: one where she takes the bus, one where she drives.

$$Pr(O) = Pr(O \,\&\, B) + Pr(O \,\&\, D).$$

We can use the General Multiplication Rule to calculate the probability she'll take the bus and be on time:

$$Pr(O \,\&\, B) = Pr(O \mid B)Pr(B).$$

And we can do the same for the probability she'll drive and be on time:

$$Pr(O \,\&\, D) = Pr(O \mid D)Pr(D)$$

Putting all the pieces together:

$$\begin{aligned} Pr(O) &= Pr(O \,\&\, B) + Pr(O \,\&\, D) \\ &= Pr(O \mid B)Pr(B) + Pr(O \mid D)Pr(D) \\ &= (9/10)(1/4) + (8/10)(3/4) \\ &= 33/40. \end{aligned}$$

Notice that we didn't just get the same answer, we ended up doing the same calculation too. Our second approach just reconstructed from scratch the reasoning behind the Law of Total Probability. It's a very good idea to understand the rationale behind the Law of Total Probability. But once you get used to the formula, it's also fine to skip straight to applying it directly.

You can also use a tree diagram. Again, the calculation will be the same. But the diagram may help you get started, and it helps you check that you've applied the formula correctly too.

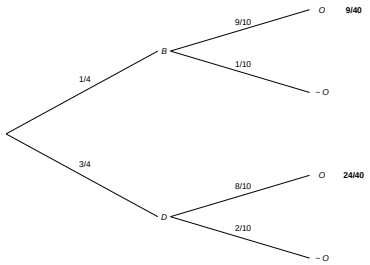


Figure 7.8: A probability tree for Professor X

Exercises

1. Suppose you have an ordinary deck of 52 playing cards, and you draw one card at random. What is the probability you will draw:

   a. A face card (king, queen, or jack)?
   b. A card that is not a face card?
   c. An ace or a spade?
   d. A queen or a heart?
   e. A queen or a non-spade?

2. Suppose that $Pr(A) = 1/3$, $Pr(B) = 1/4$, and that A and B are independent. What is $Pr(\sim A \,\&\, \sim B)$?

3. What is $Pr(X \vee B)$ in the first version of the urn problem? (The first version is the one where we start with a fair coin flip to choose between Urn X and Urn Y, then draw one marble at random.)

4. Recall Laplace's version of the urn puzzle: we select either Urn X or Urn Y at random, then we do two random draws from it, with replacement. What is $Pr(X \vee BB)$?

5. Suppose we add a third urn to Laplace's puzzle: Urn Z contains 2 black marbles and 2 white ones. We choose one of the three urns at random, and then do two random draws with replacement. What is $Pr(BB)$ then?

6. The Law of Total probability calculates $Pr(A)$ by considering two cases, B and $\sim B$. Notice that B and $\sim B$ form a partition: they are mutually exclusive and exhaustive possibilities.

   Suppose we had a partition of three propositions instead: B, C, and D. Would the following extension of the Law of Total Probability hold then?

   $$Pr(A) = Pr(A \mid B)Pr(B) + Pr(A \mid C)Pr(C) + Pr(A \mid D)Pr(D).$$

   Justify your answer.

7. Suppose there are two urns with the following contents:

   - Urn I has 8 black balls, 2 white.
   - Urn II has 2 black balls, 3 white.

   A fair coin will be flipped. If it comes up heads, a ball will be drawn from Urn I at random. Otherwise a ball will be drawn from Urn II at random. What is the probability a black ball will be drawn?

8. Suppose you have an ordinary deck of 52 cards. A card is drawn and is **not replaced**, then another card is drawn. Assume that on each draw all the cards then in the deck have an equal chance of being drawn.

   a. What is the probability of getting an ace on draw 1?
   b. What is the probability of a ten on draw 2 given ace on draw 1?
   c. What is the probability of an ace on draw 1 and a ten on draw 2?
   d. What is the probability of a ten on draw 1 and an ace on draw 2?
   e. What is the probability of an ace and a ten?
   f. What is the probability of 2 aces?

9. The probability that George will study for the final is 4/5. The probability he will pass given that he studies is 3/5. The probability he will pass given that he does not study is 1/10. What is the probability George will pass?

10. Calculate each of the following probabilities:

a. $Pr(P) = 1/2$, $Pr(Q) = 1/2$, $Pr(P \,\&\, Q) = 1/8$. What is $Pr(P \vee Q)$?
b. $Pr(R) = 1/2, Pr(S) = 1/4, Pr(R \vee S) = 3/4$. What is $Pr(R \,\&\, S)$?
c. $Pr(U) = 1/2, Pr(T) = 3/4, Pr(U \,\&\, \sim T) = 1/8$. What is $Pr(U \vee \sim T)$?

11. $Pr(P) = 1/2, Pr(Q) = 1/2$, and P and Q are independent.

    a. What is $Pr(P \,\&\, Q)$?
    b. Are P and Q mutually exclusive?
    c. What is $Pr(P \vee Q)$?

12. Suppose A, B, and C are all mutually exclusive, and they each have the same probability: $1/5$. What is $Pr(\sim(A \,\&\, B) \,\&\, C)$?

13. Researchers are studying the safety of drug X. They enroll 60 subjects in a study and give drug X to 35 of them. By the end of the study, 5 subjects have developed stomach cancer: 3 who were taking drug X, 2 who were not.

    Draw a Venn diagram and use it to answer the following questions about a randomly selected subject:

    a. What is the probability they developed stomach cancer?
    b. What is the probability they developed stomach cancer given that they were taking drug X?
    c. What is the probability they developed stomach cancer given that they were not taking drug X?
    d. Based on this study, would you conclude that drug X increases or decreases the risk of stomach cancer?

14. There is a room filled with two types of urns.

    - Type A urns contain 30 yellow marbles, 70 red.
    - Type B urns contain 20 green marbles, 80 yellow.

    The two types of urn look identical, but 80% of them are Type A.

    a. You pick an urn at random and draw a marble from it at random. What is the probability the marble will be yellow?
    b. You look at the marble: it is yellow. What's the probability the urn is a Type B urn?

15. Suppose A, B, and C are independent of one another. Does it follow that $Pr(B \mid A \,\&\, C) = Pr(B)$? Justify your answer.

16. Is the following combination of probabilities possible? $Pr(A) = 2/5$, $Pr(B) = 4/5$, and $Pr(A \vee B) = 3/5$. Justify your answer.

17. Which of the following situations is impossible? Justify your answer.

    a. $Pr(A) = 4/5$, $Pr(B) = 1/5$, $Pr(\sim A \,\&\, B) = 3/5$.
    b. $Pr(\sim X) = 1/3$, $Pr(\sim Y) = 2/3$, $Pr(X \,\&\, Y) = 0$.

18. If $Pr(A) = 0$, what is $Pr(A \mid B)$? Justify your answer.

19. Do Exercise 2 from p. 67 of Hacking.

20. Do Exercise 3 from p. 67 of Hacking.

21. If A and B are logically equivalent, what is $Pr(A \mid B)$? Justify your answer.

22. Suppose A, B, and C all have the same probability, namely $1/4$. Suppose they are also independent of one another. What is $Pr(\sim A \vee \sim B \vee \sim C)$?

    Hint: $\sim A \vee \sim B \vee \sim C$ is logically equivalent to $\sim(A \,\&\, B \,\&\, C)$. Why?

23. If $Pr(A) = 1/2$ and $Pr(B) = 3/5$, are A and B mutually exclusive? Justify your answer.

24. Suppose $Pr(A) = 1/4$, $Pr(B) = 1/3$, and A and B are independent. What is $Pr(A \vee (B \,\&\, \sim A))$?

25. Suppose A logically entails C, and A and B are independent. If $Pr(A) = 1/7$, $Pr(B) = 1/3$, and $Pr(C) = 1/3$, what is $Pr((A \,\&\, \sim B) \vee \sim C)$?

26. If A and B are mutually exclusive, must the following hold?

$$Pr(A \vee B \mid C) = Pr(A \mid C) + Pr(B \mid C).$$

    Assume the conditional probabilities are all well-defined, and justify your answer.

    Hint: apply the definition of conditional probability and use the following fact: $(A \vee B) \,\&\, C$ is logically equivalent to $(A \,\&\, C) \vee (B \,\&\, C)$.

27. Prove that if A and B are mutually exclusive, then

$$Pr(A \mid A \vee B) = \frac{Pr(A)}{Pr(A) + Pr(B)}.$$

28. If $Pr(C \mid B \,\&\, A) = Pr(C \mid B)$, does this follow?

$$Pr(A \mid B \,\&\, C) = Pr(A \mid B).$$

    Assume all conditional probabilities are well-defined, and justify your answer.

29. Justify the claim from Chapter 6 that independence extends to negations: if A is independent of B, then it's also independent of $\sim B$ (provided $Pr(\sim A) > 0$).

    Warning: this one is hard. I suggest starting with the equation:

    $$Pr(A \,\&\, B) = Pr(A)Pr(B).$$

    Then use the Negation Rule which tells us:

    $$Pr(B) = 1 - Pr(\sim B).$$

    And use the Addition Rule to get:

    $$Pr(A \,\&\, B) = Pr(A) - Pr(A \,\&\, \sim B).$$

8 Bayes' Theorem

In a famous psychology experiment, subjects were asked to solve the following problem.

> A cab was involved in a hit and run accident at night. Two cab companies, the Green and the Blue, operate in the city. You are given the following data:
>
> 1. 85% of the cabs in the city are Green and 15% are Blue.
> 2. A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.
>
> What is the probability that the cab involved in the accident was blue rather green?

The experiment was first published in 1971. It was performed by Daniel Kahneman and Amos Tversky. Their work on human reasoning reshaped the field of psychology, and eventually won a Nobel prize in 2002.

Most people answer 80%, because the witness is 80% reliable. But the right answer is 12/29, or about 41%.

How could the probability be so low when the witness is 80% reliable? The short answer is: because blue cabs are rare. So most of the time, when the witness says a cab is blue, it's one of the 20% of green cabs they mistakenly identify as blue.

But this answer still needs some explaining. Let's use a diagram.

Imagine there are just 100 cabs in town, 85 green and 15 blue. The dashed blue line represents the cabs the witness identifies as "blue", both right or wrong. Because the witness is 80% accurate, that line encompasses 80% of the blue cabs, which is 12 cabs. But it also encompasses 20% of the green cabs, which is 17. That's a total of 29 cabs identified as "blue", only 12 of which actually are blue.

So out of the 29 cabs the witness calls "blue", only 12 really are blue. The probability a cab really is blue given the witness says so is only 12/29, about 41%.

Another way to think about the problem is that there are *two* pieces of information relevant to whether the cab is blue. The witness says

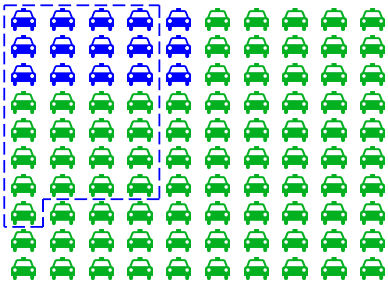


Figure 8.1: The taxicab problem. There are 85 blue cabs, 15 green. The dashed region indicates those cabs the witness identifies as "blue". It includes 80% of the blue cabs (12), and only 20% of the green ones (17). Yet it includes more green cabs than blue.

the cab is blue, but also, most cabs are not blue. So there's evidence for the cab being blue, but also strong evidence against it. The diagram in Figure 8.1 shows us how to balance these two, competing pieces of evidence and come to the correct answer.

What trips people up so much in the taxicab problem? Remember how order matters with conditional probability. In this problem, we're asked to find $Pr(B \mid W)$, the probability the cab is blue given that the witness says it is. That's not the same as $Pr(W \mid B)$, the probability the witness will say the cab is blue if it really is. The problem tells us $Pr(W \mid B) = 8/10$, but it doesn't tell us a number for $Pr(B \mid W)$. We have to figure that out.

We saw back in Chapter 6 that $Pr(A \mid B)$ is usually a different number from $Pr(B \mid A)$. A university student will probably be young, but a young person will probably not be a university student. That's an example where it's easy to see that order matters. The taxicab example makes it much less obvious, in fact it tempts us to confuse $Pr(B \mid W)$ with $Pr(W \mid B)$.

8.1 Bayes' Theorem

PROBLEMS where we're given $Pr(B \mid A)$ and we have to figure out $Pr(A \mid B)$ are extremely common. Luckily, there's a famous formula for solving them.

Thomas Bayes (1701–1761) was an English minister and mathematician, the first to formulate the theorem that now bears his name.

Bayes' Theorem If $Pr(A), Pr(B) > 0$, then

$$Pr(A \mid B) = \frac{Pr(A)Pr(B \mid A)}{Pr(B)}.$$

Notice two things here. First, we need $Pr(A)$ and $Pr(B)$ to both be positive, because otherwise $Pr(A \mid B)$ and $Pr(B \mid A)$ aren't well-defined. Second, we need to know more than just $Pr(B \mid A)$ to apply the formula. We also need numbers for $Pr(A)$ and $Pr(B)$.

In the taxicab problem we're given two of the three pieces of information we need. Here's Bayes' theorem for the taxicab example:

$$Pr(B \mid W) = \frac{Pr(B)Pr(W \mid B)}{Pr(W)}.$$

Whereas the problem gives us the following information:

- $Pr(W \mid B) = 80/100$.
- $Pr(W \mid \sim B) = 20/100$.
- $Pr(B) = 15/100$.
- $Pr(\sim B) = 85/100$.

What's missing for Bayes' Theorem is $Pr(W)$. Fortunately, we can calculate it with the Law of Total Probability!

$$\begin{aligned} Pr(W) &= Pr(W \mid B)Pr(B) + Pr(W \mid \sim B)Pr(\sim B) \\ &= (80/100)(15/100) + (20/100)(85/100 \\ &= 29/100. \end{aligned}$$

And now we have everything we need to plug into Bayes' Theorem:

$$\begin{aligned} Pr(B \mid W) &= \frac{Pr(B)Pr(W \mid B)}{Pr(W)} \\ &= \frac{(15/100)(80/100)}{29/100} \\ &= 12/29. \end{aligned}$$

This is the same answer we got with our grid diagram (Figure 8.1). And notice, it's also the answer we'd get from a tree diagram too (Figure 8.2).

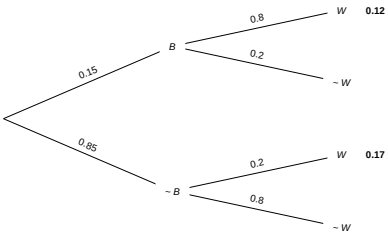


Figure 8.2: Tree diagram for the taxicab problem. Since $Pr(B \,\&\, W) = .12$ and $Pr(W) = .12 + .17$, the definition of conditional probability yields $Pr(B \mid W) = 12/29$.

8.2 *Understanding Bayes' Theorem*

WHY does Bayes' theorem work? One way to think about it is to start by recalling the definition of conditional probability:

$$Pr(A \mid B) = \frac{Pr(A \,\&\, B)}{Pr(B)}.$$

Then apply the General Multiplication Rule to the numerator:

$$Pr(A \,\&\, B) = Pr(B \mid A)Pr(A).$$

And plug that back into the first equation:

$$Pr(A \mid B) = \frac{Pr(A)Pr(B \mid A)}{Pr(B)}.$$

This proves that Bayes' theorem is correct. But it also suggests a way of understanding it visually, with an Euler diagram.

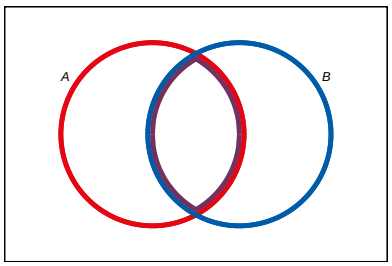


Figure 8.3: An Euler diagram for visualizing Bayes' theorem

WE'VE TALKED before about $Pr(A \mid B)$ being the portion of the B region that's inside the A region. The purple portion of the blue B circle, in other words:

$$Pr(A \mid B) = \frac{Pr(A \,\&\, B)}{Pr(B)}.$$

So when we apply the General Multiplication Rule to the purple region we get:

$$Pr(A \mid B) = \frac{Pr(B \mid A)Pr(A)}{Pr(B)}.$$

We'll come back to another, non-visual way of understanding Bayes' theorem in Chapter 10.

There are lots more visual explanations of Bayes' theorem you can find online. There's even one using Legos! But they all tend to come down to the same thing. A two step explanation that goes:

1. Start with the definition of conditional probability, which we learned how to visualize in Chapter 6.
2. Then apply the General Multiplication Rule, which we learned how to visualize in Chapter 7.

This is a perfectly good and helpful way to think about Bayes' theorem. But it's not really a visualization of the theorem itself. It's two separate visualizations of the ingredients we use to derive the theorem.

In any case, Bayes' theorem comes up so often it's good to memorize the formula itself. The visualization in Figure 8.4 is probably about as helpful as it gets for this purpose.

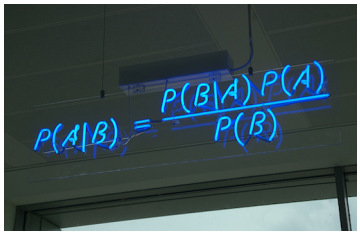


Figure 8.4: Bayes' theorem on display at the offices of HP Autonomy, in Cambridge, UK

8.3 *Bayes' Long Theorem*

We had to apply the Law of Total Probability first, before we could solve the taxicab problem with Bayes' theorem, to calculate the denominator. This is so common that you'll often see Bayes' theorem written with this calculation built in. That is, the denominator $Pr(B)$ is expanded out using the Law of Total Probability.

Bayes' Theorem (long version) If $1 > Pr(A) > 0$ and $Pr(B) > 0$, then

$$Pr(A \mid B) = \frac{Pr(A)Pr(B \mid A)}{Pr(A)Pr(B \mid A) + Pr(\sim A)Pr(B \mid \sim A)}.$$

Notice how there's some repetition in the numerator and the denominator. The term $Pr(A)Pr(B \mid A)$ appears both above and below. So, when you're doing a calculation with this formula, you can just do that bit once and then copy it in both the top and bottom. Then you just have to do the bottom-right term to complete the formula.

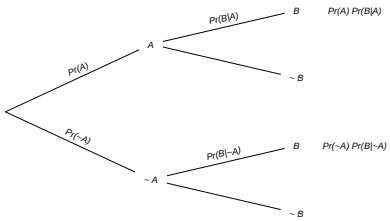


Figure 8.5: A tree diagram for the long form of Bayes' theorem. The definition of conditional probability tells us $Pr(A \mid B)$ is the first leaf divided by the sum of the first and third leaves.

A tree diagram helps illuminate the long version of Bayes' theorem. To calculate $Pr(A \mid B)$, the definition of conditional probability directs us to calculate $Pr(A \,\&\, B)$ and $Pr(B)$:

$$Pr(A \mid B) = \frac{Pr(A \,\&\, B)}{Pr(B)}.$$

Looking at the tree diagram in Figure 8.5, we see that this amounts to computing the first leaf for the numerator, and the sum of the first and third leaves for the denominator. Which yields the same formula as in the long form of Bayes' theorem.

8.4 Example

LET's practice the long form of Bayes' theorem.

> Joe has just heard about the Zika virus and wonders if he has it. His doctor tells him only 1% of the population has the virus, but he's still worried. There's a blood test he can take, but it's not perfect. The test always comes up either negative or positive, but:
>
> - 95% of people who have the virus test positive.
> - 85% of people who don't have the virus test negative.
>
> If Joe tests positive, what is the probability he really has the Zika virus?

We're asked to calculate $Pr(Z \mid P)$, and we're given the following:

- $Pr(Z) = 1/100$.
- $Pr(P \mid Z) = 95/100$.
- $Pr(P \mid \sim Z) = 15/100$.

So we can calculate $Pr(Z \mid P)$ using the long form of Bayes' theorem:

$$\begin{aligned} Pr(Z \mid P) &= \frac{Pr(Z)Pr(P \mid Z)}{Pr(Z)Pr(P \mid Z) + Pr(\sim Z)Pr(P \mid \sim Z)} \\ &= \frac{(1/100)(95/100)}{(1/100)(95/100) + (99/100)(15/100)} \\ &= \frac{95}{95 + 1,485} \\ &= 19/316 \\ &\approx .06. \end{aligned}$$

The calculation is also diagrammed in Figure 8.6.

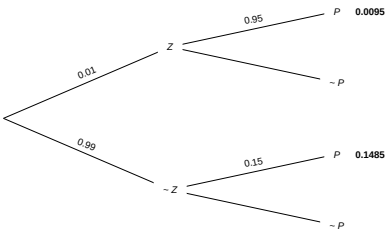


Figure 8.6: Tree diagram of the Zika problem

It turns out there's only about a 6% chance Joe has the virus. Even though he tested positive with a fairly reliable blood test! It's counterintuitive, but the reason is the same as with the taxicab problem. There are two, conflicting pieces of evidence: the blood test is positive but the virus is rare. It turns out the virus is so rare that the positive blood test doesn't do much to increase Joe's chances of being infected.

BAYES' THEOREM doesn't always give such surprising results. In fact the results are often very intuitive. Professors just like to use the counterintuitive examples to demonstrate how important Bayes' theorem is. Without it, it's easy to go astray.

8.5 The Base Rate Fallacy

In the Zika example, the rate of infection in the general population is very low, just 1%. The rate at which something happens in general is called the *base rate*. In the taxicab example, the base rate for blue cabs was 15%.

One lesson of this chapter is that you have to take the base rate into account to get the right answer, via Bayes' theorem. Humans have a tendency to ignore the base rate, and focus only on the "test" performed: the blood test in the Zika example, the testimony of the witness in the taxicab example. This mistake is called *base rate neglect*, or *the base rate fallacy*.

The base rate fallacy is so tempting, even trained professionals are susceptible to it. In one famous study, 160 medical doctors were given a problem similar to our Zika example (but with cancer instead of Zika). The question was multiple choice, and it used easier numbers than we did. Yet only 34 of the doctors got it right.

Hence the quote that opens Chapter 1 of this book: "in no other branch of mathematics is it so easy for experts to blunder as in probability theory."

Exercises

1. Recall this problem from Chapter 6:

   > Five percent of tablets made by the company Ixian have factory defects. Ten percent of the tablets made by their competitor company Guild do. A computer store buys 40% of its tablets from Ixian, and 60% from Guild.

   Use Bayes' theorem to find $Pr(I \mid D)$, the probability a tablet from this store is made by Ixian, given that it has a factory defect?

2. Recall this problem from Chapter 6:

   > In the city of Elizabeth, the neighbourhood of Southside has lots of chemical plants. 2% of Elizabeth's children live in Southside, and 14% of those children have been exposed to toxic levels of lead. Elsewhere in the city, only 1% of the children have toxic levels of exposure.

   Use Bayes' theorem to find $Pr(S \mid L)$, the probability that a randomly chosen child from Elizabeth who has toxic levels of lead exposure lives in Southside?

3. The probability that Nasim will study for her test is $4/10$. The probability that she will pass, given that she studies, is $9/10$.

The probability that she passes, given that she does not study, is 3/10. What is the probability that she has studied, given that she passes?

4. At the height of flu season, roughly 1 in every 100 people have the flu. But some people don't show symptoms even when they have it: only half the people who have the virus show symptoms.

   Flu symptoms can also be caused by other things, like colds and allergies. So about 1 in every 20 people who don't have the flu still have flu-like symptoms.

   If someone has flu-like symptoms at the height of flu season, what is the probability that they actually have the flu?

5. A magic shop sells two kinds of trick coins. The first kind are biased towards heads: they come up heads 9 times out of 10 (the tosses are independent). The second kind are biased towards tails: they comes up tails 8 times out of 10 (tosses still independent). Half the coins are the first kind, half are the second kind. But they don't label the coins, so you have to experiment to find out which are which.

   You pick a coin at random and flip it twice. Given that it comes up heads both times, what is the probability it's the first kind of coin?

6. There is a room filled with two types of urns.

   - Type A urns have 30 yellow marbles, 70 red.
   - Type B urns have 20 green marbles, 80 yellow.

   The two types of urn look identical, but 80% of them are Type A. You pick an urn at random and draw a marble from it at random.

   a. What is the probability the marble will be yellow?

   Now you look at the marble: it is yellow.

   b. What is the probability the urn is a Type B urn, given that you drew a yellow marble?

   Suppose now you put the yellow marble back, shake hard, and draw another marble at random from the same urn.

   c. If this marble is also yellow, what is the probability the urn is a Type B urn?
   d. If this marble is instead green, what is the probability the urn is a Type B urn?

7. In the long form of Bayes' theorem, the denominator is broken down by applying the Law of Total Probability to a partition of two propositions, A and $\sim A$. We can extend the same idea to a partition of three propositions, X, Y, and Z. Here is a start on the formula:

$$Pr(X \mid B) = \frac{Pr(X)Pr(B \mid X)}{Pr(X)Pr(B \mid X) + \ldots}.$$

   Fill in the rest of the formula, then justify it.

8. A company makes websites, always powered by one of three server platforms: Bulldozer, Kumquat, or Penguin. Bulldozer crashes 1 out of every 10 visits, Kumquat crashes 1 in 50 visits, and Penguin only crashes 1 out of every 200 visits.

   Half of the websites are run on Bulldozer, 30% are run on Kumquat, and 20% are run on Penguin.

   You visit one of their sites for the first time and it crashes. What is the probability it was run on Penguin?

   This problem is based on Exercise 6 from p. 78 of Ian Hacking's *An Introduction to Probability & Inductive Logic*.

9. You and Carlos are at a party, which means there's a 2/3 chance he's been drinking. You decide to experiment to find out: you throw a tennis ball to Carlos and he misses the catch. Five minutes later you try again and he misses again. Assume the two catches are independent.

   When he's sober, Carlos misses a catch only two times out of ten. When he's been drinking, Carlos misses catches half the time.

   What is the probability that Carlos has been drinking, given that he missed both catches?

10. The Queen Gertrude Hotel has two kinds of suites: singles have one bed, royal suites have three beds. There are 80 singles and 20 royals.

    In a single, the probability of bed bugs is 1/100. But every additional bed put in a suite doubles the chance of bed bugs.

    If a suite is inspected at random and bed bugs are found, what is the probability it's a royal?

11. Willy Wonka Chocolates Inc. makes two kinds of boxes of chocolates. The "wonk box" has four caramel chocolates and six regular chocolates. The "zonk box" has six caramel chocolates, two regular chocolates, and two mint chocolates. A third of their boxes are wonk boxes, the rest are zonk boxes.

    They don't mark the boxes. The only way to tell what kind of box you've bought is by trying the chocolates inside. In fact, all

the chocolates look the same; you can only tell the difference by tasting them.

If you buy a random box, try a chocolate at random, and find that it's caramel, what is the probability you've bought a wonk box?

12. A room contains four urns. Three of them are Type X, one is Type Y.

    - The Type X urns each contain 3 black marbles, 2 white marbles.
    - The Type Y urn contains 1 black marble, 4 white marbles.

    You are going to pick an urn at random and start drawing marbles from it at random *without* replacement.

    What is the probability the urn is Type X if the first draw is black?

9 *Multiple Conditions*

We often need to account for multiple pieces of evidence. More than one witness testifies about the colour of a taxicab; more than one person responds to our poll about an upcoming election; etc.

How do we a calculate conditional probability when there are multiple conditions? In other words, how do we handle quantities of the form $Pr(A \mid B_1 \,\&\, B_2 \,\&\, \ldots)$?

9.1 Multiple Draws

Imagine you're faced with another one of our mystery urns. There are two equally likely possibilities:

A : The urn contains 70 black marbles, 30 white marbles.

$\sim A$: The urn contains 20 black marbles, 80 white marbles.

Now suppose you draw a marble at random and it's black. You put it back, give the urn a good shake, and then draw another: black again. What's the probability the urn has 70 black marbles?

We need to calculate $Pr(A \mid B_1 \,\&\, B_2)$, the probability of A given that the first and second draws were both black. We already know how to do this calculation for one draw, $Pr(A \mid B_1)$. We use Bayes' theorem to get:

$$\begin{aligned} Pr(A \mid B_1) &= \frac{Pr(B_1 \mid A)Pr(A)}{Pr(B_1 \mid A)Pr(A) + Pr(B_1 \mid \sim A)Pr(\sim A)} \\ &= \frac{(70/100)(1/2)}{(70/100)(1/2) + (20/100)(1/2)} \\ &= 7/9. \end{aligned}$$

But for two draws, Bayes' theorem gives us:

$$Pr(A \mid B_1 \,\&\, B_2) = \frac{Pr(B_1 \,\&\, B_2 \mid A)Pr(A)}{Pr(B_1 \,\&\, B_2 \mid A)Pr(A) + Pr(B_1 \,\&\, B_2 \mid \sim A)Pr(\sim A)}.$$

To fill in the values on the right hand side, we need to know these quantities:

- $Pr(B_1 \,\&\, B_2 \mid A)$,

- $Pr(B_1 \,\&\, B_2 \mid \sim A)$.

To get the first quantity, remember that we replaced the first marble before doing the second draw. So, given A, the second draw is independent of the first. There are still 70 black marbles out of 100 on the second draw, so the chance of black on the second draw is still 70/100. In other words:

$$\begin{aligned} Pr(B_1 \,\&\, B_2 \mid A) &= Pr(B_1 \mid A)Pr(B_2 \mid A) \\ &= (70/100)^2. \end{aligned}$$

The same reasoning applies given $\sim A$, too. Except here the chance of black on each draw is 20/100. So:

$$\begin{aligned} Pr(B_1 \,\&\, B_2 \mid \sim A) &= Pr(B_1 \mid A)Pr(B_2 \mid \sim A) \\ &= (20/100)^2. \end{aligned}$$

Returning to Bayes' theorem, we can now finish the calculation:

$$\begin{aligned} Pr(A \mid B_1 \,\&\, B_2) &= \frac{Pr(B_1 \,\&\, B_2 \mid A)Pr(A)}{Pr(B_1 \,\&\, B_2 \mid A)Pr(A) + Pr(B_1 \,\&\, B_2 \mid \sim A)Pr(\sim A)} \\ &= \frac{(70/100)^2(1/2)}{(70/100)^2(1/2) + (20/100)^2(1/2)} \\ &= 49/53. \end{aligned}$$

THE same solution can also be captured in a probability tree. The tree will have an extra stage now, because there's a second draw. And it will have many more leaves, but luckily we can ignore most of them. We just need to worry about the two leaves where both draws have come up black. And we only need to fill in the probabilities along the paths that lead to those two leaves. The result is Figure 9.1.

So $Pr(A \mid B_1 \,\&\, B_2) = 0.245/(0.245 + 0.2)$, which is the same as 49/53, the answer we got with Bayes' theorem.

YOU might be able to guess now what would happen after three black draws. Instead of getting squared probabilities in Bayes' theorem, we'd get cubed probabilities. And using the same logic, we could keep going. We could use Bayes' theorem to calculate $Pr(A \mid B_1 \,\&\, \ldots \,\&\, B_n)$ for as many draws n as you like.

9.2 *Multiple Witnesses*

LET'S TRY a different sort of problem with multiple conditions. Recall the taxicab problem from Chapter 8:

> A cab was involved in a hit and run accident at night. Two cab companies, the Green and the Blue, operate in the city. You are given the following data:

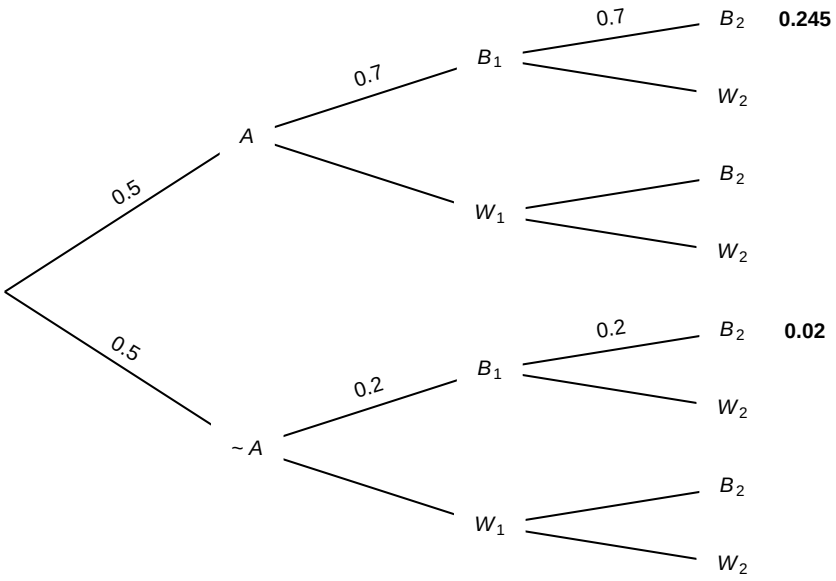


Figure 9.1: Tree diagram for two draws with replacement

1. 85% of the cabs in the city are Green and 15% are Blue.
2. A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.

What is the probability that the cab involved in the accident was blue rather green?

We saw it's only about 41% likely the cab was really blue, even with the witness' testimony. But what if there had been two witnesses, both saying the cab was blue?

Let's use Bayes' theorem again:

$$Pr(B \mid W_1 \,\&\, W_2) = \frac{Pr(B)Pr(W_1 \,\&\, W_2 \mid B)}{Pr(W_1 \,\&\, W_2)}.$$

We have one of the terms here already: $Pr(B) = 15/100$. What about the other two:

- $Pr(W_1 \,\&\, W_2 \mid B)$,
- $Pr(W_1 \,\&\, W_2)$.

