Market Concentration, Growth, and Acquisitions

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January 24, 2023

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Abstract

I study an oligopolistic growth model in which firms can innovate by creatively destroying their competitors' goods, innovating on their own varieties, and developing new varieties. To avoid cannibalization, larger firms innovate disproportionately through creative destruction, which generates little growth but deters other firms from innovating. A fall in large firm fixed costs, calibrated to match the recent rise in industry concentration in the US, explains almost half the fall in growth from the 1990s to the 2010s, as well as the burst in growth during the late 1990s, the positive across-industry correlation between changes in concentration and growth, and the fall in growth relative to R&D expenditures as a share of GDP. Despite this link between the rise of dominant firms and the fall in growth, a substantial reduction in taxes on large firm acquisitions of their competitors' goods *increases* growth and welfare: to preserve valuable acquisition opportunities, large firms engage in less creative destruction. Dispersion in large firm innovation costs across industries yields a novel theory of the inverted-U relationship between concentration and growth.

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1 Introduction

Many authors have documented a rising share of revenue going to the top firms in industries at the national level in the US since the 1990s.[1] This trend has spurred research into its connection to the recent decline in growth, as well as the policy implications.[2] Over a similar time period, there was a dramatic rise in the rate at which venture capital backed startups are acquired relative to the rate at which they go public.[3] Two questions emerge: Is large firm behavior behind the fall in growth? If so, should antitrust authorities limit acquisitions to reduce industry concentration and promote growth?

To answer these questions, I study an oligopolistic growth model. I find that if firms can creatively destroy their competitors' goods and if small firm innovation is sufficiently responsive to the value of innovating, then a shock that induces a rise in large firm innovation leads to a fall in the growth rate. In that case, policies that limit acquisitions may backfire by encouraging large firms to innovate more, which leads to lower growth.

I begin in Section 2 with a qualitative model of an industry with a single large firm and a continuum of small firms that can innovate only by creatively destroying their competitors' goods. I provide analytical conditions so that a rise in large firm innovation incentives leads to a fall in the long-run growth rate. I decompose the effect of large firm innovation incentives on the growth rate into the product of the effect on the creative destruction rate small firms face—the discount rate effect—and the effect on the growth rate relative to the creative destruction rate—the composition effect. The discount rate effect is positive because small firms only reduce innovation in response to an increase in the creative destruction rate. The composition effect is negative because large firm innovation only targets small firm goods, leaving large firm goods to stagnate. If small firm innovation is sufficiently elastic, then a rise in large firm innovation incentives leads to a fall in the long-run growth rate: a higher elasticity implies a weaker discount rate effect because a slight rise in the creative destruction rate leads to a substantial fall in small firm innovation, and a stronger composition effect because the large firm's share of economic activity increases faster with its innovation rate. Moreover, further increases in large firm innovation incentives have increasingly negative effects on the long-run growth rate: in the limit, the large firm deters all small firm innovation with a rapid creative destruction rate, but generates no growth because there are no small firm goods to creatively destroy.

[1] See Autor, Dorn, Katz, Patterson, and Van Reenen (2020), Barkai (2020).

[2] See Cavenaile, Celik, and Tian (2021), Aghion, Bergeaud, Boppart, Klenow, and Li (2022), Akcigit and Ates (2021), De Ridder (2021), Liu, Mian, and Sufi (2022), Olmstead-Rumsey (2022), and Garcia-Macia, Hsieh, and Klenow (2019).

[3] See Pellegrino (2021).

The large firm can search for acquisition opportunities in which it makes a take-it-or-leave-it offer to a small firm for its good subject to a tax. I provide analytical conditions so that a rise in the tax on acquisitions leads to a fall in the long-run growth rate. I decompose the effect of a rise in the tax on the growth rate into the effect on growth through the acquisition rate and the effect on growth through the large firm innovation rate, taking into account the response of optimal small firm innovation. The acquisition rate effect is positive because a rise in the tax leads to fewer acquisitions, which reduces the large firm's revenue share and leaves fewer goods stagnating. The innovation effect is negative if a rise in large firm innovation incentives leads to a lower growth rate because from the large firm's perspective, creatively destroying its competitors' goods and acquiring them are substitutes. If the innovation effect is negative and the elasticity of the acquisition rate is sufficiently low relative to the elasticity of large firm innovation, then a *fall* in the acquisition tax leads to a *higher* growth rate. Moreover, if the innovation effect is negative, then a rise in large firm innovation incentives makes a fall in the acquisition tax more beneficial to growth because a higher large firm revenue share implies a bigger benefit from reducing large firm innovation.

To conduct a quantitative analysis and address the empirical finding in Garcia-Macia, Hsieh, and Klenow (2019) that creative destruction is a minority share of growth in the US, I develop a quantitative macroeconomic model in Sections 3 and 4 in which firms can also innovate by improving on their own goods and creating new ones. Crucially, Garcia-Macia, Hsieh, and Klenow (2019) find that creative destruction is responsible for the majority of labor flows because relative to other forms of innovation, it generates more business stealing and less growth. The Arrow (1962) "replacement effect" thus implies a new component of the composition effect. A higher large firm revenue share decreases the large firm's incentive to improve on its own goods and develop new ones because the gains from doing so come from generating growth, which cannibalizes the sales of its other goods. The same does not hold for its incentive to creatively destroy its competitors' goods because the gains from doing so come mostly from taking sales directly from its competitors. An increase in large firm innovation incentives therefore leads mostly to a rise in creative destruction of small firms' goods, which deters their innovation without generating much growth.

In Section 5, I demonstrate that a rise in large firm innovation incentives—a fall in large firm per-good fixed costs, innovation costs, or revenue taxes—calibrated to match the observed rise in concentration from the mid 1990s to the late 2010s can explain 41% of the long-run fall in growth, as well as a number of other related empirical observations: the short-run burst in growth in the late 1990s, the positive correlation across industries between changes in concentration and growth, the fall in the entry rate, and the fall in large negative labor flows. Moreover, considering a fall in large firm per-good fixed costs or revenue taxes in particular, the theory provides a novel explanation for a puzzle based on empirical work in Olmstead-Rumsey (2022): R&D relative to GDP rose, R&D shifted toward firms with low R&D to revenue ratios, i.e., efficient innovators, and the growth rate fell. In the model, due to the Arrow (1962) "replacement effect", large firms innovate relatively little and their innovation is more efficient at generating revenue on the margin. Nonetheless, it is less efficient at creating *growth* because it is tilted toward creative destruction, which mostly entails business stealing rather than growth generation.

In Section 6, I analyze the effects of a tax on acquisitions and confirm the predictions of the qualitative model. A sufficiently high acquisition subsidy leads to faster growth because large firms innovate less to preserve valuable acquisition opportunities. Moreover, following the rise in concentration from the experiment in Section 5, a smaller subsidy and acquisition rate is required to get faster growth than in the balanced growth path without acquisitions. It follows that policy should not place unnecessary road blocks in the way of high surplus acquisitions, particularly in concentrated industries, but should block acquisitions that generate little surplus. Allowing high value acquisitions is an effective way to limit large firm innovation because unlike a tax, it does not require intimate knowledge of industry boundaries or of which firms are large.

These results highlight an important subtlety in optimal competition and innovation policy. In theories that focus only on *how much* large firms produce or innovate, high markups or the Arrow (1962) "replacement effect" imply that it is optimal to subsidize high market shares and encourage *more* large firm production or innovation.[4] In these models, reducing large firm innovation is a cost of acquisitions, rather than a benefit.[5] Instead, taking into account *the way in which* large firms innovate, policies that encourage large firm production or innovation backfire. Policies that offer large firms a tempting alternative to innovation increase growth by shifting activity to smaller firms that innovate in a more socially optimal way.

Testable empirical predictions of the model's main mechanism are that firms can direct their efforts at improving goods toward their competitors' goods, and that smaller firms face higher discount rates on the profits from their innovations. The theory implies that this disparity increased as market concentration rose and growth fell in the US since the 1990s. Argente, Lee, and Moreira (2021) show that the revenues of high sales products depreciate more quickly than the revenues of low sales products, in line with the prediction that such products would be creatively destroyed more quickly if firms can direct their creative destruction efforts. Akcigit, Alp, and Peters (2021) show that a relatively high creative destruction rate for goods produced by firm types that innovate less, and tend to be small, can explain the high employment shares of old firms in US and Indian data. My theory provides an explanation for this disparity: firms focus their goods improvement

[4] See Edmond, Midrigan, and Xu (2021).

[5] See Fons-Rosen, Roldan-Blanco, and Schmitz (2022).

efforts on their competitors' goods, so more innovative firms avoid a larger fraction of total creative destruction.

Large Firm Market Shares and Innovation:

Previous work on industries with oligopolistic competition and innovation mostly focuses on the impact of a large firm's market share on its *magnitude* of innovation.[6] A notable exception is the theory put forward in Argente, Baslandze, Hanley, and Moreira (2021) and mentioned in Akcigit and Ates (2021) that large firms use patents to deter competition. Although the mechanism is different, the result is similar in that large firm innovation generates little growth relative to the rate at which it deters small firm innovation. This theory is complementary to the one proposed here because creative destruction is most relevant in industries with weak patent protection.

The model generates an inverted-U relationship between concentration and growth across industries, where variation in concentration is driven by variation in large firm innovation incentives. Previous theories of the inverted-U relationship such as in Aghion, Bloom, Blundell, Griffith, and Howitt (2005) and Cavenaile, Celik, and Tian (2021), are based on the relative sizes of the "escape competition" and the Schumpeterian effects of competition. Smaller market shares increase pressure for firms to innovate to escape their competitors, but the expectation of the competition that pushes down market shares discourages innovation. In this paper, the magnitude of small firm innovation depends only on the Schumpeterian effect: as the expected rate of innovation increases, small firms innovate less. Instead, the inverted-U arises because the composition effect is zero if the large firm's revenue share is insignificant, and dominates if the large firm is a monopolist. An implication is that if small firms are more responsive to aggregate rather than industry-specific shocks, then widespread higher concentration can be associated with lower growth even if across industries or within an industry over time, higher concentration is associated with higher growth.

Large Firm Acquisitions of Small Competitors' Goods:

"Entry for buyout", described in Rasmusen (1988) and more recently, Letina, Schmutzler, and Seibel (2021) and Fons-Rosen, Roldan-Blanco, and Schmitz (2022), is another theory in which a high expected value of future acquisitions increases growth even though the distributional consequences of actual acquisitions do not: firms enter and innovate to be acquired because they receive a fraction of the surplus. Yet, if large firm innovation is one-dimensional, then the entry for buyout effect needs to be weighed against the negative effect of valuable acquisition opportunities on large firm innovation. In this paper, acquisitions are made with take-it-or-leave-it offers, so the entry for buyout effect is absent. Instead, acquisitions may be useful *because* they reduce large firm innovation, not in spite of it. A distinct implication is that acquisitions have more potential to

[6] See Shapiro (2012) for a discussion, and Cavenaile, Celik, and Tian (2021) for a recent example.

increase growth if concentration is higher.

The Rise in Concentration and The Fall in Growth:
Aghion, Bergeaud, Boppart, Klenow, and Li (2022) and De Ridder (2021) study the effects of productivity dispersion across small firms, whereas I abstract from productivity differences and study how a large firm's innovation is shaped by its size. They find that increased competition from high productivity competitors reduces less productive firms' incentives to grow. The channel I study in this paper is complementary in the sense that they focus on the flow profits a small firm receives from innovating, whereas I focus on the discount rate on small firm profits. Moreover, they do not study the effects of acquisitions. Finally, in De Ridder (2021), larger firms have higher ratios of R&D to revenue rather than lower, at odds with the finding in Olmstead-Rumsey (2022), which explains the fall in growth despite a rise in R&D relative to GDP. In this paper, larger firms have lower ratios of R&D to revenue, but higher ratios of R&D to growth generated.

Liu, Mian, and Sufi (2022) study a growth model with two large firms in each industry, and find that a large firm can reduce growth by building a substantial productivity advantage over its competitor: a greater advantage implies that the large firm will optimally cut its price by more in response to innovation by its competitor. My mechanism does not rely on a large firm responding directly to the actions of a single competitor. Thus, it may be more relevant when thinking about an industry with both large and small firms. Cavenaile, Celik, and Tian (2021) study a growth model with large firms, but the pressure those large firms place on small firms has no effect on growth because small firms always have zero profits.

Akcigit and Ates (2021) and Olmstead-Rumsey (2022) propose theories in which exogenous changes in the economy's innovation technology cause a decline in growth, as well as a rise in market concentration. The theory I propose reverses the causality and suggests that changes in industry structure—a rise in concentration—drive a decline in growth. Moreover, it provides an alternative explanation for the fall in the effect of a patent on a firm's market value documented in Olmstead-Rumsey (2022): the innovation is more quickly creatively destroyed by a large firm.

My theory is complementary to previous work in the sense that it does not rely on a specific shock, but suggests that any shock that shifts innovation and sales from small firms to large firms will have the additional effect of shifting innovation toward business stealing and away from growth.

Model Building Blocks:
The model builds on two different strands of the growth literature, one focused on models of

creative destruction[7], and one on expanding varieties models[8]. Recent work combines the two, but without large firms with positive market shares.[9]

2 Qualitative Industry Model

I first study an oligopolistic growth model with a single industry in which firms can only innovate through creative destruction. The model illustrates the key mechanisms of the theory and is sufficiently tractable to yield analytical results.

2.1 Model

2.1.1 Overview

Time is continuous and indexed by $t \in [0, \infty)$. At each time t, there is a unit measure of small firms indexed by $i \in [0,1]$, and a single large firm denoted by $i = L$. They produce a unit measure of intermediate goods, indexed by $j \in [0,1]$, which they sell to a representative household with R to spend. Firms hire labor at wage W to produce and innovate, and maximize expected discounted profits with discount rate r.

This simplified model is in partial equilibrium: I take as given expenditures on each good R, the wage W, and the discount rate r.

2.1.2 Static Block

Demand:
At each time t, the household takes as given a price offered by each firm for each intermediate good, $\{p_t(i,j)\}$. All versions of a good j are perfect substitutes, so the household purchases only the cheapest one, with price $p_t(j) = \min\{p_t(i,j)\}_{i \in [0,1] \cup \{L\}}$, and splits purchases evenly if multiple versions have price $p_t(j)$. The household chooses consumption of each version of each good $\{c_t(j)\}$

[7] See Grossman and Helpman (1991b), Aghion and Howitt (1992), Klette and Kortum (2004), and Aghion, Bergeaud, Boppart, Klenow, and Li (2022).

[8] See Romer (1990) and Grossman and Helpman (1991a).

[9] See Atkeson and Burstein (2019).

to maximize a Cobb-Douglas aggregate C_t defined by

$$\ln(C_t) = \int_0^1 \ln(c_t(j))dj,$$

subject to spending R:

$$\int_0^1 p_t(j)c_t(j)dj = R.$$

It follows that the household spends R on each good: $c_t(j) = R/p_t(j)$. Consumption of each firm's output of each good, $\{c_t(i,j)\}$ is implied.

Production and Competition:

At each time t, production occurs in two stages. Each firm can potentially produce a version of each good in its industry with a version specific productivity $z_t(i,j)$. Let $z_t(j) \equiv \max\{z_t(i,j)\}_{i\in[0,1]\cup\{L\}}$ be the highest productivity version of good j at time t. Given the innovation process that I describe in the next subsection, for each j,t, there is a unique firm with $z_t(i,j) = z_t(j)$.