Let's make things easy on ourselves by assuming our two witnesses are reporting independently. They don't talk to each other, or influence one another in any way. They're only reporting what they saw (or think they saw). Then we can "factor" these probabilities like we did

when sampling with replacement:

$$\begin{aligned} Pr(W_1 \,\&\, W_2 \mid B) &= Pr(W_1 \mid B)Pr(W_2 \mid B) \\ &= (80/100)^2. \end{aligned}$$

And for the denominator we use the Law of Total Probability:

$$\begin{aligned} Pr(W_1 \,\&\, W_2) &= Pr(W_1 \,\&\, W_2 \mid B)Pr(B) + Pr(W_1 \,\&\, W_2 \mid \sim B)Pr(\sim B) \\ &= (80/100)^2(15/100) + (20/100)^2(85/100) \\ &= 96/1000 + 34/1000 \\ &= 13/100. \end{aligned}$$

Now we can return to Bayes' theorem to finish the problem:

$$\begin{aligned} Pr(B \mid W_1 \,\&\, W_2) &= \frac{(15/100)(80/100)^2}{13/100} \\ &= 96/130 \\ &\approx .74. \end{aligned}$$

So, with two witnesses independently agreeing that the cab was blue, the probability goes up from less than $1/2$ to almost $3/4$.

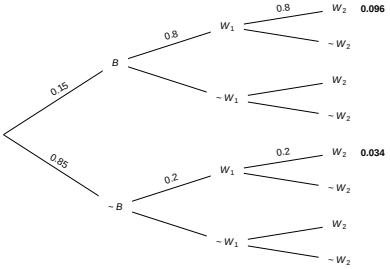


Figure 9.2: Tree diagram for the two-witness taxicab problem

We can use a tree here too, similar to the one we made when sampling two black marbles with replacement. As before, we only need to worry about the $W_1 \,\&\, W_2$ leaves, the ones where both witnesses say the cab was blue. The result is Figure 9.2, which tells us that $Pr(B \mid W_1 \,\&\, W_2) = 0.096/(0.096 + 0.034)$, which is approximately 0.74.

9.3 *Without Replacement*

The problems we've done so far were simplified by assuming independence. We sampled with replacement in the urn problem, and we assumed our two witnesses were independently reporting what they saw in the taxicab problem. What about when independence doesn't hold?

Let's go back to our urn problem, but this time suppose we don't replace the marble after the first draw. How do we calculate $Pr(A \mid B_1 \,\&\, B_2)$ then?

We're still going to start with Bayes' theorem:

$$Pr(A \mid B_1 \,\&\, B_2) = \frac{Pr(B_1 \,\&\, B_2 \mid A)Pr(A)}{Pr(B_1 \,\&\, B_2 \mid A)Pr(A) + Pr(B_1 \,\&\, B_2 \mid \sim A)Pr(\sim A)}.$$

But to calculate terms like $Pr(B_1 \,\&\, B_2 \mid A)$ now, we need to think things through in two steps.

We know the first draw has a $70/100$ chance of coming up black if A is true:

$$Pr(B_1 \mid A) = 70/100.$$

And once the first draw has come up black, if A is true then there are 69 black balls remaining and 30 white. So:

$$Pr(B_2 \mid B_1 \,\&\, A) = 69/99.$$

So instead of multiplying 70/100 by itself, we're multiplying 70/100 by *almost* 70/100:

$$\begin{aligned} Pr(B_1 \,\&\, B_2 \mid A) &= (70/100)(69/99) \\ &= 161/300. \end{aligned}$$

Using similar reasoning for the possibility that $\sim A$ instead, we can calculate

$$\begin{aligned} Pr(B_1 \,\&\, B_2 \mid \sim A) &= (20/100)(19/99) \\ &= 19/495. \end{aligned}$$

Returning to Bayes' theorem to finish the calculation:

$$\begin{aligned} Pr(A \mid B_1 \,\&\, B_2) &= \frac{Pr(B_1 \,\&\, B_2 \mid A)Pr(A)}{Pr(B_1 \,\&\, B_2 \mid A)Pr(A) + Pr(B_1 \,\&\, B_2 \mid \sim A)Pr(\sim A)} \\ &= \frac{(161/300)(1/2)}{(161/300)(1/2) + (19/495)(1/2)} \\ &= 5313/5693 \\ &\approx .93. \end{aligned}$$

Notice how similar this answer is to the .92 we got when sampling with replacement. With so many black and white marbles in the urn, taking one out doesn't make much difference. The second draw is almost the same as the first, so the final answer isn't much affected.

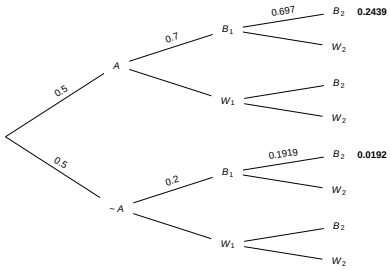


Figure 9.3: Tree diagram for two draws with replacement, values rounded

THE tree diagram for this problem will also be similar to the with-replacement version. The key difference is the probabilities at the last stage of the tree. Without independence, the probability of a B_2 branch is affected by the B_1 that precedes it. The result is Figure 9.3, though note that some values are rounded. Still we find that:

$$\begin{aligned} Pr(A \mid B_1 \,\&\, B_2) &\approx \frac{0.2439}{0.2439 + 0.0192} \\ &\approx 0.93. \end{aligned}$$

9.4 *Multiplying Conditional Probabilities*

THE calculation we just did relied on a new rule, which we should make explicit. Start by recalling a familiar rule:

The General Multiplication Rule $Pr(A \,\&\, B) = Pr(A \mid B)Pr(B)$.

Our new rule applies the same idea to situations where some proposition C is taken as a given.

The General Multiplication Rule (Conditional Version) $Pr(A \,\&\, B \mid C) = Pr(A \mid B \,\&\, C)Pr(B \mid C)$.

In a way, the new rule isn't really new. We just have to realize that the probabilities we get when we take a condition C as given are still probabilities. They obey all the same rules as unconditional probabilities, and this includes the General Multiplication Rule.

Another example which illustrates this point is the Negation Rule. The following conditional version is also valid:

The Negation Rule (Conditional Version) $Pr(\sim A \mid C) = 1 - Pr(A \mid C)$.

We could go through all the rules of probability we've learned and write out the conditional version for each one. But we've already got enough rules and equations to keep track of. So let's just remember this mantra instead:

- Conditional probabilities are probabilities.

So if we have a rule of probability, the same rule will hold if we add a condition C into each of the $Pr(\ldots)$ terms.

9.5 Summary

WE'VE LEARNED TWO STRATEGIES for calculating conditional probabilities with multiple conditions.

The first strategy is easier, but it only works when the conditions are appropriately independent. Like when we sample with replacement, or when two witnesses independently report what they saw.

In this kind of case, we first use Bayes' theorem, and then "factor" the terms:

$$\begin{aligned} Pr(A \mid B_1 \,\&\, B_2) &= \frac{Pr(B_1 \,\&\, B_2 \mid A)Pr(A)}{Pr(B_1 \,\&\, B_2 \mid A)Pr(A) + Pr(B_1 \,\&\, B_2 \mid \sim A)Pr(\sim A)} \\ &= \frac{Pr(B_1 \mid A)Pr(B_2 \mid A)Pr(A)}{Pr(B_1 \mid A)Pr(B_2 \mid A)Pr(A) + Pr(B_1 \mid \sim A)Pr(B_2 \mid \sim A)Pr(\sim A)} \\ &= \frac{(Pr(B_1 \mid A))^2 Pr(A)}{(Pr(B_1 \mid A))^2 Pr(A) + (Pr(B_1 \mid \sim A))^2 Pr(\sim A)}. \end{aligned}$$

Our second strategy is a little more difficult. But it works even when the conditions are **not** independent. We still start with Bayes' theorem. But then we apply the conditional form of the General Multiplication Rule:

$$\begin{aligned} Pr(A \mid B_1 \,\&\, B_2) &= \frac{Pr(B_1 \,\&\, B_2 \mid A)Pr(A)}{Pr(B_1 \,\&\, B_2 \mid A)Pr(A) + Pr(B_1 \,\&\, B_2 \mid \sim A)Pr(\sim A)} \\ &= \frac{Pr(B_2 \mid B_1 \,\&\, A)Pr(B_1 \mid A)Pr(A)}{Pr(B_2 \mid B_1 \,\&\, A)Pr(B_1 \mid A)Pr(A) + Pr(B_2 \mid B_1 \,\&\, \sim A)Pr(B_1 \mid \sim A)Pr(\sim A)}. \end{aligned}$$

These are some pretty hairy formulas, so memorizing them probably isn't a good idea. It's better to understand how they flow from Bayes' theorem or a tree diagram.

Exercises

1. Recall the following problem from Chapter 8.

   Willy Wonka Co. makes two kinds of boxes of chocolates. The "wonk box" has four caramel chocolates and six regular chocolates. The "zonk box" has six caramel chocolates, two regular chocolates, and two mint chocolates. A third of their boxes are wonk boxes, the rest are zonk boxes.

   They don't mark the boxes. The only way to tell what kind of box you've bought is by trying the chocolates inside. In fact, all the chocolates look the same; you can only tell the difference by tasting them.

   Previously you calculated the probability a randomly chosen box is a wonk box given that a chocolate randomly selected from it is caramel. This time, suppose you randomly select two chocolates.

   a. What is the probability it's a wonk box given that both chocolates are caramel?
   b. What is the probability it's a wonk box given that the first is caramel and the second is regular?
   c. What is the probability it's a wonk box given that the first is regular and the second is caramel?

2. Recall the following problem from Chapter 8.

   A room contains four urns. Three of them are Type X, one is Type Y.

   - The Type X urns each contain 3 black marbles, 2 white marbles.
   - The Type Y urn contains 1 black marble, 4 white marbles.

   You are going to pick an urn at random and start drawing marbles from it at random *without* replacement.

   Previously you calculated the probability the urn is Type X given that the first draw is black.

   a. What is the probability the urn is Type X if the first draw is black and the second is white?
   b. What is the probability the urn is Type X if the first draw is white and the second is black?

c. What is the probability the third draw will be black, if the first draw is black and the second is white?

3. The order in which conditions are given doesn't matter. More precisely, the following equation always holds:

$$Pr(A \mid B \,\&\, C) = Pr(A \mid C \,\&\, B).$$

Use the rules of probability to prove that it always holds.

4. The order in which things happen often matters. If the light was red but is now green, the intersection is probably safe to drive through. But if the light was green and is now red, it's probably not safe.

We just saw, though, that the order in which conditions are given doesn't make any difference to the probability.

Explain why these two observations do not conflict.

5. Above we observed that $Pr(\sim A \mid C) = 1 - Pr(A \mid C)$. Prove that this equation holds. Hint: start with the definition of conditional probability, and then recall that $1 = Pr(C)/Pr(C)$.

10 Probability & Induction

We met some common types of inductive argument back in Chapter 2. Now that we know how to work with probability, let's use what we've learned to sharpen our understanding of how those arguments work.

10.1 Generalizing from Observed Instances

Generalizing from observed instances was the first major form of inductive argument we encountered. Suppose you want to know what colour a particular species of bird tends to be. Then you might go out and look at a bunch of examples:

> I've seen 10 ravens and they've all been black.
> Therefore, all ravens are black.

How strong is this argument?

Observing ravens is a lot like sampling from an urn. Each raven is a marble, and the population of all ravens is the urn. We don't know what nature's urn contains at first: it might contain only black ravens, or it might contain ravens of other colours too. To assess the argument's strength, we have to calculate $Pr(A \mid B_1 \,\&\, B_2 \,\&\, \ldots \,\&\, B_{10})$: the probability that all ravens in nature's urn are black, given that the first raven we observed was black, and the second, and so on, up to the tenth raven.

We learned how to solve simple problems of this form in the previous chapter. For example, imagine you face another of our mystery urns, and this time there are two equally likely possibilities:

> A : The urn contains only black marbles.
> $\sim A$: The urn contains an equal mix of black and white marbles.

If we do two random draws with replacement, and both are black, we

calculate $Pr(A \mid B_1 \,\&\, B_2)$ using Bayes' theorem:

$$\begin{aligned} Pr(A \mid B_1 \,\&\, B_2) &= \frac{Pr(B_1 \,\&\, B_2 \mid A)Pr(A)}{Pr(B_1 \,\&\, B_2 \mid A)Pr(A) + Pr(B_1 \,\&\, B_2 \mid {\sim}A)Pr({\sim}A)} \\ &= \frac{(1)^2(1/2)}{(1)^2(1/2) + (1/2)^2(1/2)} \\ &= 4/5. \end{aligned}$$

If we do a third draw with replacement, and it too comes up black, we replace the squares with cubes. On the fourth draw we'd raise to the fourth power. And so on. When we get to the tenth black draw, the calculation becomes:

$$\begin{aligned} Pr(A \mid B_1 \,\&\, B_2) &= \frac{(1)^{10}(1/2)}{(1)^{10}(1/2) + (1/2)^{10}(1/2)} \\ &= 1,024/1,025 \\ &\approx .999. \end{aligned}$$

So after ten black draws, we can be about 99.9% certain the urn contains only black marbles.

But that doesn't mean our argument that all ravens are black is 99.9% strong!

10.2 Real Life Is More Complicated

THERE are two major limitations to our urn analogy.

THE first limitation is that the ravens we observe in real life aren't randomly sampled from nature's "urn". We only observe ravens in certain locations, for example. But our solution to the urn problem relied on random sampling. For example, we assumed $Pr(B_1 \mid {\sim}A) = 1/2$ because the black marbles are just as likely to be drawn as the white ones, if there are any white ones.

If there are white ravens in the world though, they might be limited to certain locales.[1] So the fact we're only observing ravens in our part of the world could make a big difference to what we find. It matters whether your sample really is random.

[1] In fact there are white ravens, especially in one area of Vancouver Island.

THE second limitation is that we pretended there were only two possibilities: either all the marbles in the urn are black, or half of them are. And, accordingly, we assumed there was already a 1/2 chance all the marbles are black, before we even looked at any of them.

In real life though, when we encounter a new species, it could be that 90% of them are black, or 31%, or 42.718%, or any portion from 0% to 100%. So there are many, many more possibilities. The possibility

that *all* members of the new species (100%) are black is just one of these many possibilities. So it would start with a much lower probability than 1/2.

We could make our analysis more realistic by taking these complications into account. But the calculations would get ugly, and we'd have to use calculus to solve the problem. This is the kind of technical topic you'd cover in a math or statistics class on probability. But it's not the kind thing this book is about.

For our purposes, the key ideas that matter are as follows. First, the strength of an inductive argument is a question of conditional probability: how probable is the argument's conclusion given its premises? Second, an argument that generalizes from observed instances is similar to an urn problem, where we guess the contents of the urn by repeated sampling. And third, we know how to solve simple versions of these urn problems using Bayes' theorem. In principle, we could also solve more complicated and realistic versions using the same fundamental ideas, the math would just be harder.

In a famous calculation, Laplace showed how to solve the second problem. The result was his famous rule of succession: if you do n random draws and they're all black, the probability the next draw will be black is $(n+1)/(n+2)$.

Laplace's calculation is too advanced for this book. But statistician Joe Blitzstein gives a nice explanation in this video, for students who have more background in probability (specifically random variables and probability density functions).

10.3 Inference to the Best Explanation

Another common form of inductive argument we met in Chapter 2 was Inference to the Best Explanation. An example:

> My car won't start and the gas gauge reads 'empty'.
> Therefore, my car is out of gas.

My car being out of gas is a very good explanation of the facts that it won't start and the gauge reads 'empty'. So this seems like a pretty strong argument.

How do we understand its strength using probability? This is actually a controversial topic, currently being studied by researchers. There are different, competing theories about how Inference to the Best Explanation fits into probability theory. So we'll just look at one, popular way of understanding things.

Let's start by thinking about what makes an explanation a good one.

A good explanation should account for all the things we're trying to explain. For example, if we're trying to explain why my car won't start and the gauge reads 'empty', I'd be skeptical if my mechanic said it's because the brakes are broken. That doesn't account for any of the symptoms! I'd also be skeptical if they said the gas gauge was broken. That might fit okay with one of the symptoms (the gauge reads 'empty'), but it doesn't account for the fact the car won't start.

The explanation that my car is out of gas, however, fits both symptoms. It would account for both the 'empty' reading on the gauge and the car's refusal to start.

A good explanation should also fit with other things I know. For example, suppose my mechanic tries to explain my car troubles by saying that both the gauge and the ignition broke down at the same time. But I know my car is new, it's a highly reliable model, and it was recently serviced. So my mechanic's explanation doesn't fit well with the other things I know. It's not a very good explanation.

We have two criteria now for a good explanation:

1. it should account for all the things we're trying to explain, and
2. it should fit well with other things we know.

These criteria match up with terms in Bayes' theorem. Imagine we have some evidence E we're trying to explain, and some hypothesis H that's meant to explain it. Bayes' theorem says:

$$Pr(H \mid E) = \frac{Pr(H)Pr(E \mid H)}{Pr(E)}.$$

How probable is our explanation H given our evidence E? Well, the larger the terms in the numerator are, the higher that probability is. And the terms in the numerator correspond to our two criteria for a good explanation.

1. $Pr(E \mid H)$ corresponds to how well our hypothesis H accounts for our evidence E. If H is the hypothesis that the car is out of gas, then $Pr(E \mid H) \approx 1$. After all, if there's no gas in the car, it's virtually guaranteed that it won't start and the gauge will read 'empty'. (It's not perfectly guaranteed because the gauge could be broken after all, though that's not very likely.)

2. $Pr(H)$ corresponds to how well our hypothesis fits with other things we know. For example, suppose I know it's been a while since I put gas in the car. If H is the hypothesis that the car is out of gas, this fits well with what I already know, so $Pr(H)$ will be pretty high.

   Whereas if H is the hypothesis that the gauge and the ignition both broke down at the same time, this hypothesis starts out pretty improbable given what else I know (it's a new car, a reliable model, and recently serviced). So in that case, $Pr(H)$ would be low

So the better H accounts for the evidence, the larger $Pr(E \mid H)$ will be. And the better H fits with my background information, the larger $Pr(H)$ will be. Thus, the better H is as an explanation, the larger

$Pr(H \mid E)$ will be. And thus the stronger E will be as an argument for H.

What about the last term in Bayes' theorem though, the denominator $Pr(E)$? It corresponds to a virtue of good explanations too!

Scientists love theories that explain the unexplained. For example, Newton's theory of physics is able to explain why a heavy object and a light object, like a hammer and feather, fall to the ground at the same speed as long as there's no air resistance. If you'd never performed this experiment before, you'd probably expect the hammer to fall faster. You'd be surprised to find that the hammer and feather actually hit the ground at the same time. That Newton's theory explains this surprising fact strongly supports his theory.

So the ability to explain surprising facts is a third virtue of a good explanation. And this virtue corresponds to our third term in Bayes' theorem:

3. $Pr(E)$ corresponds to how surprising the evidence E is. If E is surprising, then $Pr(E)$ will be low, since E isn't something we expect to be true.

And since $Pr(E)$ is in the denominator of Bayes' theorem, a smaller number there means a *bigger* value for $Pr(H \mid E)$. So the more surprising the finding E is, the more it supports a hypothesis H that explains it.

According to this analysis then, each term in Bayes' theorem corresponds to a virtue of a good explanation. And that's why Inference to the Best Explanation works as a form of inductive inference.

Figure 10.1: The hammer/feather experiment was performed on the moon in 1971. See the full video here.

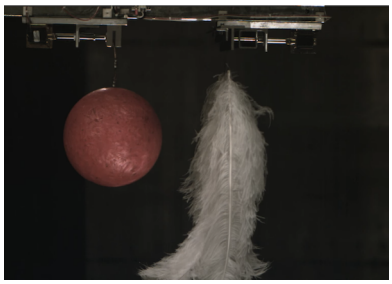


Figure 10.2: It's also been performed in vacuum chambers here on earth. A beautifully filmed example is available on YouTube, courtesy of the BBC.

Part II

11 *Expected Value*

We've been thinking a lot about what conclusions our evidence supports. But a lot of reasoning is *practical*: it's not just about what to believe, but about what you should *do*. Once you know how likely the various possible outcomes of a decision are, how do you make your choice?

> Suppose you have a free ticket to play one game at the local casino. You can choose either the Coin Game or the Die Game.
>
> - Coin Game: a fair coin is flipped. You win $1 if it lands heads, nothing otherwise.
> - Die Game: a fair die is rolled. You win $3 if it lands five or six, nothing otherwise.
>
> Which game should you play?

The Coin Game has better odds: a 1/2 chance of winning vs. just 1/3 in the Die Game. But the Die Game promises a bigger payoff: $3 rather than just $1. So there are conflicting arguments here. The odds favour the first choice, while the potential payoffs favour the second. How do you reconcile these considerations and come to a decision?

Well, let's think about what would happen in the long run if you played each game over and over.

Suppose you played the Coin Game over and over, a hundred times. Half the time you'd win $1, and half the time you'd win nothing. So after a hundred plays, you'd probably win about $50, which is an average of $0.50 per play.

Now suppose you played the Die Game a hundred times instead. A third of the time you'd win $3, and the other two thirds of the time you'd win nothing. So after a hundred plays you'd have won about $100, an average of about $1 per play.

In technical terms, this means the Die Game has a higher *expected value*. In the long run, you'd expect to win about $1 per play in the Die Game, as opposed to $0.50 for the Coin Game.

So the Die Game is the more advantageous one, on this analysis. On average, people tend to win more playing that game.

11.1 Expected Monetary Values

To calculate the expected value of an option, we multiply each payoff by its probability, and then sum up the results. For example, our analysis of the Coin Game can be written this way:

$$\begin{aligned} Pr(\$1) \times \$1 + Pr(\$0) \times \$0 &= (1/2)(\$1) + (1/2)(\$0) \\ &= \$0.50. \end{aligned}$$

And for the Die Game:

$$\begin{aligned} Pr(\$3) \times \$3 + Pr(\$0) \times \$0 &= (1/3)(\$3) + (2/3)(\$0) \\ &= \$1. \end{aligned}$$

Sometimes a gamble has the potential to lose money, so we use a negative number for the "payoff". For example, suppose the Die Game is modified so that you have to pay \$3 if the die lands 1, 2, 3, or 4. Then we calculate:

$$\begin{aligned} Pr(\$3) \times \$3 + Pr(-\$3) \times -\$1 &= (1/3)(\$3) + (2/3)(-\$3) \\ &= -\$1. \end{aligned}$$

In general, the formula for a game with possible payoffs $\$x$ and $\$y$ is:

$$Pr(\$x) \times \$x + Pr(\$y) \times \$y.$$

Notice that expected values aren't necessarily what you might think from the name. The expected value of a game isn't necessarily the amount you expect to win playing it. If you play the Die Game, you'll either win \$3 or nothing. But the expected value is \$1. So you actually know for certain ahead of time that you won't win the expected value, \$1.

⊘ The expected value isn't the amount you expect to win from a game. It's the amount you expect to win *on average, in the long run*, if you play the game over and over.

11.2 Visualizing Expectations

We can visualize these long run averages as the area in a diagram. Figure 11.1 shows a gamble with a 1/3 chance of paying \$2, and a 2/3 chance of paying \$6.

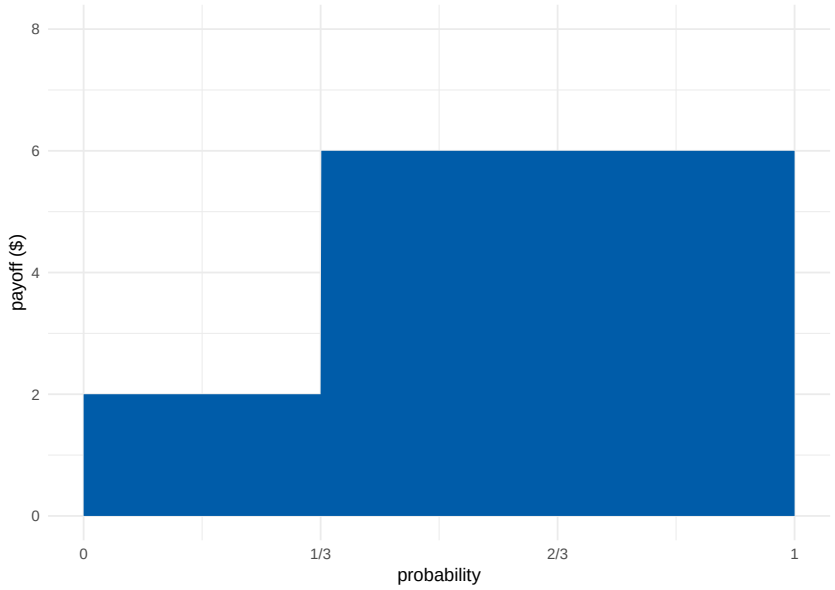


Figure 11.1: A gamble with a 1/3 chance of paying $2, and a 2/3 chance of paying $6

There are two rectangular regions, one for each possible outcome. The width of each rectangle is the probability of the corresponding outcome, and the height is the payoff. So each rectangle's area is one term in the expected value's sum:

$$1/3 \times \$2 + 2/3 \times \$6.$$

The expected value is thus the area of the two rectangles together, i.e. the area of the blue region (about $4.67 in this example).

The Coin Game and the Die Game are visualized in Figure 11.2.

We started this chapter wondering how to reconcile conflicting arguments. The Coin Game has better odds, but the Die Game has a greater potential payoff.

Well, a rectangle's area can be enlarged by increasing either its width or its height. Likewise, a payoff's contribution to expected value can be increased either by increasing its amount or by increasing its probability.

So the expected value formula answers our opening question by giving equal weight to the two competing factors. Width (probability) and height (payoff) are treated the same: neither one is given greater weight. We just multiply one against the other and take the total area that results.

Sometimes a decision loses money. We use negative dollar amounts to represent losses. For example, imagine the Coin Game is modified

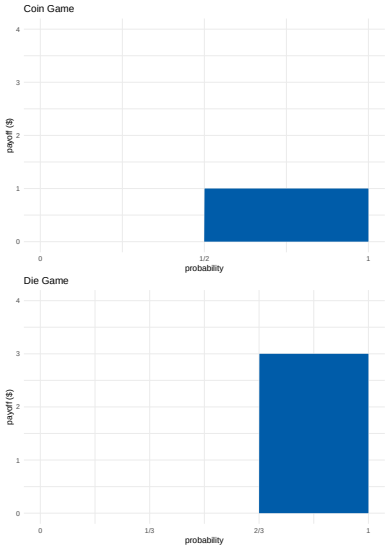


Figure 11.2: The Coin Game (top) and the Die Game (bottom)

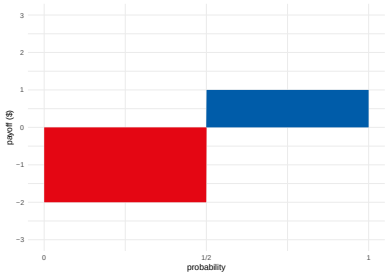


Figure 11.3: A gamble with one negative payoff, one positive

so that you lose $2 if the coin lands tails. Then the expected value is:

$$1/2 \times \$1 + 1/2 \times -\$2 = -\$1/2.$$

In our diagrams, potential losses extend below the x-axis. And we colour them red as a reminder that they are to be subtracted from the area above the x-axis. To get the expected value, we subtract the red area from the blue area.

11.3 More Than Two Outcomes

Most casino games have mor than two possible payoffs. For that matter, most decisions in life have more than two possible outcomes. If you buy insurance for your phone, it could end up saving you $0, or it could save you $100, or it might save you $150, etc.

So the official definition of expected monetary value is:

Expected Monetary Value Given an act A with possible consequences $\$x_1, \$x_2, \ldots, \$x_n$, the *expected monetary value* of A is:

$$E(A) = Pr(\$x_1) \times \$x_1 + Pr(\$x_2) \times \$x_2 + \ldots + Pr(x_n) \times \$x_n.$$

For example, suppose a fair die will be rolled and you will win $1 if it lands on a low number, $2 if it's a middling number. If it's a high number, you lose $3. Then:

$$\begin{aligned} E(A) &= (1/3)(\$1) + (1/3)(\$2) + (1/3)(-\$3) \\ &= \$1/3 + \$2/3 - \$1 \\ &= \$0. \end{aligned}$$

Gambles with more than two outcomes are visualized the same as before. We just use three or more reactangles, one for each possible outcome.

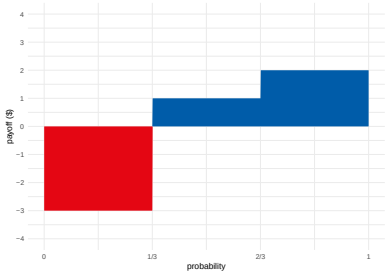


Figure 11.4: A gamble with three possible outcomes

11.4 Fair Prices

Imagine a variation of the Coin Game, where you win $1 if the coin lands heads, and you pay $1 if it lands tails. You might be able to guess, the expected value of this game is $0:

$$\begin{aligned} Pr(\$1) \times \$1 + Pr(-\$1) \times -\$1 &= (1/2)(\$1) + (1/2)(-\$1) \\ &= 0. \end{aligned}$$

If you played over and over again, you'd break even in the long run. Half the time you'd win a dollar, and the other half of the time you'd pay a dollar back.

So if someone asked you to pay in order to play this game, that would seem unfair. Why pay to play if you don't expect to win?

But for a game like the Die Game, asking people to pay to play does seem fair. In the long run, the casino is going to lose an average of $1 per play. So it's only fair that they ask people to pay $1 to play that game. In the long run they'll break even. (Of course, a real casino will demand more so they can turn a profit.)

In general, a game's fair price is thought to be its expected value. If the expected value is $3, it's fair to pay $3 to play. If the expected value is −$3, it's fair to "pay" −$3 to play. Meaning *they should pay you* $3 to play.

NOTICE that the expected value of paying the fair price is always $0. Suppose you pay $1 to play the original Die Game, for example. Because $1 is automatically deducted from your winnings, the possible consequences are now $3 − $1 = $2 or $0 − $1 = −$1. So the expected value of paying $1 to play is:

$$\begin{aligned}(1/3)(\$2) + (2/3)(-\$1) &= \$2/3 - \$2/3\\ &= \$0.\end{aligned}$$

That makes sense: if it's a fair price, you shouldn't have any advantage over the casino. You don't stand to gain or lose money in the long run, if you pay the fair price to play over and over.

The casino is in the same position. For them, the losses and gains are reversed: a roll of five or six loses them $3, and otherwise they lose nothing. So if you pay them $1 to play, they either lose $2 in the end or gain $1:

$$\begin{aligned}(1/3)(-\$2) + (2/3)(\$1) &= -\$2/3 + \$2/3\\ &= \$0.\end{aligned}$$

Since nobody has an advantage, it's a fair deal.

11.5 Other Goods

MONEY isn't everything, of course. People value lots of other things, like their health, their family and friends, and their free time. The concept of expected value can be applied to other goods besides money, as long as they're quantifiable.

> Imagine you're taking a trip to Ottawa and you can either drive or go by train. If you take the train, it's pretty predictable: it's always a five or six hour ride, with equal probability. If you drive, it's less predictable. Nine times out of ten it's just a four hour drive. But one time in ten, an accident causes a traffic jam and it ends up being a ten hour drive.

If all you care about is spending as little time on the road as possible, should you take the train or drive?

$$\begin{aligned} E(\text{Train}) &= Pr(-5 \text{ hr}) \times -5 \text{ hr} + Pr(-6 \text{ hr}) \times -6 \text{ hr} \\ &= (1/2)(-5 \text{ hr}) + (1/2)(-6 \text{ hr}) \\ &= -11/2 \text{ hr} \\ &= -5.5 \text{ hr}, \\ E(\text{Drive}) &= Pr(-4 \text{ hr}) \times -4 \text{ hr} + Pr(-10 \text{ hr}) \times -10 \text{ hr} \\ &= (9/10)(-4 \text{ hr}) + (1/10)(-10 \text{ hr}) \\ &= -46/10 \text{ hr} \\ &= -4.6 \text{ hr}. \end{aligned}$$

So driving has a higher expected value—measured in terms of free time, rather than money.

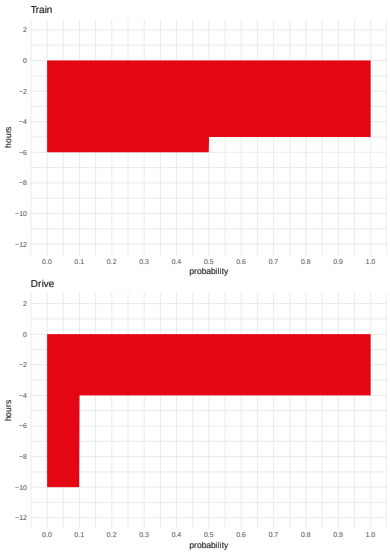


Figure 11.5: Take the train to Ottawa or drive?

Notice that both options have negative expected value in this example. After all, you're bound to lose time travelling no matter what. Sometimes you don't have any choice but to accept a loss. But you can still choose the option that minimizes your expected loss, i.e. maximizes expected value.

11.6 Decision Tables

MANY decision problems can be represented in a table. Suppose you're choosing between the following two gambles, based on the roll of a fair, six-sided die.

- *A*: you win $3 if it lands 1 or 2, you win $1 if it lands 3 or 4, and you lose $1 if it lands 5 or 6.
- *B*: you win $3 if it lands 1 or 2, you lose $3 if it lands 3 or 4, and nothing happens if it lands 5 or 6.

A decision table has a row for each option and a column for each possible outcome. Each cell of the table then contains the payoff for that combination of act and outcome:

	1 or 2	3 or 4	5 or 6
A	$3	$1	−$1
B	$1	$2	$0

We can also add in the probability for each cell:

	1 or 2	3 or 4	5 or 6

	1 or 2	3 or 4	5 or 6
A	1/3, \$3	1/3, \$1	1/3, −\$1
B	1/3, \$3	1/3, −\$3	1/3, \$0

Now calculating expected values is just a matter of multiplying the numbers within each cell, and then adding up across each row:

	1 or 2	3 or 4	5 or 6	Expected Value
A	1/3, \$3	1/3, \$1	1/3, −\$1	\$1
B	1/3, \$3	1/3, −\$3	1/3, \$0	\$0

NOTE that the probabilities don't have to be the same from row to row. The casino might choose to use a different die depending on how you bet, for example. Suppose if you choose A, they'll use a loaded die to eliminate your edge:

	1 or 2	3 or 4	5 or 6	Expected Value
A	1/6, \$3	1/6, \$1	4/6, −\$1	\$0
B	1/3, \$3	1/3, −\$3	1/3, \$0	\$0

Now that the die is loaded towards 5 or 6 if you choose A, the expected value of choosing A is nil: \$0.