In the first stage of production, firms simultaneously choose for which goods they will pay a fixed cost to access their version of the good. If firm i pays the fixed cost for good j, then they can produce it in the second stage using labor with production function

$$q_t(i,j) = z_t(i,j)l_t(i,j).$$

Otherwise, they can produce good j with productivity $z_t(j)/\sigma$, where $\sigma > 1$ captures the ability of firms to imitate each other's versions of a good. The fixed cost is f_S and f_L units of labor for small firms and the large firm, respectively. In the second stage of production, fixed cost payments are common knowledge and firms simultaneously choose prices for each good to maximize static profits subject to the household's demand given prices: $q_t(i,j) \leq c_t(i,j)$.

Finally, the large firm's revenue is subject to a tax τ_R so that it is multiplied by $1 - \tau_R$.

I study equilibria in which only the most productive producer of good j pays the fixed cost, and for any first stage actions by their competitors, they set their price equal to the marginal cost of the second-most productive producer. Thus, in equilibrium, they set a markup of price over marginal cost equal to σ. These strategies form an equilibrium if and only if the most productive producer has a sufficiently low fixed cost so that they earn positive profits. Note that there are no equilibria in which any firm other than the most productive producer pays the fixed cost as long as σ is less than the innovation step size, which I define in the next subsection. This is always the case in the calibrated model.

To ensure the desired equilibrium exists, I make the following assumption.

Assumption 1. *The most productive producer of a good earns strictly positive profits:*

$$\left(1-\sigma^{-1}\right)R > max\{\tau_R R + f_L W, f_S W\}.$$

Aggregation:
Given that all goods are sold with a markup σ, household industry consumption is $C_t = Z_t R/(\sigma W)$, where industry productivity is given by

$$\ln(Z_t) = \int_0^1 \ln(z_t(j))dj.$$

2.1.3 Dynamic Block

Innovation:
Each firm contains entrepreneurs it uses to innovate. Let $J_t(i) \subseteq [0,1]$ be the set of goods for which firm i does not have the most productive version at time t. At each time t, for each of its entrepreneurs, firm i chooses a creative destruction rate $\kappa_t(i,j)$ for each good $j \in J_t(i)$. Each of its entrepreneurs creatively destroys a good at Poisson arrival rate $\int_{J_t(i)} \kappa_t(i,j)dj$, and the relative probability it creatively destroys good j is proportional to $\kappa_t(i,j)$. If a firm creatively destroys a good, then the productivity of its version of the good jumps to $\lambda z_t(j)$, where $\lambda > 1$ is the innovation step size, i.e., the firm becomes the sole producer of the good.

A small firm contains a single entrepreneur, creatively destroys goods at a Poisson arrival rate, and in equilibrium produces a finite number of goods. A small firm's flow labor cost of innovation is $\int_{J_t(i)} X_S(\kappa_t(i,j))dj$, where $X_S(\cdot)$ is twice continuously differentiable, strictly increasing, and weakly convex, with $X_S(0) = 0$. Moreover, the elasticity of the marginal innovation cost,

$$\epsilon_S(\kappa) \equiv \frac{\kappa X_S''(\kappa)}{X_S'(\kappa)},$$

is well-defined and continuous for all $\kappa \geq 0$.

The large firm contains a measure of entrepreneurs, creatively destroys *each good* at a Poisson arrival rate, and so creatively destroys goods at a continuous rate, and in equilibrium produces a finite *measure* of goods. If the large firm creatively destroys each good j at Poisson arrival rate $\kappa_t(L,j)$, then its flow labor cost of innovation is $\beta \int_{J_t(L)} X_L(\kappa_t(L,j))dj$, where $\beta > 0$ is a cost shifter, and $X_L(\cdot)$ is twice continuously differentiable, strictly increasing, and weakly convex, with $X_L(0) = 0$. Moreover, the elasticity of the marginal innovation cost,

$$\epsilon_L(\kappa) \equiv \frac{\kappa X_L''(\kappa)}{X_L'(\kappa)},$$

is well-defined and continuous for all $\kappa \geq 0$.

Acquisitions:
At each time t, the large firm chooses an acquisition rate $A_t(j)$ for each good $j \in J_t(i)$. At Poisson arrival rate $A_t(j)$, the large firm encounters the small firm with the most productive version of good j. The firms play a two stage game in which the large firm makes a take-it-or-leave-it offer subject to an acquisition tax rate τ_A, and the small firm chooses whether to accept. If the small firm accepts, then its version is transferred to the large firm, the large firm pays the offered price to the small firm, and pays $\tau_A > -1$ times the price to the tax authority. If the surplus of an acquisition is strictly positive, then there is a unique equilibrium in which the large firm offers the small firm's value, and the small firm accepts. If the opposite holds, then in all equilibria, there is no acquisition. Finally, if the surplus is zero, then I suppose there is no acquisition; this choice does not affect the results that follow.

The flow labor cost of searching for potential acquisitions is $\int_{J_t(i)} X_A(A_t(j))dj$, where $X_A(\cdot)$ is twice continuously differentiable, strictly increasing, and weakly convex, with $X_A(0) = 0$. Moreover, the elasticity of the marginal search cost,

$$\epsilon_A(A) \equiv \frac{A X_A''(A)}{X_A'(A)},$$

is well-defined and continuous for all $A \geq 0$.

Equilibrium:
At each time t, firms choose fixed cost payments and prices to maximize static profits in a subgame perfect Nash equilibrium of the production game. The large firm chooses innovation rates, acquisition search rates, and acquisition price offers, and small firms choose innovation rates and acquisition price acceptance strategies to maximize expected present discounted profits. I focus on Markov perfect equilibria in which all small firms creatively destroy all goods at a constant common rate κ_S, the large firm creatively destroys all goods at a constant common rate κ_L, and the large firm searches for all acquisition opportunities at a constant common rate A. Moreover, if $A > 0$, then the large firm makes an offer the small firm accepts with probability 1.

Denote by $\mathcal{L}_t \in [0, 1]$ the measure of goods for which the large firm has the most productive version at time t, which is also the large firm's share of industry revenue. The large firm revenue share evolves over time according to

$$\dot{\mathcal{L}}_t = (\kappa_L + A)(1 - \mathcal{L}_t) - \kappa_S \mathcal{L}_t, \tag{1}$$

where a dot over a variable indicates the derivative with respect to time t.

The growth rate of industry output, $\dot{C}_t/C_t$, is the growth rate of industry productivity:

$$g_t \equiv \dot{Z}_t/Z_t = (\lambda - 1)((1 - \mathcal{L}_t)\kappa_L + \kappa_S). \tag{2}$$

The growth rate reflects that the large firm potentially creatively destroys only a fraction of the goods in the economy, whereas each small firm potentially creatively destroys all but a set with measure zero of the goods.

I focus on balanced growth path equilibria in which the large firm revenue share, and thus the growth rate, are constant over time. In that case,

$$\mathcal{L} = (\kappa_L + A)/(\kappa_L + A + \kappa_S). \tag{3}$$

2.1.4 Firm Optimization

Small Firm Innovation:
The value of a small firm currently producing n goods is the sum of two components: n times the expected present discounted profits from producing a single good, Π, and the value of an entrepreneur's innovation technology at a small firm producing zero goods. Only Π is relevant to a small firm's innovation decision, and it is given by the Hamilton-Jacobi-Bellman (HJB) equation:

$$r\Pi = \left(1 - \sigma^{-1}\right) R - f_S W - (\kappa_L + \kappa_S)\Pi, \tag{4}$$

where the terms on the right-hand side are flow profits and the rate at which the good is creatively destroyed. The possibility of an acquisition does not affect a small firm's value because it receives exactly its value of the good.

At each time t, a small firm chooses innovation rates $\{\kappa(j)\}$ to maximize

$$\int_0^1 (\kappa(j)\Pi - W X_S(\kappa(j)))dj.$$

The First Order Condition for each $\kappa(j)$ shows that a small firm creatively destroys each good at the same rate κ_S given by

$$X_S'(\kappa_S) \geq \Pi/W, \tag{5}$$

where the inequality is an equality if $\kappa_S > 0$, and the expected present discounted profits from producing a single good are

$$\Pi = \frac{\left(1-\sigma^{-1}\right)R - f_S W}{r + \kappa_L + \kappa_S}. \tag{6}$$

To allow for the possibility of an equilibrium with $\kappa_S > 0$, I make the following assumption.

Assumption 2. *If innovation rates are zero, small firms find it profitable to innovate:*

$$X_S'(0) < \frac{\left(1-\sigma^{-1}\right)R - f_S W}{r}.$$

Large Firm Innovation and Acquisitions:
I split the large firm optimization problem into two steps. First, taking as given a rate at which its revenue share increases, $\partial\mathcal{L}_t/\partial t$, the large firm chooses $\kappa(j)$, $A(j)$, and $\tilde{A}(j)$ for all $j \in J_t(L)$ to minimize cost, where $\tilde{A}(j)$ is the probability, conditional on finding a potential acquisition, the large firm offers Π and acquires the good; otherwise, the large firm makes an offer that is rejected. The large firm optimally creatively destroys and searches for all its competitors' goods at single rates κ and A, and sets $\tilde{A}(j) = 1$ if $A(j) > 0$ because $X_L(\cdot)$ and $X_A(\cdot)$ are strictly increasing and convex. Second, the large firm optimally chooses κ and A. The HJB equation for the large firm's value of producing a fraction $\mathcal{L}_t$ of the industry goods is

$$\begin{aligned} rV(\mathcal{L}_t) = &\left(\left(1-\sigma^{-1}-\tau_R\right)R - f_L W\right)\mathcal{L}_t \\ &+ \max_{\kappa, A}\left\{\dot{\mathcal{L}}_t(\kappa, A)V'(\mathcal{L}_t) - W\beta X_L(\kappa) - W X_A(A) - A(1+\tau_A)\Pi\right\}, \end{aligned} \tag{7}$$

where $\mathcal{L}_t(\cdot,\cdot)$ is the rate at which $\mathcal{L}_t$ increases as a function of the large firm's actions, taking as given small firm innovation. The terms on the right-hand side of the HJB equation are flow profits and the innovation/acquisition optimization problem.

Taking as given creative destruction and acquisition rates that do not depend on $\mathcal{L}_t$, κ_L and A, I guess and verify that the marginal value of a good does not either. Setting $V'(\mathcal{L}_t) = \bar{V}$ and differentiating each side of equation (7) yields

$$\bar{V} = \frac{\left(1-\sigma^{-1}-\tau_R\right)R - f_L W + W\beta X_L(\kappa_L) + W X_A(A) + A(1+\tau_A)\Pi}{r + \kappa_L + A + \kappa_S}. \tag{8}$$

The First Order Conditions for κ and A show that the large firm creatively destroys and acquires each competitor's good at rates κ_L and A, respectively, which do not depend on $\mathcal{L}_t$ and are given by

$$\beta X_L'(\kappa_L) \geq \bar{V}/W \tag{9}$$

which holds with equality if $\kappa_L > 0$, and

$$X'_A(A) \geq (\bar{V} - (1 + \tau_A)\Pi)/W, \tag{10}$$

which holds with equality if $A > 0$.

2.2 Results

I characterize the effects of various shocks or policy changes on the long-run growth rate. I begin with changes in parameters that only directly affect large firm innovation incentives—the large firm's innovation cost shifter β, fixed cost f_L, and revenue tax rate τ_R. I then consider changes in the acquisition tax rate τ_A.

Some results depend on taking derivatives of equilibrium outcomes with respect to exogenous parameters. Thus, I begin with the following proposition concerning the uniqueness of equilibria.

Proposition 1. *Small firm optimization—inequality* (5) *and equation* (6)*—implicitly define a continuous decreasing function $\kappa_S(\kappa_L)$ that does not depend on large firm variables β, $X_L(\cdot)$, f_L, or τ_R. There is a $\kappa_L^* > 0$, possibly infinite, such that $\kappa_S(\cdot)$ is continuously differentiable on $[0, \kappa_L^*)$, and $\kappa_S(\kappa_L) > 0$ if and only if $\kappa_L < \kappa_L^*$.*

Suppose τ_A is sufficiently large so that the acquisition rate A is always zero, $X_L(\cdot)$ is strictly convex, and for all κ_L, $\epsilon_L(\kappa_L)(\epsilon_S(\kappa_S(\kappa_L)) + 1) \geq 1$. There is a unique equilibrium κ_L, which is a continuous increasing function of $\left(\left(1 - \sigma^{-1} - \tau_R\right) R - f_L W\right)/\beta$ that is strictly increasing if $\kappa_L > 0$, and is continuously differentiable everywhere except at $\kappa_L = \kappa_L^*$.

2.2.1 Large Firm Innovation and Growth

I study the long-run effects of shocks and policy changes that only directly affect large firm innovation incentives: the large firm's innovation cost shifter, β, fixed cost, f_L, and revenue tax rate τ_R. Throughout, I suppose the acquisition tax rate τ_A and search cost function $X_A(\cdot)$ are such that $A = 0$ is optimal, i.e., there are no acquisitions. Using the first part of Proposition 1, one way to understand the effects of changes in large firm variables is to study the effects of exogenous changes in κ_L on small firm optimal innovation. For example, if an increase in κ_L leads to a higher long-run growth rate g taking into account small firm optimization, and if a combination of changes to β, f_L, and τ_R leads to an equilibrium with a higher κ_L, then that equilibrium has a higher g.

I now characterize the effect of large firm innovation on the long-run growth rate. I focus on $\kappa_L < \kappa_L^*$, defined in Proposition 1, so that $\kappa_S(\kappa_L)$ is continuously differentiable, which implies that the derivatives I take throughout are well-defined. In Theorem 1, I cover the case $\kappa_L \geq \kappa_L^*$ (for finite κ_L^*) as well.

To build intuition, I write the long-run growth rate as the product of two terms:

$$g = \underbrace{\bar{\kappa}}_{\text{Discount}} \underbrace{\frac{g}{\bar{\kappa}}}_{\text{Composition}}, \tag{11}$$

where the first term, $\bar{\kappa} \equiv \kappa_L + \kappa_S$, is the rate at which a small firm's good is creatively destroyed, which is the non-interest component of the discount rate on a small firm's good. If small firms can tolerate a higher discount rate, then there is more innovation and growth. The second term is the growth rate relative to the discount rate, or the discount rate's "composition", which depends only on relative innovation rates:[10] $g/\bar{\kappa} = (\lambda - 1)(1 - \mathcal{L}^2)$. Large firm innovation produces little growth relative to its effect on the discount rate because it only targets small firms' goods.

I use equation (11) to decompose the effect of a change in large firm innovation:

$$\frac{\partial g}{\partial \kappa_L} = \underbrace{\frac{\partial \bar{\kappa}}{\partial \kappa_L}\frac{g}{\bar{\kappa}}}_{\text{Discount Effect}} + \underbrace{\bar{\kappa}\frac{\partial (g/\bar{\kappa})}{\partial \kappa_L}}_{\text{Composition Effect}}.$$

The "Discount Effect" is positive.[11] An increase in large firm innovation increases the small firm discount rate, and so the growth rate, because small firms face convex innovation costs. The "Composition Effect" is negative.[12] An increase in large firm innovation shifts the source of creative destruction to the large firm, which in the long-run increases the large firm's share of industry goods. Each reduces the growth rate because large firm innovation only targets small firm goods, whereas small firm innovation targets all goods.

Combining the two effects, $\partial g/\partial \kappa_L$ is the product of a strictly positive function and[13]

$$\epsilon_S(\kappa_S) - \frac{2\mathcal{L}}{1 - \mathcal{L}}\frac{\bar{\kappa}}{r + \bar{\kappa}}, \tag{12}$$

which is therefore a sufficient statistic based on measurable equilibrium outcomes for whether an increase in large firm innovation leads to an increase in the long-run growth rate. In particular,

[10] Divide equation (2) by $\bar{\kappa}$, and use equation (3) to replace $\kappa_L/\bar{\kappa}$ with $\mathcal{L}$ and $\kappa_S/\bar{\kappa}$ with $1 - \mathcal{L}$.

[11] This follows from equation (26) in the proof of Proposition 1 in the appendix, which shows that $\kappa_S'(\kappa_L) \geq -1$, and strictly so if $\epsilon_S(\kappa_S) > 0$.

[12] This holds because $\kappa_S(\kappa_L)$ is decreasing.

[13] Use $\kappa_S'(\kappa_L)$ given in equation (26) in the proof of Proposition 1 in the appendix to take the derivatives of $\mathcal{L}$ and $\bar{\kappa}$ with respect to κ_L. The strictly positive function multiplying expression (12) is $(\lambda - 1)(1 - \mathcal{L})^2 \frac{r+\bar{\kappa}}{\epsilon_S(\kappa_S)(r+\bar{\kappa})+\kappa_S}$.

$\epsilon_S(\kappa_S)$ is the *inverse elasticity of small firm innovation with respect to the value of innovating.* If the elasticity is sufficiently high, then an increase in large firm innovation leads to lower growth; a higher elasticity means an increase in large firm innovation leads to a bigger decrease in small firm innovation, which implies a weaker discount effect and a stronger composition effect. In one limit, if the elasticity is zero, then small firm innovation is fixed, so an increase in large firm innovation simply implies more innovation and always leads to higher growth. In the other limit, if the elasticity if infinity, then an increase in large firm innovation simply replaces small firm innovation (the discount effect is zero), and always leads to lower growth.

The following theorem characterizes the relationship between large firm innovation and the effect of a marginal increase in large firm innovation. If variation in large firm innovation drives variation across industries, economies, or over time, then the theorem also characterizes the relationship between concentration and growth.

Theorem 1. *Vary κ_L and use $\kappa_S(\kappa_L)$ defined in Proposition 1 to determine small firm innovation.*

g is a continuous function of κ_L that is continuously differentiable everywhere except at κ_L^* with the following properties:

1. *$g > 0$ if and only if $\kappa_L < \kappa_L^*$.*
2. *$\partial g/\partial \kappa_L \geq 0$ at $\kappa_L = 0$, and strictly so if and only if $\epsilon_S(\kappa_S(0)) > 0$.*
3. *There exists a $\kappa_L' < \kappa_L^*$ such that for all $\kappa_L \in (\kappa_L', \kappa_L^*)$, $\partial g/\partial \kappa_L < 0$.*
4. *If $\epsilon_S(\cdot)$ is constant, then there is a single $\kappa_L < \kappa_L^*$ at which $\partial g/\partial \kappa_L = 0$.*

The first three properties of g, taken together, state that as large firm innovation starts at 0 and increases, initially the long-run growth rate increases, but then eventually decreases and goes to 0 as the large firm takes over the industry. The final property of g states that if the elasticity of small firm innovation is constant, then this pattern fully characterizes the relationship between large firm innovation and growth in the sense that g exhibits an inverse-U shape.

Intuitively, if the large firm produces an insignificant share of industry goods, then it innovates like a small firm and targets all goods, so the composition effect is zero. In the other limit, if the large firm produces all the industry goods, then the growth rate is zero because the large firm maintains its dominance by creatively destroying small firm goods at a high rate, which deters small firm innovation, and implies that in equilibrium, there are no small firm goods to creatively destroy.

Theorem 1 has normative and positive implications. Suppose variation in large firm innovation drives variation in concentration and growth. To maximize the long-run growth rate, a policymaker should discourage innovation by sufficiently large firms. If a small firm free entry condition fixes the value of a small firm innovation ($\epsilon_S^{-1} = \infty$), then higher concentration is associated with lower growth. If small firm innovation is less elastic ($\epsilon_S^{-1} < \infty$), then growth as a function of concentration has an inverted-U shape.

Finally, the following proposition maps the results thus far more precisely to the effects of varying the large firm innovation cost shifter, fixed cost, or revenue tax rate.

2.2.2 Acquisition Policy

Proposition 2. *Suppose $X_L(\cdot)$ and $X_A(\cdot)$ are strictly convex and for all κ_L, $\epsilon_L(\kappa_L)(1+\epsilon_S(\kappa_S(\kappa_L)) \geq 1$. There is a unique equilibrium, and κ_L, A, and g are continuous functions of τ_A with the following properties:*

1. *κ_L is increasing in τ_A and strictly so if $\kappa_L, A > 0$.*
2. *A is decreasing in τ_A and strictly so if $A > 0$.*
3. *If $\kappa_L, \kappa_S, A > 0$, then κ_L, A, and g are continuously differentiable with respect to τ_A.*

If $A = 0$ (and not at the boundary), then changes in the acquisition tax rate have no effect. If $\kappa_L = 0$, then the large firm has no effect on growth, which is $(\lambda - 1)\kappa_S(0)$. If $\kappa_S = 0$, then growth is 0. Thus, I focus on the case in which $\kappa_S, \kappa_L, A > 0$, so that changes in the acquisition tax rate can have an effect on the long-run growth rate.

I study the long-run effects of changes to the acquisition tax rate τ_A. To guarantee a unique equilibrium and that large firm decisions are continuously differentiable with respect to relevant exogenous parameters, I use Proposition 1 and impose throughout that $X_L(\cdot)$ and $X_A(\cdot)$ are strictly convex and for all κ_L, $\epsilon_L(\kappa_L)(1 + \epsilon_S(\kappa_S(\kappa_L))) \geq 1$. I decompose the effect of a change in the tax rate on the long-run growth rate (see Appendix BLANK for derivations):

$$\frac{\partial g}{\partial \tau_A} = \underbrace{\frac{\partial \left(\bar{V} - (1+\tau_A)\Pi\right)}{\partial \tau_A} \frac{\partial A}{\partial \left(\bar{V} - (1+\tau_A)\Pi\right)} \frac{\partial g}{\partial A}}_{\text{Acquisition Effect}} + \underbrace{\frac{\partial \bar{V}}{\partial \tau_A} \frac{\partial \kappa_L}{\partial \bar{V}} \frac{\partial g}{\partial \kappa_L}}_{\text{Innovation Effect}},$$

where $\partial g / \partial A$ is taken holding innovation rates fixed, and $\partial g / \partial \kappa_L$ is taken holding the acquisition rate fixed, but taking into account optimal small firm innovation $\kappa_S(\kappa_L)$. In each term, the first derivative is the equilibrium response of the value of acquiring or innovating, the second derivative is the response of the acquisition or innovation rate to the value of acquiring or innovating determined by the First Order Condition for each (equations (9) and (10)), and the third derivative is the effect on growth of changes in the acquisition or innovation rate.

The "Acquisition Effect" is positive. An increase in the tax rate decreases the surplus from each acquisition, which decreases the acquisition rate, reduces the large firm's share of industry goods, and increases the long-run growth rate. The sign of the "Innovation Effect" is the sign of $\partial g/\partial\kappa_L$. The decrease in the surplus from acquisitions leads to an increase in the marginal value of a good, and so in large firm innovation because the large firm's desire to creatively destroy its competitors' goods depends on the outside option, which is acquiring those goods instead. Thus, *encouraging acquisitions is good for growth exactly when large firm innovation is bad for growth.*

The large firm may be worse off following a decrease in the acquisition tax rate. It wants to commit to faster innovation to discourage small firm innovation, but cannot. If the acquisition tax rate is lower, then small firms innovate more because they anticipate that the large firm will optimally innovate less.

If $\kappa_L, \kappa_S(\kappa_L), A > 0$ and $\epsilon_S(\kappa_S) = 0$, then $\partial g/\partial\tau_A$ is the product of a strictly positive function and

$$\frac{1}{\epsilon_A(A)} - \frac{1}{1-\mathcal{L}}\frac{2\kappa_L + A}{\epsilon_L\left(r+\bar{\kappa}\right) - \kappa_L}\left(1 + \frac{(1+\tau_A)\Pi}{\bar{V} - (1+\tau_A)\Pi}\right)^{-1}, \tag{13}$$

which is therefore a sufficient statistic based on measurable equilibrium outcomes for whether an increase in the acquisition tax rate leads to an increase in the long-run growth rate. In particular, $\epsilon_L(\kappa_L)$ and $\epsilon_A(A)$ are the *inverse elasticities of large firm innovation with respect to the value of innovating and of the acquisition rate with respect to the value from an acquisition*, respectively. If the elasticity of large firm innovation is sufficiently high or the elasticity of the acquisition rate is sufficiently low, then an increase in the tax rate reduces growth; a higher innovation elasticity implies a stronger innovation effect, and a lower acquisition elasticity implies a weaker acquisition effect. For example, as the acquisition elasticity goes to 0, an increase in the acquisition tax rate always reduces growth because it has no effect on the acquisition rate, but lowers the value from acquisitions, which reduces the large firm's outside option to innovating, and results in more large firm innovation. In the appendix, I develop the more complicated analog to the expression in (13) in the general $\epsilon_S(\kappa_S)$ case. The intuition and results concerning $\epsilon_A(A)$ and $\epsilon_L(\kappa_L)$ are the same, and a lower $\epsilon_S(\kappa_S)$ lowers the threshold for $\epsilon_A(A)$, i.e., makes it easier for it to be the case that encouraging acquisitions increases growth.

The following theorem, similar to Theorem 1 supposes that variation in large firm innovation drives variation in industry concentration, and characterizes the relationship between concentration and the effect of an increase in the acquisition tax rate.

Theorem 2. *Suppose $\epsilon_S(\cdot)$, $\epsilon_L(\cdot)$, and $\epsilon_A(\cdot)$ are constants, with $\epsilon_S = 0$. Vary the large firm fixed cost f_L and revenue tax rate τ_R, holding other parameters fixed. There exists a long-run large firm revenue share cutoff $\mathcal{L}^*$ such that the acquisition rate A is strictly positive if and only if $\mathcal{L} > \mathcal{L}^*$. If $\mathcal{L}^* < 1$, then there exists a cutoff $\mathcal{L}^{**} \in (max\{\mathcal{L}^*, 0\}, 1)$ such that $\partial g/\partial \tau_A > 0$ if $\mathcal{L} < \bar{\mathcal{L}}$ and $\partial g/\partial \tau_A < 0$ if $\mathcal{L} > \bar{\mathcal{L}}$.*

Intuitively, if the large firm's share of the industry goods is high, then its innovation generates particularly little growth relative to its effect on the small firm discount rate. It is thus particularly valuable to encourage the large firm to acquire its competitors' goods instead of creatively destroying them, which shifts innovation to small firms.

3 Quantitative Model

I now develop the richer quantitative model. There is a unit measure of industries, indexed by $n \in [0, 1]$, each of which consists of a measure of differentiated intermediate goods, indexed by $j \in [0, M_{n,t}]$. There is a representative household who consumes the intermediate goods in each industry, and inelastically supplies $\bar{L}$ units of labor. In each industry, a single large firm and a continuum of small firms use labor to produce, develop new goods, and improve on old goods. The household owns all firms and has access to a risk-free bond in zero net supply.

The wage, interest rate, and expenditures on goods are determined in general equilibrium.

3.1 Representative Household Problem

The household chooses a path of consumption bundles to maximize the present discounted value of its utility, taking prices as given. The household problem can be split into two steps, where first the household chooses final good consumption to maximize the present discounted value of its utility, and then at each time t, chooses a consumption bundle to minimize cost subject to achieving the specified level of final good consumption. Thus, in the first stage, the household maximizes

$$\int_0^\infty e^{-\rho t} \ln(C_t) dt,$$

subject to the budget constraint

$$P_t C_t + \dot{B}_t = W_t \bar{L} + D_t + r_t B_t,$$

where $\rho > 0$ is the time discount rate, C_t is final good consumption, W_t is the wage, D_t is flow profits from firms, B_t is bond holdings, r_t is the net interest rate, and P_t is the final good price that the household anticipates as the outcome of the second stage of its optimization problem. In the second stage, at each time t, the household chooses consumption of each good in each industry to minimize cost

$$\int_0^1 \int_0^{M_{n,t}} p_{n,t}(j) c_{n,t}(j) dj dn$$

subject to the aggregation functions

$$ln(C_t) = \int_0^1 ln(C_{n,t}) dn \qquad C_{n,t}^{\frac{\gamma-1}{\gamma}} = \int_0^{M_{n,t}} c_{n,t}(j)^{\frac{\gamma-1}{\gamma}} dj \quad \text{for all } n \in [0,1],$$

where $c_{n,t}(j)$ is consumption of good j in industry n, $p_{n,t}(j)$ is that good's price, and $\gamma > 1$ is the elasticity of substitution across goods within an industry. Thus, the final good is a Cobb-Douglass aggregate of industry goods, each of which is a Constant Elasticity of Substitution aggregate of the differentiated intermediate goods within the industry.

3.2 Household Optimization and Demand

From the first stage of the household problem, the stochastic discount factor is $e^{-\rho t}/C_t$. For the bond market to clear with zero net supply, i.e., $B_t = 0$ for all t, the net interest rate must equal the negative rate of change of the stochastic discount factor over time:

$$r_t = \rho + \dot{C}_t / C_t. \tag{14}$$

The second stage of the household problem yields the demand curve for each intermediate good at each time t. The First Order Condition for good j in industry n, along with aggregation, implies the demand curve[14]

$$c_{n,t}(j) = p_{n,t}(j)^{-\gamma} P_{n,t}^{\gamma-1} P_t C_t, \tag{15}$$

[14]The First Order Condition yields $p_{n,t}(j) = \zeta C_{n,t}^{\frac{1-\gamma}{\gamma}} c_{n,t}(j)^{\frac{-1}{\gamma}}$, where ζ is the Lagrange multiplier on the constraint that final good consumption equals C_t. Industry aggregation and setting aggregate expenditures to $P_t C_t$ implies that $\zeta = P_t C_t$. Industry aggregation then implies that $C_{n,t} = P_t C_t / P_{n,t}$.

where the industry and final good price indices are given by

$$ln(P_t) = \int_0^1 ln(P_{n,t})dn \qquad P_{n,t}^{1-\gamma} \equiv \int_0^{M_{n,t}} p_{n,t}(j)^{1-\gamma}dj \quad \text{for all } n \in [0,1]. \tag{16}$$

3.3 Intermediate Goods Producers

3.3.1 Production and Competition

Each industry consists of a measure of small firms and a single large firm. Static production and competition are as in Section 2.

Let $Z_{n,t}$ be an aggregate of productivity in industry n:

$$Z_{n,t}^{\gamma-1} \equiv \int_0^{M_{n,t}} z_{n,t}(j)^{\gamma-1}dj,$$

and define the relative productivity of firm i's version and the most productive version of a good:

$$\tilde{z}_{n,t}(i,j) \equiv (z_{n,t}(i,j)/Z_{n,t})^{\gamma-1} \qquad \tilde{z}_{n,t}(j) \equiv (z_{n,t}(j)/Z_{n,t})^{\gamma-1}.$$

The difference from Section 2 is that the fixed cost a firm must pay for a good scales with its version's relative productivity, and the large firm's fixed cost varies across industries: in units of labor, the fixed cost for a small firm is $f_S\tilde{z}_{n,t}(i,j)$, and for a large firm is $f_{L,n}\tilde{z}_{n,t}(i,j)$.