ONCE in a while you can avoid calculating expected values altogether. Consider this decision problem:

	1 or 2	3 or 4	5 or 6
A	\$3	\$1	−\$1
B	\$1	\$0	−\$3

We don't even have to put in the probabilities to make our decision here. Option A is clearly better, because it has a better payoff no matter how the die lands. In every column, row A has a larger payoff than row B. So A is a better decision no matter what.

When one option has a better payoff than another no matter what happens, we say the first option ***dominates*** the second. In terms of a table, the dominant option has a higher payoff in every column.

If A dominates B, then A must have higher expected value than

B.

Dominant options don't come up very often in practice. Real decision problems tend to be harder than that. But they do come up sometimes. (They're also important in more advanced, theoretical topics: Chapter 17 covers one example.)

Exercises

1. Which of the following best describes expected monetary value?

   a. The amount you would expect to gain if you chose the same option over and over again for a long, long time.
   b. The amount you expect to gain/lose.
   c. The average amount you would expect to gain/lose each time if you chose the same option over and over again many times.
   d. The amount of profit a business expects to garner in a fiscal year.

2. What is the expected monetary value of playing a slot machine that costs \$100 to play, and has a $1/25$ chance of paying out \$500? (The rest of the time it pays nothing.)

3. Suppose a slot machine pays off \$25 a fiftieth of the time and costs a \$1 to play, and a video poker machine pays off \$10 a twentieth of the time and costs \$2 to play. Which machine is the better bet in terms of expected monetary value?

4. You're considering downloading a new game for your phone. The game costs \$0.99. But as a promotion, the first $50,000$ downloaders are being entered in a fair lottery with a $\$10,000$ cash prize.

   If you know you'll be one of the first $50,000$ downloaders, what is the expected monetary value of downloading the game?

   This exercise is based on Exercise 2 from p. 95 of Ian Hacking's *An Introduction to Probability & Inductive Logic*.

5. A local casino offers a game which costs \$2 to play. A fair coin is flipped up to three times, and the payouts work as follows:

   - If the coin lands heads on the first toss, you win \$2 and the game is over.
   - If the coin lands heads on the second toss, you win \$4 and the game is over.
   - If the coin lands heads on the third toss, you win \$8 and the game is over.
   - If the coin lands tails all three times, you win \$0.

   What is the expected monetary value of this game?

6. When is the following statement true: the expected monetary value of a gamble is also one of its possible payoffs.

   a. Always true
   b. Sometimes true
   c. Never true

7. Suppose you can bet on either of two dogs: Santa's Little Helper or She's the Fastest. If you bet on Santa's Little Helper and he wins, you get \$5. If he loses you pay \$2. If you bet on She's the Fastest and she loses, you pay \$10. The two dogs have the same chance of winning.

   How much would a winning bet on She's the Fastest have to pay for her to be the better bet?

This problem is based on Exercise 1 from p. 85 of Michael D. Resnik's excellent book, *Choices*.

8. Suppose the government is deciding whether to create a vaccination program to prevent a potential epidemic of Zika virus. If there is an epidemic, it will cost the country \$1,000 million (for medical care, lost work-hours, etc.). The vaccination program would cost \$40 million.

   If they don't create the vaccination program, there is a 9/10 chance of an epidemic. If they do create the vaccination program, there is still a 1/10 chance of an epidemic.

   a. Draw a 2×2 table and label the rows with the available actions (program, no program). Label the columns with the possible outcomes (epidemic, no epidemic). In each cell write the monetary cost and the corresponding probability.
   b. What is the expected monetary value of creating the vaccine program?
   c. What is the expected monetary value of not creating the program?
   d. If the government makes their decision based on expected monetary values, will they create the program or not?

This problem is based on an example from p. 86 of Michael D. Resnik's *Choices*.

9. Suppose Ontario is deciding whether to enact a new tax. If the tax is enacted, it will bring in \$700 million in revenue.

   But it could also hurt the economy. The chance of harm to the economy is small, just 1/5. But it would cost the country \$1,200 million in lost earnings. (The \$700 million in revenue would still be gained, partially offsetting this loss.)

   Treat gains as positive and losses as negative.

   a. What is the expected monetary value of enacting the new tax?

The government also has the option of conducting a study before deciding whether to enact the new tax. If the study's findings are bad news, that means the chance of harm to the economy is actually double what they thought. If its findings are good news, then the chance of harm to the economy is actually half of what they thought.

b. Suppose the government conducts the study and its findings are good news. What will the expected monetary value of enacting the tax be then?
c. Suppose the government conducts the study and its findings are bad news. What will the expected monetary value of enacting the tax be then?
d. Suppose conducting the study would cost $\$5,000$. Will the government conduct the study? Explain your answer. (Assume they make decisions by maximizing expected monetary value.)

10. Suppose your university is considering a tuition increase. If the increase happens, it will bring in $7 million.

    But it could also hurt the university's reputation. The chance of harm to the university's reputation is $3/5$, and it would cost the university $20 million in lost donations if it happened. The $7 million in tuition would still be gained though, partially offsetting this loss.

    Treat gains as positive and losses as negative.

    a. What is the expected monetary value of enacting the tuition increase?

    The university has the option of conducting a study before deciding whether to increase tuition. If the study's findings are bad, the chance of harm to the university's reputation is increased by $1/5$. If its findings are good, the chance of harm is decreased by $1/5$.

    b. Suppose the university conducts the study and its findings are good. What will the expected monetary value of enacting the increase be then?
    c. Suppose the university conducts the study and its findings are bad. What will the expected monetary value of increasing tuition be then?
    d. Suppose the study would cost a thousand dollars. Will the university conduct the study? Assume they make decisions by maximizing expected monetary value.

11. Noree has a phone worth $60 and a tablet worth $240. The probability she'll have to replace the phone (because of damage/loss/theft/etc.) is 1/5. The probability she'll have to replace the tablet is 1/6. These probabilities are independent.

    An insurance policy that costs $60 will cover the cost of replacing both items.

    a. What is the expected monetary value of buying the insurance?
    b. What is the expected monetary value of declining the insurance?

12. William has a phone worth $60 and a tablet worth $540. The probability he'll have to replace the phone (because of damage/loss/theft/etc.) is 1/3. The probability he'll have to replace the tablet is 1/5. These probabilities are independent.

    An insurance policy that costs $100 will cover the cost of replacing both items.

    a. What is the expected value of buying the insurance?
    b. What is the expected value of declining the insurance?

13. Zhi has a netbook worth $350 and a desktop computer worth $1400. The probability that she'll spill liquid on the desktop is 1/7 and the probability that she'll spill liquid on the netbook is 1/5. These probabilities are independent, and in either case she will have to replace the damaged device.

    An insurance policy that costs $200 will cover the cost of replacing either/both items if they are damaged from a spill. Treat all losses as negative.

    a. What is the expected monetary value of buying the insurance?
    b. What is the expected monetary value of declining the insurance?

14. Suppose you play poker with five of your friends every week, but you're not very good. For every week you win, there are eight weeks where you lose. You don't play for real money though, so it's usually no big deal.

    But next week your friends want to have a real game where everyone puts in $15. The winner keeps all the money, everyone else loses their $15.

    a. What is the expected monetary value of playing next week?

Now suppose you see a book that teaches people to improve their poker game. The book costs $30, but it will increase your odds of winning.

b. How much would the book have to increase your odds of winning to make the expected value of buying it positive?

15. Suppose you've accumulated enough loyalty points at your local drug store to trade them in for $100 worth of merchandise.

    But the pharmacy offers you a second option. You can trade your points for an opportunity to join a lottery along with all the other customers who've accumulated the same number of points. The lottery will have one randomly selected winner, and the prize is $10,000 worth of merchandise. There would be 300 people in the lottery (including you).

    Treat losing the lottery as a monetary loss: −$100.

    a. What is the expected monetary value of joining the lottery?
    b. How small would the lottery need to be for it to be a fair trade for your points? In other words: how few people would there need to be in the lottery (including you) for it to have the same expected value as just trading your points for $100 of merchandise?

16. Consider the following game: I'm going to flip a fair coin up to three times. If it comes up heads on the first toss, the game is over and you win $2. If it comes up heads for the first time on the second toss, you win $40 and the game is over. If the first heads comes up on the third toss, you win $800 and the game is over. If it comes up tails every time, you have to pay me $$x$. What does x have to be to make the game fair.

17. Consider the following game. I'm going to roll a fair die up to two times. If it comes up 1 on the first roll you win $$x$ and the game is over. Otherwise, I roll it again. If it comes up 2 on the second roll, you win 6x$ and the game is over. Otherwise you pay me $6 and the game is over. What does x have to be for this game to be fair?

18. Consider the following game.

    I draw a card from a fair deck. If it's a 2 or 3, you pay me $$x$ times the face value: 2x$ if it's a 2, 3x$ if it's a 3.

    If the first card is neither a 2 nor a 3, I draw a second card. If it's a 7, you win 7x$. Otherwise you pay me $1 and the game is over.

    What does x have to be for this game to be fair?

19. Suppose there are three urns, A, B, and C. Each urn has 100 marbles: some are black, the rest are white. You don't know how many black marbles there are in each urn. But you do know two things:

    - B has twice as many black marbles as A, and C has twice as many black marbles as B.
    - \$1 is a fair price for the following game. I pick an urn at random and draw a marble at random. If I pick a black marble from urn A you win \$100. If I pick a black marble from urn B you win \$50. If I pick a black marble from urn C, you win \$25. Otherwise you win nothing.

    How many black marbles are there in urn A?

20. You're on a game show where there are three closed doors. One of the doors has a large cash prize behind it, though you don't know how much exactly. The other two doors have nothing. You have two options:

    - Option A: Choose a door at random and keep whatever you find, if anything.
    - Option B: First ask the host to open one of the doors with nothing behind it. Then choose one of the remaining doors at random. The catch is that now you only get to keep *half* of what you find, if anything.

    Which option has higher expected value? Justify your answer.

    Hint: first calculate the expected value of each option if the cash prize is \$100. Then do the same calculations for a prize of \$1,000. Then explain how to generalize this reasoning to any cash prize.

21. Some workplaces hold a weekly lottery. Suppose there are 30 people in your workplace lottery, and each person pays in \$5 every Monday. A finalist is chosen at random every Friday, for three weeks. Then, on the fourth Friday, one of the three finalists from the previous three weeks is chosen at random. That person gets all the prize money.

    What is the expected value of being in this lottery? Explain the reasoning behind your answer.

    Tip: it's possible to answer this question without doing any calculations.

22. Prove that paying the fair price for an option always results in expected value of \$0. In other words, if $E(A) = \$x$, then $E(\text{Pay } \$x \text{ for } A) = \$0$.

23. Prove that if A dominates B, then A has higher expected value: $E(A) > E(B)$.

12 Utility

Here, have a dollar.
In fact, no brotherman here, have two.
Two dollars means a snack for me
But it means a big deal to you.
—Arrested Development, "Mr. Wendal"

IMAGINE your university's student union is holding a lottery. There are a thousand tickets, and the prize is $500. Tickets cost $1. Would you buy one?

The expected monetary value of buying a ticket is $-\$0.50$:

$$\begin{aligned} E(T) &= (1/1,000)(\$499) + (999/1,000)(-\$1) \\ &= -\$0.50. \end{aligned}$$

So it seems like a bad deal.

But some people do buy tickets in lotteries like this one, and not necessarily because they're just bad at math. Maybe they enjoy the thrill of winning, or even just having a shot at winning. Or maybe they want to support the student union, and foster fun and community on campus.

SUPPOSE your friend is one of these people. How might they respond to the calculation above? One thing they could do is replace the dollar amounts with different numbers, numbers that more accurately capture their personal values.

Instead of just a $499 gain for winning, for example, they might use 599, to capture the added value that comes from the thrill of winning. And instead of $-\$1$ for losing, they might put 10 to capture the contribution to campus community that comes from participating in the event.

Then the calculation looks like this:

$$\begin{aligned} E(T) &= (1/1,000)(599) + (999/1,000)(9) \\ &= 9.59 \end{aligned}$$

So buying a ticket has positive expected value now.

Notice how there are no dollar signs anymore. In fact there's no explicit unit of measurement at all. So what do these numbers represent, if not monetary value?

They represent how good or bad an outcome is, from your friend's point of view. In decision theory we call these numbers ***utilities***, or ***utils*** for short.

If the real value of something is its utility, we should calculate *expected utilities* instead of expected monetary values:

Expected Utility Suppose act A has possible consequences $C_1, C_2, \ldots, C_n$. Denote the utility of each outcome $U(C_1)$, $U(C_2)$, etc. Then the *expected utility* of A is defined:

$$EU(A) = Pr(C_1)U(C_1) + Pr(C_2)U(C_2) + \ldots + Pr(C_n)U(C_n).$$

So the formula for expected utility is exactly the same as for expected monetary value, except we replace dollars with utils.

According to standard decision theory, the right choice is the act that has the highest expected utility. Or if there's a tie for highest, then any of the acts tied for highest is a good choice.

12.1 *Subjectivity & Objectivity*

People have different desires, needs, and interests. Some value friendship most, others value family more. Some prioritize professional success, others favour activism. Some people like '80s music, others prefer hip-hop—and still others, to my constant surprise, enjoy reggae.

Figure 12.1: Frank Ramsey (1903–1930) died at the age of 26, before his discovery could become widely known. Luckily the idea was rediscovered by economists and statisticians in the 1940s.

So the value of a decision's outcome depends on the person. It is *subjective*. And that raises a big question: if utility is subjective, how can we measure it objectively? Does it even make sense to treat personal tastes and preferences using mathematical equations?

The classic solution to this problem was discovered by the philosopher Frank Ramsey in the 1920s. The key idea is to quantify how much a person values a thing by considering how much they'd risk to get it.

For example, imagine you have three options for lunch: pizza, salad, or celery. And let's suppose you prefer them in that order: pizza $>$ salad $>$ celery. We want to know, how does salad compare to your best option (pizza) and your worst option (celery)? Is salad almost as good as pizza, for you? Or is it not so great, almost as bad as celery perhaps?

Here's how we find out. We offer you a choice, you can either have a salad or you can have a gamble. The gamble might win you pizza, but you might end up with celery instead.

- S: Salad.

- G: A gamble, with probability $Pr(P)$ of pizza and $1 - Pr(P)$ of celery.

The question is, how high does the probability $Pr(P)$ have to be before you are *indifferent* between these two options? How good does your shot at pizza have to be before you'd trade the salad for the gamble, and risk winding up with celery instead?

This is a question about your personal tastes, notice. Some people really like salad, they might even like it as much as pizza. So they'd need the shot at pizza to be pretty secure before they'd give up the salad. Others don't care much for salad. Like me: I'd be willing to trade the salad for the gamble even if the chances of pizza weren't very high. For me, salad isn't much better than celery, but pizza is way better than both of them.

Let's suppose for the sake of example that your answer is $Pr(P) \geq 1/3$. You'd need at least a $1/3$ chance at the pizza to risk getting celery instead of salad. Since celery is the worst option, let's say it has utility zero: $U(C) = 0$. And since pizza is the best option, let's say it has utility one: $U(P) = 1$. What is the utility of salad $U(S)$ then?

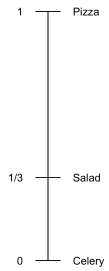


Figure 12.2: A utility scale for lunch options

We know that you're willing to trade the salad for the gamble when the probability of winning pizza is at least $1/3$. So if $Pr(P) = 1/3$, the utility of salad is the same as the expected utility of the gamble: $U(S) = EU(G)$. Therefore:

$$\begin{aligned} U(S) &= EU(G) \\ &= Pr(P)U(P) + Pr(C)U(C) \\ &= (1/3)(1) + (2/3)(0) \\ &= 1/3. \end{aligned}$$

So salad has utility $1/3$ for you.

Hey look! We just took something personal and subjective and gave it a precise, objective measurement. We've managed to quantify your lunch preferences.

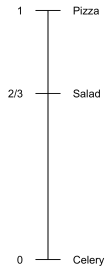


Figure 12.3: A friend's utility scale for lunch options

Now imagine your friend gives a different answer. They'd need at least a $2/3$ chance at the pizza to risk ending up with celery instead of salad. So for them, $U(S) = EU(G)$ when $Pr(P) = 2/3$. So now the calculation goes:

$$\begin{aligned} U(S) &= Pr(P)U(P) + Pr(C)U(C) \\ &= (2/3)(1) + (1/3)(0) \\ &= 2/3. \end{aligned}$$

So for your friend, salad is closer to pizza than to celery. For them it has utility $2/3$.

12.2 *The General Recipe*

LUNCH is important, but how do we apply this recipe more broadly? How do we quantify the value a person places on other things, like career options, their love life, or their preferred music?

Here's the general recipe. We start with our subject's best possible consequence B, and their worst possible consequence W. We assign these consequences the following utilities:

$$U(B) = 1,$$
$$U(W) = 0.$$

Then, to find the utility of any consequence C, we follow this recipe. We find the *lowest* probability $Pr(B)$ at which they would be indifferent between having C for sure vs. accepting the following gamble:

- G: probability $Pr(B)$ of getting B, probability $1 - Pr(B)$ of getting W.

Then we know consequence C has utility $Pr(B)$ for them.

Why does this method work? Because our subject is indifferent between having option C for sure, and taking the gamble G. So for them, the utility of C is the same as the expected utility of the gamble G. And the expected utility of gamble G is just $Pr(B)$:

$$\begin{aligned} U(C) &= EU(G) \\ &= Pr(B)U(B) + (1 - Pr(B))U(W) \\ &= Pr(B). \end{aligned}$$

I find it helps to think of this method as marking every point on our utility scale with a gamble, which we can use for comparisons.

For every utility value u between 0 and 1, there's a gamble whose expected utility is also u. To find such a gamble, we make the possible payoffs B and W, and we set $Pr(B) = u$ and $Pr(W) = 1 - u$.

These gambles give us points of comparison for every possible utility value u on our utility scale. Figure 12.4 shows a sample of a few such points. We can't show them all of course; there are infinitely many, one for every number between 0 to 1.

Now to find the utility of an outcome—whether it's a salad, a Carribbean vacation, or a million dollars—we find which of these gambles it's comparable to. To locate a Carribbean vacation on our scale, for example, we move up the scale from o until we hit a gamble where our subject says, "There! I'd be willing to trade a Carribbean vacation for that gamble there." Then we look at what $Pr(B)$ is equal to in that gamble, and that tells us their utility for a Carribbean vacation.

What are B and W? What are the best and worst options a person could have? I like to think of them as heaven and hell: B is an eternity of bliss, W is an eternity of suffering.

That way we can look at the big picture. We can quantify a person's utilities in the grand scheme of things. We can compare all kinds of life-outcomes, big and small. We can measure the utility of everything from raising happy and healthy children, to landing your dream job, to going on a Carribean vacation, to having a cup of coffee.

But it's important to remember that the best and worst outcomes vary from person to person. In fact, for most people eternal bliss (heaven) isn't the best possible outcome. They'll want their friends and family to go to heaven too, for example. They may even want everyone else to go to heaven—even their enemies, if they're a particularly forgiving sort of person.

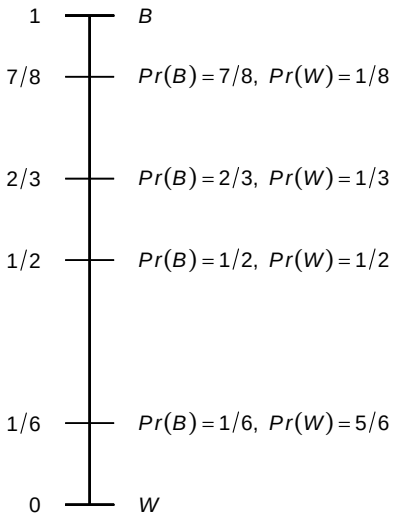


Figure 12.4: A utility scale with some arbitrarily selected points of comparison chosen for display: $u = 7/8$, $2/3$, $1/2$, and $1/6$

12.3 Choosing Scales

This method for quantifying utility assumes that $U(W) = 0$ and $U(B) = 1$. What justifies this assumption? It's actually a bit arbitrary. We could measure someone's utilities on a scale from 0 to 10 instead. Or -1 to 1, or any interval.

It's just convenient to use 0 and 1. Because then, once we've identified the relevant probability $Pr(B)$, we can immediately set $U(C) = Pr(B)$. If we used a scale from 0 to 10 instead, we'd have to multiply $Pr(B)$ by 10 to get $U(C)$. That would add some unnecessary work.

It's also inuitive to use the same 0-to-1 scale for both probability and utility. It makes our visualizations of expected utility especially tidy. Expected utility can then be thought of as portion of the unit square. The best possible option is a 100% chance of the best outcome, i.e. probability 1 of getting utility 1. So the best possible choices have area 1. Whereas the worst gamble has no chance of getting any outcome with positive utility, and has area 0.

You can think of the 0 and 1 endpoints like the choices we make when measuring temperature. On the celsius scale, the freezing point of water is 0 degrees and the boiling point is 100. On the Fahrenheit scale we use 32 and 212 instead. The celsius scale is more intuitive, so that's what most people use. But the Fahrenheit scale works fine too, as long as you don't get confused by it.

Notice, by the way, that a person's body temperature is another example of something personal and subjective, yet precisely and objectively measurable. Different people have different body temperatures. But that doesn't mean we can't quantify and measure them.

12.4 A Limitation: The Expected Utility Assumption

A measurement is only as good as the method used to generate it. An oral thermometer only works when the temperature in your mouth is the same as your general body temperature. If you've got ice cubes tucked in your cheeks, or a mouthful of hot tea, even the best thermometer will give the wrong reading.

The same goes for our method of measuring utility. It's based on the assumption that our subject is making their choices using the expected utility formula. When the probability of pizza $Pr(P)$ was 1/3, we assumed you were indifferent between keeping the salad and taking gamble on pizza/celery because $U(C) = EU(G)$.

But people don't always make their choices according to the expected utility formula. Some famous psychology experiments demonstrate this, as we'll see in Chapter 13. So it's important to keep in mind

Question: how would we calculate $U(C)$ on a scale from -1 to 1?

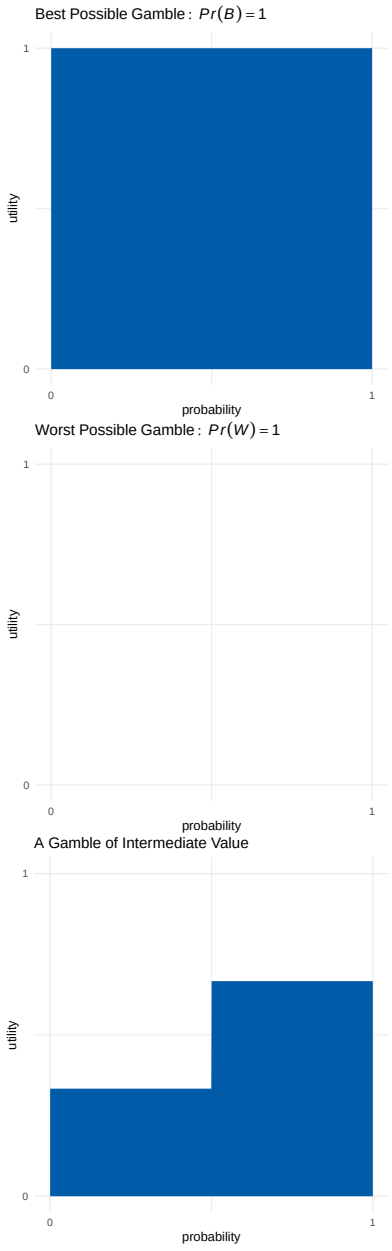


Figure 12.5: The best possible gamble (top), the worst possible gamble (middle), and an intermediate gamble (bottom)

that our "utility thermometer" doesn't always give the right reading.

⊘ Our method for quantifying utilities only works when the subject is following the expected utility formula.

12.5 *The Value of Money*

If you saw 50 cents on the ground while walking, would you stop to pick it up? I probably wouldn't. But I would have when I was younger, back then I had a lot less money. So an additional 50 cents made a bigger difference to my day. The more money you have, the less an additional 50 cents will be worth to you.

Imagine you won a million dollars in the lottery. Your life would be changed radically (unless you already happen to be a millionaire). But if you win another million dollars in another lottery, that extra million wouldn't make as big a difference. It's still a big deal, but not as big a deal as the first million.

For most people, gaining more money adds less and less value as the gains increase. In graphical terms, the relationship between money and utility looks something like Figure 12.6.

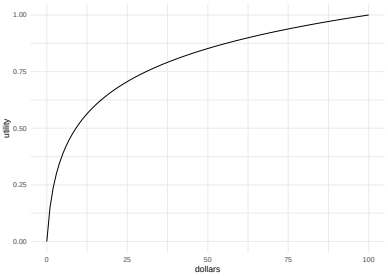


Figure 12.6: The diminishing value of additional money

We can quantify this phenomenon more precisely using our technique for measuring utilities. For example, let's figure out how much utility a gain of $50 has, compared with $0 and $100.

As usual we set the end-points of our scale at 0 and 1:

$$U(\$0) = 0,$$
$$U(\$100) = 1.$$

Then we ask: how high would $Pr(\$100)$ have to be before you would trade a guaranteed gain of $50 for the following gamble?

- Chance $Pr(\$100)$ of winning $100, chance $1 - Pr(\$100)$ of winning $0.

For me, I'd only be willing to take the gamble if $Pr(\$100) \geq .85$. So, for me, $U(\$50) = .85$.

$$\begin{aligned} U(\$50) &= Pr(\$100)U(\$100) + Pr(\$0)U(\$0) \\ &= (.85)(1) + (.15)(0) \\ &= .85. \end{aligned}$$

Notice how this fits with what we said earlier about the value of additional money. At least for me, $100 isn't twice as valuable as $50. In fact it's not even really close to being twice as valuable. A gain of $50 is almost as good as a gain of $100: it's .85 utils vs. 1 util.

But what about you: how does a gain of $50 compare to a gain of $100 in your life?

Exercises

1. According to the theory of utility presented in this chapter:

   a. Utility is subjective: people differ in their priorities, and in the value they place on things like money or food.
   b. Utility is objective: there is a fact of the matter about how much a given person values a good like money or food, and this can be quantified.
   c. Both
   d. Neither

2. Which of the following best describes the concept of utility used in decision theory?

   a. Utility measures an option's chances of having a good outcome.
   b. Utility is the moral value of an outcome: how good it is from an ethical point of view.
   c. Utility is the social value of an outcome: how good it is from society's point of view.
   d. Utility is the value a person places on an outcome, whether it involves money, food, friendship, or something else.

3. Lee's favourite ice cream flavour is chocolate, her least favourite is strawberry. Vanilla is somewhere in between. She would trade a vanilla ice cream for a gamble on chocolate vs. strawberry, but only as long as the chance of chocolate is at least 3/5. How much utility does vanilla have for her on a scale from 0 (strawberry) to 1 (chocolate)?

4. Sonia has tickets to see The Weeknd tomorrow night. Her friend has tickets to see Beyoncé, and also tickets to Katy Perry. Beyoncé is Sonia's favourite performer, in fact she would rather see Beyoncé than The Weeknd.

   Sonia's friend offers a gamble in exchange for her tickets to The Weeknd. The gamble has a 9/10 chance of winning, in which case Sonia gets the Beyoncé tickets (utility 1). Otherwise she gets the Katy Perry tickets (utility 0).

   If Sonia declines the gamble, what can we conclude?

   a. For Sonia, the utility of seeing The Weeknd is 9/10.
   b. For Sonia, the utility of seeing The Weeknd is greater than 9/10.
   c. For Sonia, the utility of seeing The Weeknd is less than 9/10.
   d. For Sonia, the utility of seeing The Weeknd is 1/10.

5. After giving her calculus midterm, Professor X always offers her students a chance to improve their grade by trying to solve an optional "challenge" problem. If they get it right, their grade is increased by one letter grade: F changes to D, D changes to C, etc. But if they get it wrong, their grade goes down by one letter-grade: A changes to B, B changes to C, etc.

   Hui got a C on his midterm. He asks the professor how often students get the challenge problem right and she says they get it right half the time. Hui decides to stick with his C. But he would be willing to try the challenge problem if the chances of getting it right were higher: 2/3 or more.

   Suppose getting a D has utility 4/10 for Hui, while a B has utility 8/10.

   a. What is the expected utility for Hui of trying the challenge problem?
   b. How much utility does a C have for Hui?

6. Let's explore how much value you place on money. We'll only consider amounts between \$0 and \$100, and we'll set our scale as usual: $U(\$100) = 1$ and $U(\$0) = 0$.

   a. Now suppose you are offered a gamble that has probability $Pr(\$100)$ of paying \$100, and probability $1 - Pr(\$100)$ of paying \$0. How high would $Pr(\$100)$ have to be for you to be willing to trade a guaranteed \$50 for this gamble? (This is a question about your personal preferences.)
   b. Based on your answer to part (a), how much utility does gaining \$50 have for you?
   c. Now consider a gamble that has probability $Pr(\$100)$ of paying \$100, and probability $1 - Pr(\$100)$ of paying \$50. How high would $Pr(\$100)$ have to be for you to be willing to trade a guaranteed \$75 for this gamble? (This is another question about your personal preferences.)
   d. Based on your previous answers, how much utility does gaining \$75 have for you?
   e. In terms of dollars, a gain of \$75 is 1.5 times as large a gain as \$50. In terms of your utilities, how does $U(\$75)$ compare to $U(\$50)$? Is it more than 1.5 times as large? Less? The same?
   f. Make a graph with dollars on the x-axis and your utilities on the y-axis. Plot the four points established in this problem. Then use them to draw a rough sketch of how your utilities increase per dollar.

7. Martha is trying to decide where to go to university. She applied to three schools: UTM, Western, and Queens. UTM is her first choice, Queens is her last choice.

   So far Martha has only heard back from Western. They are offering her early admission: they'll admit her but only if she agrees right now to go there (she can't wait until she finds out if the other two schools will admit her).

   Based on her grades, Martha knows that if she waits she'll be admitted to Queens for sure. But her chance of getting into UTM is only 6/10.

   After thinking about it a while, she can't decide: a guaranteed spot at Western seems just as good to her as the gamble on UTM vs. Queens.

   a. If the utility of going to Queens is 5/10 for Martha, and the utility of going to UTM is 9/10, what is the utility of going to Western?

   b. Martha's friend is considering York University. Martha didn't apply to York, but if she had she would be indifferent between these options:

      - Accept an early-admissions offer from York and go there.
      - Gamble on a 3/4 chance at going to Western vs. a 1/4 chance of having to go to Queens.

      How much utility does going to York have for Martha?

8. Eleanor wants to get a job at Google so she's going to university to study computer science. She has to decide between UTM and Western.

   Suppose 1/100 of UTM's computer science students get jobs at Google and the rest get jobs at RIM. For Eleanor, a job at Google has utility 200 while a job at RIM has utility 50.

   a. What is the expected utility of going to UTM for Eleanor?

   Suppose Western students have better odds of getting a job at Google: 5/400. And 360/400 students go to work at Amazon, which Eleanor would prefer to RIM. On the other hand, the remaining 35/400 of them don't get a job at all, which has utility zero for Eleanor.

   After thinking about it, she can't decide: UTM and Western seem like equally good options to her.

   b. How much utility does working at Amazon have for Eleanor?

Suppose Eleanor ends up going to UTM, and now she's about to graduate. Unfortunately, Google isn't hiring any more. The only jobs available are at Amazon and RIM.

She would have to take a special summer training program to qualify for a job at Amazon, though. And that would mean she can't get a job at RIM. RIM is offering her a job, but she has to take it now or never.

So, she has to either take the guaranteed job at RIM right now, or gamble on the summer program. The summer program could get her a job at Amazon, or it could leave her unemployed.

c. How high would the probability of getting a job at Amazon have to be for the special summer program to be the better option?

9. Farad wants to go to law school so he's going to university to study philosophy. He is deciding between Queens and Western.

Suppose 2/10 of Western philosophy students who apply to law school get in, the rest go to teacher's college or medical school. For Farad, going to law school has utility 250, and going to teacher's college or medical school has utility 50.

a. What is the expected utility of going to Western for Farad?

Suppose Queens students have better odds of getting into law school: 3/10. And 6/10 go to work for the government, which Farad would prefer to being unemployed. On the other hand, 1/10 of them don't get a job at all, which has utility -50 for Farad.

Farad can't decide: Western and Queens seem equally good choices to him.

b. How much utility does working for the government have for Farad?

Farad ends up going to Western and now he's about to graduate. Unfortunately, his grades aren't very good, so he would have to do a special summer program to get into law school. Alternatively he can apply to medical school or teacher's college, where he would definitely get in.

Farad has to choose between (i) taking the summer program, and (ii) going to medical school or teacher's college. He won't have time to do both. So if the summer program doesn't get him into law school, he'll end up unemployed.

c. How high would his chances of getting into law school have to be for him to risk taking the summer program?

10. Prove the following statement: if an action has only two possible consequences, C_1 and C_2, and they are equally probable, with $U(C_1) = -U(C_2)$, then the expected utility is zero.

13 Challenges to Expected Utility

We've learned two key elements of modern decision theory. First, we can quantify people's desires, values, and priorities, a quantity we call *utility*. Second, we should make our decisions according to the expected value formula. We should choose the option with highest *expected utility*.

These ideas have been extremely popular and influential in the last hundred years. But there have also been challenges.

13.1 The Allais Paradox

Suppose you have to choose between two options:

- 1A: $1 million dollars, guaranteed.
- 1B: a gamble...
  - 1% chance of $0,
  - 89% chance of $1 million,
  - 10% chance of $5 million.