I make the following assumption, which implies, as I discuss in Section 4.1, that a firm's price for a good is always constrained by the ability of its competitors to imitate its version, and that in equilibrium all firms set a markup of σ.

Assumption 3. $\sigma \leq \gamma/(\gamma-1)$.

Moreover, Assumption 1 no longer applies, but I suppose f_S and f_L are sufficiently low so that there is an equilibrium in which only the most productive producer of a good pays the fixed cost.[15]

[15] In practice, it is sufficient to impose this is the case on the initial calibrated balanced growth path because I consider shocks that raise expenditures on goods, and so the return to paying the fixed cost.

3.3.2 Innovation

As in Section 2, each firm contains entrepreneurs it uses to innovate. Now, at each time t, each firm chooses a creative destruction rate for all goods in its industry—not just those for which it does not currently have the most productive version—and a new good development rate. Conditional on developing a new good, a firm's productivity z for that good is drawn so that the expected value of $z^{\gamma-1}$ is equal to $Z_{n,t}^{\gamma-1}$, i.e., the expected relative productivity of the good is 1. All other firms' versions of a new good have productivity zero.

Each small firm contains a single entrepreneur. It chooses a creative destruction rate for each good in the industry $\{\kappa(j)\}$ and a new good development rate δ. It creatively destroys a good at Poisson arrival rate $\int_0^{M_{n,t}} \kappa(j)dj$, where the relative probability it creatively destroys good j is proportional to $\kappa(j)$, and develops a new good at Poisson arrival rate δ. The flow labor cost is

$$(\epsilon+1)^{-1}\left(\alpha\int_0^{M_{n,t}} \tilde{z}_{n,t}(j)\kappa(j)^{\epsilon+1}dj+\delta^{\epsilon+1}\right),$$

where $\epsilon > 0$ is the marginal innovation cost elasticity from the simple model (now constant), and $\alpha > 0$ is the relative cost of creative destruction. The costs of each type of innovation is independent and scales with the expected relative productivity of the innovation.

The large firm in industry n, contains a measure $\chi_n^{-1/\epsilon}$ of entrepreneurs. It chooses a Poisson arrival rate at which it creatively destroys *each* good in its industry, $\kappa(j)$, and a *continuous rate* at which it develops a new good, δ. The flow labor cost is

$$(\epsilon+1)^{-1}\chi_n\left(\alpha\int_0^{M_{n,t}} \tilde{z}_{n,t}(j)\kappa(j)^{\epsilon+1}dj+\delta^{\epsilon+1}\right),$$

which takes as given the cost minimizing distribution of innovation across the firm's entrepreneurs.

To be clear, a firm can creatively destroy a good for which it already has the most productive version. For a small firm, this possibility is irrelevant because it produces finitely many goods and creatively destroys each at an infinitesimal rate. For a large firm, this possibility is meaningful, and implies that unlike in the simple model, the innovative capacity of the economy is not mechanically reduced as the large firm's share of goods grows. Nonetheless, a large firm will endogenously choose a slower creative destruction rate for its own goods than for its competitors'.

3.3.3 Acquisitions

Acquisitions are as in Section 2. A large firm can only acquire goods within its industry.

3.3.4 Entry and Exit

The measure of small firm entrepreneurs is no longer exogenous. At each time t, there is an infinite mass of potential entrants that face an entry cost of $\xi > 0$ units of labor. Each one that pays the cost then draws an industry from the uniform distribution and enters as a single small firms with a 0 productivity version of each good and an entrepreneur they can use to innovate. Each one that does not pay the cost receives value 0.

Each small firm entrepreneur exits exogenously at Poisson arrival rate $\eta > 0$. After losing its entrepreneur, a small firm is still able to produce, but can no longer innovate. Thus, the measure of small firm entrepreneurs in an industry, which is relevant, is not equal to the measure of small firms producing in equilibrium, which is not relevant.

The measure of small firm entrepreneurs at time t, N_t, is the same in each industry because entry is *undirected*. It evolves over time according to

$$\dot{N}_t = e_t - \eta N_t,$$

where e_t is the entry rate.

3.4 Equilibrium

At each time t, firms choose fixed cost payments and prices to maximize static profits in a subgame perfect Nash equilibrium of the two stage production game. The large firm chooses innovation rates, acquisition search rates, and acquisition price offers, small firms choose innovation rates and acquisition price acceptance strategies, and potential entrants choose entry strategies to maximize expected present discounted profits using the interest rate to discount future payoffs. In the two stage production game, I focus on the equilibrium in which for each good, only the firm with the most productive version pays the fixed cost, and they set a markup of price over marginal cost equal to σ. In the dynamic game, I focus on Markov perfect equilibria in which the industry state is the large firm's share of industry relative productivity:

$$\mathcal{L}_{n,t} \equiv \int_{j \notin J_{n,t}(L)} \tilde{z}_{n,t}(j)dj,$$

which is also the large firm's share of industry revenue in equilibrium, and the aggregate state is the measure of small firm entrepreneurs N_t, the distribution of industry states, and aggregate productivity Z_t. Firm actions can be a function of the industry and aggregate state, *other than* Z_t. Moreover, in industry n, all small firms creatively destroy all goods at the same rate $\kappa_{n,t}(S;\mathcal{L}_{n,t})$, develop new goods at the same rate $\delta_{n,t}(S;\mathcal{L}_{n,t})$, and accept acquisition offers weakly above their values; and the large firm creatively destroys all its competitors' goods at the same rate $\kappa_{n,t}(L,S;\mathcal{L}_{n,t})$, creatively destroys all its own goods at the same rate $\kappa_{n,t}(L,L;\mathcal{L}_{n,t})$, develops new goods at rate $\delta_{n,t}(L;\mathcal{L}_{n,t})$, and searches for each acquisition opportunity at the same rate $A_{n,t}(\mathcal{L}_{n,t})$—and always offers the small firm's value of the good, which is accepted. In each case, the t subscript captures the dependence on the aggregate state other than Z_t.

At each t, the goods market must clear, i.e., the amount each firm supplies of each good is equal to the representative household's demand for that good, and the labor market must clear, i.e., the labor used for production, fixed costs, innovation, acquisition search, and entry costs must equal the labor the household inelastically supplies.

I focus on balanced growth path equilibria and the convergence to a balanced growth path following unanticipated shocks. A balanced growth path is an equilibrium in which the industry state in each industry, $\mathcal{L}_{n,t}$, and the aggregate state Γ_t are constant over time.

3.5 Evolution of the Industry State and Industry Growth

In equilibrium, the industry state evolves over time according to

$$\begin{aligned}\dot{\mathcal{L}}_{n,t} =& (\kappa_{n,t}(L,S;\mathcal{L}_{n,t}) + (\gamma-1)g_{n,t}(L;\mathcal{L}_{n,t}) + A_{n,t}(\mathcal{L}_{n,t}))(1-\mathcal{L}_{n,t}) \\ &- N_t(\kappa_{n,t}(S;\mathcal{L}_{n,t}) + (\gamma-1)g_{n,t}(S;\mathcal{L}_{n,t}))\mathcal{L}_{n,t}, \end{aligned} \tag{17}$$

where $g_{n,t}(\mathcal{L}_{n,t}) \equiv \dot{Z}_{n,t}/Z_{n,t}$ is the growth rate of industry productivity, which is the sum of growth due to large firm innovation, $g_{n,t}(L;\mathcal{L}_{n,t})$, and due to small firm innovation, $N_t g_{n,t}(S;\mathcal{L}_{n,t})$:

$$\begin{aligned}(\gamma-1)g_{n,t}(L;\mathcal{L}_{n,t}) &= \left(\lambda^{\gamma-1}-1\right)(\kappa_{n,t}(L,S;\mathcal{L}_{n,t})(1-\mathcal{L}_{n,t}) + \kappa_{n,t}(L,L;\mathcal{L}_{n,t})\mathcal{L}_{n,t}) + \delta_{n,t}(L;\mathcal{L}_{n,t}) \\ (\gamma-1)g_{n,t}(S;\mathcal{L}_{n,t}) &= \left(\lambda^{\gamma-1}-1\right)\kappa_{n,t}(S;\mathcal{L}_{n,t}) + \delta_{n,t}(S;\mathcal{L}_{n,t}).\end{aligned} \tag{18}$$

The difference between equation (17) and equation (1) in Section 2 is that now growth affects the distribution of sales. If the large firm generates novel productivity either through developing a new good or creatively destroying an old one, it takes relative productivity from all the old goods in the industry, a fraction $1 - \mathcal{L}_{n,t}$ of which comes from small firms. Equation (18) reflects that when a firm creatively destroys a good, only the improvement contributes to growth.

3.6 Aggregation and Welfare

At each time t, all industries use the same quantity of labor in production, L_t^p, and industry n output is $C_{n,t} = Z_{n,t} L_t^p$.[16] On a balanced growth path, L_t^p is constant over time, and the growth rate of industry n output is g_n. Aggregating across industries implies that aggregate output is $C_t = Z_t L_t^p$, where aggregate productivity is given by

$$ln(Z_t) = \int_0^1 ln(Z_{n,t}) dn.$$

The growth rate of aggregate productivity is $g_t = \int_0^1 g_{n,t} dn$, which is the growth rate of aggregate output on a balanced growth path. The real wage is $W_t/P_t = Z_t/\sigma$.[17]

Household welfare is

$$\int_0^\infty e^{-\rho t} \left(ln(Z_t) + ln\left(L_t^p\right) \right) dt = \frac{ln(Z_0)}{\rho} + \frac{\int_0^\infty \rho e^{-\rho t} g_t dt}{\rho^2} + \frac{\int_0^\infty \rho e^{-\rho t} \ln\left(L_t^p\right) dt}{\rho}, \tag{19}$$

which depends on current productivity and weighted averages of future growth and labor used in production. Since growth in one period raises consumption in all future periods, it is discounted by ρ^2 rather than ρ.

On a balanced growth path, the labor used in production and the growth rate are constant at L^p and g, respectively. Welfare is $ln(Z_0)/\rho + g/(\rho^2) + ln(L^p)/\rho$.

3.7 Model Discussion

Before solving the model, I discuss some of the main modeling choices.

[16] Constant markups and industry aggregation (expression (16)) imply that $P_{n,t} = \sigma W_t / Z_{n,t}$, so it follows from footnote 14 that $C_{n,t} = (\sigma W_t)^{-1} Z_{n,t} P_t C_t$. The production function and the demand curve (equation (15)) for good j, along with the expression for $P_{n,t}$ imply that $l_{n,t}(j) = c_{n,t}(j)/z_{n,t}(j) = (\sigma W_t)^{-1} \tilde{z}_{n,t}(j)^{\gamma-1} P_t C_t$. Aggregating up yields the result.

[17] Aggregating the expression for $P_{n,t}$ in footnote 16 across industries using expression (16) yields the result.

We can interpret new good development as firms innovating on the goods for which they already have the most productive version. In either case, any gains enjoyed by a firm come from adding productivity to the industry, not from taking productivity from another firm.

A consequence of imposing that all entrants are small is that the value of being large does not factor into the value of entering. This choice makes sense if large firms exit at much lower rates because they are then over represented in the cross section relative to their salience for a potential entrant. For example, if 1% of firms in a steady state are large and they exit half as quickly, then only 0.5% of entrants are large. Moreover, as large firm exit rates go to 0, their discounted profits do not become arbitrarily big as long as the interest rate is above 0. A similar point stands if new firms take time to become large. Finally, ignoring large firm profits in the entry decision implies that in the main experiment in Section 5.3, it is not particularly important to consider whether large firms have to pay higher firm level fixed costs to lower their per-good fixed costs.

4 Firm Optimization

Before describing the firm problem, note that since small firms take industry aggregates as given, we can split their static profit maximization problem into a separate problem for each good. Moreover, when innovating, a small firm's problem is the same regardless of the goods it produces.

4.1 Static Profit Maximization

At each time t, firms play the following subgame perfect Nash equilibrium of the two stage production game. In the second stage, conditional on fixed cost payments, if a firm has the most productive active version of good j, then it sets its price for good j equal to the minimum marginal cost across all other firms' active versions; otherwise, it sets its price equal to its own marginal cost. In the first stage, only the firm with the most productive version of good j pays the fixed cost for good j.

In the second stage, a firm without the most productive active version of good j has no hope of earning strictly positive profits, so it is optimal to set price equal to marginal cost. A firm with the most productive active version would set a markup of at least $\gamma/(\gamma-1)$ if unconstrained by other producers of good j, so by Assumption 3, pricing below other producers' marginal costs is a binding constraint.[18] In the first stage, a firm without the most productive version of good j will earn zero profits in the second stage if it pays the fixed cost, so it is optimal not to. A small firm with the most productive version earns positive profits across both stages from paying the fixed cost, and so finds it optimal to do so, if $(1-\sigma^{-1})\, P_t C_t / W_t \geq f_S$. A large firm's fixed cost decision is more complicated because paying the fixed cost for some of its goods reduces the relative productivity of its other goods. If the large firm pays fixed costs for a fraction x of its relative productivity, then its share of industry relative productivity is

$$\tilde{\mathcal{L}}_{n,t}(x) \equiv \frac{x\mathcal{L}_{n,t}}{1-(1-x)\left(1-\sigma^{1-\gamma}\right)\mathcal{L}_{n,t}}$$

because its versions are replaced by their imitations, and it earns total profits across both stages $\tilde{\mathcal{L}}_{n,t}(x)\left(1-\sigma^{-1}-\tau_R\right)P_t C_t - x\mathcal{L}_{n,t} f_{L,n} W_t$, which is strictly concave in x. The large firm finds it optimal to pay the fixed cost for all goods for which it has the most productive version if the first derivative of total profits at $x=1$ is positive: $\left(1-\left(1-\sigma^{1-\gamma}\right)\mathcal{L}_{n,t}\right)\left(1-\sigma^{-1}-\tau_R\right)P_t C_t / W_t \geq f_{L,n}$.

4.2 Dynamic Profit Maximization

At each moment in time, firms simultaneously choose innovation rates: a creative destruction rate for each good, and a new good development rate. In the dynamic problem, a firm takes as given its profit function from static optimization.

4.2.1 Small Firms

As in Section 2, the value of a small firm is the sum of the expected discounted profits from each of the goods for which it currently has the most productive version, and if the small firm has an entrepreneur, the value of a small firm with an entrepreneur that does not have the most productive version of any goods. The former determines the value of innovating, and the latter determines the value of entry. The expected discounted profits from being the most productive producer of good j in industry n at time t is $\tilde{z}_{n,t}(j)\Pi_{n,t}(\mathcal{L}_{n,t})$, where the time t subscript in $\Pi_{n,t}(\cdot)$ captures the dependence on the aggregate state, and where $\Pi_{n,t}(\cdot)$ is given by the HJB equation:

$$\begin{aligned} r_t \Pi_{n,t}(\mathcal{L}_{n,t}) = &\left(1-\sigma^{-1}\right)P_t C_t - f_S W_t - \left(N_t \kappa_{n,t}(S;\mathcal{L}_{n,t}) + \kappa_{n,t}(L,S;\mathcal{L}_{n,t}) + (\gamma-1)g_{n,t}\right)\Pi_{n,t}(\mathcal{L}_{n,t}) \\ &+ \dot{\mathcal{L}}_{n,t}\Pi'_{n,t}(\mathcal{L}_{n,t}) + \dot{\Pi}_{n,t}(\mathcal{L}_{n,t}). \end{aligned} \tag{20}$$

[18] See Edmond, Midrigan, and Xu (2021) for a derivation of the optimal markup with oligopoly, nested CES demand, and Bertrand competition.

Relative to equation (4) in Section 2, equation (20) includes the rate at which growth in industry productivity depreciates the good's relative productivity, and the effects of changes in the industry and aggregate state.

At each time t, a small firm chooses innovation rates $\{\kappa(j)\}, \delta$ to maximize

$$\left(\int_0^{M_{n,t}} \kappa(j)\lambda^{\gamma-1}\tilde{z}_{n,t}(j)dj + \delta \right) \Pi_{n,t}(\mathcal{L}_{n,t}) - W_t(\epsilon+1)^{-1} \left(\alpha \int_0^{M_{n,t}} \tilde{z}_{n,t}(j)\kappa(j)^{\epsilon+1}dj + \delta^{\epsilon+1} \right).$$

The First Order Conditions give the optimal single rate at which a small firm creatively destroys each good in its industry, and the optimal new good development rate:

$$\kappa_{n,t}(S;\mathcal{L}_{n,t}) = \left(\lambda^{\gamma-1}\Pi_{n,t}(\mathcal{L}_{n,t})/(W_t\alpha)\right)^{1/\epsilon} \qquad \delta_{n,t}(S;\mathcal{L}_{n,t}) = (\Pi_{n,t}(\mathcal{L}_{n,t})/W_t)^{1/\epsilon}. \tag{21}$$

A small firm values equally—conditional on relative productivity—a good gained through new good development, creatively destroying a small competitor's good, and creatively destroying a large competitor's good because it does not internalize the different effects these innovations have on the industry state or growth. Moreover, it creatively destroys all goods at the same rate because both the cost and benefit of an innovation scale with its relative productivity.

The value function of a small firm with an entrepreneur that does not have the most productive version of any goods is given by the HJB equation:

$$\begin{aligned} r_t E_{n,t}(\mathcal{L}_{n,t}) = & \left(\kappa_{n,t}(S;\mathcal{L}_{n,t})\lambda^{\gamma-1} + \delta_{n,t}(S;\mathcal{L}_{n,t})\right)\Pi_{n,t}(\mathcal{L}_{n,t}) - \eta E_{n,t}(\mathcal{L}_{n,t}) \\ & - W_t(\epsilon+1)^{-1}\left(\alpha\kappa_{n,t}(S;\mathcal{L}_{n,t})^{\epsilon+1} + \delta_{n,t}(S;\mathcal{L}_{n,t})^{\epsilon+1}\right) + \dot{\mathcal{L}}_{n,t}E'_{n,t}(\mathcal{L}_{n,t}) + \dot{E}_{n,t}(\mathcal{L}_{n,t}). \end{aligned} \tag{22}$$

The right-hand side is the benefit and cost from innovation, the risk of exit, and the effects of changes over time in the industry and aggregate state.

At each time t, the value of entry net of the cost, $\int_0^1 E_{n,t}(\mathcal{L}_{n,t})dn - \xi W_t$, is weakly negative. If it is strictly negative, then the entry rate e_t is 0.

4.2.2 Large Firms

I split the large firm optimization problem into two steps, as in Section 2. First, taking as given $\dot{\mathcal{L}}_{n,t}$, the large firm chooses δ and $\kappa(j)$ and $A(j)$ for all goods j in the industry to minimize cost. As in Section 2, the large firm always acquires a good conditional on getting the opportunity because search is costly. The large firm optimally creatively destroys and searches for all its competitors' goods at single rates, and creatively destroys all its own goods at a single rate because the respective costs are strictly increasing and convex, and because both the benefit and cost of creatively destroying a good scale with its relative productivity. Moreover, if ζ is the Lagrange multiplier on the $\dot{\mathcal{L}}_{n,t}$ constraint, then the First Order Conditions yield the optimal innovation rates:

$$\begin{aligned}
\delta_{n,t}(L;\mathcal{L}_{n,t}) &= ((1-\mathcal{L}_{n,t})\zeta/(W_t\chi_n))^{1/\epsilon} \\
\kappa_{n,t}(L,L;\mathcal{L}_{n,t}) &= \left(\left(\lambda^{\gamma-1}-1\right)(1-\mathcal{L}_{n,t})\zeta/(W_t\chi_n\alpha)\right)^{1/\epsilon} \\
\kappa_{n,t}(L,S;\mathcal{L}_{n,t}) &= \left(\left(\left(\lambda^{\gamma-1}-1\right)(1-\mathcal{L}_{n,t})+1\right)\zeta/(W_t\chi_n\alpha)\right)^{1/\epsilon},
\end{aligned} \tag{23}$$

and the optimal acquisition rate:

$$X_A'(A_{n,t}(\mathcal{L}_{n,t})) \geq (\zeta-(1+\tau_A)\Pi_{n,t}(\mathcal{L}_{n,t}))/W_t,$$

which holds with equality if $(A_{n,t}(\zeta;\mathcal{L}_{n,t})>0$.

Developing a new good or creatively destroying one of its own generates novel productivity, which the large firm discounts by $1-\mathcal{L}_{n,t}$ because it adds to industry productivity and depreciates other goods' relative productivities. Creatively destroying a small competitor's good generates novel productivity, but also transfers the good's pre-innovation productivity to the large firm. The large firm does not discount the latter because it does not add to industry productivity.

Second, the large firm chooses a Lagrange multiplier ζ. The HJB equation for the large firm's expected discounted profits, $V_{n,t}(\mathcal{L}_{n,t})$, is

$$\begin{aligned}
r_tV_{n,t}(\mathcal{L}_{n,t}) =& \mathcal{L}_{n,t}\left(\left(1-\sigma^{-1}-\tau_R\right)P_tC_t-f_{L,n}W_t\right)+\dot{V}_{n,t}(\mathcal{L}_{n,t}) \\
&+\max_{\zeta}\left\{\dot{\mathcal{L}}_{n,t}(\zeta;\mathcal{L}_{n,t})V_{n,t}'(\mathcal{L}_{n,t})-X_{n,t}(\zeta;\mathcal{L}_{n,t})-(1-\mathcal{L}_{n,t})A(1+\tau_A)\Pi_{n,t}(\mathcal{L}_{n,t})\right\},
\end{aligned}$$

where $\dot{\mathcal{L}}_{n,t}(\cdot;\cdot)$ and $X_{n,t}(\cdot;\cdot)$ are the rate at which the large firm gains revenue share and the flow innovation/search cost implied by ζ and optimal small firm innovation, respectively, and $\dot{V}_{n,t}(\cdot)$ is the effect of changes in the aggregate state over time. The Lagrange multiplier is $\zeta=V_{n,t}'(\mathcal{L}_{n,t})$.

5 Results: The Effects of Large Firm Innovation

I characterize the effect of changes in large firm innovation incentives on industry concentration, growth, and welfare. In particular, as in Section 2, variation in the large firm fixed cost, revenue tax rate, or innovation cost, which only affect small firms through their effects on large firms, drive variation in outcomes. I begin with qualitative results concerning long-run effects. I then calibrate the model to get quantitative results, which I compare to the data. Throughout, I suppose the acquisition rate is 0.

5.1 Qualitative Results: Concentration and Growth in the Long-Run

Similar to in Section 2, on a balanced growth path, using only small firm optimization and the large firm's *relative* innovation rates given by equations (23), without solving for a dynamic Markov Perfect Equilibrium, I can characterize the relationship between large firm revenue shares and growth. I show that the results from Section 2 hold under certain conditions, and provide intuition for the quantitative exercises that follow. Throughout, I omit time t subscripts for variables that are constant over time.

The following theorem, displayed graphically in Figure 1, shows that if innovation costs are quadratic, then across industries, the industry growth rate is a function of the large firm's industry revenue share that exhibits an **inverted-U shape**. On the other hand, an aggregate increase in large firm innovation incentives always leads to a fall in the long-run growth rate, leaving aside effects on labor demand, which in the calibrated model are not significant.

Theorem 3. *There are two continuously differentiable functions $g_I(\cdot)$ and $g_A(\cdot)$ on $[0,1)$. First, on a balanced growth path, the industry n growth rate is $g_I(\mathcal{L}_n)$. Second, if all industries are the same, the large firm fixed cost f_L, revenue tax rate τ_R, and innovation cost shifter χ vary, and labor supply $\bar{L}$ adjusts so that balanced growth path production labor L^p is constant, then the long-run aggregate growth rate is $g_A(\mathcal{L})$, where $\mathcal{L}$ is the large firm revenue share. The following hold:*

1. *$g_I(0) > 0$, $g_I'(0) > 0$, $g_A(0) > 0$, and $g_A'(0) = 0$.*
2. *If $\epsilon = 1$, then there exists a threshold $\mathcal{L}^* \in (0,1)$ such that $g_I'(\mathcal{L}) > 0$ for $\mathcal{L} < \mathcal{L}^*$ and $g_I'(\mathcal{L}) < 0$ for $\mathcal{L} > \mathcal{L}^*$, and $\lim_{\mathcal{L}\to 1}(g_I(\mathcal{L})) = 0$.*
3. *For all $\mathcal{L} > 0$, $g_A'(\mathcal{L}) < 0$, and $\lim_{\mathcal{L}\to 1}(g_A(\mathcal{L})) = 0$.*

The theorem is intuitive in light of Theorem 1 because across industries, ϵ takes the place of a constant ϵ_S, and across balanced growth paths, the free entry condition implies that small firms are infinitely responsive to a change in the value of innovating, as if $\epsilon_S = 0$. To understand the result better in this richer setting, again decompose the effect of large firm innovation on the long-run growth rate, taking into account small firm optimization, into the discount rate and composition effects. On a balanced growth path, using equation (20), a small firm's value of innovating relative to the wage is

$$\tilde{\Pi}_n \equiv \frac{\Pi_{n,t}(\mathcal{L}_n)}{W_t} = \frac{(\sigma-1)L^p - f_S}{\rho + N(\kappa_n(S;\mathcal{L}_n) + (\gamma-1)g_n(S;\mathcal{L}_n)) + \kappa_n(L,S;\mathcal{L}_n) + (\gamma-1)g_N(L;\mathcal{L}_n)}, \quad (24)$$

where the denominator is the discount rate. The discount rate effect is the same: across industries, higher large firm innovation implies a higher discount rate and, holding its composition fixed, a higher growth rate because N is fixed and small firms face convex innovation costs; across balanced growth paths, the free entry condition implies a fixed discount rate.

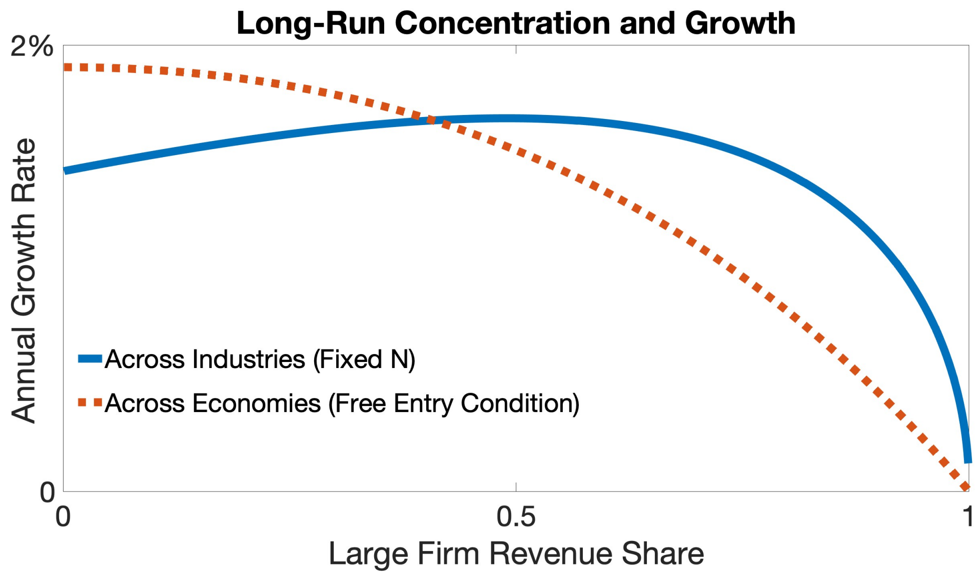

Figure 1: The solid blue line and the dotted orange line display the functions $g_I(\cdot)$ and $g_A(\cdot)$, respectively, from Theorem 3, using the calibration in Section 5.2.

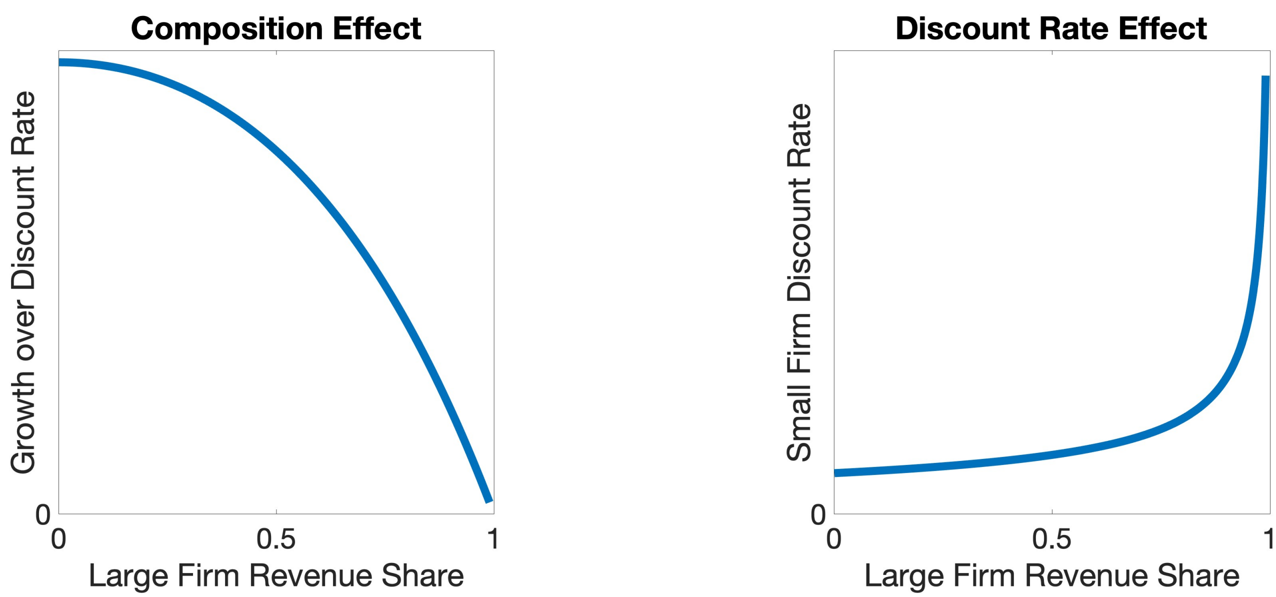

Figure 2: The figure uses the balanced growth path calibration in Section 5.2. Left panel: industry growth rate relative to the small firm discount rate (the denominator of the final expression in equation (24) minus ρ) as a function of the large firm's revenue share. Right panel: the discount rate.