Which would you choose, 1A or 1B? Write your answer down and set it aside, we'll come back to it in a moment.

Now, what if you had to choose instead between these two options:

- 2A: a gamble...
  - 90% chance of $0,
  - 10% chance of $5 million.
- 2B: a gamble...
  - 89% chance of $0,
  - 11% chance of $1 million.

Most people choose 1A over 1B in the first decision. With the safe option of walking away $1 million richer, they don't want to take the chance at $5 million. Even though there's only a 1% risk of walking away empty handed if they take the gamble, it's not worth it to them.

The 10% shot at $5 million isn't enough to risk losing out on the guaranteed $1 million.

But in the second decision, most people choose $2B$ over $2A$. There's no safe option now, in fact you'll probably walk away empty handed whatever you choose. Under these circumstances, most people prefer to take on an extra 1% risk of empty-handedness in exchange for a 10% shot at $5 million.

But here's the thing: these choices contradict the expected utility rule! It's not obvious at first. But a few lines of algebra will prove that it's so.

Suppose someone did choose $1A$ over $1B$, and $2B$ over $2A$, by applying the expected utility formula. Then we would know that $EU(1A) > EU(1B)$ and $EU(2B) > EU(2A)$. In other words:

$$EU(1A) - EU(1B) > 0,$$
$$EU(2A) - EU(2B) < 0.$$

But it turns out that's impossible, because:

$$EU(1A) - EU(1B) = EU(2A) - EU(2B).$$

To see why, let's first write out the expected utility formula for each option:

$$EU(1A) = U(\$1M),$$
$$EU(1B) = .89 \times U(\$1M) + .1 \times U(\$5M) + .01 \times U(\$0M),$$
$$EU(2A) = .9 \times U(\$0M) + .1 \times U(\$5M),$$
$$EU(2B) = .89 \times U(\$0M) + .11 \times U(\$1M).$$

Now a little arithmetic will show that we get the same result when we subtract the first two formulas and the second two:

$$EU(1A) - EU(1B) = -.01 \times U(\$0M) + .11 \times U(\$1M) - .1 \times U(\$5M),$$
$$EU(2A) - EU(2B) = -.01 \times U(\$0M) + .11 \times U(\$1M) - .1 \times U(\$5M).$$

Notice how both formulas are exactly the same. The difference in expected value between the first two options is exactly the same as the difference in value between the second two options. Which means if you're making decisions using the expected utility rule, you can't prefer the A option in the first decision and the B option in the second.

IT'S IMPORTANT TO UNDERSTAND that these calculations don't depend on the utility of money.

In Chapter 12 we noted that the difference in utility between $0 and $1 million might be larger for some people than for others. Could these individual priorities explain why some people prefer $1A$ over $1B$ yet $2B$ over $2A$?

No. In the Allais paradox, the way you personally value money is actually irrelevant. It doesn't matter how much you value $1 million vs. $0. Our calculations made no assumptions about the numerical values of $U(\$0)$, $U(\$1M)$, and $U(\$5M)$. We left those terms untouched, treating them as unknown placeholders. Whatever your personal utilities are, there's just no way to prefer $1A$ over $1B$ and $2B$ over $2A$, if you're following the expected utility rule.

Figure 13.1: Maurice Allais (1911–2010), photograph by Harcourt Studios

Leonard Savage (1917–1971) was an American staistician and mathematician, and a leading advocate of the expected utility rule. There doesn't seem to be any public-domain photograph of him available unfortunately, but you can find one here.

This challenge is called the *Allais paradox*. Maurice Allais was a French economist, writing at a time when some French thinkers disliked the idea of reducing decisions to a simple equation. The American statistician Leonard Savage, on the other hand, very much liked the idea of expected utility. So Allais cooked up this example to prove Savage wrong.

When Savage first encountered Allais' example, he did exactly what most people do. He chose $1A$ over $1B$, but $2B$ over $2A$. He violated the principles of his own theory!

Savage responded by acknowledging that people are tempted to make exactly the choices Allais predicted. But, he pointed out, people sometimes make irrational choices. And Allais' example brings out our irrational temptations, Savage argued.

To make his case, Savage cast the example in concrete terms. Imagine a random ticket will be drawn from a hat containing tickets numbered #1 through #100. The outcome of each options is determined according to the following table:

Table 13.1: Savage's version of the Allais paradox, using a lottery with 100 tickets

	#1	#2–11	#12–100
$1A$	$1M	$1M	$1M
$1B$	$0	$5M	$1M
$2A$	$1M	$1M	$0
$2B$	$0	$5M	$0

In the first row you're guaranteed to get $1 million. In the second row you have a 1% chance of getting nothing, an 89% chance of getting $1 million, and a 10% chance of getting $5 million. And so on.

Viewed this way, we can see that there's no difference between $1A$ and $1B$ if the ticket drawn is one of #12–100. And likewise for $2A$ vs. $2B$. So you must choose based on what will happen if the ticket drawn is #1 or one of #2–11. In other words, you should ignore the third column and just look at the first two.

But in the first two columns, choosing 1A over $1B$ is the same as

choosing 2A over 2B. So, to be consistent, you must choose 2A if you chose 1A.

Thus, Savage argued, we can see that the expected utility rule is correct once we frame the problem correctly.

HERE'S ANOTHER WAY of thinking about Savage's argument. If you choose 2B over 2A, then you must be willing to accept a 1% chance of getting nothing in exchange for a 10% chance at $5 million. But if you're willing to make that trade, then you should be willing to give up option 1A and take option 1B instead.

VISUALIZING the problem is also informative. Figure 13.2 shows Allais' first choice, between 1A and 1B. Notice how the riskiness of choosing 1B is much less noticeable in this format than the potential upside. The 10% chance of winning $5 million jumps right out at you. Whereas the 1% chance of winding up with nothing is barely noticable.

If you were presented with the Allais choices in this graphical format, instead of with numbers and words, you might choose very differently. Psychologists call this a "framing effect". The way a decision is framed can make a big difference to what people will choose.

Advertisers know a lot about framing effects.

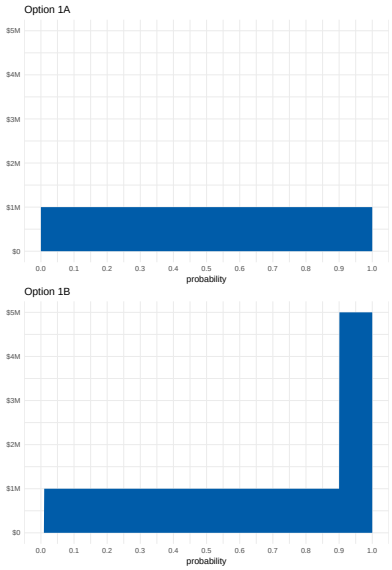


Figure 13.2: A graphical depiction of options 1A and 1B in the Allais paradox

13.2 *The Sure-thing Principle*

SAVAGE'S ANSWER to the Allais paradox is based on a famous principle of decision theory:

The Sure-thing Principle If you would choose X over Y if you knew that E was true, and you'd also choose X over Y if you knew E wasn't true, then you should choose X over Y when you don't know whether E is true or not.

How does this principle apply to the Allais paradox? Interpret E this way:

- E: One of tickets #12–100 will be drawn.

Then imagine someone who chos 1A over 1B. Would it make sense for them to choose 2B over 2A?

Well, first imagine they know E is true: they know one of tickets #12–100 will be drawn. Then they wouldn't prefer 2B over 2A. They wouldn't care about 2B vs. 2A because they'll get $0 either way.

Next imagine they know E isn't true: they know one of tickets #1–11 will be drawn instead. Then they still wouldn't prefer 2B over 2A. They chose 1A over 1B, so they prefer not to take a small risk of getting

$0 in order to have a chance at $5 million. And that's exactly the same tradeoff they're considering with 2*A* vs. 2*B*.

So the Sure-thing Principle says they should choose 2*A* over 2*B* if they chose 1*A* over 1*B*. They wouldn't choose 2*B* if they knew *E* was true, and they wouldn't choose it if they knew *E* was false. So, even when they don't know whether *E* is true or false, they should still choose 2*A*.

13.3 *Prescriptive vs. Descriptive*

SOME decision theorists use the expected utility formula to *describe* the way people make choices. If you want to predict when people will buy/sell a stock, for example, you might use the expected utility formula to describe what people will do.

But others use the formula to *prescribe* decisions: to determine what we *ought* to do. Sometimes people do what they should, but sometimes they make mistakes or do foolish things.

Savage's answer to the Allais paradox is based on the prescriptive approach to decision theory. According to Savage, people *should* make decisions according to the expected utility formula. But sometimes they don't, as Allais' example demonstrates.

13.4 *The Ellsberg Paradox*

HERE'S ANOTHER famous challenge to expected utility and the Sure-thing Principle:

> An urn contains 90 balls, 30 of which are red. The other 60 are either black or white, but the proportion of black to white is not known. A ball will be drawn at random, and you must choose between the following:
>
> - 1*A*: win $100 if the ball is red,
> - 1*B*: win $100 if the ball is black.
>
> You also face a second choice:
>
> - 2*A*: win $100 if the ball is either red or white,
> - 2*B*: win $100 if the ball is either black or white.

Most people choose 1*A* over 1*B*, since you know what you're getting with 1*A*: a 1/3 chance at the $100. Whereas 1*B* might give worse odds; it may even have no chance at all of winning, if there are no black balls.

At the same time, most people choose 2*B* over 2*A*, and for a similar reason. With 2*B* you know you're getting a 2/3 chance at the $100.

While 2*A* might give much worse odds, maybe even as low as 1/3 if there are no white balls in the urn.

Like in the Allais paradox, this popular combination of choices violates the expected utility rule. The calculation that shows this is pretty similar to the one we did with Allais, so we won't rehearse it here.

Instead let's think about what this puzzle is showing us.

13.5 *Ellsberg & Allais*

Figure 13.3: Daniel Ellsberg (b. 1931) is most famous as the leaker of the Pentagon Papers. The 2017 movie *The Post* tells that story, with Ellsberg portrayed by actor Matthew Rhys. (Photograph by Bernd Gross).

Ellsberg's paradox is strongly reminiscent of Allais'. Both exploit a human preference for the known. In the Allais paradox we prefer the sure million, and in the Ellsberg paradox we prefer to know our chances.

But the kind of risk at work in each paradox is different. In the Allais paradox, all the probabilities are known, and in one case we can even know the outcome. If you choose the safe million, you know what your fate will be.

But in the Ellsberg paradox, you never know the outcome. The most you can know is the chance of each outcome. And yet, our preference for the known still takes hold. We still prefer to go with what we know, even when all we can know is the chance of each outcome.

Is this preference for known risks rational? Well, it violates Savage's Sure-thing Principle. View Ellsberg's dilemma as a table:

Table 13.2: The Ellsberg paradox

	Red	Black	White
1*A*	$100	$0	$0
1*B*	$0	$100	$0
2*A*	$100	$0	$100
2*B*	$0	$100	$100

If you knew a white ball was going to be drawn, you wouldn't care which option you chose. In the first decision, you'd get $0 either way, and in the second you'd get $100 either way.

And if you knew a white ball wouldn't be drawn, then options 1A and 2*A* would be equivalent. The first two rows are identical to the second two rows, if we ignore the "White" column. So consistency seems to demand selecting 2*A* if you selected 1A.

Most decision theorists find this reasoning compelling. But some turn it on its head. They say: so much the worse for the Sure-thing Principle. The debate has yet to settle into a universal consensus.

Exercises

1. Which of the following best describes the Allais paradox? Choose one.

   a. Most people choose $1B$ over $1A$, and $2B$ over $2A$. But the expected monetary value of $2A$ is greater than the expected monetary value of $2B$.
   b. Most people choose $1A$ over $1B$, and $2A$ over $2B$. But the expected monetary value of $2B$ is greater than the expected monetary value of $2A$.
   c. Most people choose $1A$ over $1B$, and $2B$ over $2A$. But $1B$ and $2B$ have greater expected utilities.
   d. Most people choose $1A$ over $1B$, and $2B$ over $2A$. But it's impossible for the expected utility to be greater for $1A$ than for $1B$ and for $2B$ than for $2A$.

2. True or false: the usual choices in the Allais paradox conflict with the expected utility rule no matter what utilities we assign to \$0, \$$1M$, and \$$5M$.

   a. True
   b. False
   c. Neither

3. According to Savage's response to the Allais paradox, the rule of expected utility is:

   a. descriptive
   b. prescriptive
   c. both
   d. neither

4. Which of the following best describes Savage's answer to the Allais paradox? Choose one.

   a. People value money differently. For some people \$5 million is a lot better than \$1 million, for others it's only a littler better.
   b. Expected utility is a descriptive formula, not a prescriptive one. The Sure-thing Principle explains why most peoples' choices are wrong.
   c. Expected utility is a prescriptive formula, not a descriptive one. The Sure-thing Principle explains why most peoples' choices are wrong.
   d. Expected utility is only a rough guide. Other factors affect what choice you should make, like whether there is a "safe" (risk-free) option.

5. In the Allais Paradox, people tend to choose $1A$ over $1B$, but $2B$ over $2A$. This suggests that:

   a. for many people, the utility of $5 million is not five times that of $1 million.
   b. people maximize expected utility, not expected monetary value.
   c. people are more willing to accept a small risk when things are already risky.
   d. the Sure-thing Principle is descriptive, not prescriptive.

6. In Savage's description of the Allais paradox, we imagine the gambles depending on the random draw of a ticket. When applying the Sure-thing Principle to this analysis...

   a. the unknown event E must be: the ticket drawn is one of #12–#100.
   b. the unknown event E must be: the ticket drawn is one of #1–#11.
   c. the unknown event E can be either as in (a) or as in (b).
   d. None of the above

7. When applying Savage's Sure-thing Principle to the Ellsberg paradox, which of the following is the unknown event E?

   a. The ball drawn is red.
   b. The ball drawn is red or white.
   c. The ball drawn is white.
   d. None of the above

8. In the Ellsberg Paradox, people tend to choose $1A$ over $1B$, but $2B$ over $2A$. Which of the following does this suggest?

   a. People sometimes violate the Sure-thing Principle.
   b. People don't always maximize expected utility.
   c. People tend to prefer known chances over unknown chances.
   d. All of the above

14 *Infinity & Beyond*

Infinity poses a different sort of challenge to expected utility, and in this chapter we'll meet two famous examples.

14.1 The St. Petersburg Paradox

Suppose I'm going to flip a fair coin, and I'm going to keep flipping it until it lands heads. When it does, the game is over.

- If it comes up heads on the first flip, you win $2.
- If it comes up heads on the second flip, you win $4.
- If it comes up heads on the third flip, you win $8.
- Etc.

How much would you be willing to pay to play this game? Most people aren't willing to pay very much. After all, you probably won't win more than a few dollars. Most likely you'll only win $2 or $4 dollars. The chance you'll win more than say $32 is only about .03.

And yet, bizarrely, the expected value of this game is infinite! There's a $1/2$ chance you'll win $2, a $1/4$ chance you'll win $4, a $1/8$ chance you'll win $8, and so on. So:

$$\begin{aligned} E(G) &= Pr(\$2) \times \$2 + Pr(\$4) \times \$4 + Pr(\$8) \times \$8 + \ldots \\ &= (1/2)(\$2) + (1/4)(\$4) + (1/8)(\$8) + \ldots \\ &= \$1 + \$1 + \$1 + \ldots \\ &= \$\infty. \end{aligned}$$

And doesn't that mean a fair price for the game would be infinity dollars? So even if you only have a finite bankroll, you should be willing to stake it all to play!

Visually, the St. Petersburg game looks something like Figure 14.1. We can't show every possible outcome, because the payoffs get larger and larger without limit. So we have to cut things off at some point. But we can get the general idea by displaying just the first few possible outcomes. (The rest are shown faded and only partially.)

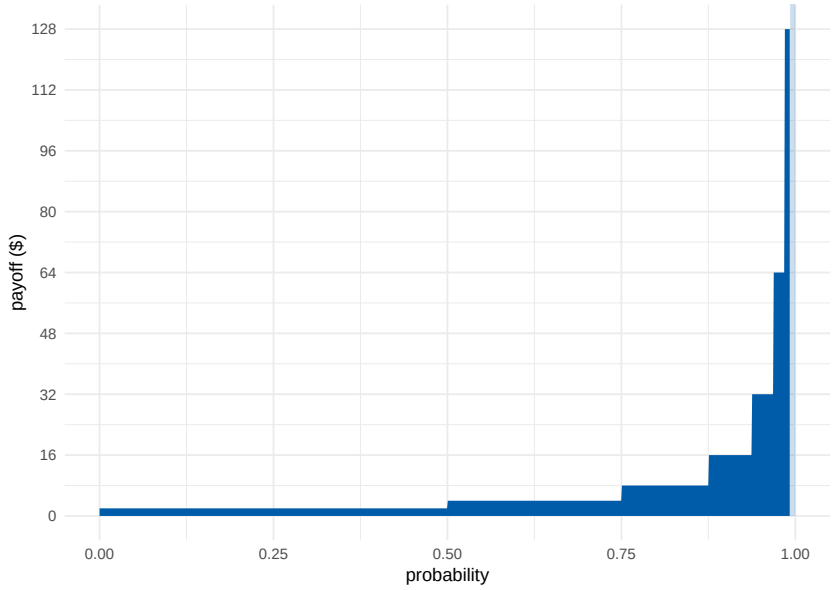


Figure 14.1: The St. Petersburg game's first seven outcomes. The rest extend off the chart and are shown faded.

As the number of potential flips grows, the rectangles get narrower because the outcomes become less probable. But they also get taller. The payoff doubles every time the probability is cut in half. The result is an infinite sequence of rectangles, each with the same area, namely 1. So the total area of all the rectangles is infinite.

Most people, when they first encounter this puzzle, try to take the easy way out. "The game is impossible," they say. "Nobody could actually fund this game in real life. No casino has unlimited money, not even the government does. Besides, the coin would wear out after a while. So the house couldn't even guarantee they'd be able to finish the game."

There are two serious problems with this answer.

First, consider whether you'd be willing to risk your life to play the St. Petersburg game if the casino did have enough cash on hand, and enough coins to keep the game going as long as it takes? Imagine if God descended from heaven and offered to run the game. Would you gamble your life to play then?

Second, even if the game had a finite limit—let's suppose it was 100 flips—would you be willing to pay $100 to play? The chance you'd make any money at all is less than 1%.

Figure 14.2: Daniel Bernoulli (1700–1782)

14.2 *Bernoulli's Solution*

The St. Petersburg game was invented by the mathematician Nicolaus Bernoulli in the 18th Century. He discussed it with his cousin Daniel Bernoulli, who published a famous solution in the *St. Petersburg Academy Proceedings*. (Hence the name of the game.)

His solution: replace monetary value with real value, *utility* in other words. Recall that the more money you have, the less value additional money brings. So even though the payoffs in the St. Petersburg game double from \$2 to \$4 to \$8 etc., the *real* value doesn't double. It grows more slowly.

What is the real value of money then? How much utility does a gain of $\$x$ bring? Bernoulli proposed that utility increases "logarithmically" with money. In terms of dollars, $U(\$x) = \log(x)$. Look at Figure 14.3 to see what this looks like.

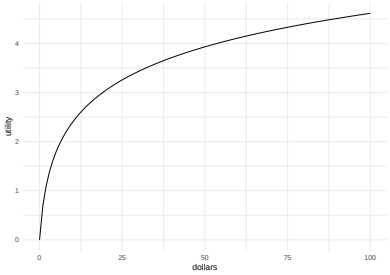


Figure 14.3: Bernoulli's logarithmic utility function

Notice how, for example, \$16 doesn't have twice the utility of \$8. In fact you have to go all the way up to \$64 to get twice the utility of \$8. That's because $64 = 8^2$, and logarithms have the property that $\log(a^b) = b \times \log(a)$.

The difference this makes to the St. Petersburg game is displayed in Figure 14.4. The rectangles don't all have the same area of 1 anymore, they get smaller instead. In fact, Bernoulli showed that the total area of all the rectangles is only about 1.39. In other words, the expected utility of the St. Petersburg game is about 1.39, the equivalent of a \$4 gain (because $\log(4) \approx 1.39$).

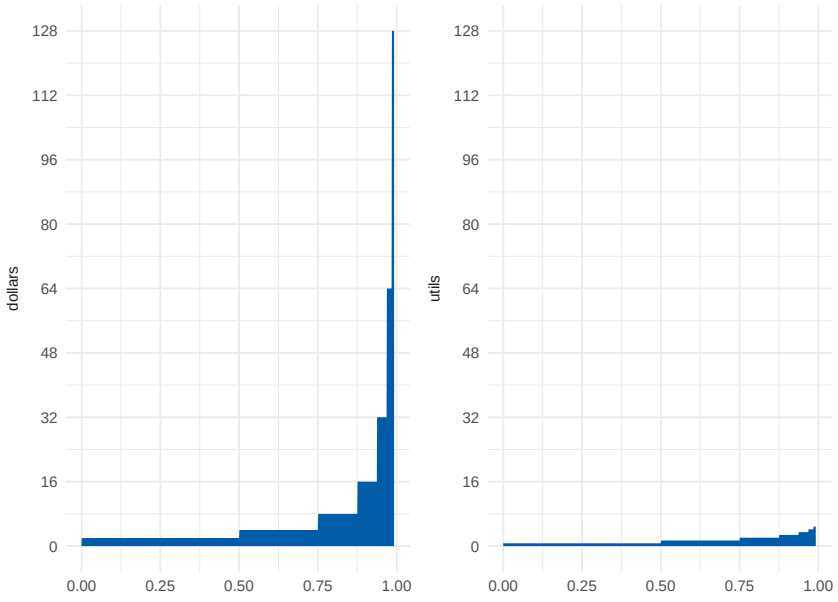
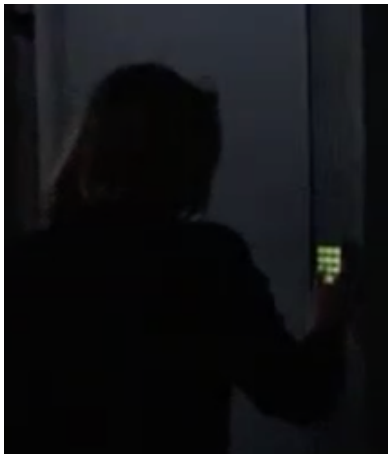


Figure 14.4: Dollars vs. utils in the St. Petersburg game

So if Bernoulli is right about the utility of money, the fair price for the St. Petersburg game is only about $4. And in fact that's about what most people say they would pay.



The decimal value of e was used as the key code for a locked room in the *X-Files* episode "Paper Clip" (except they seem to have forgotten the 1). The same scene also references Monty Hall.

14.3 St. Petersburg's Revenge

UNFORTUNATELY, although Bernoulli was probably right that money decreases in value the more of it you have, that doesn't actually solve the paradox. Because we can just modify the game so that the monetary payoffs grow even faster.

Instead of the payoffs increasing like this:

$$\$2, \$4, \$8, \ldots$$

Let's change the game so they increase like this:

$$\$e^2, \$e^4, \$e^8, \ldots$$

You may remember from math class that e is a special number, approximately equal to 2.71828. It has the special property that:

$$\log(e^x) = x.$$

So the utility of winning $\$e^2$ is $U(\$e^2) = 2$. The utility of winning $\$e^4$ is $U(\$e^4) = 4$. And so on.

Now the utilities are the same as the dollar payoffs were in the original version of the game:

$$2, 4, 8, \ldots$$

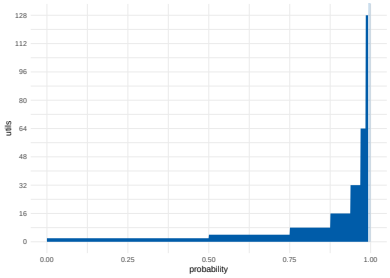


Figure 14.5: St. Petersburg's revenge

So the expected *utility* is the same as the expected monetary value of the original game, namely infinite:

$$\begin{aligned} E(G) &= Pr(\$e^2)U(\$e^2) + Pr(\$e^4)U(\$e^4) + Pr(\$8)U(\$e^8) + \ldots \\ &= (1/2)(2) + (1/4)(4) + (1/8)(8) + \ldots \\ &= 1 + 1 + 1 + \ldots \\ &= \infty. \end{aligned}$$

So, once again, you should be willing to give anything to play. Or so the expected utility rule seems to demand.[1]

WHAT'S THE RIGHT SOLUTION to the St. Petersburg paradox then? Nobody knows, really. Once infinities get involved, the whole expected value framework seems to go off the rails.

Some decision theorists respond by insisting that there's a finite limit on utility. There's only so good an outcome can be, they say.

[1] Still worried there's not enough money in the world to guarantee this game? Try another variation. Imagine God offers to substitute days in heaven for dollars. Unlike dollars, days in heaven don't lose value the more you have.

But others don't find this response plausible. There may be a limit on how much good you can get out of money, because there's only so much money can buy. But money is only a means to an end, a medium we can exchange for the things we really want—things of intrinsic value like pleasure, happiness, beauty, and love. Is there really a finite limit on how much value these things can bring into the world? If so, what is that limit, and why?

14.4 *Pascal's Wager*

Figure 14.6: Blaise Pascal (1623–1662)

THE modern theory of probability has a curious origin. It started with Blaise Pascal, a French mathematician and philosopher living in the 1600s. Pascal had a friend who has an avid gambler, and who came to him for advice on gambling problems. Pascal discussed these problems in letters with another famous mathematician, Pierre de Fermat. Out of their correspondence, the concept of expected value was born.

Pascal was a devout Catholic. So, once he developed the tools of decision theory, he applied them to religious questions. Like: is it rational to believe in God?

Pascal realized he could think of this question as a decision problem. If God exists, then believing gets you into heaven, which is very good. Whereas not believing gets you a one-way ticket to hell, which is terribly bad. So believing in God looks like the better option.

But Pascal also realized that probabilities matter as much as the potential payoffs when making decisions. Playing the lottery might win you millions, but the odds are very poor. So spending your money on a cup of coffee might be the smarter choice.

Likewise, believing in God might be such a long shot that it's not worth it, even if the potential payoff of heaven is fantastic. Of course, Pascal himself already believed in God. But he wanted to convince others to do the same, even if they thought it very unlikely that God exists.

The potential payoff of believing in God is special though, Pascal realized: it's not just very good, it's *infinitely* good. If you believe in God and you're right, you go to heaven for eternity, a neverending existence of pure ecstacy. Whereas if you don't believe in God and you're wrong, the payoff is infinitely bad: an eternity in hell.

So Pascal figured the decision problem looks something like this:

Table 14.1: Pascal's Wager

	God Exists	God Doesn't Exist
Believe	$+\infty$	-10

	God Exists	God Doesn't Exist
Don't Believe	$-\infty$	$+10$

The -10 in the top-right cell is somewhat arbitrary. There's some finite cost to believing in God if he doesn't exist. People generally prefer to believe the truth. And if there's no God, many people would prefer not to spend their time and energy on religious matters. But is -10 a good representation of these losses?

It doesn't matter, as it turns out. Whether we use -10, -20, or any other finite number for those losses, it ends up getting drowned out by the $+\infty$ in the neighbouring cell. And likewise for the $+10$ in the bottom-right (which represents the value of being right about God's existence, and maybe also of living life your own way).

Why do these finite values not matter in the end? How do they get drowned out?

Well, said Pascal, even atheists must admit that there's some small chance God exists. Nobody can be 100% sure there is no God. So when we calculate the expected utility of believing in God, we get the following result:

$$\begin{aligned} E(\text{Believe}) &= Pr(G) \times \infty + Pr(\sim G) \times -10 \\ &= \infty - \text{something finite} \\ &= \infty. \end{aligned}$$

As long as $Pr(G)$ is some positive number, multiplying it by ∞ results in ∞. No matter how small a number is, adding it to itself infinitely many times results in an infinite quantity. No matter how small your steps, if you take infinitely many of them, you can travel any distance.

Also, ∞ minus any finite quantity is still ∞. If I start with an infinite collection of objects, say an infinite list of numbers for example:

$$1, 2, 3, 4, 5, \ldots$$

and I remove finitely many of them, say the numbers 1 throught 10, I'm still left with an infinite list:

$$11, 12, 13, 14, 15, \ldots$$

Therefore, Pascal argued, believing in God has infinite expected value.

What about not believing? It has infinitely negative expected value:

$$\begin{aligned} E(\text{Don't Believe}) &= Pr(G) \times -\infty + Pr(\sim G) \times 10 \\ &= -\infty + \text{something finite} \\ &= -\infty. \end{aligned}$$

So believing in God is an infinitely better decision than not believing! Or so Pascal thought.

14.5 Responses to Pascal's Wager

Some people criticized Pascal's argument on the grounds that belief is not a decision. Whether you believe in something isn't voluntary, like deciding what shirt to wear in the morning. You can't just decide to believe in God, you can only believe what seems plausible to you based on what you know.

But, Pascal famously replied, you can decide how you spend your days. And you can decide to spend them with religious people, reading religious books, and going to a house of worship. So you can decide to take steps that will, eventually, make you a believer. And since believing is so much better than not, that's how you should spend your days.

A more serious problem with Pascal's argument is known as *the many gods* problem.

In Pascal's day, in France, Catholicism dominatd the religious landscape. So for him, believing in God just meant believing in *Catholicism's* conception of God. But there are many possible gods besides the Catholic god, like the god of Islam, the god of Judaism, the gods of Hinduism, and so on.

What happens if you choose the wrong God? You might go to hell! The god of Mormonism might send you to hell for believing in Catholicism, for example. There might even be an anti-Catholic god, who sends all Catholics to hell and everyone else to heaven!

So the correct decision table looks more like this:

Table 14.2: Pascal's Wager with many gods. The ... stands in for the many different gods that might exist, each of which has its own column. There is also a row for each of these possible gods, since we have the option to believe in that god.

	Catholic God Exists	Anti-Catholic God Exists	...	No God Exists
Believe Catholic	$+\infty$	$-\infty$	...	-10
Believe Anti-Catholic	$-\infty$	$+\infty$	...	-10
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Don't Believe	$-\infty$	$+\infty$	...	$+10$

What's the expected utility of believing in the Catholic god now? It

turns out there's no answer! The calculation comes out undefined:

$$\begin{aligned} E(\text{Believe Catholic}) &= Pr(\text{Catholic})U(\text{Catholic}) + Pr(\text{Anti-Catholic})U(\text{Anti-Catholic}) + \ldots \\ &= Pr(\text{Catholic}) \times \infty + Pr(\text{Anti-Catholic}) \times -\infty + \ldots \\ &= \infty - \infty + \ldots \\ &= \text{undefined} \end{aligned}$$

Why is this undefined? Why isn't $\infty - \infty = 0$? Because taking an infinite quantity away from an infinite quantity can result in any quantity left over.

Imagine we start with an infinite list, a list of all the counting numbers for example:

$$1, 2, 3, 4, 5 \ldots$$

Now remove all the even numbers from that list. There's still an infinite list of numbers remaining:

$$1, 3, 5 \ldots$$

After removing an infinite quantity from an infinite quantity, we still have a neverending list left over. So it *looks* at first as though $\infty - \infty = \infty$.

But not so fast! Suppose we start again with all the counting numbers:

$$1, 2, 3, 4, 5 \ldots$$

But this time we remove *all* the numbers. Not just the evens, but the odds too. Then there would be nothing left on our list. So now it looks like $\infty - \infty = 0$ instead of ∞!

The moral: there are many ways to take an infinite quantity away from an infinite quantity. Some of these leave an infinite quantity remaining. Others leave nothing remaining. There are still others that would leave just one, two, or three items remaining. (Can you think of your own examples here?)

So $\infty - \infty$ is not a well-defined quantity.

THE expected value framework doesn't seem to work well when infinities show up. The St. Petersburg problem gave us similar trouble. Researchers are still trying to figure out how to make decisions that involve infinite quantities.

Exercises

1. In the St. Petersburg game, what is the probability of winning $16 or less?

   a. 1/16

b. 1/2
c. 7/8
d. 15/16

2. Suppose we modify the St. Petersburg game by capping the number of flips at 50: if the coin lands tails 50 times in a row, the game is over and you win nothing. What is the expected *monetary* value of this game?

   a. \$25
   b. \$50
   c. \$100
   d. $\$2^{50}$

3. According to Daniel Bernoulli's solution to the St. Petersburg paradox, the utility of the coin landing heads on the $(n+1)$-th flip isn't twice that of landing on the n-th flip, because...

   a. when the payouts get very large, it becomes less and less likely you'll actually be paid the amount promised.
   b. the longer the game goes on, the greater the chance it will be interrupted before the $(n+1)$-th flip can happen.
   c. infinity isn't a real number.
   d. the more money you have, the less value additional money adds.

4. According to Daniel Bernoulli, the utility of $\$2x$ is not twice that of $\$x$ because utility is logarithmic: $U(\$x) = \log(\$x)$. So the expected monetary value of the St. Petersburg game may be infinite, but its expected utility is finite.

   In the text we discussed one reason this doesn't resolve the paradox. What was the reason?

   a. It's not very plausible that utility is logarithmic.
   b. Even on a logarithmic scale, the expected utility is still infinite.
   c. Even if Bernoulli is right, the expected monetary value is still infinite.
   d. The paradox appears all over again when we double the utilities instead of the dollar amounts.

5. Some people respond to the St. Petersburg paradox by arguing that there's a limit on how good an outcome can be. Utilities have an upper bound, they say.

   Bernoulli's logarithmic utility function does not have an upper bound. In other words: for any real number y, no matter how large, there is some real number x such that $\log(x) = y$. Give a

formula for y in terms of x. In other words, what is the function f such that $x = f(y)$, if $y = \log(x)$?