The composition effect is again negative, but differs in two key ways from the effect in Section 2. With $\gamma > 1$, there are two abstract ways a firm can take (or lose) relative productivity, and so sales: by taking productivity from another firm or by increasing total industry productivity, i.e., generating growth. If a firm improves on its own good or develops a new one, then it gains relative productivity only through the latter channel, whereas if a firm creatively destroys a competitor's good, then it gains through both channels. One implication is that if innovation is as in Section 2, i.e., $\kappa_n(L, L; \mathcal{L}_n) = \delta_n(S; \mathcal{L}_n) = \delta_n(L; \mathcal{L}_n) = 0$, then the composition effect is weaker with $\gamma > 1$. Holding fixed the discount rate, as the source of creative destruction shifts from small to large firms and the growth rate falls, the total creative destruction rate a small firm faces must go up. A second implication is that the optimal relative innovation rates of large firms strengthen the composition effect. A bigger large firm revenue share has no effect on small firm relative innovation rates, i.e., $\delta_n(S; \mathcal{L}_n)/\kappa_n(S; \mathcal{L}_n) = \left(\alpha/\lambda^{\gamma-1}\right)^{1/\epsilon}$, but shifts the large firm's innovation toward creative destruction of its competitors' goods:

$$\begin{aligned}
\frac{\kappa_n(L, L; \mathcal{L}_n)}{\kappa_n(L, S; \mathcal{L}_n)} &= \left(\frac{1-\mathcal{L}_n}{1-\mathcal{L}_n\left(1-1/\lambda^{\gamma-1}\right)}\right)^{1/\epsilon}\left(1-1/\lambda^{\gamma-1}\right)^{1/\epsilon} \\
\frac{\delta_n(L, \mathcal{L}_n)}{\kappa_n(L, S; \mathcal{L}_n)} &= \left(\frac{1-\mathcal{L}_n}{1-\mathcal{L}_n\left(1-1/\lambda^{\gamma-1}\right)}\right)^{1/\epsilon}\left(\alpha/\lambda^{\gamma-1}\right)^{1/\epsilon}
\end{aligned}$$

because taking relative productivity without generating growth avoids cannibalization. Thus, as innovation shifts from small to large firms, the same small firm discount rate is achieved with a lower growth rate.

5.2 Calibration

I first calibrate the model to a balanced growth path in which all industries are identical, and small and large firms have the same fixed cost. I set some parameters externally, listed in Table 1, and internally calibrate the rest to jointly match a set of moments in the data, listed in Tables 2 and 3. I set the household's labor supply $\bar{L}$ so that output relative to productivity, $C_t/Z_t = L^p$, is 1. The units of time are years.

The relative cost of creative destruction α, the innovation step size λ, and the demand elasticity γ are important for the size of the composition effect. The latter two determine the fraction of a creative destruction innovation that improves on the replaced good and is thus novel, which in the calibration is $\left(1-1/\lambda^{\gamma-1}\right) = 0.1$. As $\lambda^{\gamma-1}$ increases and the novel fraction goes to 1, the composition effect goes to 0 because the distinction between creative destruction and new good development disappears. I calibrate γ externally and α and λ internally, in particular using the labor flows data from Garcia-Macia, Hsieh, and Klenow (2019). Informally, the identification is as follows: a higher level of large negative labor flows implies a shift in the small firm discount rate from growth to creative destruction, which requires a higher α or a lower λ to get less growth from a given creative destruction rate; a higher level of large positive labor flows implies a shift in small firm innovation toward creative destruction or a bigger gain from each creative destruction innovation, i.e., a higher α or λ.

I calibrate the marginal innovation cost elasticity ϵ to 1 based on the studies Acemoglu, Akcigit, Alp, Bloom, and Kerr (2018) discuss. The elasticity ϵ encompasses two elasticities: the response of a firm's total innovation to a change in the value or cost of innovating, and the response of a firm's *relative* innovation rates to a change in the value or cost of one type of innovation *relative* to another. The studies in Acemoglu, Akcigit, Alp, Bloom, and Kerr (2018) relate to the former. Theorem 3 thus provides an argument that $\epsilon = 1$ for the latter: Proposition 4 in Appendix A.6 shows that to match the inverted-U relationship between growth and concentration across industries documented in Cavenaile, Celik, and Tian (2021), the elasticity of relative innovation rates cannot be much greater than 1 (even if the elasticity is greater than 1.5, then an industry's growth rate diverges to infinity as its large firm's revenue share goes to 1).

The mapping between the remaining internally calibrated parameters and the target data moments is largely as follows. The TFP growth rate and the real interest rate exactly determine ρ. The large firm industry revenue share and the TFP growth rate pin down χ and ξ, which affect large and small firm innovation, respectively. The aggregate markup exactly determines σ. R&D expenditures relative to GDP determines firm profits relative to sales, and thus f_S and f_L given σ. The distinction between the fixed cost and the imitation discount, identified by the aggregate markup given R&D expenditures relative to GDP, is not significant; the fixed cost is only present so that I can lower the fixed cost for large firms in the main experiment in Section 5.3.

I calibrate the revenue share of large firms, which is the same in each industry, as well as the shock in Section 5.3, to match the average industry revenue share of the largest firm in 4-digit industries in Compustat. This measure likely overstates the size of the largest firm because Compustat

Table 1: Externally Calibrated Parameters

Parameter	Description	Value
η	Exit Rate	0.04
γ	Demand Elasticity	3.1
ϵ	Innovation Cost Elasticity	1

η is from Boar and Midrigan (2022), γ from Broda and Weinstein (2006) using the median estimate from 1990-2001 at the most disaggregated level, and ϵ from Acemoglu, Akcigit, Alp, Bloom, and Kerr (2018).

Table 2: Internally Calibrated Parameters

Parameter	Description	Value
α	Relative Creative Destruction Cost	0.265
λ	Innovation Step Size	1.05
ρ	Time Discount Rate	0.0194
χ^{-1}	Large Firm Innovation Cost	0.066
ξ	Entry Cost	2.22
σ	Imitation Discount	1.3
f_S, f_L	Fixed Cost	0.183

Table 3: Calibration Targets

Moment Description	Data Average from 1983-1993	Model
Large job destruction	25.0%	25.0%
Large job creation	29.5%	29.5%
TFP Growth Rate	1.66%	1.66%
Real Interest Rate	3.6%	3.6%
Large Firm Revenue Share	40.7%	40.7%
Aggregate Markup	1.3	1.3
R&D Relative to GDP	1.81%	1.81%

Large job destruction and creation are firm-level decreases greater than two-thirds and increases greater than tripling, respectively, over 5-year intervals as a percentage of all labor, and TFP growth rate is from the BLS measure, all from Garcia-Macia, Hsieh, and Klenow (2019) (excluding public, educational, agricultural, and mining sectors). Real interest rate is the 1-year real interest rate from FRED. Large firm revenue share is the sales-weighted average across 4-digit industries of the largest firm's revenue share in Compustat from Olmstead-Rumsey (2022). Aggregate markup is the cost-weighted average markup estimated from Compustat in De Loecker, Eeckhout, and Unger (2020). R&D relative to GDP is the Business Enterprise Expenditure on R&D (BERD) relative to GDP from the OECD MSTI database.

does not include all firms. An alternative measure is the Census data on industry concentration measures, which show a smaller level of industry concentration, but a similar rise over the same time period. One downside of the Census data is that it only lists the revenue share of the top 4 firms in each industry, not the top firm. Moreover, while the Census data provide a more accurate measure of revenue shares, they may include too many small firms that are not relevant to the mechanism in the model, which depends on a large firm's share of innovations. Olmstead-Rumsey (2022) shows that the average share of R&D expenditures by the largest firm in 4-digit industries in Compustat closely tracks the average sales share.

5.3 Quantitative Experiment: A Rise in Large Firm Innovation

I ask whether and to what extent a rise in concentration driven by a fall in large firm fixed costs can explain changes in US data since the mid-1990s. I interpret the fall in large firm fixed costs as capturing a shift from per-good costs to firm wide fixed costs due to the rise in information technology.[19] I discuss the effects of falls in the large firm innovation cost or revenue tax, which are similar, in Section 5.3.3.

The economy begins on the balanced growth path calibrated in Section 5.2. There is an unanticipated permanent fall in f_L to 0.171 in all industries, which is calibrated so that the large firm revenue share in the new balanced growth path is 0.51, the sales-weighted average across 4-digit industries of the largest firm's revenue share in 2018 in Compustat from Olmstead-Rumsey (2022).

5.3.1 Industry Concentration and Aggregate Growth

The large firm revenue share, growth, and output, depicted in Figure 3, converge over a similar time interval as the gap between the years in the initial calibration, 1983-1993, and the target year for the shock, 2018. The main results are in Table 4. The shock explains 41% of the fall in the long-run growth rate, due entirely to a change in the productivity growth rate because C_t/Z_t is constant on a balanced growth path. The shock explains all of the increase in the short-run growth rate of output relative to inputs, mostly due to a temporary large increase in real output that is necessary to clear the labor market because of the reduction in labor used for entry costs. The growth burst does not last as long in the model as in the data: the peak difference in output along the transition path from the original balanced growth path occurs after 4 years and at that point is 42% of the difference in the data.

I decompose the change in the productivity growth rate over time from the initial balanced growth path into the composition and discount rate effects in Figure 4. Growth is higher in the short-run because the small firm discount rate is higher: the measure of small firms is slow to fall (see Figure

[19] See Aghion, Bergeaud, Boppart, Klenow, and Li (2022) for a discussion.

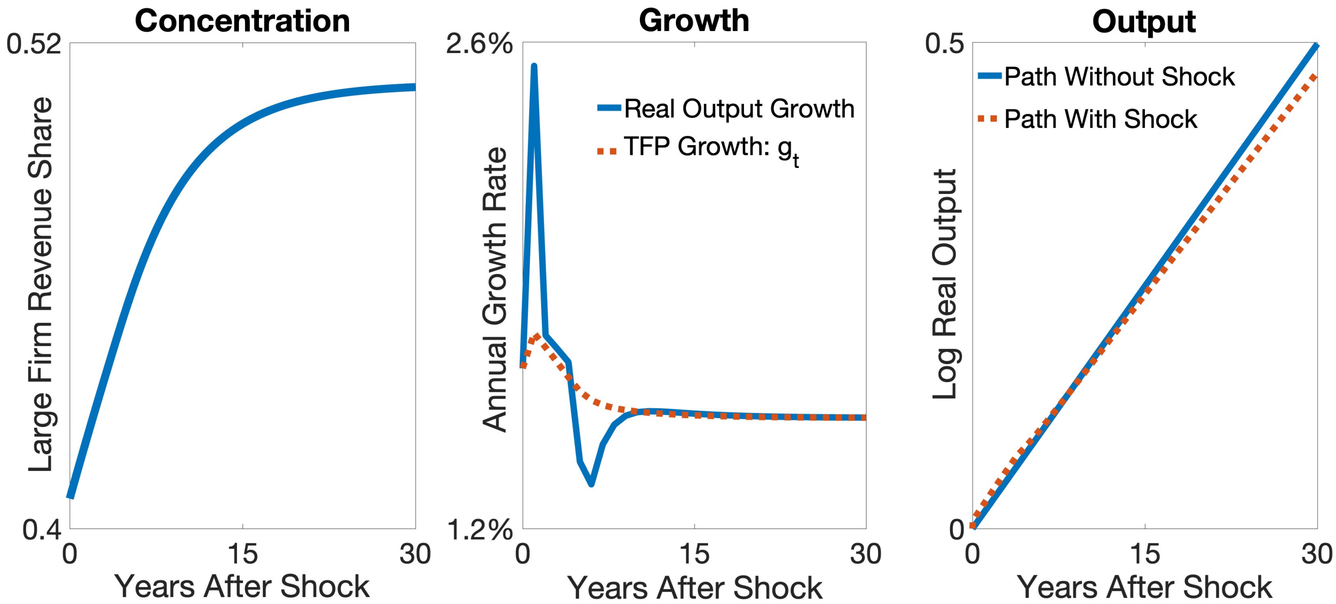

Figure 3: Transition paths following the shock to f_L. Left panel: the revenue share of each large firm over time, $\mathcal{L}_t$. Middle panel: the dotted orange line is g_t, and the solid blue line is the growth rate of C_t, which includes changes in L_t^p. Right panel: the solid blue line is the original path of $ln(C_t)$ without the shock; the dotted orange line is the actual path following the shock.

Table 4: Growth After a Fall in f_L

Moment Description	Data	Model
Growth Rate Burst	+0.64 ppt (38.6%) (1993-2003)	Output: +0.87 ppt (52.4%) (first year) TFP: +0.1 ppt (6.0%) (first year)
Cumulative Burst	+6.4 ppt (38.6%) (1993-2003)	Output: +1.07 ppt (16.1%) (4 years) TFP: +0.18 ppt (2.7%) (3 years)
Growth Rate Fall	-0.34 ppt (-20.5%) (2003-2013)	-0.14 ppt (-8.4%) (New BGP)

For each value, ppt is the percentage point rise, and in parentheses is the percent rise. The data are from Garcia-Macia, Hsieh, and Klenow (2019). Growth rate burst is the peak growth rate following the shock. Cumulative burst is the peak difference between the new and old paths.

5), and in the meantime, the fall in small firm innovation pushes down the cost of innovating (convex innovation costs), and the temporary large increase in real output relative to the wage pushes up the return to innovating. In the long-run, growth is lower because the composition effect dominates: Theorem 3 kicks in, and the small firm discount rate is only slightly higher due to a small increase in real output relative to the wage.

Welfare:

Taking into account the transition path, household welfare falls by the equivalent of a permanent 5.8% drop in consumption. The decline in the long-run growth rate is the dominant effect.

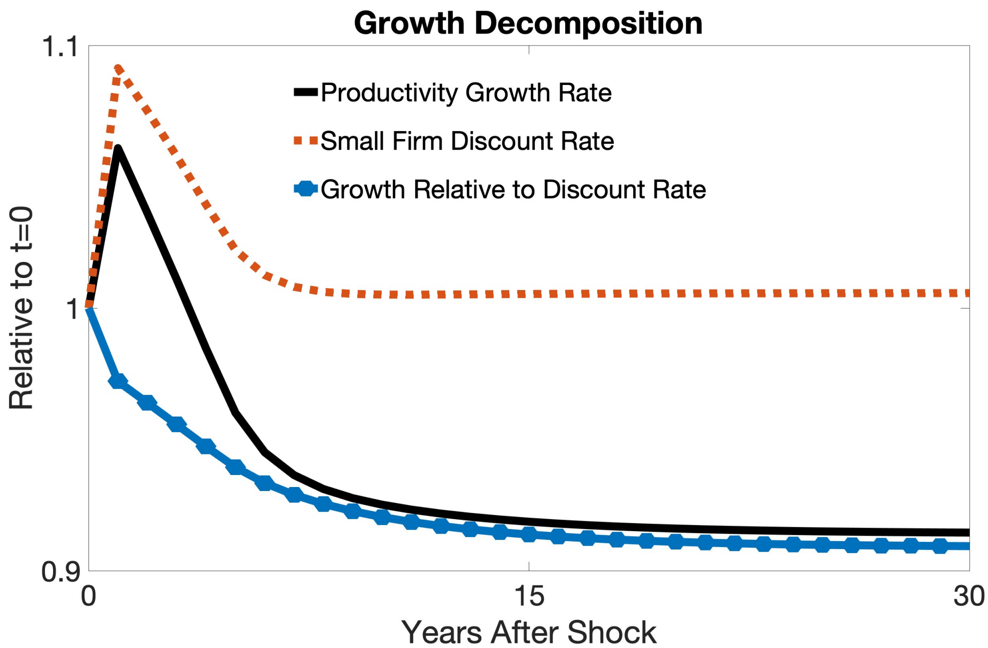

Figure 4: The solid black line is g_t/g, where g is on the original balanced growth path. The dotted orange and textured blue lines decompose g_t/g into the small firm discount rate (minus ρ) and growth over the discount rate, respectively, relative to their values on the original balanced growth path.

5.3.2 Comparing Model Predictions to the Data

Entry:

The large fall in entry in the short-run and the smaller fall in the long-run, depicted in Figure 5, match the data in Decker, Haltiwanger, Jarmin, and Miranda (2016), which show that the entry rate declined sharply in the mid-to-late 1990s followed by a partial recovery before a large drop during the Great Recession.