6. Consider the first form of Pascal's Wager, displayed in Table 14.1. What is the expected utility of believing in God if the probability God exists is 0?

   a. -10
   b. 0
   c. ∞
   d. Undefined

7. Consider the first form of Pascal's Wager, displayed in Table 14.1. Suppose the probability God exists is almost zero, but not quite: one in ten trillion. What is the expected utility of believing in God?

   a. ∞
   b. $-\infty$
   c. 0
   d. one in ten trillion

8. According to the many gods objection, Pascal's Wager argument fails to account for the possibility of other gods besides the Catholic conception. What happens to Pascal's decision table when other gods are included? Circle one answer.

   a. The expected utilities can't be calculated because there are infinitely many possible Gods.
   b. The expected utilities can't be calculated because you can't subtract ∞ from ∞.
   c. The expected utilities can't be calculated because we don't know the probabilities.
   d. The expected utility of believing in God ends up being the same as the expected utility of not believing in God.

9. Consider the "many gods" version of Pascal's Wager, displayed in Table 14.2. What is the expected utility of believing in Catholicism?

   a. ∞
   b. $-\infty$
   c. Undefined
   d. 0

10. Suppose an atheist responds to Pascal's Wager as follows: "Belief isn't something we choose. I can't just decide to believe in God any more than you can decide to believe unicorns exist."

How would Pascal reply?

a. Belief *is* a choice when it comes to abstract questions, like the existence of God. Because such questions can't be settled by just looking and seeing.
b. Belief *is* a choice when it comes to questions where the evidence is conflicting. And the evidence for/against God's existence does conflict.
c. Maybe you can't control what you believe (descriptive), but it's still true that you *should* believe (normative).
d. Maybe you can't control what you believe directly. But you can influence your beliefs indirectly, by choosing how and where you spend your time.

11. Explain why the expected utility of believing in God is undefined in Pascal's Wager, if we include other possible gods like the "anti-Catholic" god.

    Write your answer using complete sentences. You can include equations, tables, or diagrams, but you must explain what they mean in words. Your answer should include an explanation why $\infty - \infty$ is undefined.

Part III

15 Two Schools

Figure 15.1: Ned Flanders informs us that, well sir, there are two schools of thought on the matter.

WHAT does the word "probability" mean? There are two competing philosophies of probability, and two very different schools of statistics to go with them.

15.1 Probability as Frequency

IN statistics, the dominant tradition has been to think of probability in terms of "frequency". What's the probability a coin will land heads? That just depends on how often it lands heads—the *frequency* of heads.

If a coin lands heads half the time, then the probability of heads on any given toss is $1/2$. If it lands heads $9/10$ of the time, then the probability of heads is $9/10$.

This is probably the most common way of understanding "probability". You may even be thinking to yourself, *isn't it obvious that's what probability is about?*

15.2 Probability as Belief

BUT many statements about probability don't fit the frequency mold, not very well at least.

Consider the statement, "the probability the dinosaurs were wiped out by a meteor is 90%." Does this mean 90% of the times dinosaurs existed on earth, they were wiped out? They only existed once! This probability is about an event that doesn't repeat. So there's no frequency with which it happens.

Here's another example: "humans are probably the main cause of our changing climate." Does that mean most of the time, when climate change happens, humans are the cause? Humans haven't even been around for most of the climate changes in Earth's history. So again: this doesn't seem to be a statement about the frequency with which humans cause global warming.

These statements appear instead to be about what beliefs are supported by the evidence. When someone says it's 90% likely the dinosaurs were wiped out by a meteor, they mean the evidence warrants

being 90% confident that's what happened.[1] Similarly, when someone says humans are probably the main cause of climate change, they mean that the evidence warrants being more than 50% confident it's true.

So, some probability statements appear to be about *belief*, not frequency. If a proposition has high probability, that means the evidence warrants strong belief in it. If a proposition has low probability, the evidence only warrants low confidence.

[1] What evidence? People don't always say what evidence they're relying on. But sometimes they do: fossil records and geological traces, for example.

15.3 Which Kind of Probability?

WHICH kind of probability are scientists using when they use probability theory? Is science about the frequency with which certain events happen? Or is it about what beliefs are warranted by the evidence?

There is a deep divide among scientists on this issue, especially statisticians.

The *frequentists* think that science deals in the first kind of probability, frequency. This interpretation has the appeal of being conrete and objective, since we can observe and count how often something happens. And science is all about observation and objectivity, right?

The *Bayesians* think instead that science deals in the second kind of probability, belief-type probability. Science is supposed to tell us what to believe given our evidence, after all. So it has to go beyond just the frequencies we've observed, and say what beliefs those observations support.

Let's consider the strengths and weaknesses of each approach.

15.4 Frequentism

ACCORDING to frequentists, probability is all about how often something happens. But what if it only ever has one opportunity to happen?

For example, suppose we take an ordinary coin fresh from the mint, and we flip it once. It lands heads. Then we melt it down and destroy it. Was the probability of heads on that flip 1? The coin landed heads 1 out of 1 times, so isn't that what the frequency view implies? And yet, common sense says the probability of heads was 1/2, not 1. It was an ordinary coin, it could have landed either way.

Well, we can distinguish *actual* frequency from *hypothetical* frequency.

Actual frequency is the number of times the coin actually lands heads, divided by the total number of flips. If there's only one flip and it's a heads, then the actual frequency is 1/1, which is just 1. If there's ten flips and four are heads, then the actual frequency is 4/10.

But *hypothetical frequency* is the number of times the coin *would* land heads if you flipped it over and over for a long time, divided by the

total number of hypothetical flips. If we flipped the coin a hundred times for example, it would probably land heads about half the time, like in Figure 15.2.

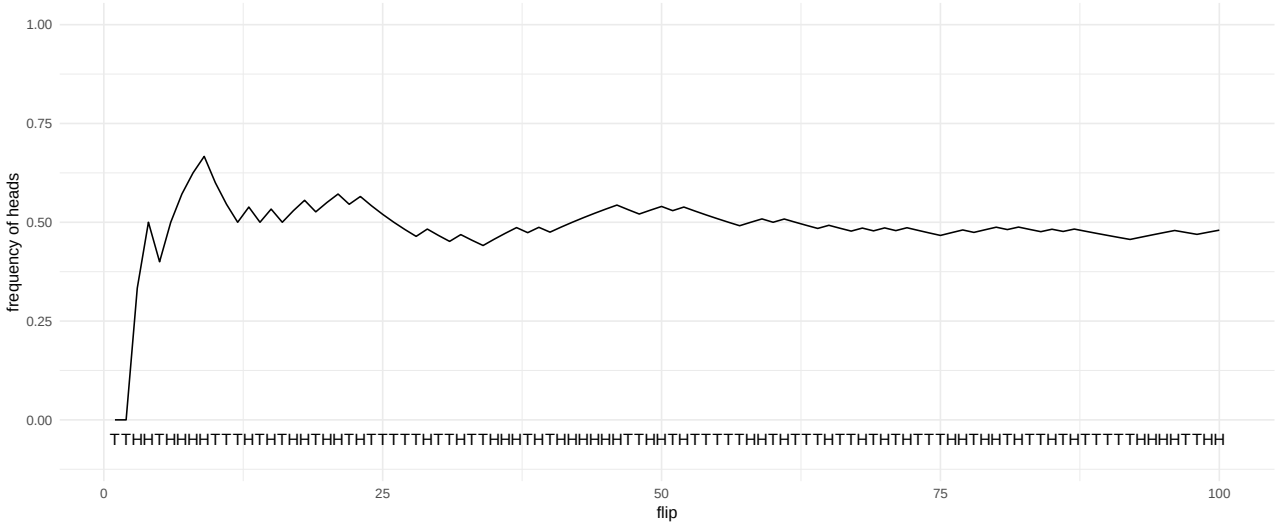


Figure 15.2: The frequency of heads over the course of 100 coin flips. This particular sequence of heads and tails was generated by a computer simulation.

Presumably, it's the *hypothetical* frequency that is the real probability of heads, according to frequentists. So doesn't that solve our problem with one-off events? Even if a coin is only ever flipped once, what matters is how it would have landed if we'd flipped it many times.

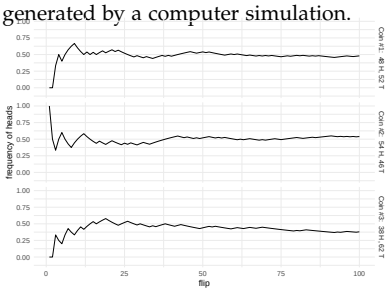


Figure 15.3: Three fair coins flipped 100 times each, yielding three different frequencies

SERIOUS problems beset the hypothetical frequency view too, however.

The first problem is that it makes our definition of "probability" circular, because hypothetical frequency has to be defined in terms of probability. If you flipped the coin over and over, say a hundred times, the most *probable* outcome is 50 heads and 50 tails. But other outcomes are perfectly possible, like 48 heads, or 54 heads. Figure 15.3 shows an example of three fair coins flipped 100 times each, yielding three different frequencies.

So the hypothetical frequency of 1/2 isn't what would necessarily happen. It's only what would *probably* happen. So what we're really saying is: "probability" = most probable hypothetical frequency. But you can't define a concept in terms of itself!

The second problem is about observability. You can observe actual frequencies, but not hypothetical frequencies. We never actually get to flip a coin more than a few hundred times. So hypothetical frequencies aren't observable and objective, which undermines the main appeal of the frequency theory.

A third problem has to do with evaluating scientific theories. Part

of the point of science is establish which theory is most probable. But theories don't have frequencies. Recall the example from earlier, about the dinosaurs being made extinct by a meteor. Or take the theory that DNA has a double-helix structure. When we say these theories are highly probable, how would we translate that into a statement about hypothetical frequencies?

A fourth and final problem is that how often an event happens depends on what you compare it to. It depends on the *reference class*. Consider Tweety, who has wings and is coloured black-and-white. What is the probability that Tweety can fly? Most things with wings can fly. Most things that are black-and-white cannot. Which reference class determines the probability that Tweety can fly? The class of winged things, or the class of black-and-white things?

It's problems like these that drive many philosophers and scientists away from frequentism, and towards the alternative offered by so-called "Bayesian" probability.

15.5 Bayesianism

ACCORDING to Bayesians, probability is ultimately about belief. It's about how certain you should be that something is true.

For example, $Pr(A) = .9$ means that A is certain to degree 0.9. We can be 90% confident that A is true. Whereas $Pr(A) = .3$ means that A is certain to degree 0.3. We can only be 30% confident A is true.

Why is this view called "Bayesianism"? Because it uses Bayes' theorem to explain how science works.

Suppose we have a hypothesis H, and some evidence E. How believable is H given the evidence E? Bayes' Theorem tells us $Pr(H \mid E)$ can be calculated:

$$Pr(H \mid E) = \frac{Pr(H)Pr(E \mid H)}{Pr(E)}.$$

And we saw in Section 10.3 how each term on the right corresponds to a rule of good scientific reasoning.

The better a theory fits with the evidence, the more believable it is. And $Pr(E \mid H)$ corresponds to how well the hypothesis explains the evidence. Since this term appears in the numerator of Bayes' theorem, it makes $Pr(H \mid E)$ larger.

The more surprising ("novel") a finding is, the more it supports a theory that explains it. The term $Pr(E)$ corresponds to how surprising the evidence is. And since it appears in the denominator of Bayes' theorem, surprising evidence makes $Pr(H \mid E)$ larger if H can successfully explain E.

Finally, new evidence has to be weighed against previous evidence and existing considerations. The term $Pr(H)$ corresponds to the prior

plausibility of the hypothesis H, and it appears in the numerator of Bayes' theorem. So the more the hypothesis fits with prior considerations, the larger $Pr(H \mid E)$ will be.

So, Bayesians say, we should understand probability as the degree of belief it's rational to have. The laws of probability, like Bayes' theorem, show us how to be good, objective scientists, shaping our beliefs according to the evidence.

THE main challenge for Bayesians is objectivity. Critics complain that science is objective, but belief is subjective. How so?

First, belief is something personal and mental, so it can't be quantified objectively. What does it even mean to be 90% confident that something is true, you might ask? How can you pin a number on a belief?

And second, belief varies from person to person. People from different communities and with different personalities bring different opinions and assumptions to the scientific table. But science is supposed to eliminate personal, subjective elements like opinion and bias.

So frequentists and Bayesians both have their work cut out for them. In the coming chapters we'll see how they address these issues.

16 Beliefs & Betting Rates

For Bayesians, probabilities are beliefs. When I say it'll probably rain today, I'm telling you something about my personal level of confidence in rain today. I'm saying I'm more than 50% confident it'll rain.

But how can we quantify something as personal and elusive as a level of confidence? Bayesians answer this question using the same basic idea we used for utility in Chapter 12. They look at people's willingness to risk things they care about.

16.1 Measuring Personal Probabilities

The more confident someone is, the more they'll be willing to bet. So let's use betting rates to quantify personal probabilities.

I said I'm more than 50% confident it'll rain today. But exactly how confident: 60%? 70%? Well, I'd give two-to-one odds on it raining today, and no higher. In other words, I'd accept a deal that pays \$1 if it rains, and costs me \$2 otherwise. But I wouldn't risk more than \$2 when I only stand to win \$1.

In this example I put 2 dollars on the table, and you put down 1 dollar. Whoever wins the bet keeps all 3 dollars. The sum of all the money on the table is called the ***stake***. In this case the stake is $\$2 + \$1 = \$3$.

If it doesn't rain, I'll lose \$2. To find my ***fair betting rate***, we divide this potential loss by the stake:

$$\begin{aligned} \text{betting rate} &= \frac{\text{potential loss}}{\text{stake}} \\ &= \frac{\$2}{\$2 + \$1} \\ &= \frac{2}{3}. \end{aligned}$$

A person's betting rate reflects their degree of confidence. The more confident they are of winning, the more they'll be willing to risk losing. In this example my betting rate is $2/3$ because I'm $2/3$ confident it will rain. That's my personal probability: $Pr(R) = 2/3$.

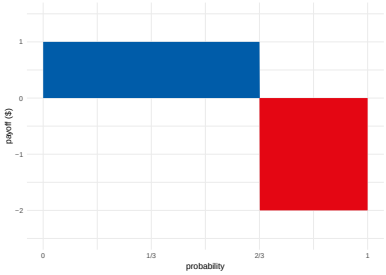


Figure 16.1: A bet that pays \$1 if you win and costs \$2 if you lose, is fair when the blue and red regions have equal size: when the probability of winning is $2/3$.

Notice that a bet at two-to-one odds has zero expected value given my personal probability of 2/3:

$$(2/3)(\$1) + (1/3)(-\$2) = 0.$$

This makes sense: it's a fair bet from my point of view, after all.

What if I were less confident in rain, say just 1/10 confident? Then I'd be willing to stake much less. I'd need you to put down at least $9 before I'd put down even $1. Only then would the bet have 0 expected value:

$$(1/10)(\$9) + (9/10)(-\$1) = 0.$$

So, for the bet to be fair in my eyes, the odds have to match my fair betting rate.

Here's the general recipe for quantifying someone's personal probability in proposition A:

1. Find a bet on A they deem fair. Call the potential winnings w and the potential losses l.
2. Because they deem the bet fair, set the expected value of the bet equal to zero:

$$Pr(A) \times w + (1 - Pr(A)) \times -l = 0.$$

3. Now solve for $Pr(A)$:

$$\begin{aligned} Pr(A) \times w + (1 - Pr(A)) \times -l &= 0 \\ Pr(A) \times w &= (1 - Pr(A)) \times l \\ Pr(A) \times w + Pr(A) \times l &= l \\ Pr(A) &= \frac{l}{w + l}. \end{aligned}$$

Notice how we got the same formula we started with: potential loss divided by total stake.

You can memorize this formula, but personally, I prefer to apply the recipe. It shows why the formula works, and it also exposes the formula's limitations. It helps us understand when the formula *doesn't* work.

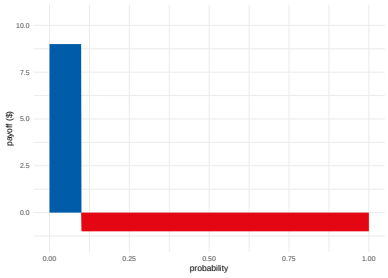


Figure 16.2: A bet that pays $9 if you win and costs $1 if you lose is fair when the probability of winning is 1/10.

16.2 *Things to Watch Out For*

PERSONAL probabilities aren't revealed by just any old betting rate a person will accept. They're exposed by the person's *fair* betting rates.

Consider: I'd take a bet where you pay me a million dollars of it rains today, and I pay you just $1 otherwise. But that's because I think this bet is *advantageous*. I don't think this is a fair bet, which is why I'd only take one side of it. I wouldn't take the reverse deal, where I win

$1 if it rains and I pay you a million dollars if it does. That's a terrible deal from my point of view!

So you can't just look at a bet a person is willing to accept. You have to look at a bet they're willing to accept *because they think it's fair*.

ANOTHER caveat is that we're cheating by using dollars instead of utils. When we learned about utility, we saw that utility and dollars can be quite different. Gaining a dollar and losing a dollar aren't necessarily comparable. Especially if it's your last dollar!

So, to really measure personal probabilities accurately, we'd have to substitute utilities for dollars. Nevertheless, we'll pretend dollars and utils are equal for simplicity. Dollars are a decent approximation of utils for many people, as long as we stick to small sums.

LAST but definitely not least, our method only works when the person is following the expected value formula. Setting the expected value equal to zero was the key to deriving the formula:

$$Pr(A) = \frac{\text{potential loss}}{\text{stake}}.$$

But we know people don't always follow the expected value formula, that's one of the lessons of the Allais paradox. So this way of measuring personal probabilities is limited.

16.3 Indirect Measurements

SOMETIMES we don't have the betting rate we need in order to apply the loss/stake formula directly. But we can still figure things out indirectly, given the betting rates we do have.

For example, I'm not very confident there's intelligent life on other planets. But I'd be much more confident if we learned there was life of any kind on another planet. If NASA finds bacteria living on Mars, I'll be much less surprised to learn there are intelligent aliens on Alpha Centauri.

Exactly how confident will I be? What is $Pr(I \mid L)$, my personal probability that there is intelligent life on other planets given that there's life of some kind on other planets at all?

Suppose I tell you my betting rates for I and L. I deem the following bets fair:

- I win $9 if I is true, otherwise I pay $1.
- I win $6 if L is true, otherwise I pay $4.

You can apply the loss/stake formula to figure $Pr(I) = 1/10$ and $Pr(L) = 4/10$. But what about $Pr(I \mid L)$? You can figure that out

by starting with the definition of conditional probability:

$$\begin{aligned} Pr(I \mid L) &= Pr(I \,\&\, L)/Pr(L) \\ &= Pr(I)/Pr(L) \\ &= 1/4. \end{aligned}$$

The second line in this calculation uses the fact that I is equivalent to $I \,\&\, L$. If there's intelligent life, then there must be life, by definition. So $I \,\&\, L$ is redundant. We can drop the second half and replace the whole statement with just I.

The general strategy here is: 1) identify what betting rates you have, 2) apply the loss/stakes formula to get those personal probabilities, and then 3) apply familiar rules of probability to derive other personal probabilities.

We have to be careful though. This technique only works if the subject's betting rates follow the familiar rules of probability. If my betting rate for rain tomorrow is $3/10$, you might expect my betting rate for no rain to be $7/10$. But people don't always follow the laws of probability, just as they don't always follow the expected utility rule. The taxicab problem from Chapter 8 illustrates one way people commonly violate the rules of probability. We'll encounter another way in the next chapter.

Exercises

1. Li thinks humans will eventually colonize Mars. More exactly, he regards the following deal as fair: if he's right about that, you pay him \$3, otherwise he'll pay you \$7.

   Suppose Li equates money with utility: for him, the utility of gaining \$3 is 3, the utility of losing \$7 is -7, and so on.

   a. What is Li's personal probability that humans will colonize Mars?

   Li also thinks there's an even better chance of colonization if Elon Musk is elected president of the United States. If Musk is elected, Li will regard the following deal as fair: if colonization happens you pay him \$3, otherwise he pays you \$12.

   b. What is Li's personal conditional probability that humans will colonize Mars, given that Elon Musk is elected U.S. president?

   Li thinks the chances of colonization are lower if Musk is not elected. His personal conditional probability that colonization will happen given that Musk is not elected is $1/2$.

c. What is Li's personal probability that Musk will be elected? (Assume his personal probabilities obey the laws of probability.)

2. Sam thinks the Saskatchewan Roughriders will win the next Grey Cup game. She's confident enough that she regards the following deal as fair: if they win, you pay her $3, otherwise she'll pay you $7.

   Suppose Sam equates money with utility: for her, the utility of gaining $3 is 3, the utility of losing $7 is -7, and so on.

   a. What is Sam's personal probability that the Roughriders will win the Grey Cup?

   Sam thinks the Roughriders will have an even better chance in the snow. If it snows during the game, she will regard the following deal as fair: if the Roughriders win, you pay her $3, otherwise she'll pay you $12.

   b. What is Sam's personal conditional probability that the Roughriders will win the Grey Cup if it snows?

   Sam thinks that the Roughriders will lose their advantage if it doesn't snow. Her personal conditional probability that the Roughriders will win if it doesn't snow is $1/2$.

   c. What is Sam's personal probability that it will snow during the Grey Cup? (Assume her personal probabilities obey all the familiar laws of probability.)

3. Sam thinks the Leafs have a real shot at the playoffs next year. In fact, she regards the following deal as fair: if the Leafs make the playoffs, you pay her $2, otherwise she pays you $10.

   Suppose Sam equates money with utility: for her, the utility of gaining $2 is 2, the utility of losing $10 is -10, and so on.

   a. What is Sam's personal probability that the Leafs will make the playoffs?

   Sam also thinks the Leafs might even have a shot at winning the Stanley Cup. She's willing to pay you $1 if they don't win the Cup, if you agree to pay her $2 if they do. That's a fair deal for her.

   b. What is Sam's personal probability that the Leafs will win the Stanley Cup?

c. What is Sam's personal conditional probability that the Leafs will win the Stanley Cup if they make the playoffs? (Assume that winning the Stanley Cup logically entails making the playoffs.)

4. Freya isn't sure whether it will snow tomorrow. For her, a fair gamble is one where she gets $10 if it snows and she pays $10 if it doesn't. Assume Freya equates money with utility.

   a. What is Freya's personal probability for snow tomorrow?

   Here's another gamble Freya regards as fair: she'll check her phone to see whether tomorrow's forecast calls for snow. If it does predict snow, she'll pay you $10, but you have to pay her $5 if it doesn't.

   b. What is Freya's personal probability that the forecast calls for snow?

   After checking the forecast and seeing that it does predict snow, Freya changes her betting odds for snow tomorrow. Now she's willing to accept as little as $5 if it snows, while still paying $10 if it doesn't.

   c. Now what is Freya's personal probability for snow tomorrow?
   d. Before she checked the forecast, what was Freya's personal probability that the forecast would predict snow and be right. (Assume that winning the Stanley Cup logically entails making the playoffs.)

5. Ben's favourite TV show is *Community*. He thinks it's so good they'll make a movie of it. In fact, he's so confident that he thinks the following is a fair deal: he pays you $8 if they don't make it into a movie and you pay him $1 if they do. Assume Ben equates money with utility.

   a. What is Ben's personal probability that *Community* will be made into a movie?

   Ben thinks the odds of a *Community* movie getting made are even higher if his favourite character, Shirly, returns to the show (she's on leave right now). If Shirly returns, he's willing to pay as much as $17 if the movie does not get made, in return for $1 if it does.

   b. What is Ben's personal conditional probability that there will be a *Community* movie if Shirly returns?

Ben also thinks the chances of a movie go down drastically if Shirly doesn't return. His personal conditional probability that the movie will happen without Shirly is only 1/3.

d. What is Ben's personal probability that Shirly will return?

17 Dutch Books

As to the speculators from the south, they had the advantage in the toss up; they said heads, I win, tails, you lose; they could not lose any thing for they had nothing at stake.
—Newspaper report, Sep. 2, 1805

Critics of Bayesianism won't be satisfied just because beliefs can be quantified with betting rates. After all, a person's betting rates might not obey the laws of probability. What's to stop someone being 9/10 confident it will rain tomorrow and also 9/10 confident it won't? How do we enforce laws of probability like the Negation Rule, if probability is just subjective opinion?

17.1 Dutch Books

The Bayesian answer uses a special betting strategy. If someone violates the laws of probability, we can use this strategy to take advantage of them. We can sucker them into a deal that will lose them money no matter what.

For example, suppose Sam violates a very simple rule of probability. His personal probability in the proposition $2 + 2 = 4$ is only 9/10, when it should be 1. (Because it's impossible for $2 + 2$ to be anything other than 4.) Sam has violated the laws of probability, so he'll be willing to accept a very bad deal, as follows.

We put two marbles into an empty bucket, and then two more. We then offer Sam the following deal: we pay him 90¢, and in exchange he pays us $1 if the bucket has a total of 4 marbles in it.

This deal should be fair according to Sam. As he sees it, there's a 90% chance he'll have to pay us $1 for a net loss of 10¢. But there's also a 10% chance he won't have to pay us anything, a net gain of 90¢. So the expected value is zero according to Sam's personal probabilities:

$$(9/10)(-\$.10) + (1/10)(\$.90) = 0.$$

And yet, Sam will lose money no matter what! There's no way for

him to win: he's bound to end up paying out $1, when we only paid him 90¢ to get in on the deal.

THIS kind of deal is called a *Dutch book*.[1] A Dutch book has two defining features.

1. The arrangement is fair according to the subject's personal probabilities.
2. The subject will lose money *no matter what*.

If you are vulnerable to a Dutch book, then it looks like there's something very wrong with your personal probabilities. A deal looks fair to you when it clearly isn't. It's impossible for you to win!

Almost nobody in real life would be foolish enough accept the deal we just offered Sam. Real people have enough sense not to violate the laws of probability in such obvious ways. But there are subtler ways to violate the laws of probability, which people do fall prey to. And we can create Dutch books for them too.

[1] Why is it called that? The 'book' part comes from gambling, where 'making a book' means setting up a betting arrangement. But the 'Dutch' part is a mystery.

17.2 *The Bankteller Fallacy*

CONSIDER a famous problem:

> Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. Which is more probable?
>
> 1. Linda is a bank teller.
> 2. Linda is a bank teller and is active in the feminist movement.

This problem was devised by the same psychologists who studied the taxicab problem: Daniel Kahneman and Amos Tversky. This one is from a 1983 study of theirs.

In psychology experiments, almost everyone chooses (2). But that can't be right, and you don't need to know anything about Linda to prove it.

In general, the probability of a conjunction $B \,\&\, F$ cannot be greater than the probability of B. There are simply more possibilities where B is true than there are possibilities where *both B and F* are true. Figure 17.1 illustrates the point.

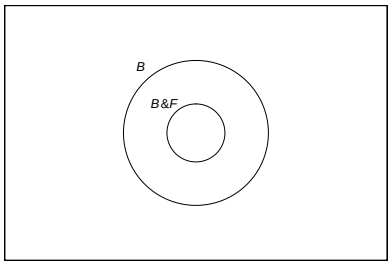


Figure 17.1: Bank tellers and feminist bank tellers

So it's a law of probability that $Pr(B \,\&\, F) \leq Pr(B)$. Which means it can't be more likely that Linda is both a bankteller and a feminist than that she is a bankteller.

Here's another way to think about it. Imagine everyone who fits the description of Linda gathered in a room. The people in the room who happen to be bank tellers are then gathered together inside a circle. Some of the people inside that circle will also be feminists. But there can't be more feminists inside that circle than people! Feminist bank tellers are just less common than bank tellers in general.

How do we Dutch book someone who mistakenly thinks B is more probable than $B \,\&\, F$? We offer them two deals, one involving a bet on B and the other a bet on $B \,\&\, F$.

Suppose for example their betting rate for B is $1/5$, and their betting rate for $B \,\&\, F$ is $1/4$. Then we offer them these deals:

1. We pay them 20¢, and in exchange they agree to pay us \$1 if B is true.
2. They pay us 25¢, and in exchange we agree to pay them \$1 if $B \,\&\, F$ is true.

Both of these deals are fair according to their betting rates. So our victim will be willing accept both. And yet, they'll lose money no matter what. Why?

Notice they've already paid us more than we've paid them before the bets are even settled. We're up 5¢ from the get-go. So for them to come out ahead, they'll have to win the second bet. But if they do win the second bet, we win the first bet. They only win the second bet if $B \,\&\, F$ is true, in which case B is automatically true and we win the first bet. Since both bets pay off \$1, they cancel each other out. So even if they win the second bet, they still suffer a net loss of 5¢.

Thinking through a Dutch book can be tricky, but a table like 17.1 can help. The columns capture the possible outcomes, the rows represent exchanges of cash. For clarity, we separate investments and returns into their own rows. The investment is the amount the person pays or receives at first, when the bet is arranged. The return is the amount they win/lose if the bet pays off. The last row sums up all the investments and returns to get the net amount won or lost.

Table 17.1: A Dutch book for the bank teller fallacy

		$B \,\&\, F$	$B \,\&\, \sim F$	$\sim B$
Bet 1	Investment	\$0.20	\$0.20	\$0.20
	Return	−\$1.00	−\$1.00	\$0.00
Bet 2	Investment	−\$0.25	−\$0.25	−\$0.25
	Return	\$1.00	\$0.00	\$0.00
Net		−\$0.05	−\$1.05	−\$0.05

Notice how the net amount is negative in every column. In a Dutch book, no matter what the victim loses money.

17.3 *Dutch Books in General*

ANYONE who violates any law of probability can be suckered via a Dutch book. But if you obey the laws of probability, it's impossible to be Dutch booked. That's why the laws of probability are objectively correct, according to Bayesians.

Bruno de Finetti (1906–1985) proved in the 1930's that the laws of probability are the only safegaurd against Dutch books. The same point was also noted by Frank Ramsey in an essay from the 1920's, though he didn't include a proof.

What's the general recipe for constructing a Dutch book when someone violates the laws of probability?

First, use bets with a \$1 stake. That makes things simple, because the victim's personal probability will be the same as their fair price for the bet. If their personal probability is $1/2$, they'll be willing to pay 50¢ for a bet with a \$1 payoff. If their personal probability is $1/4$, they'll be willing to pay 25¢ for a bet with a \$1 payoff. And so on.

Second, remember that if a bet is fair then a person should be willing to take *either side* of the bet. Suppose they think it's fair to pay 25¢ in exchange for \$1 if A turns out to be true. Then they should also be willing to *accept* 25¢ instead, and *pay you* \$1 if A turns out to be true. It's a fair bet, so either side should be acceptable.

Third is the trickiest bit. Which propositions should you get them to bet on, and which should you get them to bet against? Think about which propositions they are overconfident of, and which they are underconfident of.

For example, Sam was underconfident in $2 + 2 = 4$. He was only $9/10$ confident instead of 1. So we underpaid him for a bet on that proposition: 90¢ in exchange for \$1 if it was true. His underconfidence meant he was willing to accept too little in exchange for a \$1 payout.

Whereas in the bank teller example, the victim was overconfident of B & F, and underconfident in B. So we underpaid them for a bet on B, and got them to overpay us for a bet on B & F.

HERE'S ONE MORE example. Suppose Suzy the Scientist is conducting an experiment. The experiment could have either of two outcomes, E or $\sim E$. Suzy thinks there's a 70% chance of E, and a 40% chance of $\sim E$. She's violated the Additivity rule: $Pr(E) = 7/10$ and $Pr(\sim E) = 4/10$, which adds up to more than 1.

So we can make a Dutch book against Suzy. In this case she's overconfident in both of the propositions E and $\sim E$. So we get her to overpay us for bets on E and on $\sim E$:

1. Suzy pays us 70¢, and in exchange we pay her \$1 if E is true.
2. Suzy pays us 40¢, and in exchange we pay her \$1 if $\sim E$ is true.

The end result is that Suzy loses 10¢ no matter what. At the beginning she pays us $\$0.70 + \$0.40 = \$1.10$. Then she wins back \$1, either for the bet on E or for the bet on $\sim E$. But in the end, she still has a net

loss of 10¢.

Table 17.2: A Dutch book for Suzy the Scientist

		E	$\sim E$
Bet 1	Investment	−$0.70	−$0.70
	Return	$1.00	$0.00
Bet 2	Investment	−$0.40	−$0.40
	Return	$0.00	$1.00
Net		−$0.10	−$0.10

We now have a second way for Bayesians to argue that the concept of personal probability is scientifically respectable. Not only can beliefs be quantified. There are also objective rules that everyone's beliefs should follow: the laws of probability. Anyone who doesn't follow those laws can be suckered into a sure loss.

Exercises

1. Suppose Ronnie has personal probabilities $Pr(A) = 4/10$ and $Pr(\sim A) = 7/10$. Explain how to make a Dutch book against Ronnie. Your answer should include all of the following:

   - A list of the bets to be made with Ronnie
   - An explanation why Ronnie will regard these bets as fair
   - An explanation why these bets will lead to a sure loss for Ronnie no matter what.

2. Suppose Marco has personal probabilities $Pr(X) = 3/10$, $Pr(Y) = 2/10$, and $Pr(X \vee Y) = 6/10$. Explain how to make a Dutch book against him. Your answer should include all of the following:

   - A list of the bets to be made with Marco.
   - An explanation why Marco will regard these bets as fair.
   - An explanation why these bets will lead to a sure loss for Marco no matter what.

3. Maya is the star of her high school hockey team. My personal probably that she'll go to university on an athletic scholarship is 3/5. But because she doesn't like schoolwork, my personal probability that she'll go to university is 1/3. Explain how to make a Dutch book against me.

4. Saia isn't sure what grade she'll get in her statistics class. But she thinks the following deals are all fair:

- If she gets at least a B+, you pay her $2; otherwise she pays you $5.
- If she gets a B+, you pay her $6; otherwise she pays you $1.
- If she gets a B+, B, or B-, you pay her $5; otherwise she pays you $2.

Assume that utility and money are equal for Saia.

a. What is Saia's personal probability that she'll get at least a B+?
b. What is Saia's personal probability that she'll get a B or a B-?
c. True or false: given the information provided about Saia's fair betting rates, there is a way to make a Dutch book against her.