Creative Destruction:

Large negative and positive labor flows, as a percentage of labor supply, fall to 23.5% and 23.9%, respectively, across balanaced growth paths, in line with the data from Garcia-Macia, Hsieh, and Klenow (2019), which show a decrease to 23.1% and 21.4%. In the model, large negative labor flows fall even though creative destruction rises as a share of the small firm discount rate because the small firm share of economic activity falls, and labor flows only occur at small firms. Thus, the theory is not at odds with the estimated fall in the share of growth due to creative destruction in

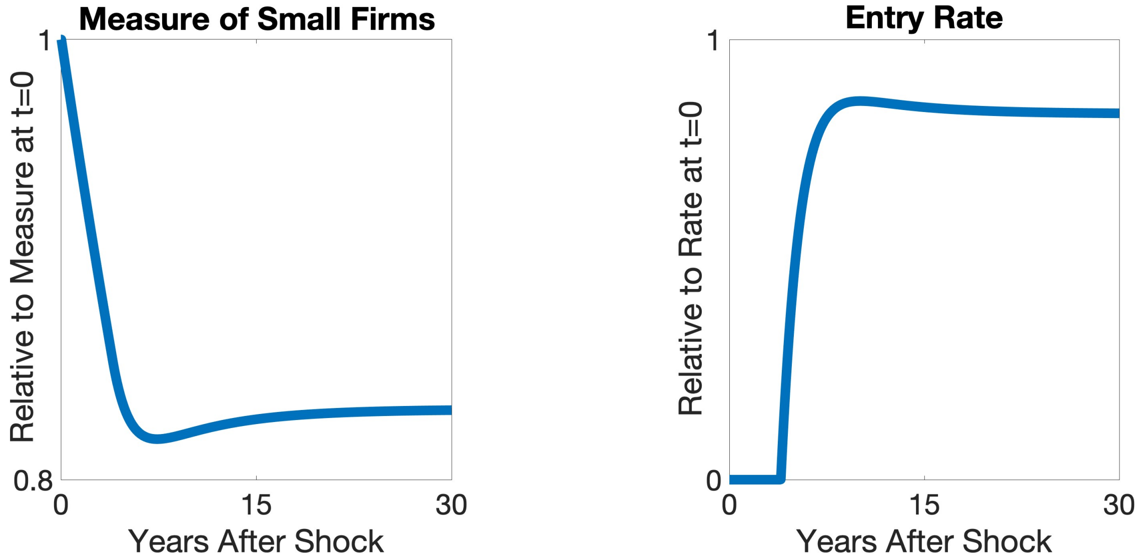

Figure 5: N_t/N (left) and E_t/E (right), where N and E are from the initial balanced growth path.

Garcia-Macia, Hsieh, and Klenow (2019); the creative destruction growth share falls in the model because large firms only creatively destroy small firm goods, and the small firm revenue share falls.

Growth Relative to R&D Expenditures:
Data in Olmstead-Rumsey (2022) presents a puzzle that aggregate R&D expenditures relative to GDP increased and shifted toward more efficient innovators, yet growth fell. Specifically, in Compustat, R&D expenditures shifted toward industry leaders over a similar time period and of a similar magnitude as the shift in revenue, and relative to other firms with R&D expenditures, industry leaders have a low ratio of R&D to revenue.

The model offers an explanation. First, in the data, the average R&D expenditures over revenue among industry leaders relative to the average among non-leaders with positive R&D expenditures is 0.4, and in the model it is 0.67. In the model, all firms generate sales through R&D. Large firm innovation is particularly effective because their innovation cost function is convex, and they innovate relatively little to avoid cannibalization. Second, in the data R&D expenditures as a share of GDP rose while growth fell, and the growth rate relative to R&D expenditures over GDP fell from 0.91 in 1983-1993 to 0.69 in 2003-2013. In the model, innovation expenditures as a share of GDP are flat while growth falls, and the growth rate relative to innovation expenditures over GDP falls from 0.91 on the initial balanced growth path to 0.83 on the balanced growth path following the shock. In the model, large firm innovation is particularly efficient at generating revenue, but not at generating growth because large firms endogenously focus their innovation on taking productivity from competitors without generating growth.

Industry Concentration and Industry Growth Rates:
The model matches the finding in Ganapati (2021) that across industries in the US from 1972-2012, rising concentration is associated with *faster* growth. Specifically, Ganapati (2021) estimates, controlling for sector and time fixed effects, that a 1 percent rise in the revenue share of the largest 4 firms in a 6-digit NAICS industry over a 5-year period is associated with a 0.1 percent rise in real output and a 0.2 percent rise in real output relative to employment.

I run the same regression in the model by creating a control group of a measure 0 of industries in which the large firm fixed cost does not change. I then regress the change in industry log real output on the change in industry log large firm revenue share and a time fixed effect over the three 5-year time periods during the first 15 years of the transition path following the shock, after which industry concentration is effectively constant. A 1 percent rise in the revenue share of the largest firm in an industry is associated with a 0.03 percent rise in real output and a 0.07 percent rise in real output relative to employment.

The theory thus generates a parsimonious explanation for the short-run burst in growth as well as the positive relationship between concentration and growth across industries: the measure of small firms is slow to adjust and doesn't respond at the industry-level.

5.3.3 Other Shocks to Large Firm Innovation Incentives

I analyze the effects of alternative shocks on the long-run growth rate. Specifically, starting from the initial balanced growth path, I calibrate separately a fall in the large firm innovation cost χ^{-1} and a fall in the large firm revenue tax τ_R to generate a rise in the large firm revenue share to 0.51 in the new balanced growth path. The results, depicted in Table 5, are nearly identical as for the main experiment. The only difference across experiments is in the labor used for fixed or innovation costs, which affects real output through the labor market clearing condition, and so the small firm discount rate implied by the free entry condition. Ultimately, the composition effect, which depends only on the rise in the large firm revenue share and not on its cause, dominates.

An implication is that a tax on firms increasing in their size will improve growth and welfare.

Table 5: Alternative Shocks

Experiment	Initial Growth Rate	New Long-Run Growth Rate
f_L	1.66%	1.52%
χ^{-1}	1.66%	1.51%
τ_R	1.66%	1.5%

For each experiment, the increase in the large firm revenue share across balanced growth paths is the same. The new long-run growth rate is g in the new balanced growth path following the shock.

6 Antitrust Policy: Acquisitions

I use the calibrated model to explore the effects of a change in the acquisition tax rate τ_A. Any taxes collected are dispersed to the representative household. If the tax is negative, then it is funded by a lump sum tax on the representative household.

I solve for balanced growth path outcomes as functions of the acquisition tax rate under three different scenarios. Using the initial calibration in Section 5.2, I use two different acquisition search cost functions. First, I set $X_A(A) = 0$ for all $A \leq 0.05$ and $X_A(A) = \infty$ for all $A > 0.05$, which is useful for distinguishing between the acquisition effect and the innovation effect. Second, I set the flow labor cost of acquisition search to be $\omega \int_0^{M_{n,t}} \tilde{z}_{n,t}(j) A_{n,t}(j)^2 dj$, where $\omega > 0$ is a cost-shifter, and the cost of searching for good j scales with $\tilde{z}_{n,t}(j)$ so that the large firm searches for all goods at the same rate. Finally, I use the quadratic cost function in the calibration following the fall in the large firm fixed cost in Section 5.3.

The main results, depicted in Figure 6, show that if acquisitions are sufficiently subsidized, then the long-run growth rate, and so welfare, is higher than without acquisitions. Moreover, the required subsidy and acquisition rate to get higher growth are much lower in the higher concentration balanced growth path following the fall in the large firm fixed cost.

I provide intuition for the results in Figure 7 using the acquisition search cost function given by $X_A(A) = 0$ for $A \leq 0.05$ and $X_A(A) = \infty$ for $A > 0.05$. As the acquisition tax rate falls, when it first crosses the threshold so that large firms acquire goods, the acquisition rate immediately jumps to the maximum, but the expectation of future acquisitions has no effect on a large firm's value. Thus, the only direct effect is an increase in the large firm revenue share, which worsens fears of cannibalization and so leads to a substantial fall in large firm own innovation and new

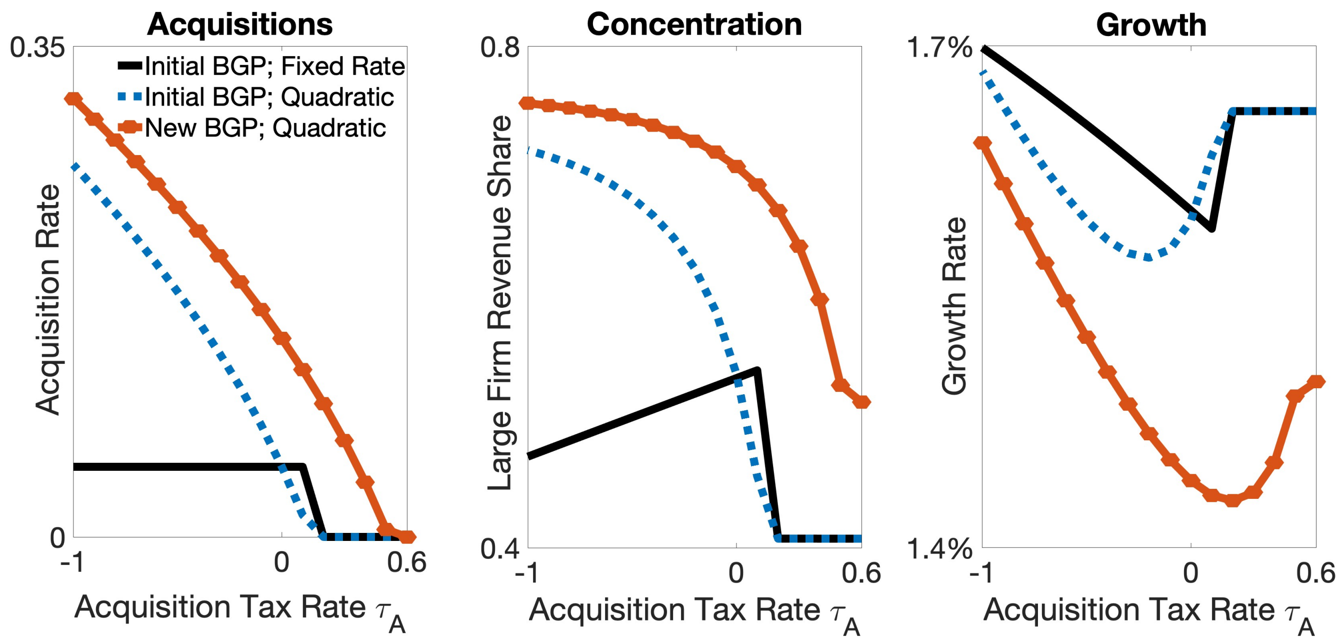

Figure 6: Fixed rate refers to the acquisition search cost function that is 0 if $A \leq 0.05$, and quadratic refers to the other cost function. Each panel displays results on a balanced growth path as a function of τ_A. Left panel: the rate A at which the large firm acquires each good. Middle panel: $\mathcal{L}$. Right panel: g.

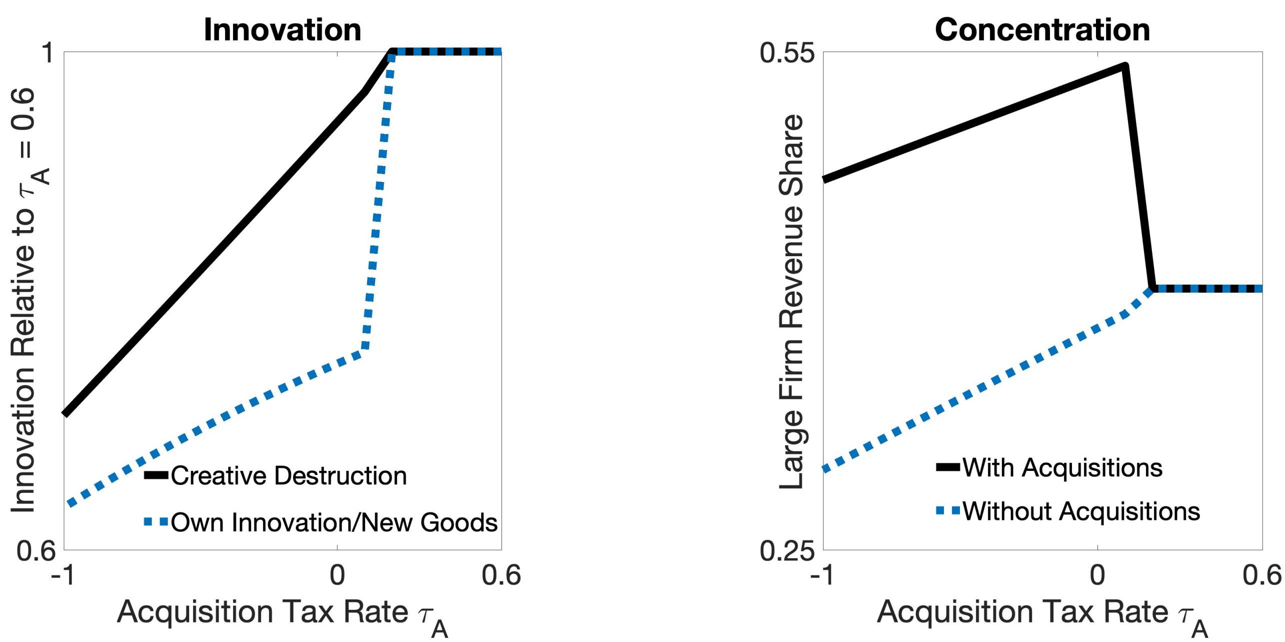

Figure 7: Each panel displays results on a balanced growth path as a function of τ_A, using the calibration from Section 5.2 and the acquisition search cost function that is 0 if $A \leq 0.05$. Left panel: the large firm creative destruction rate (solid black line) and own innovation/new good development rate (dotted blue line) relative to their respective values at $\tau_A = 0.6$. Right panel: $\mathcal{L}$ in equilibrium (solid black line) and using equilibrium innovation rates but with the acquisition rate set to 0 (dotted blue line).

good development rates, but only a slight fall in large firm creative destruction rates. The growth rate falls. As the acquisition tax rate falls further, the acquisition rate is the same, but large firms receive more surplus from acquisitions. Large firms innovate less to preserve valuable acquisition opportunities, which pushes down their revenue shares and leads to a higher growth rate.

If an econometrician used an unexpected acquisition to estimate the effect of acquisitions on economic outcomes, they would only pick up the acquisition effect, i.e., the fall in own innovation, new good development, and growth, and would miss the innovation effect, i.e., the fall in creative destruction and the rise in growth.

7 Conclusion

To understand the relationship between concentration and growth, and the policy implications, I study a model with one large firm and a continuum of small firms in each industry. Firms can innovate through creative destruction, developing new goods, and improving on their own goods. Large firms, to avoid cannibalization, have a strong relative preference for creatively destroying their competitors' goods. As a result, when large firms innovate more, small firms' innovations are discounted heavily relative to the overall innovation and growth rate. A widespread fall in large firm fixed costs stimulates large firm innovation, which increases concentration and reduces small firm entry, long-run growth, and taking into account the transition path, welfare. Growth rises in the short-run and in industries with a bigger fall in large firm fixed costs because the measure of small firms, which affects small firm innovation, is slow to fall and only responses to the aggregate environment. The aggregate growth rate falls despite flat innovation expenditures and a shift toward large firms whose innovation is relatively efficient at generating revenue; large firms focus their innovation on creative destruction, which creates relatively little growth. I show that these predictions match US data from the mid-1990s to the late 2010s.