5. Cheryl isn't sure what the weather will be tomorrow, but she thinks the following deals are both fair:

- If it rains, you pay her $3; otherwise she pays you $7.
- If it rains or snows, she pays you $8; otherwise you pay her $2.

Assume that utility and money are equal for Cheryl.

a. What is Cheryl's personal probability that it will rain?
b. What is Cheryl's personal probability that it will rain or snow?
c. True or false: given the information provided about Cheryl's fair betting rates, there is a way to make a Dutch book against her.

6. Silvio can't find his keys. He suspects they're either in his car or in his apartment. His personal probability is 6/10 that they're in one of those two places. But he searched his car top to bottom and didn't find them, so his personal probability that they're in his car is only 1/10. On the other hand, his apartment is messy and most of the time when he can't find something, it's buried somewhere in the apartment. So his personal probability that the keys are in his apartment is 3/5. Explain how to make a Dutch book against Silvio.

7. Suppose my personal probability for rain tomorrow is 30%, and my personal probability for snow tomorrow is 40%. Suppose also that my personal probability that it will either snow or rain tomorrow is 80%. Explain how to make a Dutch book against me.

8. Piz's personal probability that Pia is a basketball player is 1/4. His probability that she's a good basketball player is 1/3. Explain how to make a Dutch book against Piz.

18 The Problem of Priors

When my information changes, I alter my conclusions. What do you do, sir?
—attributed to John Maynard Keynes

THE last two chapters showed how Bayesians make personal probabilities objective. They can be quantified using betting rates. And they are bound to the laws of probability by Dutch books.

But what about learning from evidence? Observation and evidence-based reasoning are the keystones of science. They're supposed to separate the scientific method from other ways of viewing the world, like superstition or faith. So where do they fit into the Bayesian picture?

18.1 Priors & Posteriors

WHEN we observe something new, we change our beliefs. A doctor sees the results of her patient's lab test and concludes he doesn't have strep throat after all, just a bad cold.

In Bayesian terms, the beliefs you have before a change are called your ***priors***. We denote your prior beliefs with the familiar operator Pr. Your prior belief about hypothesis H is written $Pr(H)$. The new beliefs you form based on the evidence are called your ***posteriors***. We write Pr^* to distinguish them from what you believed before. So $Pr^*(H)$ is your posterior belief in H.

What's the rule for changing your beliefs? When you get new evidence, how do you go from $Pr(H)$ to $Pr^*(H)$? Let's start by thinking about an example.

Imagine you're about to test a chemical with litmus paper to determine whether it's an acid or a base. Before you do the test, you think it's probably an acid if the paper turns red, and it's probably a base if the paper turns blue. Suppose the paper turns red. Conclusion: the sample is probably an acid.

So your new belief in hypothesis H is determined by your prior *conditional* belief. Before, you thought H was probably true *if* E is true. When you learn that E in fact is true, you conclude that H is probably

true.

Conditionalization When you learn new evidence E, your posterior probability in hypothesis H should match your prior conditional probability:

$$Pr^*(H) = Pr(H \mid E).$$

For example, imagine I'm going to roll a six-sided die behind a screen so you can't see the result. But I'll tell you whether the result is odd or even. Before I do, what is your personal probability that the die will land on a high number (either 4, 5, or 6)? Let's assume your answer is $Pr(H) = 1/2$.

Also before I tell you the result, what is your personal probability that the die will land on a high number *given that it lands on an even number*? Let's assume your answer here is $Pr(H \mid E) = 2/3$.

Now I roll the die and I tell you it did in fact land even. What is your new personal probability that it landed on a high number? Following the Conditionalization rule, $Pr^*(H) = Pr(H \mid E) = 2/3$.

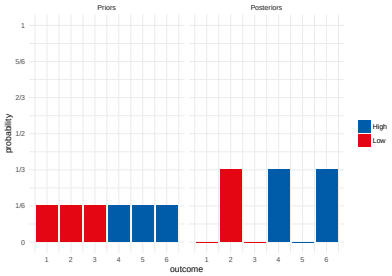


Figure 18.1: Prior vs. posterior probabilities in a die-roll problem. H = the die landed 4, 5, or 6. E = the die landed even. $Pr(H) = 1/2$, $Pr^*(H) = 2/3$.

We learned how to use Bayes' theorem to calculate $Pr(H \mid E)$. If we combine Bayes' theorem with Conditionalization we get:

$$Pr^*(H) = Pr(H)\frac{Pr(E \mid H)}{Pr(E)}.$$

Because this formula is so useful for figuring out what conclusion to draw from new evidence, the Bayesian school of thought is named after it. Bayesian statisticians use it to evaluate evidence in actual scientific research. And Bayesian philosophers use it to explain the logic behind the scientific method.[1]

[1] We caught a glimpse of this explanation back in Chapter 15.

18.2 *The Principle of Indifference*

Bayes' theorem provides an objective guide for *changing* your personal probabilities. Given the prior probabilities on the right hand side, you can calculate what your new probabilities should be on the left. But where do the prior probabilities on the right come from? Are there any objective rules for determining them? How do we calculate $Pr(H)$, for example?

Let's go back to our example where I roll a die behind a screen. Before I tell you whether the die landed on an even number, it seems reasonable to assign probability $1/2$ to the proposition that the die will land on a high number (4, 5, or 6). But what if someone had a different prior probability, like $Pr(H) = 1/10$?

That seems like a strange opinion to have. Why would they think the die is so unlikely to land on a high number, when there are just

as many high numbers as low ones? On the other hand, if you don't know whether the die is fair, it is possible it's biased against high numbers. So maybe they're on to something. And notice, assigning $Pr(H) = 1/10$ doesn't violate the laws of probability, as long as they also assign $Pr(\sim H) = 9/10$. So we couldn't make a Dutch book against them.

Where do prior probabilities come from then? How do we decide whether to start with $Pr(H) = 1/2$ or $Pr(H) = 1/10$? Here is a very natural proposal:

The Principle of Indifference If there are n possible outcomes, each outcome should have the same prior probability: $1/n$.

In the die example, there are six possible outcomes. So each would have prior probability $1/6$, and thus $Pr(H) = 1/2$:

$$\begin{aligned} Pr(H) &= Pr(4) + Pr(5) + Pr(6) \\ &= 1/6 + 1/6 + 1/6 \\ &= 1/2. \end{aligned}$$

Here's one more example. In North American roulette, the wheel has 38 pockets, 2 of which are green: zero (0) and double-zero (00). If you don't know whether the wheel is fair, what should your prior probability be that the ball will land in a green pocket?

According to the Principle of Indifference, each space has equal probability, $1/38$. So $Pr(G) = 1/19$:

$$\begin{aligned} Pr(G) &= Pr(0) + Pr(00) \\ &= 1/38 + 1/38 \\ &= 1/19. \end{aligned}$$

The Principle of Indifference dates back to the very early days of probability theory. In fact Laplace seems to have thought it was *the* central principle of probability.

For a long time it was known by a different name: "The Principle of Insufficient Reason". The idea was that, without any reason to think one outcome more likely than another, they should all get the same probability.

In 1921 it was renamed "The Principle of Indifference" by economist John Maynard Keynes (1883–1946). The idea behind the new name is that you should be indifferent about which outcome to bet on, since they all have the same probability of winning.

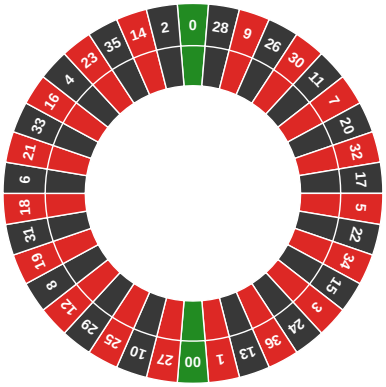


Figure 18.2: A North American roulette wheel

18.3 The Continuous Principle of Indifference

So far so good, but there's a problem. Sometimes the number of possible outcomes isn't a finite number n, it's a continuum. Suppose you had to bet on the *angle* the roulette wheel will stop at, rather than just the colour it will land on. There's a continuum of possible angles, from 0 deg to 360 deg. It could land at an angle of 3 deg, or 314.1 deg, or 100π deg, etc.

So what's the probability the wheel will stop at, say, an angle between 180 deg and 270 deg? Well, this range is $1/4$ of the whole range of possibilities from 0 deg to 360 deg. So the natural answer is $1/4$. Generalizing this idea gives us another version of the Principle of Indifference.

Principle of Indifference (Continuous Version) If there is an interval of possible outcomes from a to b, the probability of any subinterval from c to d is:

$$\frac{d-c}{b-a}.$$

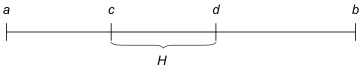


Figure 18.3: The continuous version of the Principle of Indifference: $Pr(H)$ is the length of the c-to-d interval divided by the length of the whole a-to-b interval.

The idea is that the prior probability of a hypothesis H is just the proportion of possibilities where H occurs. If the full range of possibilities goes from a to b, and the subrange of H possibilities is from c to d, then we just calculate how big that subrange is compared to the whole range.

18.4 Bertrand's Paradox

Unfortunately, there's a serious problem with this way of thinking. In fact it's so serious that the Principle of Indifference is not accepted as part of the modern theory of probability. You won't find it in a standard mathematics or statistics textbook on probability.

What's the problem? Imagine a factory makes square pieces of paper, whose sides always have length somewhere between 1 and 3 feet. What is the probability the sides of the next piece of paper they manufacture will be between 1 and 2 feet long?

Applying the Principle of Indifference we get 1/2:

$$\frac{d-c}{b-a} = \frac{2-1}{3-1} = \frac{1}{2}.$$

That seems reasonable, but now suppose we rephrase the question. What is the probability that the *area* of the next piece of paper will be between 1 ft^2 and 4 ft^2? Applying the Principle of Indifference again, we get a different number, 3/8:

$$\frac{d-c}{b-a} = \frac{4-1}{9-1} = \frac{3}{8}.$$

But the answer should have been the same as before: it's the same questions, just rephrased! If the sides are between 1 and 2 feet long, that's the same as the area being between 1 ft^2 and 4 ft^2.

So which answer is right, 1/2 or 3/8? It depends on which dimension we apply the Principle of Indifference to: length vs. area. And there doesn't seem to be any principled way of deciding which dimension to use. So we don't have a principled way to apply the Principle of Indifference.

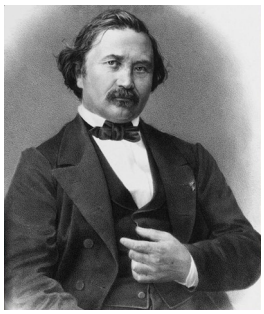
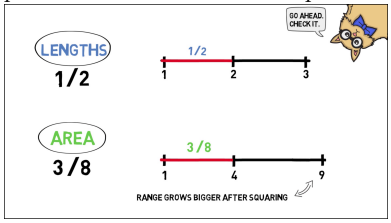


Figure 18.4: Joseph Bertrand (1822–1900) presented this paradox in his 1889 book *Calcul des Probabilités*. He used a different example though. Our example is a bit easier to understand, and comes from the book *Laws and Symmetry* by Bas van Fraassen.

There's nothing special about the example of the paper factory, the same problem comes up all the time. Take the continuous roulette wheel. Suppose the angle is stops at depends on how hard it's spun. The wheel's starting speed can be anywhere between 1 and 10 miles

Here's a video explaining Bertrand's paradox thanks to wi-phi.com:



per hour, let's suppose. And if it's between 2 and 5 miles per hour, it lands at an angle between 180 deg and 270 deg degrees. Otherwise it lands at an angle outside that range.

If we apply the Principle of Indifference to the wheel's starting speed we get a probability of 1/3 that it will land at an angle between 180 deg and 270 deg:

$$\frac{d-c}{b-a} = \frac{5-2}{10-1} = \frac{1}{3}.$$

But we got an answer of 1/4 when we solved the same problem before. Once again, what answer we get depends on how we apply the Principle of Indifference. If we apply it to the final angle we get 1/4, if we apply it to the starting speed we get 1/3. And there doesn't seem to be any principled way of deciding which way to go.

18.5 The Problem of Priors

THERE is no accepted solution to Bertrand's paradox.

Some Bayesians think it shows that prior probabilities should be somewhat subjective. Your beliefs have to follow the laws of probability to avoid Dutch books. But beyond that you can start with whatever prior probabilities seem right to you. (The Principle of Indifference should be abandoned.)

Others think the paradox shows that Bayesianism is too subjective. The whole idea of "prior" and "posterior" probabilities was a mistake, say the frequentists. Probability isn't a matter of personal beliefs. There are objective rules for using probability to evaluate a hypothesis, but Bayes' theorem is the wrong way to go about it.

So what's the right way, according to frequentism? The next two chapters introduce the frequentist method.

Exercises

1. Suppose a carpenter makes circular tables that always have a diameter between 40 and 50 inches. Use the Principle of Indifference to answer the following questions. (Give exact answers, not decimal approximations.)

   a. What is the probability the next table the carpenter makes will have a diameter of at least 43 inches?
   b. What is the probability the next table the carpenter makes will have either a diameter between 41 and 42 inches, or between 47 and 49 inches?

c. Recalculate the probabilities from parts (a) and (b), but this time apply the Principle of Indifference to circumference instead of diameter. Does this change what answers you get?
d. Recalculate the probabilities from parts (a) and (b), but this time apply the Principle of Indifference to area instead of diameter. Does this change what answers you get?

2. Joe spends his afternoons whittling cubes that have a side length between 2 and 10 centimetres. Use the Principle of Indifference to answer the following questions. (Give exact answers, not decimal approximations.)

   a. What is the probability that Joe's next cube will have sides at least 6 centimetres long?
   b. What is the probability Joe's next cube will either have sides shorter than 4 centimetres or longer than 7 centimetres?
   c. Recalculate the probabilities from parts (a) and (b), but this time apply the Principle of Indifference to the area of each face, rather than the length of each side. Does this change what answers you get?
   d. Recalculate the probabilities from parts (a) and (b), but this time apply the Principle of Indifference to volume. Is the result the same or different compared to the answers from parts (a), (b), and (c)?

3. Joel is in New York and he needs to be in Montauk by 4:00 to meet Clementine. He boards a train departing at 3:00 and asks the conductor whether they'll be in Montauk by 4:00. The conductor says the train will arrive some time between 3:50 and 4:12, but she refuses to be more specific.

   a. According to the Principle of Indifference, what is the probability that Joel will be in Montauk in time to meet Clementine?

   After thinking it over, Joel realizes that his odds may actually be better than that. It's a 60 mile trip to Montauk, so the train must travel at an average speed between a and b miles per hour.

   b. What are a and b?
   c. How fast must the train travel to get to Montauk by 4:00?
   d. According to the Principle of Indifference, what is the probability that the train will travel fast enough to get to Montauk by 4:00?

4. A factory makes triangular traffic signs. The height of their signs is always the same as the width of the base. And the base is always between 3 and 6 feet.

a. According to the Principle of Indifference, what is the probability that the next sign produced will be between 5 and 6 feet high?
b. Explain how to reformulate the problem in part (a) so that the probability given by the Principle of Indifference changes.
c. Explain the challenge that cases like this pose for the theory of personal probability. What do critics of Bayesianism say these examples demonstrate about prior probabilities?

5. A factory makes circular dartboards whose diameter is always between 1 and 2 feet.

a. According to the Principle of Indifference, what is the probability that the next dartboard produced will have a diameter between 1 and 5/3 feet?
b. If we reformulate part (a) in terms of the dartboard's area, what is the probability given by the Principle of Indifference then? (Reminder: the area of a circle with diameter d is $A = \pi/4 \times d^2$.)
c. Explain the challenge that cases like this pose for the theory of personal probability. What do critics of Bayesianism say these examples demonstrate about prior probabilities?

6. Some bars water down their whisky to save money. Suppose the proportion of whisky to water at your local bar is always somewhere between 1/2 and 2. That is, there's always at least 1 unit of whisky for every 2 units of water. But there's never more than 2 units of whisky for every 1 unit of water. Suppose you order a "whisky".

a. According to the Principle of Indifference, what is the probability that it will be mostly whisky? ("Mostly" means more than half.)
b. What is the maximum possible proportion water to whisky?
c. What is the minimum possible proportion water to whisky?
d. Now calculate the probability that your drink will be less than 1/2 water again, but this time apply the Principle of Indifference to the proportion of water to whisky. Is the result the same or different?

19 *Significance Testing*

How do we evaluate a hypothesis, according to frequentism? The short answer is, we look for things that would be too much of a coincidence if the hypothesis were true.

19.1 Coincidence

Suppose we flip a coin ten times and it lands heads every single time. That would be too much of a coincidence if the coin were fair. So the hypothesis that it is fair has been tested, and failed. Conclusion: the coin is biased towards heads.

Or imagine we divide a thousand patients with Disease X into two, equal-sized groups. We give the first group Drug Y, the second group gets a placebo. After a month, 90% of the patients who got the drug are cured, compared to only 10% of the patients who didn't. That would be too much of a coincidence if the drug were ineffective. So the hypothesis that Drug Y has no effect has been tested, and it has failed the test. Conclusion: Drug Y helps cure Disease X.

That's the rough idea behind the most popular frequentist approach to hypothesis testing. Now let's take a closer look.

19.2 Making it Precise

Suppose your friend likes to do magic tricks. Their favourite trick involves flipping a special coin and predicting how it will land. You're curious how the trick is done, and you suspect the coin is biased. So you flip it 100 times to see what happens. It lands heads 67 out of 100 times. Does that mean the coin is biased after all? Or is 67 heads out of 100 flips within the realm of plausibility for a fair coin?

Figure 19.1 shows the probability for each number of heads we might get, if the coin is fair. Notice how unlikely 67 heads is, compared to say 50 heads, or even compared to 60 heads. So it seems like a stretch to just write off our 67 heads as a coincidence. Plausibly, the coin is biased towards heads.

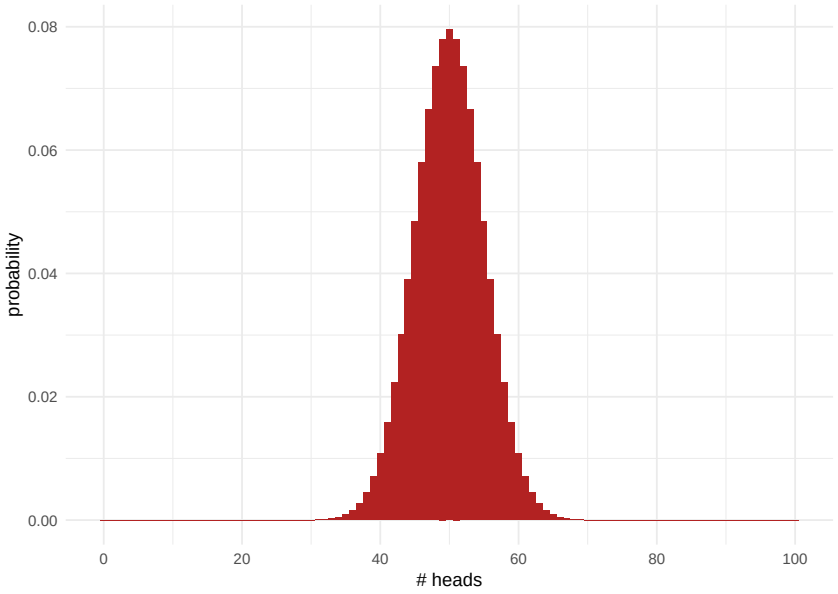


Figure 19.1: The probability of getting x heads out of 100 flips of a fair coin.

What if we'd gotten only 65 heads though? Or just 60? At what point do we just say, "oh well, that's a normal outcome for a fair coin"?

A common convention is to draw the line at 95% probability. If a coin is fair, then 95% of the time it will land heads between 40 and 60 times out of 100 tosses. That's not obvious by the way: I used a computer to find that 40-to-60 range. But we'll come back to these calculations in a bit.

The key idea for now is this. The hypothesis the coin is fair fails the test if the number of heads falls *outside* the 40-to-60 range. As Figure 19.2 illustrates, we'll reject the hypothesis if the outcome of our experiment is one of the red ones.

We call the result of an experiment ***statistically significant*** when it falls outside the 95% range. So the red outcomes in Figure 19.2 are statistically significant, the blue ones are not.

The idea behind this terminology is that a red outcome tells us something about our hypothesis that the coin is fair. In particular it tells us something negative: the hypothesis is not looking good. Because 67 heads would be a big coincidence if the hypothesis were true.

19.3 *Levels of Significance*

But why should we choose 95% probability as our cutoff? Why not 90%, or 99%?

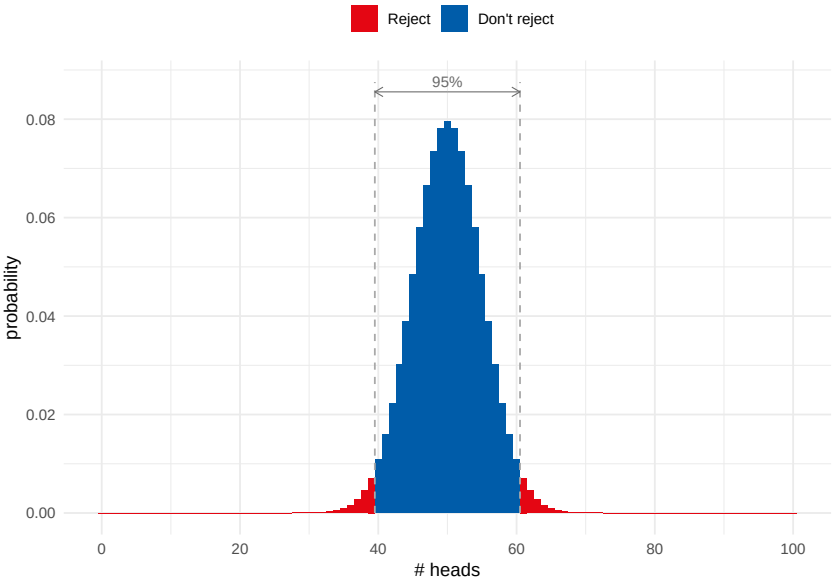


Figure 19.2: In 95% of cases, a fair coin will land heads between 40 and 60 times out of 100 flips. If the number of heads falls outside that range, we conclude the coin is not fair.

In fact we don't have to make 95% the cutoff. In some sciences it's customary to make 99% the cutoff instead, or even 99.9%.

To be explicit about what cutoff we're using, we describe the result as statistically significant *at such-and-such a level*. For example, if we're using the 95% cutoff, we say the result is ***significant at the*** 0.05 ***level***. And if we're using the 99% cutoff, we say the result is ***significant at the*** 0.01 ***level***. And so on.

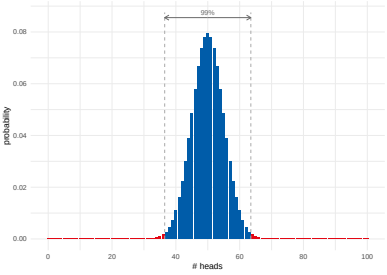


Figure 19.3: In 99% of cases, a fair coin will land heads between 37 and 63 times out of 100 flips.

Different sciences have different conventions about where to put the cutoff. Social sciences like Psychology typically use the 95% cutoff. Only when the outcome of a study is significant at the .05 level is the hypothesis rejected. But medical and physical sciences often use a stricter cutoff like 99%. A finding has to be significant at the .01 level to disprove a hypothesis then.

There's nothing special or magical about the 95% and 99% cutoffs, though. So why have scientists adopted these conventions? This is a deep and important question, with no easy answer.

But part of the answer is that it's actually just kind of a historical and mathematical accident. Before computers, it was hard to calculate significance levels exactly. There's a trick though for estimating the 95% and 99% cutoffs, which we'll learn in the next section. So scientists adopted these conventions before computers came along, partly just because they were easy to work with. And now they've kind of just stuck.

We'll come back in the next chapter for a deeper look at the question of where to put the cutoff for statistical significance. But, for now, let's

learn a quick and easy way of finding the 95% and 99% cutoffs.

19.4 *Normal Approximation*

You may have noticed that the probabilities in our hundred-flips experiment look a lot like the famous "bell curve". In fact they line up almost perfectly, as Figure 19.4 illustrates.

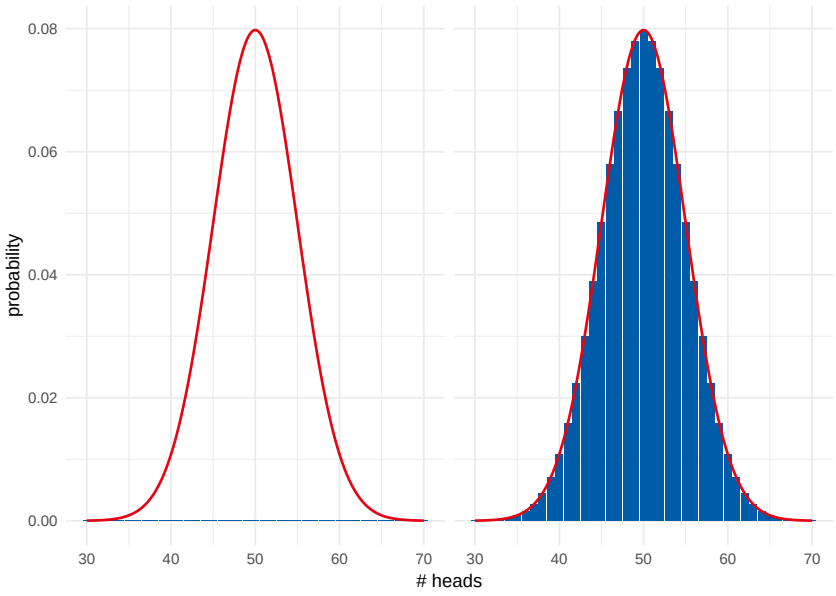


Figure 19.4: A bell curve (left) overlayed on the probability of getting x heads out of 100 flips of a fair coin (right).

The bell curve's official name is the ***normal distribution***. And it has some very handy mathematical properties, which make it easy to estimate when the number of heads falls outside the 95% range. Two features of the normal distribution are required for the calculation.

The normal distribution actually has a very complicated mathematical formula. You *really* don't need to know it for this book, but if you're curious you can find it on Wikipedia.

First we need to know where the bell is located: where is it centred? This is called the ***mean***, or μ for short. In our example $\mu = 50$. That's the most likely outcome for 100 flips of a fair coin. The general formula is

$$\mu = np,$$

where n is the number of tosses, and p is the probability of heads on each toss. So in our example $np = (100)(1/2) = 50$.

The symbol μ is from Greek and is called *mu* (rhymes with *stew*). It looks like the English letter *u*, but it actually corresponds to the letter *m*, as in *m*ean. It helps to picture it as a cursive *m*, written very sloppily by someone in a hurry.

Second, how wide is the bell? This is called the ***standard deviation***, or σ for short. The formula for σ is a bit mysterious:

$$\sigma = \sqrt{np(1-p)}.$$

Deriving this formula is pretty advanced, so we'll have to just take it on faith. We only need to understand that the larger the standard deviation, the wider the bell. Figure 19.5 gives some examples to illustrate.

The Greek symbol σ is called *sigma*. It corresponds to the English letter *s*, as in *s*tandard deviation.

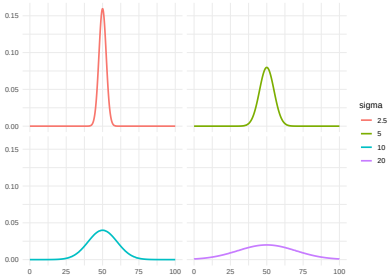


Figure 19.5: Four bell curves with the same mean of 50, but different standard deviations. The larger the standard deviation, the wider the bell.

Applying the formula for standard deviation to our example, we get:

$$\begin{aligned}\sigma &= \sqrt{(100)(1/2)(1/2)} \\ &= 5.\end{aligned}$$

Now for the punchline: we can use the numbers μ and σ to get a pretty accurate estimate of the 95% cutoff.

Mathematicians have proved that about 95% of the time, the number of heads will be within the range $\mu \pm 2\sigma$. In our example $2\sigma = (2)(5) = 10$, so $\mu \pm 2\sigma$ is the range from 40 to 60. Notice how this is the same as the computer-based answer we used earlier.

For a 99% cutoff we just multiply σ by 3 instead of 2. In other words, 99% of the time the number of heads will be within the range $\mu \pm 3\sigma$. In our example $3\sigma = (3)(5) = 15$, so $\mu \pm 3\sigma$ is the range from 35 to 65.

We got 67 heads in our experiment, which falls outside the 35-to-65 range. So our result wasn't just significant at the .05 level. It was significant at the .01 level too.

19.5 The 68-95-99 Rule

THERE are actually three cutoffs we can estimate with this method. The one we haven't talked about is 68%, because it's not used much in actual practice. But the general rule is:

- With about 68% probability, the number of heads will be in the range $\mu \pm \sigma$.
- With about 95% probability, the number of heads will be in the range $\mu \pm 2\sigma$.
- With about 99% probability, the number of heads will be in the range $\mu \pm 3\sigma$.

This is called ***The 68-95-99 Rule***. We can illustrate it with a diagram like Figure 19.6.

It's customary to use k for the number of heads we actually get in our experiment. So we can also state the rule formally like this:

$$\begin{aligned}Pr(\mu - \sigma \leq k \leq \mu + \sigma) &\approx .68, \\ Pr(\mu - 2\sigma \leq k \leq \mu + 2\sigma) &\approx .95, \\ Pr(\mu - 3\sigma \leq k \leq \mu + 3\sigma) &\approx .99.\end{aligned}$$

19.6 Binomial Probabilities

WHEN do the true probabilities closely match the normal distribution, though? A lot of the time, actually. But we'll just focus on one common case.

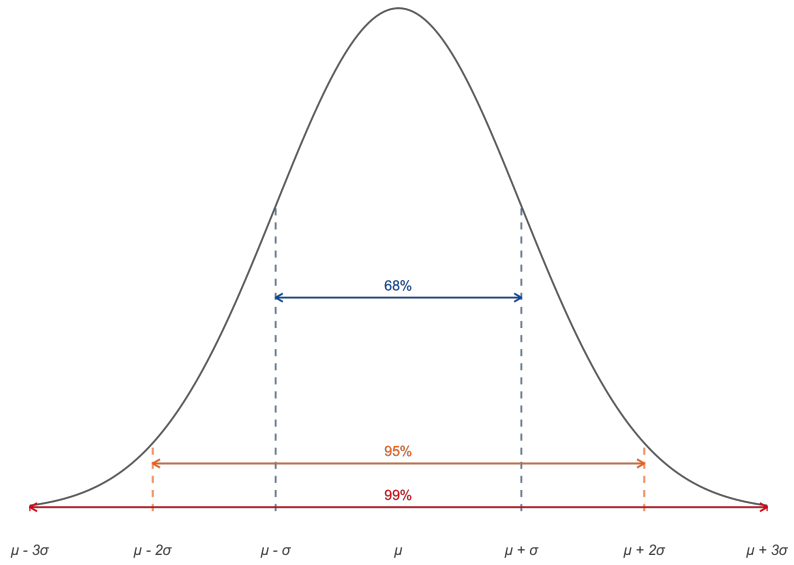


Figure 19.6: The 68-95-99 Rule

Suppose a certain kind of event will be repeated over and over, and there are two possible outcomes. If the probabilities of these two outcomes stay the same from repetition to repetition, and the repetitions are independent of one another, then the probabilities are called ***binomial***.

Binomial probabilities are very common. We saw that they apply to flips of a fair coin. Another example could be patients in a drug study who either get better or don't. Or subjects in a psychology study answering yes/no questions on a survey. Or respondents to a poll about a political race.

If the event is repeated enough times, the binomial probabilities will closely match the bell curve. But you have to be careful: the more extreme the probability in your hypothesis, the larger the study will need to be. If the probability p is close to either 0 or 1, the normal approximation won't be very good without a pretty large number of trials n.

Suppose for example your hypothesis is that the coin is heavily biased towards tails: $p = 0.05$, to be exact. Then we see from Figure 19.7 that we need a sample of around $n = 50$ before the normal approximation starts to become reasonably accurate. Whereas $n = 30$ is already quite accurate for $p = 0.5$.

In general, the more extreme the value of p is, the larger we need n to be for the approximation to be accurate. But if you do a big enough study, the 68-95-99 rule will give a reliable estimate.

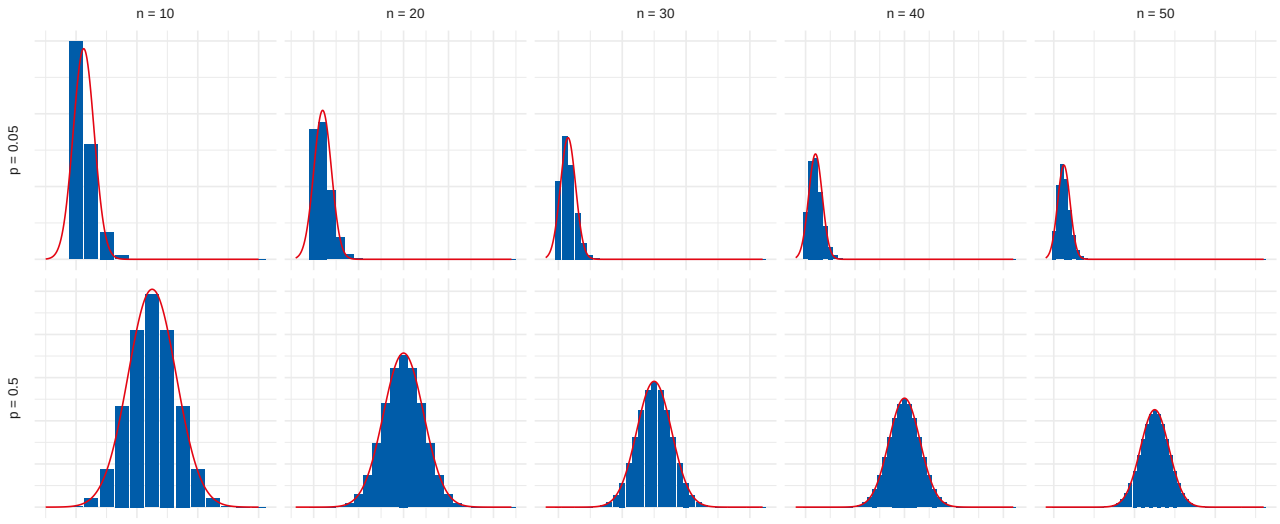


Figure 19.7: When p is close to 0 or 1, we need a larger n to make the normal approximation accurate.

19.7 *Significance Testing*

Figure 19.8: Ronald A. Fisher (1890–1962) established significance testing as a standard scientific method with his influential 1925 book *Statistical Methods for Research Workers*.

THE method of evaluating hypotheses we've been explaining is called ***significance testing***. Here is the general recipe for significance testing with binomial probabilities:

1. State the hypothesis you want to test: the true probability of outcome X is p. This is called the ***null hypothesis***.[1]
2. Repeat the event over and over n times, and count the number of times k you get outcome X.
3. Calculate $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.
4. Use the 68-95-99 rule to figure out how much of a coincidence your finding k must be if the null hypothesis is true.
5. If it's too much of a coincidence, conclude that the null hypothesis is false. The true probability isn't p.

[1] Why is it called the "null" hypothesis? The idea is that our default assumption should be that things are random, until we discover a pattern. For example, we assume our sequence of heads/tails is random unless our observations prove otherwise. And we assume a drug has no effects unless we discover that it does. So the default assumption is "null": no pattern, no effect. It's not a great piece of terminology, but unfortunately we're stuck with it.

LET'S DO ONE more example for practice.

Example 19.1. A company named VisioPerfect makes "long-life" lightbulbs. According to their ads, 96% of their bulbs outlast their competitors' average lifespan.

The magazine *Consumers' Advocate* decides to run a test of 2,400 VisioPerfect bulbs. They find that 133 of them weren't "long-life".

This example is from page 205 of Ian Hacking's excellent textbook, *An Introduction to Probability & Inductive Logic*.

Is the result of this test bad news for VisioPerfect? Are their ads just hype? Let's test their claim.

The hypothesis we're testing is that each bulb has probability .96 of being long-life, which means it has probability $p = .04$ of being "short-life". The magazine found $k = 133$ short-life bulbs out of $n = 2,400$. Is this about what you'd expect if the ads are honest? Let's start by finding the center and width of our normal approximation:

$$\mu = np = (2,400)(.04) = 96,$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{(2,400)(.04)(.96)} = 9.6.$$

Then we use the 68-95-99 rule to figure out where $k = 133$ falls. In this case the 99% range is $96 \pm (3)(9.6)$, which is 67.2 to 124.8. Since $k = 133$ falls outside this range, it's a pretty far out result. It's the kind of thing you'd expect to happen less than 1% of the time if each bulb really had a 96% chance of being long-life.

So VisioPerfect's claim has failed our test: the results are significant at the .01 level.

19.8 Warnings

The American Statistical Association recently released a statement to clarify the ideas behind significance testing, and prevent their misuse. It's also the punchline of one of my favourite cartoons.

SIGNIFICANCE testing is very confusing. So confusing that scientists, and even statisticians, often misunderstand and misuse it. Dangerous medical treatments have been approved and administered as a result. Scientific careers have even been built on the misuse of significance testing.

So what exactly does it mean for a result to be "significant at level α"? It means exactly this:

The Greek letter α is called *alpha* and corresponds to the English letter *a*. It's customarily used for significance levels, I don't know why.

- ⊘ *If* the hypothesis we are testing is true, *then* a result this unusual was less than α likely to occur.

Notice how this is an *if/then* statement, where the *if* part supposes that the hypothesis we're testing is true. We're considering what things look like from the hypothetical point of view where the hypothesis is true. Then we assess how probable various outcomes are, given that assumption.

"What's wrong with NHST [null hypothesis significance testing]? Well, among many other things, it does not tell us what we want to know…" — Jacob Cohen

But notice, what we really want to know is the reverse thing. We want to know: *if* we get k heads, *then* how likely is it the hypothesis is true? You might think if the result is significant at the .05 level, then the hypothesis is less than .05 probable. Not so!

- ⊘ Just because the outcome of an experiment is significant at the α level doesn't mean the probability of the hypothesis is α (or less than α).

We learned back in Chapter 8 about how $Pr(H \mid E)$ and $Pr(E \mid H)$ are different things. In fact they can be very different; one can be high and the other low. The taxicab problem is one famous illustration. And it's a very common mistake to make a similar confusion with significance testing.

We'll deepen our understanding of this point in the next chapter.

Exercises

1. Suppose 20% of the marbles in an urn are green. We are going to randomly draw 100 marbles, with replacement, and then count the number of draws that are green.

   Let's use a normal distribution to approximate the probability of getting k green balls.

   a. What are μ and σ in the normal approximation?
   b. Draw a rough graph of the corresponding normal curve.
   c. The probability is about .68 that the number of green draws will fall between a and b. What are a and b?

d. What are a and b for an approximate probability of .95?
e. What are a and b for an approximate probability of .99?
f. If you get fewer than the expected 20 greens, how few do you have to get for the results to be significant at the .01 level? (Assume the estimates above are accurate.)
g. If you get more than the expected 20 greens, how many do you have to get for the results to be significant at the .01 level? (Assume the estimates above are accurate.)
h. If you got one of these "significant" results in this experiment, would you reject the hypothesis that the urn contains 20% green marbles? (This is a question about your personal judgment.)

2. Suppose 60% of the marbles in an urn are green, and we are going to randomly draw 150 marbles, with replacement.

a. What are μ and σ in the normal approximation?
b. Draw a rough graph of the corresponding normal curve.
c. The probability is about .95 that the number of green draws will fall between a and b. What are a and b?
d. What are a and b for an approximate probability of .99?
e. If you do the 150 draws and you only get 80 greens, is that significant at the .05 level? (Assume the estimates above are accurate.)
f. If instead you do the 150 draws and you only get 70 greens, is that significant at the .05 level? (Assume the estimates above are accurate.)
g. Would you reject the hypothesis that the urn contains 60% green marbles if you only got 70 green draws?

3. Dr. Colbert claims to have a miraculous new weight-loss product. According to him, 75% percent of people who use it lose weight. But the Ministry of Health is suspicious so they run a study. They recruit 192 subjects to try Dr. Colbert's new treatment. 135 of them lose weight.

   Use a normal approximation to assess Dr. Colbert's claim:

a. What are μ and σ in the normal approximation?
b. Fill in the blanks: the results of the study are significant at the .05 level if the number of subjects who lose weight is either smaller than _____ or greater than _____.
c. Fill in the blanks: the results of the study are significant at the .01 level if the number of subjects who lose weight is either smaller than _____ or greater than _____.

d. Are the results of the Health Canada study significant at the .05 level? Are they significant at the .01 level?
e. True or false: given the results of the study, the probability that Dr. Colbert's product works as advertised is less than .05.

4. Your professor says 80% of the class passed the midterm, but that seems high since the test was so hard. So you ask all the people sitting in your row if they passed: 4 of them did, 3 didn't. Since you didn't pass either, that's a pass-rate of only 50% in your sample.

   Use a normal approximation to answer the following questions:

   a. Is your finding significant at the .05 level?
   b. Is it significant at the .01 level?
   c. True or false: given your findings, the probability that your professor's claim is true is at least .05.

5. Suppose you arrange seven dates next week, one for each night of the week. You ask each date the same question: which was your favourite grade, out of grades 1 through 8? Your first and last dates both give the same answer as you: [insert your favourite grade here]. Is this result significant? Check both the .05 and .01 levels using a normal approximation.

   As a null hypothesis, assume that each person's answer is independent (including yours), and that each person is equally likely to name any one grade as any other (including you).

6. Suppose 10% of all wine bottles have corks (instead of screw-tops). A restaurant opens 400 random bottles in a month and counts the number that are corked.

   Use a normal distribution to approximate the probability that the restaurant will open k corked bottles.

   a. What are μ and σ in the normal approximation?
   b. The probability is about .68 that the number of corked bottles will fall between a and b. What are a and b?
   c. What are a and b for an approximate probability of .95?
   d. What are a and b for an approximate probability of .99?

   The restaurant wants the head waiter to avoid serving corked bottles since they're more expensive. So they offer her a bonus if fewer than the expected number of corked bottles gets opened, provided the results are significant at the .01 level. On the other hand they'll fire her if more than the expected number of corked bottles gets opened, assuming the results are significant at the .01 level.

e. How few corked bottles have to be opened for the head waiter to get the bonus? How many corked bottles have to opened for the head waiter to lose her job? (Assume the estimates above are accurate.)

7. A few years ago Arkansas passed a law sparking a debate about a new treatment. The treatment is supposed to help a person's pregnancy continue under conditions that usually end it.

   Normally under those conditions, only about 40% of people's pregnancies continue. But in a small case study, six pregnant people were given the new treatment, and four of them continued their pregnancies.

   Critics say this result is not significant: it doesn't show the treatment has an effect. (If you're curious, you can read more about the controversy here.)

   Use a normal approximation to assess this criticism. The null hypothesis is that the treatment has no effect, so each of the six pregnancies has a 0.4 chance of continuing.

   a. What are μ and σ in the case study?
   b. Are the results of the case study significant at the .05 level? (Assume the normal approximation is accurate.)
   c. Suppose a much larger study were done, with 150 subjects getting the treatment, and two-thirds of them continuing their pregnancies. What would μ and σ be then? Would the results be significant at the .01 level?

8. Medical researchers are testing a new cancer treatment. Ordinarily, a patient's chance of going into remission is only $1/10$. The null hypothesis is that the drug will have no effect on patients' chances of going into remission.

   They select 100 patients at random and give each one the new treatment. The result: 18 of them go into remission.

   a. What are μ and σ in a normal approximation here?
   b. According to the normal approximation, is the result significant at the .01 level?
   c. True or false: the fact that the result is significant at the .05 level tells us that the null hypothesis is less than .05 probable.
   d. True or false: the fact that the result is significant at the .05 level tells us that, if the null hypothesis is true, then the result was .05 probable.

9. You've seen Gonzo the Great doing magic tricks with a coin. You suspect he's using a biased coin, so you sneak into his dressing room and steal the coin.

   Your hypothesis is that the coin has a 8/10 bias towards heads. You flip it 100 times and it comes up heads 65 times.

   a. What are μ and σ in the normal approximation here?
   b. According to the normal approximation, is the result significant at the .01 level?
   c. Explain, in terms of frequencies, what it means for a result to be "significant at the .01 level".

10. Suppose the registrar has a list of all the student numbers from a class, but they've lost the data that says which class it is. They know it's either Philosophy 101 or Economics 101. But they have no idea which of those two it is: the two hypotheses are equally probable, 50%.

    If it's Philosophy 101, then 40% of the students are philosophy majors. If it's Economics 101, then only 25% of the class are philosophy majors.

    The registrar picks ten student numbers from the list at random and looks up the students' majors. The result: all ten of them are philosophy majors.

    a. Suppose our null hypothesis is that the list is for Philosophy 101. What are μ and σ in the normal approximation? (You may use a decimal approximation here.)
    b. Is the result significant at the .01 level for this null hypothesis?
    c. Suppose our null hypothesis is that the list is for Economics 101. What are μ and σ in the normal approximation then? (You may use a decimal approximation here.)
    d. Is the result significant at the .01 level for this null hypothesis?

11. Explain in your own words what it means for a result to be significant at the .05 level. See if you can do it without looking back over the text of this chapter.

20 *Lindley's Paradox*

Another reason for the popularity of statistical significance testing is probably that complicated mathematical procedures lend an air of scientific objectivity to conclusions.
—Ronald P. Carver

Significance testing is all about whether the outcome would be too much of a coincidence for the hypothesis to be true. But how much of a coincidence is too much? Should we reject a hypothesis when we find results that are significant at the .05-level? At the .01-level?

20.1 Significance & Subjectivity

Critics say this decision is subjective. In the social sciences, researchers have mainly adopted .05 as the cutoff, while in other sciences the convention is to use .01. But we saw there's nothing special about these numbers. Except that they're easy to estimate without the help of a computer.

It makes a big difference what cutoff we choose, though. The lower the cutoff, the less often a result will be deemed significant. So a lower cutoff means fewer hypotheses ruled out, which means fewer scientific discoveries.

Think of all the studies of potential cancer treatments being done around the world. In each study, the researcher is testing the hypothesis that their treatment has no effect, in the hopes of disproving that hypothesis. They're hoping enough patients get better in their study to show the treatment genuinely helps. If just a few more patients improve with their treatment compared to a placebo, it could just be a fluke. But if a lot more improve, that means the treatment is probably making a real difference.

The lower the significance level, the more patients have to improve before they can call their study's results "significant". So a lower significance level means fewer treatments will be approved by the medical community.

That might seem like a bad thing, but it also has an upside. It means fewer *bogus* treatments being adopted.

Medical treatments are often expensive, painful, or dangerous. So it's a serious problem to approve ones that don't actually help.

But sometimes it just leads to silly dietary choices.

After all, occasionally a study of a useless treatment will have lots of patients improving anyway, just by luck. When that happens, the medical community adopts a treatment that doesn't actually work. And there might be lots of studies out there experimenting with treatments that don't actually help. So if we do enough studies, we're bound to get flukey results in some of them, and adopt a treatment that doesn't actually work by mistake.

20.2 *Making It Concrete*

SUPPOSE for example there are only two types of potential treatment being studied. The useless kind just leave the patient's recovery up to chance: it's a 50% shot. But the effective kind increase their chances of recovery by 15%: they have a 65% chance of getting better with these treatments.

Let's also imagine we're studying 200 different treatments, 160 of which are actually useless and 40 of which are effective. Since we don't know which are which, we'll run 200 studies, one for each treatment. And just to make things concrete, let's suppose each study has $n = 100$ subjects enrolled.

What will happen if our researchers set the cutoff for statistical significance at .05? Only a few of the bogus treatments will be approved. Just 5% of those studies will get flukey, statistically significant results. And half of those will look like they're harming patients' chances of recovery, rather than helpful. So only 2.5% of the 160 bogus treatments will be approved, which is 4 treatments.

But also, a good number of the genuine treatments will be missed. Only about 85% will be discovered as it turns out: see Figure 20.1.

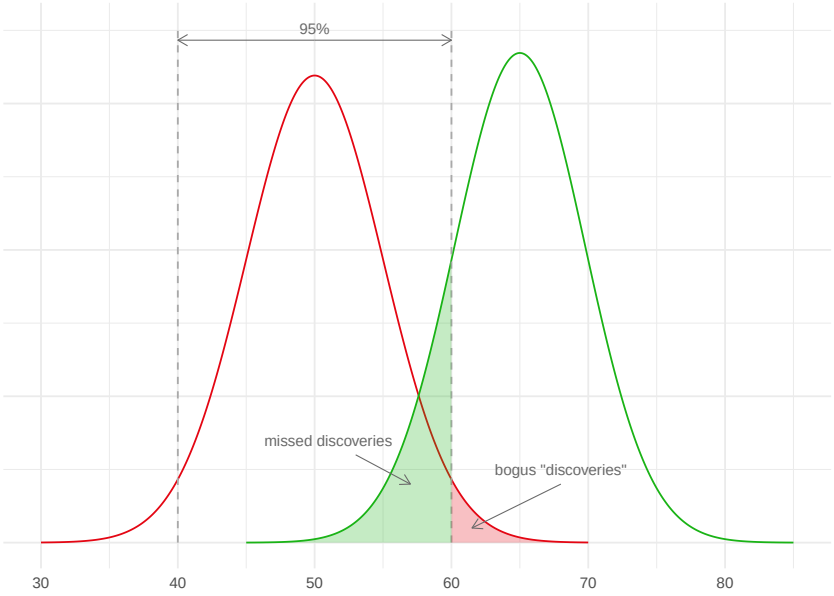


Figure 20.1: A significance cutoff of .05, when running studies with 100 patients in each. The red curve represents the probable outcomes when studying useless treatments (50% chance of recovery), the green curve represents the probable outcomes when studying effective treatments (65% chance of recovery).

As a result, only about 89% of the treatments we approve will actually be genuinely effective. We can't see this from Figure 20.1 because it doesn't show the base rates. We have to turn to the kind of reasoning we did in the taxicab problem instead.

Figure 20.2 shows the results. Since 2.5% of the 160 useless treatments (red pills) will be approved, that's 4 bogus "discoveries". And since 85% of the 40 genuine ones (green pills) will be approved, that's about 34 genuine treatments discovered. So only about $34/38 \approx 89\%$ of our approved treatments actually work.

We could improve this percentage by lowering the threshold for significance to .01. But then only half of the genuine treatments would be identified by our studies (Figure 20.3).

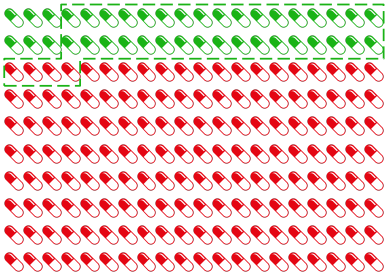


Figure 20.2: The results of our 200 studies. Green pills represent genuinely effective treatments, red pills represent useless treatments. The dashed green line represents the treatments we approve: only 34 out of 38 of these are genuinely effective.

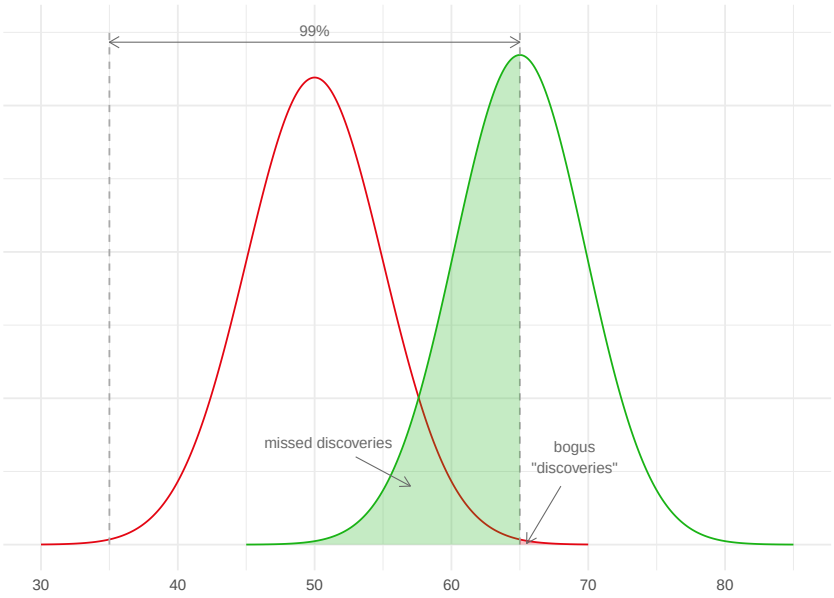


Figure 20.3: A significance cutoff of .01 when running studies with 100 patients in each. The red curve represents the probable outcomes when studying useless treatments (50% chance of recovery), the green curve represents the probable outcomes when studying effective treatments (65% chance of recovery).

Figure 20.4 shows what happens now: about 95% of our approved treatments will be genuine. We discover half of the 40 genuine treatments, which is 20. And .005 of the 160 useless treatments is .8, which we'll round up to 1 for purposes of the diagram. So $20/21 \approx .95$ of our approved treatments are genuinely effective.

But we've paid a dear price for this increase in precision: we've failed to identify half the treatments there are to be discovered. We've missed out on a lot of potential benefit to our patients.

Notice, however, that if bogus treatments were rarer the problem wouldn't be so pressing. For example, Figure 20.5 shows what would happen if half the treatments being studied were bogus, instead of 80%. Then we could have stuck with the .05 threshold and still had very good results: we would discover about 85 genuine treatments and only approved about 3 bogus ones, a precision of about 97%.

There are two lessons here. First, lowering the threshold for significance has both benefits and costs. A lower threshold means fewer false discoveries, but it means fewer genuine discoveries too. Second, the base rate informs our decision about how to make this tradeoff. The more false hypotheses there are to watch out for, the stronger the incentive to use a lower threshold.

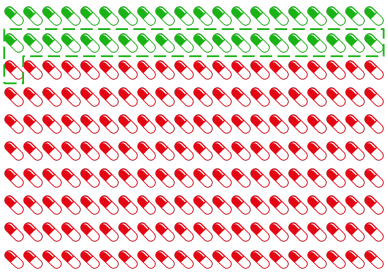


Figure 20.4: With a stricter significance cutoff of .01, we make fewer "bogus" discoveries. But we miss out on a lot of genuine discoveries too.

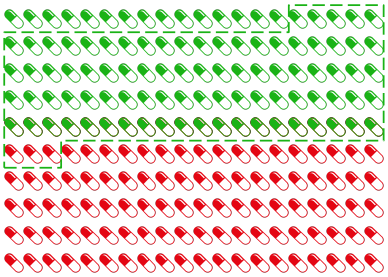


Figure 20.5: A significance cutoff of .05 does much better if the base rate is more favourable.

20.3 The Role of Priors in Significance Testing

BAYESIAN critics of significance testing conclude that, when we choose where to set the cutoff, we're choosing based on our "priors". We're relying on assumptions about the base rate, the prior probability of the null hypothesis.

So, these critics say, frequentism faces exactly the same subjectivity problem as Bayesianism. Bayesians use Bayes' theorem to evaluate hypothesis H in light of evidence E:

$$Pr(H \mid E) = Pr(H)\frac{Pr(E \mid H)}{Pr(E)}.$$

But we saw there's no recipe for calculating the prior probability of H, $Pr(H)$. You just have to start with your best guess about how plausible H is. Likewise, frequentists have to start with their best guess about how many of the potential cancer treatments being studied are bogus, and how many are real. That's how we decide where to set the cutoff for statistical significance.

From a Bayesian point of view, significance testing focuses on just one term in Bayes' theorem, the numerator $Pr(E \mid H)$. We suppose the hypothesis is true, and then consider how likely the sort of outcome we've observed is. But Bayes' theorem tells us we need more information to find what we really want, namely $Pr(H \mid E)$. And for that, we need to know $Pr(H)$, how plausible H is to begin with.

So significance testing is based on a mistake, according to Bayesian critics. It ignores essential base rate information, namely how many of the hypotheses we study are true. And this can lead to very wrong results, as we learned from the taxicab problem in Chapter 8.

This critique of frequentism is sharpened by a famous problem known as Lindley's paradox.

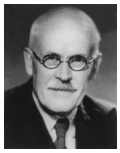


Figure 20.6: Harold Jeffreys (1891–1989) first raised the problem known as Lindley's paradox in 1939. Dennis Lindley labeled it a paradox in 1957, hence the name. It is sometimes called the Jeffreys-Lindley paradox.

20.4 Lindley's Paradox

SUPPOSE a florist receives a large shipment of tulip bulbs with the label scratched off. The company that sent the shipment only sends two kinds of shipments. The first kind contains 25% red bulbs, the second kind has 50% red bulbs.

The two kinds of shipment are equally common. So the store owner figures this shipment could be of either kind, with equal probability.

To figure out which kind of shipment she has, she takes a sample of 48 bulbs and plants them to see what colour they grow. Of the 48 planted, 36 grow red. What should she conclude?

Intuitively, this result fits much better with the 50% hypothesis than the 25% hypothesis. So she should conclude she got the second, 50%

The tulip example is based on an example from Henry Kyburg's book *Logical Foundations of Statistical Inference*. It also appears in Howson & Urbach's *Scientific Reasoning: A Bayesian Approach*.

kind of shipment. It's just a coincidence that well over half the bulbs in her experiment were red.

But if she uses significance testing, she won't get this result. In fact she'll get an *impossible* result. Let's see how.

Our florist starts by testing the hypothesis that 25% of the bulbs in the shipment are red. She calculates μ and σ:

$$\mu = np = (48)(1/4) = 12,$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{(48)(1/4)(3/4)} = 3.$$

The 99% range is $12 \pm (3)(3)$, or from 3 to 21. Her finding of $k = 36$ is nowhere close to this range, so the result is significant at the .01 level. She rejects the 25% hypothesis.

So far so good, but what if she tests the 50% hypothesis too? She calculates the new μ and σ:

$$\mu = np = (48)(1/2) = 24,$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{(48)(1/2)(1/2)} \approx 3.5.$$

So her result $k = 36$ is also significant at the .01 level! The 99% range is 13.5 to 34.5, which doesn't include $k = 36$. So she rejects the 50% hypothesis also.

But now she has rejected the only two possibilities. There are only two kinds of shipment, and she's ruled them both out. Something seems to have gone wrong!

How did things go so wrong? Figure 20.9 shows what's happening here. Neither hypothesis fits the finding $k = 36$ very well: it's an extremely improbable result on either hypothesis. This is why both hypotheses end up being rejected by a significance test.

But one of these hypotheses still fits the finding much better than the other. The blue curve ($p = .25$) flatlines long before it gets to $k = 36$, while the red curve ($p = .5$) is only close to flatlining. So the most plausible interpretation is that the shipment is half red bulbs ($p = .5$), and it's just a fluke that we happen to have gotten much more than half red in our sample.

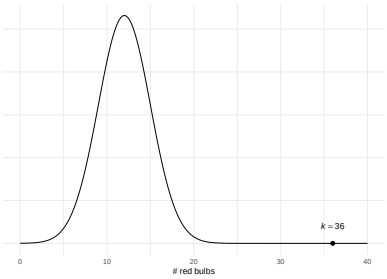


Figure 20.7: The result $k = 36$ out of $n = 48$ is easily statistically significant for the null hypothesis $p = .25$.

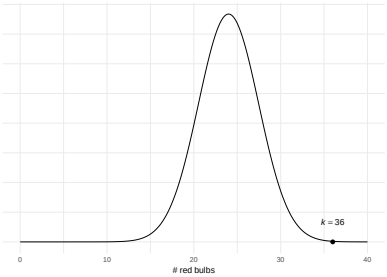


Figure 20.8: The result $k = 36$ out of $n = 48$ is also statistically significant for the null hypothesis $p = .5$.

20.5 *A Bayesian Analysis*

BAYESIANS will happily point out the source of the trouble: our florist has ignored the prior probabilities. If we use Bayes' theorem instead of significance testing, we'll find that the store owner should believe the second hypothesis, which seems right. 36 red bulbs out of 48 fits much better with the 50% hypothesis than with the 25% hypothesis.

How do we apply Bayes' theorem in the tulip example? First we

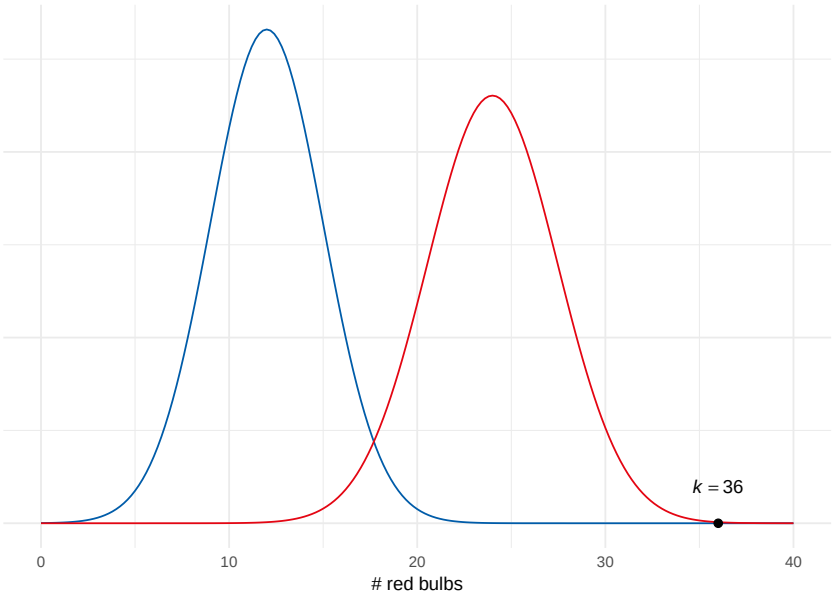


Figure 20.9: An illustration of Lindley's paradox. The finding $k = 36$ doesn't fit well with either hypothesis, so both are rejected. But it fits one of them (red) much better than the other (blue). So the red hypothesis is more plausible from an intuitive perspective.

label our two hypotheses and the evidence:

$$
\begin{aligned}
H &= 25\% \text{ of the bulbs are red,} \\
\sim H &= 50\% \text{ of the bulbs are red,} \\
E &= \text{Out of 48 randomly selected bulbs, 36 grew red.}
\end{aligned}
$$

Because the two kinds of shipment are equally common, the prior probabilities of our hypotheses are:

$$
\begin{aligned}
Pr(H) &= 1/2, \\
Pr(\sim H) &= 1/2.
\end{aligned}
$$

So we just need to calculate $Pr(E \mid H)$ and $Pr(E \mid \sim H)$. That's actually not so easy, but with the help of a computer we get:

$$
\begin{aligned}
Pr(E \mid H) &\approx 4.7 \times 10^{-13}, \\
Pr(E \mid \sim H) &\approx 2.5 \times 10^{-4}.
\end{aligned}
$$

So we plug these numbers into Bayes' theorem and get:

$$\begin{aligned} Pr(H \mid E) &= \frac{Pr(E \mid H)Pr(H)}{Pr(E \mid H)Pr(H) + Pr(E \mid \sim H)Pr(\sim H)} \\ &\approx \frac{(4.7 \times 10^{-13})(1/2)}{(4.7 \times 10^{-13})(1/2) + (2.5 \times 10^{-4})(1/2)} \\ &\approx .000000002, \\ Pr(\sim H \mid E) &= \frac{Pr(E \mid \sim H)Pr(\sim H)}{Pr(E \mid \sim H)Pr(\sim H) + Pr(E \mid H)Pr(H)} \\ &\approx \frac{(2.5 \times 10^{-4})(1/2)}{(2.5 \times 10^{-4})(1/2) + (4.7 \times 10^{-13})(1/2)} \\ &\approx .999999998. \end{aligned}$$

Conclusion: the probability of the first hypothesis H has gone way down, from $1/2$ to about .000000002. But the probability of the second hypothesis $\sim H$ has gone way up from $1/2$ to about .999999998! So we should believe the second hypothesis, not reject it.

According to Bayesian critics, this shows that significance testing is misguided. It ignores crucial background information. In this example, there were only two possible hypotheses, and they were equally likely. So whichever one fits the results best is supported by the evidence. In fact, the second hypothesis is *strongly* supported by the evidence, even though it fits the result quite poorly! Sometimes it makes more sense to reconcile ourselves to a coincidence than to reject the null hypothesis.

Exercises

1. Which of the following statements are true? Select all that apply.

   a. The higher the cutoff for statistical significance, the more true hypotheses will be discovered.
   b. The higher the cutoff for statistical significance, the more false hypotheses will be "discovered".
   c. The higher the cutoff for statistical significance, the more true hypotheses will go undiscovered.
   d. The higher the cutoff for statistical significance, the more false hypotheses will be correctly rejected.

2. Suppose we have $1,000$ coins and we are going to conduct an experiment on each one to determine which are biased. Each coin will be tossed 100 times. In each experiment, the null hypothesis is always that the coin is fair, and we will reject this hypothesis when the results of the experiment are significant at the .05 level.

Suppose half the coins are fair and half are not. Suppose also that when a coin is unfair, the probability of getting a result that is not statistically significant is 0.2.

a. How many fair coins should we expect will incorrectly end up labeled "unfair".
b. How many unfair coins should we expect will correctly end up abeled "unfair".
c. What percentage of the coins labeled "unfair" should we expect to actually be unfair?

3. Suppose we are going to investigate 1,000 null hypotheses by running an experiment on each. In each experiment, we will reject the null hypothesis when the results are significance at the .01 level.

   Suppose 90% of our hypotheses are false. Suppose also that when a null hypothesis is false, the results will not be statistically significant 25% of the time.

   a. How many false hypotheses should we expect will fail to be rejected?
   b. What percentage of the rejected hypotheses should we expect to actually be false?

4. True or false: it is possible for the results of an experiment to be significant at the .05 level even though the posterior probability of the null hypothesis is $Pr(H \mid E) = .99$.

5. Suppose there are two types of urns, Type A and Type B. Type A urns contain 1/5 black balls, the rest white. Type B urns contain 1/10 black balls, the rest white.

   You have an urn that could be either Type A or Type B, you aren't sure. You think it's equally likely to be Type A as Type B. So you decide to use a significance test to find out.

   Your null hypothesis is that it's a Type A urn. You draw 25 marbles at random, and 13 of them are black.

   a. What are μ and σ in the normal approximation?
   b. Is the result significant at the .01 level for this null hypothesis?
   c. Suppose our null hypothesis had been instead that the urn is Type B. What would μ and σ be then?
   d. Is the result significant at the .01 level for the Type B hypothesis?

6. Suppose the government is testing a new education policy. There are only two possibilities: either the policy will work and it will

help high school students learn to write better 3/4 of the time, or it will have no effect and students' writing will only improve 1/2 of the time, as usual.

The government does a study of 432 students and finds that, under the new policy, 285 of them improved in their writing.

a. If the null hypothesis is that each student has a 1/2 chance of improving, are the results of the study significant at the .01 level?
b. If the null hypothesis is that each student has a 3/4 chance of improving, are the results of the study significant at the .01 level?

7. Lindley's paradox occurs when a significance test will direct us to reject the hypothesis even though the posterior probability of the hypothesis $Pr(H \mid E)$ is high. Describe your own example of this kind of case.

A Cheat Sheet

Deductive Logic

Validity An argument is valid if it is impossible for the premises to be true and the conclusion false.

Soundness An argument is sound if it is valid and all the premises are true.

Connectives There are three connectives: $\sim$ (negation), & (conjunction), and $\vee$ (disjunction).

Their truth tables are as follows

A	B	$\sim A$	A & B	$A \vee B$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

Logical Truth A proposition that is always true.

Contradiction A proposition that is never true.

Mutually Exclusive Two propositions are mutually exclusive if they cannot both be true.

Logical Entailment One proposition logically entails another if it is impossible for the first to be true and the second false.

Logical Equivalence Two propositions are logically equivalent if they entail one another.

Probability

Independence Proposition A is independent of proposition B if the truth (or falsity) of B makes no difference to the probability of A.

Fairness A repeating process is fair if each repetition has the same probability and the repetitions are independent of one another.

Multiplication Rule If A and B are independent then $Pr(A \,\&\, B) = Pr(A) \times Pr(B)$.

Addition Rule If A and B are mutually exclusive then $Pr(A \vee B) = Pr(A) + Pr(B)$.

Tautology Rule If A is a tautology then $Pr(A) = 1$.

Contradiction Rule If A is a contradiction then $Pr(A) = 0$.

Equivalence Rule If A and B are logically equivalent then $Pr(A) = Pr(B)$.

Conditional Probability

$$Pr(A \mid B) = \frac{Pr(A \,\&\, B)}{Pr(B)}.$$

Independence (Formal Definition) A is independent of B if $Pr(A \mid B) = Pr(A)$.

Negation Rule $Pr(\sim A) = 1 - Pr(A)$.

General Multiplication Rule $Pr(A \,\&\, B) = Pr(A \mid B)Pr(B)$ if $Pr(B) > 0$.

General Addition Rule $Pr(A \vee B) = Pr(A) + Pr(B) - Pr(A \,\&\, B)$.

Law of Total Probability If $1 > Pr(B) > 0$ then

$$Pr(A) = Pr(A \mid B)Pr(B) + Pr(A \mid \sim B)Pr(\sim B).$$

Bayes' Theorem If $Pr(A), Pr(B) > 0$ then

$$Pr(A \mid B) = Pr(A)\frac{Pr(B \mid A)}{Pr(B)}.$$

Bayes' Theorem (Long Version) If $1 > Pr(A) > 0$ and $Pr(B) > 0$ then

$$Pr(A \mid B) = \frac{Pr(B \mid A)Pr(A)}{Pr(B \mid A)Pr(A) + Pr(B \mid \sim A)Pr(\sim A)}.$$

Decision Theory

Expected Monetary Value Suppose act A has possible payoffs $\$x_1, \$x_2, \ldots, \$x_n$. Then the *expected monetary value* of A is defined:

$$E(A) = Pr(\$x_1) \times \$x_1 + Pr(\$x_2) \times \$x_2 + \ldots + Pr(x_n) \times \$x_n.$$

Expected Utility Suppose act A has possible consequences $C_1, C_2, \ldots, C_n$. Denote the utility of each outcome $U(C_1)$, $U(C_2)$, etc. Then the *expected utility* of A is defined:

$$EU(A) = Pr(C_1)U(C_1) + Pr(C_2)U(C_2) + \ldots + Pr(C_n)U(C_n).$$

Measuring Utility Suppose an agent's best and worst possible outcomes are B and W. Let $U(B) = 1$ and $U(W) = 0$. And suppose $Pr(B)$ be the lowest probability such that they are indifferent between outcome O and a gamble with probability $Pr(B)$ of outcome B, and probability $1 - Pr(B)$ of outcome W. Then, if the agent is following the expected utility rule, $U(O) = Pr(B)$.

Sure-thing Principle If you would choose X over Y if you knew that E was true, and you'd also choose X over Y if you knew E wasn't true, then you should choose X over Y when you don't know whether E is true or not.

Bayesianism

Measuring Personal Probability Personal probabilities are measured by fair betting rates, if the agent is following the expected value rule. More concreteley, suppose an agent regards as fair a bet where they win w if A is true, and they lose l if A is false. Then, if they are following the expected value rule, their personal probability for A is:

$$Pr(A) = \frac{l}{w + l}.$$

Dutch book A Dutch book is a set of bets where each bet is fair according to the agent's betting rates, and yet the set of bets is guaranteed to lose them money. Agents who violate the laws of probability can be Dutch booked. Agents who obey the laws of probability cannot be Dutch booked.

Principle of Indifference If there are n possible outcomes, each outcome should have the same prior probability: $1/n$.

If there is an interval of possible outcomes from a to b, the probability of any subinterval from c to d is:

$$\frac{d - c}{b - a}.$$

Frequentism

Significance Testing A significance test at the .05 level can be described in three steps:

1. State the hypothesis you want to test: the true probability of outcome X is p. This is called the *null hypothesis*.
2. Repeat the event over and over and count the number of times k that outcome X occurs.
3. If the number k falls outside the range of outcomes expected 95% of the time, reject the null hypothesis. (Otherwise, draw no conclusion.)

For a test at the .01 level, follow the same steps but check instead whether k falls outside the range of outcomes expected 99% of the time.

Normal Approximation Suppose an event has two possible outcomes, with probabilities p and $1 - p$. And suppose the event will be repeated n independent times. We define the mean $\mu = np$ and the standard deviation $\sigma = \sqrt{np(1-p)}$. Let k be the number of times the first outcome occurs. Then, if n is large enough:

- The probability is about .68 that k will be between $\mu - \sigma$ and $\mu + \sigma$ times.
- The probability is about .95 that k will be between $\mu - 2\sigma$ and $\mu + 2\sigma$ times.
- The probability is about .99 that k will be between $\mu - 3\sigma$ and $\mu + 3\sigma$ times.

B The Axioms of Probability

Theories and Axioms

In mathematics, a theory like the theory of probability is developed axiomatically. That means we begin with fundamental laws or principles called *axioms*, which are the assumptions the theory rests on. Then we derive the consequences of these axioms via *proofs*: deductive arguments which establish additional principles that follow from the axioms. These further principles are called *theorems*.

In the case of probability theory, we can build the whole theory from just three axioms. And that makes certain tasks much easier. For example, it makes it easy to establish that anyone who violates a law of probability can be Dutch booked. Because, if you violate a law of probability, you must also be violating one of the three axioms that entail the law you've violated. And with only three axioms to check, we can verify pretty quickly that violating an axiom always makes you vulnerable to a Dutch book.

The axiomatic approach is useful for lots of other reasons too. For example, we can program the axioms into a computer and use it to solve real-world problems. Or, we could use the axioms to verify that the theory is consistent: if we can establish that the axioms don't contradict one another, then we know the theory makes sense. Axioms are also a useful way to summarize a theory, which makes it easier to compare it to alternative theories.

In addition to axioms, a theory typically includes some *definitions*. Definitions construct new concepts out of existing ones, ones that already appear in the axioms. Definitions don't add new assumptions to the theory. Instead they're useful because they give us new language in which to describe what the axioms already entail.

So a theory is a set of statements that tells us everything true about the subject at hand. There are three kinds of statements:

1. Axioms: the principles we take for granted.
2. Definitions: statements that introduce new concepts or terms.
3. Theorems: statements that follow from the axioms and the defi-

nitions.

In this appendix we'll construct probability theory axiomatically. We'll learn how to derive all the laws of probability discussed in Part I from three simple statements.

The Three Axioms of Probability

PROBABILITY theory has three axioms, and they're all familiar laws of probability. But they're fundamental laws in a way. All the other laws can be derived from them.

The three axioms are:

Normality For any proposition A, $0 \leq Pr(A) \leq 1$.

Tautology Rule If A is a logical truth then $Pr(A) = 1$.

Additivity Rule If A and B are mutually exclusive then $Pr(A \vee B) = Pr(A) + Pr(B)$.

Our task now is to derive from these three axioms the other laws of probability. We do this by stating each law, and then giving a proof of it: a valid deductive argument showing that it follows from the axioms and definitions.

First Steps

LET'S START with one of the easier laws to derive.

The Negation Rule $Pr(\sim A) = 1 - Pr(A)$

Proof. To prove this rule, start by noticing that $A \vee \sim A$ is a logical truth. So we can reason as follows:

$$\begin{aligned} Pr(A \vee \sim A) &= 1 && \text{by Tautology} \\ Pr(A) + Pr(\sim A) &= 1 && \text{by Additivity} \\ Pr(\sim A) &= 1 - Pr(A) && \text{by algebra.} \end{aligned}$$

□

The black square indicates the end of the proof. Notice how each line of our proof is justified by either applying an axiom or using basic algebra. This ensures it's a valid deductive argument.

Now we can use the Negation rule to establish the flipside of the Tautology rule: the Contradiction rule.

The Contradiction Rule If A is a contradiction then $Pr(A) = 0$.

Proof. Notice that if A is a contradiction, then $\sim A$ must be a tautology. So $Pr(\sim A) = 1$. Therefore:

$$\begin{aligned} Pr(A) &= 1 - Pr(\sim A) && \text{by Negation} \\ &= 1 - 1 && \text{by Tautology} \\ &= 0 && \text{by arithmetic.} \end{aligned}$$

□

Conditional Probability & the Multiplication Rule

OUR next theorem is about conditional probability. But the concept of conditional probability isn't mentioned in the axioms, so we need to define it first.

Definition: Conditional Probability The conditional probability of A given B is written $Pr(A \mid B)$ and is defined:

$$Pr(A \mid B) = \frac{Pr(A \,\&\, B)}{Pr(B)},$$

provided that $Pr(B) > 0$.

From this definition we can derive the following theorem.

Multiplication Rule If $Pr(B) > 0$, then $Pr(A \,\&\, B) = Pr(A \mid B)Pr(B)$.

Proof.

$$\begin{aligned} Pr(A \mid B) &= \frac{Pr(A \,\&\, B)}{Pr(B)} && \text{by definition} \\ Pr(A \mid B)Pr(B) &= Pr(A \,\&\, B) && \text{by algebra} \\ Pr(A \,\&\, B) &= Pr(A \mid B)Pr(B) && \text{by algebra.} \end{aligned}$$

□

Notice that the first step in this proof wouldn't make sense if we didn't assume from the beginning that $Pr(B) > 0$. That's why the theorem begins with the qualifier, "If $Pr(B) > 0 \ldots$".

Equivalence & General Addition

NEXT we'll prove the Equivalence rule and the General Addition rule. These proofs are longer and more difficult than the ones we've done so far.

Equivalence Rule When A and B are logically equivalent, $Pr(A) = Pr(B)$.

Proof. Suppose that A and B are logically equivalent. Then $\sim A$ and B are mutually exclusive: if B is true then A must be true, hence $\sim A$ false. So B and $\sim A$ can't both be true.

So we can apply the Additivity axiom to $\sim A \vee B$:

$$\begin{aligned} Pr(\sim A \vee B) &= Pr(\sim A) + Pr(B) && \text{by Additivity} \\ &= 1 - Pr(A) + Pr(B) && \text{by Negation.} \end{aligned}$$

Next notice that, because A and B are logically equivalent, we also know that $\sim A \vee B$ is a necessary truth. If B is false, then A must be false, so $\sim A$ must be true. So either B is true, or $\sim A$ is true. So $\sim A \vee B$ is always true, no matter what.

So we can apply the Tautology axiom:

$$Pr(\sim A \vee B) = 1 \quad \text{by Tautology.}$$

Combining the previous two equations we get:

$$\begin{aligned} 1 &= 1 - Pr(A) + Pr(B) && \text{by algebra} \\ Pr(A) &= Pr(B) && \text{by algebra.} \end{aligned}$$

□

Now we can use this theorem to derive the General Addition rule.

General Addition Rule $Pr(A \vee B) = Pr(A) + Pr(B) - Pr(A \,\&\, B)$.

Proof. Start with the observation that $A \vee B$ is logically equivalent to:

$$(A \,\&\, \sim B) \vee (A \,\&\, B) \vee (\sim A \,\&\, B).$$

This is easiest to see with an Euler diagram, but you can also verify it with a truth table. (We won't go through either of these exercises here.)

So we can apply the Equivalence rule to get:

$$Pr(A \vee B) = Pr((A \,\&\, \sim B) \vee (A \,\&\, B) \vee (\sim A \,\&\, B)).$$

And thus, by Additivity:

$$Pr(A \vee B) = Pr(A \,\&\, \sim B) + Pr(A \,\&\, B) + Pr(\sim A \,\&\, B).$$

We can also verify with an Euler diagram (or truth table) that A is logically equivalent to $(A \,\&\, B) \vee (A \,\&\, \sim B)$, and that B is logically equivalent to $(A \,\&\, B) \vee (\sim A \,\&\, B)$. So, by Additivity, we also have the equations:

$$Pr(A) = Pr(A \,\&\, \sim B) + Pr(A \,\&\, B).$$
$$Pr(B) = Pr(A \,\&\, B) + Pr(\sim A \,\&\, B).$$

Notice, the last equation here can be transformed to:

$$Pr(\sim A \,\&\, B) = Pr(B) - Pr(A \,\&\, B).$$

Putting the previous four equations together, we can then derive:

$$\begin{aligned} Pr(A \vee B) &= Pr(A \,\&\, \sim B) + Pr(A \,\&\, B) + Pr(\sim A \,\&\, B) && \text{by algebra} \\ &= Pr(A) + Pr(\sim A \,\&\, B) && \text{by algebra} \\ &= Pr(A) + Pr(B) - Pr(A \,\&\, B) && \text{by algebra.} \end{aligned}$$

□

Total Probability & Bayes' Theorem

NEXT we derive the Law of Total Probability and Bayes' theorem.

Total Probability If $0 < Pr(B) < 1$, then

$$Pr(A) = Pr(A \mid B)Pr(B) + Pr(A \mid \sim B)Pr(\sim B).$$

Proof.

$$\begin{aligned} Pr(A) &= Pr((A \,\&\, B) \vee (A \,\&\, \sim B)) && \text{by Equivalence} \\ &= Pr(A \,\&\, B) + (A \,\&\, \sim B) && \text{by Additivity} \\ &= Pr(A \mid B)Pr(B) + Pr(A \mid \sim B)Pr(\sim B) && \text{by Multiplication.} \end{aligned}$$

□

Notice, the last line of this proof only makes sense if $Pr(B) > 0$ and $Pr(\sim B) > 0$. That's the same as $0 < Pr(B) < 1$, which is why the theorem begins with the condition: "If $0 < Pr(B) < 1 \ldots$".

Now for the first version of Bayes' theorem:

Bayes' Theorem If $Pr(A), Pr(B) > 0$, then

$$Pr(A \mid B) = Pr(A)\frac{Pr(B \mid A)}{Pr(B)}.$$

Proof.

$$\begin{aligned} Pr(A \mid B) &= \frac{Pr(A \,\&\, B)}{Pr(B)} && \text{by definition} \\ &= \frac{Pr(B \mid A)Pr(A)}{Pr(B)} && \text{by Multiplication} \\ &= Pr(A)\frac{Pr(B \mid A)}{Pr(B)} && \text{by algebra.} \end{aligned}$$

□

And next the long version:

Bayes' Theorem (long version) If $1 > Pr(A) > 0$ and $Pr(B) > 0$, then

$$Pr(A \mid B) = \frac{Pr(A)Pr(B \mid A)}{Pr(A)Pr(B \mid A) + Pr(\sim A)Pr(B \mid \sim A)}.$$

Proof.

$$\begin{aligned} Pr(A \mid B) &= \frac{Pr(A)Pr(B \mid A)}{Pr(B)} && \text{by Bayes' theorem} \\ &= \frac{Pr(A)Pr(B \mid A)}{Pr(A)Pr(B \mid A) + Pr(\sim A)Pr(B \mid \sim A)} && \text{by Total Probability.} \end{aligned}$$

□

Independence

FINALLY, LET'S INTRODUCE the concept of independence, and two key theorems that deal with it.

Definition: Independence A is independent of B if $Pr(A \mid B) = Pr(A)$ and $Pr(A) > 0$.

Now we can state and prove the Multiplication rule.

Multiplication Rule If A is independent of B, then $Pr(A \,\&\, B) = Pr(A)Pr(B)$.

Proof. Suppose A is independent of B. Then:

$$\begin{aligned} Pr(A \mid B) &= Pr(A) && \text{by definition} \\ \frac{Pr(A \,\&\, B)}{Pr(B)} &= Pr(A) && \text{by definition} \\ Pr(A \,\&\, B) &= Pr(A)Pr(B) && \text{by algebra.} \end{aligned}$$

□

Finally, we prove another useful fact about independence, namely that it goes both ways.

Independence is Symmetric If A is independent of B, then B is independent of A.

Proof. To derive this fact, suppose A is independent of B. Then:

$$\begin{aligned} Pr(A \,\&\, B) &= Pr(A)Pr(B) && \text{by Multiplication} \\ Pr(B \,\&\, A) &= Pr(A)Pr(B) && \text{by Equivalence} \\ \frac{Pr(B \,\&\, A)}{Pr(A)} &= Pr(B) && \text{by algebra} \\ Pr(B \mid A) &= Pr(B) && \text{by definition.} \end{aligned}$$

□

We've now established that the laws of probability used in this book can be derived from the three axioms we began with.

C The Grue Paradox

In Section 2.5 we noticed that many inductive arguments follow this pattern:

> All observed instances of X have been Y.
> Therefore, all instances of X are Y.

All observed ravens are black, so we expect all ravens to be black. All observed emeralds are green, so we expect all emeralds to be green. And so on.

So it seems like a fundamental principle of scientific inquiry that we expect the unobserved to resemble the observed. Philosophers call this *The Principle of Induction*.

Figure C.1: Nelson Goodman (1906–1988) discovered the grue paradox in the 1940s and '50s.

A Gruesome Concept

But in the 1940s, Nelson Goodman discovered a problem with the Principle of Induction. To illustrate the problem he invented a very curious concept, *grue*.

There are two ways for an object to be grue. Some green things are grue, but it depends on when we first encounter them. If our first observation of a green object happens before the year 2050, then it's grue. So the Statue of Liberty is grue: it's a green object that was first observed before the year 2050 (*long* before).

But if our first encounter with a green object happens in the year 2050 or later, then it's not grue. The same goes if we never observe it. Objects on the far side of the universe that we'll never see, or buried deep underground, are not grue.

There is a second way for an object to be grue: some blue objects are grue. Not the ones observed before 2050, though. Instead it's the ones that *aren't* observed before 2050. If a blue object is observed for the first time *after* 2049, or it's never observed at all, then it's grue. So blue sapphires that won't be mined before the year 2050 are grue, for example.

As usual, it helps to have a diagram:

Here is the formal definition of grue:

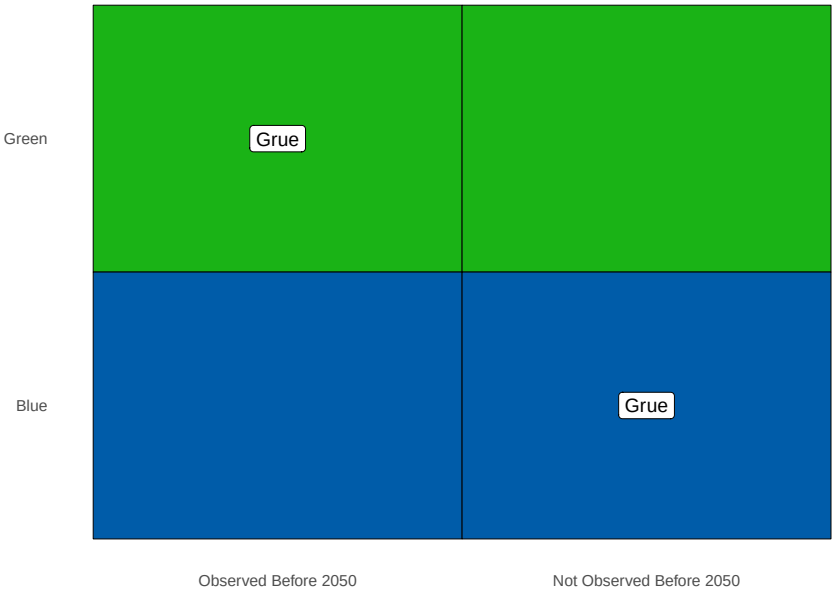


Figure C.2: The definition of grue

Grue An object is *grue* if either (a) it is green and first observed before the year 2050, or (b) it's blue and not observed before 2050.

To test your understanding, see if you can explain why each of the following are examples of grue things: the $20 bill in my pocket, Kermit the Frog, the first sapphire to be mined in 2050, and blue planets on the far side of the universe.

Then see if you can explain why these things aren't grue: fire engines, the Star of India, and the first $20 bill to be printed in 2050.

Once you've got all those down, try this question: do grue objects change colour in the year 2050? It's a common mistake to think they do.

But no, grue objects don't change colour. The Statue of Liberty is green and it always will be (let's assume). So it's grue, and always will be, because it's a green thing that was first observed before the year 2050. Part (a) of the definition of grue guarantees that.

The only way time comes into it is in determining which green things are grue, and which blue things. If a green thing is first observed before 2050, then it's grue, ever and always. Likewise if a blue thing is *not* first observed before 2050. Then it's grue—and it always has been!

The Paradox

Now ask yourself, have you ever seen a grue emerald? You probably have. In fact, every emerald everyone's ever seen has been grue.

Why? Because they're all green, and they've all been observed before the year 2050. So they're all grue the first way—they all satisfy part (a) of the definition. (Notice it's an either/or definition, so you only have to satisfy one of the two parts to be grue.)

So all the emeralds we've ever seen have been grue. Let's apply the Principle of Induction then:

All observed emeralds have been grue.
Therefore *all* emeralds are grue.

But if all emeralds are grue, then the first emeralds to be mined in 2050 will be grue. And that means they'll be blue! Because they won't have been observed before 2050, so the only way for them to be grue is to be blue.

We've reached the absurd conclusion that there are blue emeralds out there, just waiting to be pulled out of the earth. Something has gone off the rails here, but what?

Here's another way to put the challenge. We have two "patterns" in our observed data. The emeralds we've seen are uniformly green, but they're also uniformly grue. We can't project both these patterns into the future, though. They'll contradict each other starting in 2050.

Now, obviously, common sense says the green pattern is the real one. The grue "pattern" is bogus, and no one but a philosopher would even bother thinking about it. But *why* is it bogus? What's so special about green?

Apparently the Principle of Induction has a huge hole in it! It says to extrapolate from observed patterns. But *which* patterns?

Grue & Artificial Intelligence

Patterns are cheap, as any data scientist will tell you. Given a bunch of data points in an *xy*-plane, there are lots of ways to connect the dots. Even if they all lie on a straight line, you could draw an oscillating curve that passes through each point (Figure C.3). You can probably think of even sillier curves that will fit all the points.

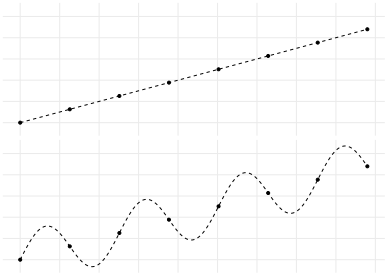


Figure C.3: The same set of points interpreted two different ways

Designing a computer program that will know which patterns to use and which to ignore is a big part of what machine learning experts do. And it's one reason humans are still essential to designing artificial intelligence. Thanks to our experience, and our genetic inheritance, we have *lots* of information about which patterns are likely to continue, and which are bogus, like grue.

But how do we pass all that wisdom on to the machines? How do we teach them the difference between green and grue, so that they can take it from here and we can all go on permanent vacation?

Disjunctivitis

HERE'S ONE very natural answer. The problem with grue is that it's *disjunctive*: it's defined using either/or. It suffers from what we might call "disjunctivitis".

But the beauty of Goodman's puzzle is the neat way it exposes the flaw in this answer. It allows us to make 'green' the disjunctive concept instead! How? We start by building grue a friend, a concept to fill in the missing spaces in our original diagram. We'll call it *bleen*:

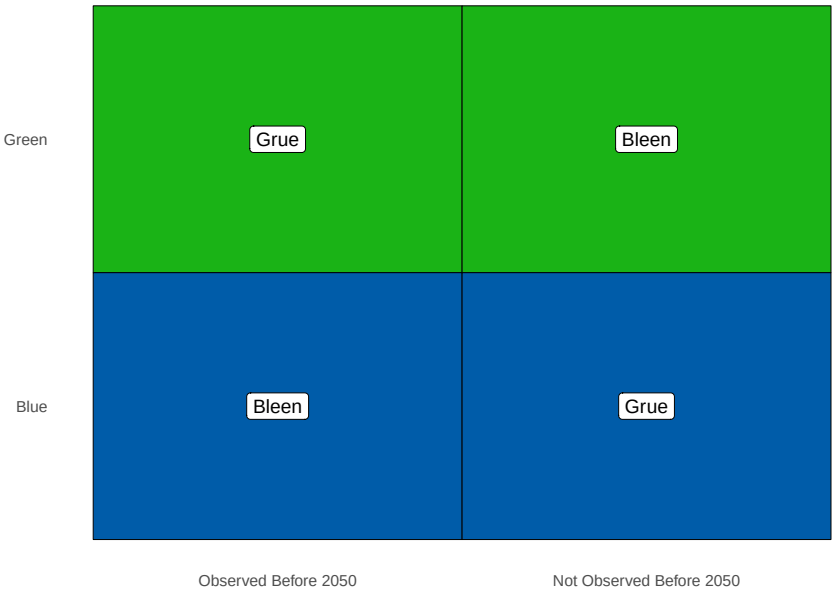


Figure C.4: Defining bleen, a counterpart to grue

Now we can define green in terms of grue and bleen:

Green An object is *green* if either (a) it's grue and first observed before the year 2050, or (b) it's bleen and not observed before 2050.

Now maybe you're thinking: you *could* define green that way, but that's not how it's *actually* defined. In reality, we already understand the concept of green, and we have to learn the concept of grue from its disjunctive definition.

The problem is, that's just a fact about *us humans*, not about the concepts grue and green. That's just the way we *homo sapiens* happen to be built (or maybe socialized, or both).

Some bizarre species of alien could grow up thinking in grue/bleen terms instead. And when they landed on Earth, we'd have to explain our green/blue language to them using an either/or definition. Then *they* would be looking at *us* thinking: you guys have a very weird, disjunctive way of thinking!

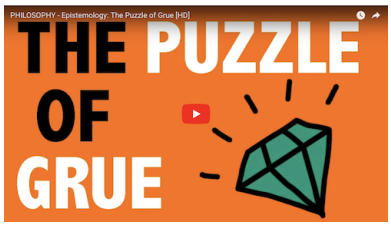
What could we say to them to establish the superiority of our way of thinking? It's been more than $7$0 years since Goodman first posed this question. Yet no answer has emerged as the clear and decisively correct one.

Time Dependence

ANOTHER natural answer to Goodman's challenge is to say that grue is defective because it's time-dependent. It means different things depending on the time an object is first observed.

But the same reversal of fortunes that toppled the "disjunctivitis" diagnosis happens here. We can define green in terms of grue and bleen. And when we do, it's green that's the time-dependent concept, not grue.

So we're left in the same spot. We need some way of showing that the "true" order of definition is the one we're used to. By what criterion can we say that green is more fundamental, more basic, than grue?



For another explanation of the grue puzzle, check out this excellent Wi-Phi video.

The Moral

GOODMAN'S PUZZLE may seem just cute at first, a mere curiosity. But it is actually quite profound.

In a way, the central question of this book is: what is the logic of science? What are the correct rules of scientific reasoning?

The laws of probability seem like a good place to start. But that path led us to a dead end at the problem of priors in Chapter 18. Perhaps then we could start with the Principle of Induction instead? But then we end up in another dead end, stopped by the grue paradox.

Just as Bertrand's paradox stops us from using the Principle of Indifference to answer the problem of priors, Goodman's paradox stops us from using the Principle of Induction.

There is even more to the similarity between these two paradoxes. Both paradoxes are problems of "language dependence". Depending on what language we work in, rules like the Principle of Indifference and the Principle of Induction give different recommendations. If we apply the Principle of Indifference to length, we get one prior probability; if we apply it to area, we get another. If we apply the Principle of Induction to green, we expect all emeralds to be green; if we apply it to grue, we expect some to be blue.

To this day, we do not have an answer to the question: which language is the right one to use, and why?

D The Problem of Induction

Many inductive arguments work by projecting an observed pattern onto as-yet unobserved instances. All the ravens we've observed have been black, so all ravens are. All the emeralds we've seen have been green, so all emeralds are.

The assumption that the unobserved will resemble the observed seems to be central to induction. Philosophers call this assumption the *Principle of Induction*.[1] But what justfies this assumption? Do we have any reason to think the parts of reality we've observed so far are a good representative of the parts we haven't seen yet?

[1] See Section 2.5 and Appendix C for previous discussions of the Principle of Induction.

Actually there are strong reasons to doubt whether this assumption can be justified. It may be impossible to give any good argument for expecting the unobserved to resemble the observed.

The Dilemma

We observed previously that there are two kinds of argument, inductive and deductive. Some arguments establish their conclusions necessarily, others only support them with high probability. If there is an argument for the Principle of Induction, it must be one of these two kinds. Let's consider each in turn.

Could we give an inductive argument for the Principle of Induction? At first it seems we could. Scientists have been using inductive reasoning for millenia, often with great success. Indeed, it seems humans, and other creatures too, have relied on it for much longer, and could not have survived without it. So the Principle of Induction has a very strong track record. Isn't that a good argument for believing it's correct?

Figure D.1: David Hume (1711–1776) raised the problem of induction in 1739. Our presentation of it here is somewhat modernized from his original argument.

No, because the argument is circular. It uses the Principle of Induction to justify believing in the Principle of Induction. Consider that the argument we are attempting looks like this:

> The principle has worked well when we've used it in the past.
> Therefore it will work well in future instances.

This is an inductive argument, an argument from observed instances to

ones as yet unobserved. So, under the hood, it appeals to the Principle of Induction. But that's exactly the conclusion we're trying to establish. And one can't use a principle to justify itself.

What about our second option: could a deductive argument establish the Principle of Induction? Well, by definition, a deductive argument establishes its conclusion with necessity. Is it necessary that the unobserved will be like the observed? It doesn't look like it. It seems perfectly possible that tomorrow the world will go haywire, randomly switching from pattern to pattern, or even to no pattern at all.

Maybe tomorrow the sun will fail to rise. Maybe gravity will push apart instead of pull together, and all the other laws of physics will reverse too. And just as soon as we get used to those patterns and start expecting them to continue, another pattern will arise. And then another. And then, just as we give up and come to have no expectation at all about what will come next, everything will return to normal. Until we get comfortable and everything changes again.

Thankfully, our universe hasn't been so mischievous. We get surprised now and again, but for the most part inductive reasoning is pretty reliable, when we do it carefully. But we're lucky in this respect, is the point.

Nature *could* have been mischievous, totally unpredictable. It is not a necessary truth that the unobserved must resemble the observed. And so it seems there cannot be a deductive argument for the Principle of Induction. Because such an argument would establish the principle as a necessary truth.

The Problem of Induction vs. the Grue Paradox

If you read Appendix C, you know of another famous problem with the Principle of Induction: the grue paradox. (If you haven't read that chapter, you might want to skip this section.)

The two problems are quite different, but it's easy to get them confused. The problem we're discussing here is about justifying the Principle of Induction. Is there any reason to believe it's true? Whereas the grue paradox points out that we don't even really know what the principle says, in a way. It says that what we've observed is a good indicator of what we haven't yet obsered. But in what respects? Will unobserved emeralds be green, or will they be grue?

So the challenge posed by grue is to spell out, precisely, what the Principle of Induction says. But even if we can meet that challenge, this challenge will remain. Why should we believe the principle, once it's been spelled out? Neither a deductive argument nor an inductive argument seems possible.

Probability Theory to the Rescue?

THE Problem of Induction is centuries old. Isn't it out of date? Hasn't the modern, mathematical theory of probability solved the problem for us?

Not at all, unfortunately. One thing we learn in this book is that the laws of probability are very weak in a way. They don't tell us much, without us first telling them what the prior probabilities are. And as we've seen over and again throughout Part III, the problem of priors is very much unsolved.

For example, suppose we're going to flip a mystery coin five times. We don't know whether the coin is fair or biased, but we hope to have some idea after a few flips.

Now suppose we get through the first four flips and they've all been tails. The Principle of Induction says we should expect the next flip to be tails too. At least, that outcome should now be more probable.

Do the laws of probability agree? Well, we need to calculate the quantity:

$$Pr(T_5 \mid T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4).$$

The definition of conditional probability tell us:

$$Pr(T_5 \mid T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4) = \frac{Pr(T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4 \,\&\, T_5)}{Pr(T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4)}.$$

But the laws of probability don't tell us what numbers go in the numerator and the denominator.

The numbers have to be between 0 and 1. And we have to be sure mutually exclusive propositions have probabilities that add up, according to the Additivity rule. But that still leaves things wide open.

For example, we could finish the calculation this way:

$$\begin{aligned} Pr(T_5 \mid T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4) &= \frac{Pr(T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4 \,\&\, T_5)}{Pr(T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4)} \\ &= \frac{1/32}{1/16} \\ &= 1/2. \end{aligned}$$

Or we could finish it this way:

$$\begin{aligned} Pr(T_5 \mid T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4) &= \frac{Pr(T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4 \,\&\, T_5)}{Pr(T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4)} \\ &= \frac{5/20}{6/20} \\ &= 5/6. \end{aligned}$$

We could even do this:

$$\begin{aligned} Pr(T_5 \mid T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4) &= \frac{Pr(T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4 \,\&\, T_5)}{Pr(T_1 \,\&\, T_2 \,\&\, T_3 \,\&\, T_4)} \\ &= \frac{0}{1} \\ &= 0. \end{aligned}$$

All these options result from different choices of prior probabilities. And the laws of probability don't tell us what prior probabilities we must choose, as we learned in Part III.

So the laws of probability don't by themselves tell us what to expect. It could be undecided, with heads and tails equally probable on the final toss (1/2). Or the pattern of tails could continue into the future with high probability (5/6). There could even be no chance of the pattern continuing (0).

The laws of probability only tell us what to expect once we've specified the necessary prior probabilities. The problem of induction challenges us to justify one choice of prior probabilities over the alternatives.

In the 280 years since this challenge was first raised by David Hume, no answer has gained general acceptance.