Large firm acquisitions of their competitors' goods have incentive effects and distributional effects with opposite implications for concentration, growth, and welfare. Acquisitions directly shift revenue to large firms, strengthening their relative preference for creative destruction, and leading to a fall in growth. The expectation of future valuable acquisitions pushes each large firm to innovate less so that more revenue share remains for it to acquire. As large firm innovation falls, it is replaced by small firm innovation, which is less geared toward creative destruction and includes creative destruction of large firm goods, ultimately facilitating more innovation and growth. If acquisitions are sufficiently valuable to large firms, then growth and welfare are higher in an economy with acquisitions than without. This positive effect is stronger when industries are more concentrated.

The theory and results highlight a novel way to think about the effects of market power and optimal competition policy. Large firms are harmful because of how they achieve their size through innovation. Research and development subsidies that target large firms may backfire by discouraging small firm innovation. Policies that increase concentration may be beneficial as long as they reduce large firm innovation. Facilitating acquisitions is a particularly useful policy because, unlike taxing large firms, it does not require knowledge of firms' relevant industries or their revenue shares in those industries.

Finally, although this paper focuses on growth, the theory has implications for other settings, and suggests potential avenues for future research. For example, suppose a firm can develop different types of goods, some of which are more novel to the industry, and others of which are close substitutes with the firm's competitors' goods. The same force that leads larger firms to set higher markups in static models of oligopolistic competition implies that larger firms have a stronger preference for producing the types of goods that are close substitutes with their competitors. Thus, subsidizing large high markup firms to produce more may be costly unlike in models in which firm production is one-dimensional.

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A Proofs and Derivations

A.1 Proof of Proposition 1

I begin with the first part of the proposition. Combining the First Order Condition for the small firm innovation rate κ_S (inequality (5)) and the expression for the expected present discounted value a small firm earns from a good Π (equation (6)) yields

$$WX_S'(\kappa_S)(r + \kappa_L + \kappa_S) \geq \left(1 - \sigma^{-1}\right) R - f_S W, \tag{25}$$

where the inequality holds with equality if $\kappa_S > 0$. The left-hand side is strictly increasing in κ_S and goes to infinity as κ_S goes to infinity because $X_S(\cdot)$ is strictly increasing and convex. Thus, there is a unique solution for the small firm innovation rate that depends on the large firm innovation rate κ_L: $\kappa_S(\kappa_L)$. Moreover, $\kappa_S(\kappa_L)$ is decreasing because the left-hand side is increasing in κ_L and κ_S.

For the remainder of the proof of the first part of the proposition, consider two cases. Suppose $X_S'(0) = 0$. Inequality (25) cannot hold at $\kappa_S = 0$ because the right-hand side is strictly positive by Assumption 1. It follows that for all κ_L, $\kappa_S(\kappa_L) > 0$, which implies that $X_S'(\kappa_S(\kappa_L)) > 0$ and that inequality (25) holds with equality. Totally differentiating each side with respect to κ_L and using the definition of $\epsilon_S(\kappa_S)$ yields the derivative

$$\kappa_S'(\kappa_L) = \frac{-\kappa_S(\kappa_L)}{\epsilon_S(\kappa_S(\kappa_L))(r + \kappa_L + \kappa_S(\kappa_L)) + \kappa_S(\kappa_L)}, \tag{26}$$

which is continuous in κ_L because $\epsilon_S(\cdot)$ is continuous by assumption and because the existence of $\kappa_S'(\kappa_L)$ implies that $\kappa_S(\kappa_L)$ is continuous. Thus, setting $\kappa_L^* = \infty$, for all $\kappa_L < \kappa_L^*$, $\kappa_S(\kappa_L) > 0$ and has a continuous derivative.

Next, suppose $X_S'(0) > 0$. Define κ_L^* by making inequality (25) hold with equality at $\kappa_S = 0$. It follows that $\kappa_L^* > 0$ because at $\kappa_L = 0$, the left-hand side is strictly less than the right-hand side by Assumption 2. Moreover, $\kappa_S(\kappa_L) > 0$ if and only if $\kappa_L < \kappa_L^*$: if $\kappa_L < \kappa_L^*$, then $\kappa_S(\kappa_L) > 0$ because the left-hand side is strictly less than the right-hand side at $\kappa_S = 0$, and if $\kappa_L \geq \kappa_L^*$, then $\kappa_S(\kappa_L) = 0$ because the inequality holds at $\kappa_S = 0$. Finally, $\kappa_S(\kappa_L)$ is continuously differentiable on $[0, \kappa_L^*)$ because there $\kappa_S(\kappa_L) > 0$ and inequality (25) holds with equality, so totally differentiating each side with respect to κ_L yields the continuous derivative in equation (26).

I now prove the second part of the proposition. Suppose there are no acquisitions ($A = 0$), that $X_L(\cdot)$ is strictly convex, and that for all κ_L, $\epsilon_L(\kappa_L)(\epsilon_S(\kappa_S(\kappa_L)) + 1) \geq 1$. I show that we can write κ_L as a continuously differentiable function of the marginal value of a good to the large firm relative to the innovation cost shifter, $\bar{V}/\beta$. I use this function and equation (8) for $\bar{V}$ to show that the equilibrium value of $\bar{V}/\beta$ is a function of $((1-\sigma^{-1}-\tau_R)R - f_L W)/\beta$ with the right properties so that the equilibrium values of κ_L and g are functions of the same with the desired properties.

As a preliminary step, equation (8) for $\bar{V}$ and Assumption 1 imply that $\bar{V} > 0$ in any equilibrium, which I take as given for the remainder of the proof.

The First Order Condition for κ_L (inequality (9)) implicitly defines a function $\kappa_L\left(\bar{V}/\beta\right)$ because, due to the strict convexity of $X_L(\cdot)$, the left-hand side is strictly increasing in κ_L and goes to infinity as κ_L goes to infinity. To see that $\kappa_L(\cdot)$ is continuously differentiable, consider two cases. If $X_L'(0) = 0$, then the First Order Condition cannot hold at $\kappa_L = 0$ because $\bar{V} > 0$. It follows that $\kappa_L\left(\bar{V}/\beta\right) > 0$, which implies that the First Order Condition holds with equality and, given $\epsilon_L(\cdot) > 0$, implies that

$$W\beta X_L''\left(\left(\kappa_L\left(\bar{V}/\beta\right)\right)\right) = \epsilon_L\left(\kappa_L\left(\bar{V}/\beta\right)\right)\frac{\bar{V}}{\kappa_L\left(\bar{V}/\beta\right)} > 0.$$

Therefore, differentiating both sides of the First Order Condition with respect to $\bar{V}/\beta$ yields the derivative

$$\kappa_L'\left(\bar{V}/\beta\right) = \frac{1}{\epsilon_L\left(\kappa_L\left(\bar{V}/\beta\right)\right)}\frac{\kappa_L\left(\bar{V}/\beta\right)}{\bar{V}/\beta}, \tag{27}$$

which is continuous in $\bar{V}/\beta$ because $\epsilon_L(\cdot)$ is continuous by assumption and because the existence of $\kappa_L'\left(\bar{V}/\beta\right)$ implies that $\kappa_L\left(\bar{V}/\beta\right)$ is continuous. On the other hand, suppose $X_L'(0) > 0$. If $\bar{V} < W\beta X_L'(0)$, then the First Order Condition holds with a strict inequality, which implies that $\kappa_L\left(\bar{V}/\beta\right) = \kappa_L'\left(\bar{V}/\beta\right) = 0$. If $\bar{V} > W\beta X_L'(0)$, then the First Order Condition holds with equality, $\kappa_L\left(\bar{V}/\beta\right) > 0$, and $\kappa_L'\left(\bar{V}/\beta\right)$ is given by equation (27). It follows that $\kappa_L'\left(WX_L'(0)\right) = 0$ because as $\bar{V}$ goes to $W\beta X_L'(0)$ from either direction, $\kappa_L'\left(\bar{V}/\beta\right)$ converges to 0. Thus, $\kappa_L(\cdot)$ is continuously differentiable.

We can write equation (8) for $\bar{V}$ (with $A = 0$) using $\kappa_L(\cdot)$ and $\kappa_S(\cdot)$:

$$\left(r + \kappa_L\left(\bar{V}/\beta\right) + \kappa_S\left(\kappa_L\left(\bar{V}/\beta\right)\right)\right)\bar{V}/\beta - WX_L\left(\kappa_L\left(\bar{V}/\beta\right)\right) = \left((1-\sigma^{-1}-\tau_R)R - f_L W\right)/\beta. \tag{28}$$

The right-hand side is constant in $\bar{V}/\beta$ and strictly greater than 0 by Assumption 1. The left-hand side is continuous in $\bar{V}/\beta$ and continuously differentiable everywhere except at the single point where $\kappa_L\left(\bar{V}/\beta\right) = \kappa_L^*$. Moreover, as $\bar{V}/\beta$ goes to 0, the First Order Condition for κ_L implies that $\kappa_L\left(\bar{V}/\beta\right)$ goes to 0 because $X_L(\cdot)$ is strictly increasing, which implies that $X_L\left(\kappa_L\left(\bar{V}/\beta\right)\right)$ goes to 0 because $X_L(\cdot)$ is continuous and $X_L(0) = 0$. Therefore, as $\bar{V}/\beta$ goes to 0, the left-hand side of equation (28) goes to 0. Finally, I show that the derivative of the left-hand side, where it exists, is bounded below by a strictly positive number, which implies that a unique solution $\bar{V}/\beta$ always exists because the left-hand side goes to infinity as $\bar{V}/\beta$ goes to infinity, implies that this solution is strictly increasing in the right-hand side, and implies that everywhere but at a single point, the left-hand side as a function of $\bar{V}/\beta$ has a continuously differentiable inverse. Taking the derivative and plugging in that if $\kappa_L'\left(\bar{V}/\beta\right) \neq 0$, then $WX_L'\left(\kappa_L\left(\bar{V}/\beta\right)\right) = \bar{V}/\beta$ yields

$$r + \kappa_L\left(\bar{V}/\beta\right) + \kappa_S\left(\kappa_L\left(\bar{V}/\beta\right)\right) + \kappa_S'\left(\kappa_L\left(\bar{V}/\beta\right)\right)\kappa_L'\left(\bar{V}/\beta\right)\bar{V}/\beta. \tag{29}$$

If the last term in the sum is not equal to 0, then $\kappa_L\left(\bar{V}/\beta\right) > 0$ and $\kappa_S\left(\kappa_L\left(\bar{V}/\beta\right)\right) > 0$, which implies that the derivatives are given by equations (26) and (27), which implies that expression (29) is weakly greater than

$$r + \kappa_L\left(\bar{V}/\beta\right) + \kappa_S\left(\kappa_L\left(\bar{V}/\beta\right)\right) - \frac{1}{\epsilon_S\left(\kappa_S\left(\kappa_L\left(\bar{V}/\beta\right)\right)\right) + 1}\frac{1}{\epsilon_L\left(\kappa_L\left(\bar{V}/\beta\right)\right)}\kappa_L\left(\bar{V}/\beta\right),$$

which is weakly greater than $r > 0$ by the premise of the proposition. Thus, the equilibrium value of $\bar{V}/\beta$ is a strictly increasing continuous function of $\left(\left(1-\sigma^{-1}-\tau_R\right)R - f_L W\right)/\beta$ that is continuously differentiable everywhere except at the single point where $\kappa_L\left(\bar{V}/\beta\right) = \kappa_L^*$.

Finally, $\kappa_L(\cdot)$ implies that the equilibrium value of κ_L is an increasing continuous function of $\left(\left(1-\sigma^{-1}-\tau_R\right)R - f_L W\right)/\beta$ that is strictly increasing if $\kappa_L > 0$, and that is continuously differentiable everywhere except at $\kappa_L = \kappa_L^*$.

A.2 Proof of Theorem 1

Equation (2) for the growth rate, evaluated on a balanced growth path, equation (3) for the long-run large firm revenue share, and the function $\kappa_S(\kappa_L)$ imply that if $A = 0$, then the long-run growth rate is a function of κ_L:

$$g(\kappa_L) = (\lambda - 1)\kappa_S(\kappa_L)\left(1 + \frac{\kappa_L}{\kappa_L + \kappa_S(\kappa_L)}\right),$$

which is well-defined for all $\kappa_L \geq 0$ because $\kappa_S(0) > 0$ by Assumption 2. Moreover, $g(\cdot)$ is continuous everywhere and continuously differentiable everywhere except at $\kappa_L = \kappa_L^*$ because that is the case for $\kappa_S(\cdot)$. The derivative is

$$g'(\kappa_L) = (\lambda - 1)\kappa_S'(\kappa_L)\left(1 + \left(\frac{\kappa_L}{\kappa_L + \kappa_S(\kappa_L)}\right)^2\right) + (\lambda - 1)\left(\frac{\kappa_S(\kappa_L)}{\kappa_L + \kappa_S(\kappa_L)}\right)^2,$$

where $\kappa_S'(\kappa_L)$ is given in equation (26).

The first property of $g(\cdot)$ in the theorem holds because $g(\kappa_L) > 0$ if and only if $\kappa_S(\kappa_L) > 0$, which holds if and only if $\kappa_L < \kappa_L^*$.

The second property of $g(\cdot)$ holds because $g'(0) = (\lambda - 1)(1 + \kappa_S'(0))$, and $\kappa_S'(0) \geq -1$ and strictly so if and only if $\epsilon_S(\kappa_S(0)) > 0$.

For the third property of $g(\cdot)$, consider two cases. If $\epsilon_S(0) = 0$, then as κ_L goes to κ_L^* from below, $\kappa_S(\kappa_L)$ converges to 0 and $\kappa_S'(\kappa_L)$ converges to -1 by the continuity of $\epsilon_S(\cdot)$ (even though $\kappa_S'(\kappa_L^*)$ does not exist), which implies that $g'(\kappa_L)$ converges to $-2(\lambda - 1)$. If $\epsilon_S(0) > 0$, then as κ_L goes to κ_L^* from below, $\kappa_S(\kappa_L)$ converges to 0, which implies that $g'(\kappa_L)$ converges to

$$-2(\lambda - 1)\frac{\kappa_S(\kappa_L)}{\epsilon_S(0)(r + \kappa_L^*)} + (\lambda - 1)\left(\frac{\kappa_S(\kappa_L)}{\kappa_L^*}\right)^2,$$

which is strictly negative for κ_L sufficiently close to κ_L^*. In either case, the property follows from the continuity of $g'(\kappa_L)$ on $[0, \kappa_L^*)$.

For the fourth property, for $\kappa_L < \kappa_L^*$, divide $g'(\kappa_L)$ by $-(\lambda - 1)\kappa_S'(\kappa_L)$, which is strictly positive, to see that $g'(\kappa_L) = 0$ if and only if

$$1 + \left(\frac{\kappa_L}{\kappa_L + \kappa_S(\kappa_L)}\right)^2 = \frac{\kappa_S(\kappa_L)}{\kappa_L + \kappa_S(\kappa_L)}\left(\epsilon_S\left(\frac{r}{\kappa_L + \kappa_S(\kappa_L)} + 1\right) + \frac{\kappa_S(\kappa_L)}{\kappa_L + \kappa_S(\kappa_L)}\right).$$

The left-hand side is strictly increasing in κ_L because $\kappa_S'(\kappa_L) < 0$, and the right-hand side is strictly decreasing in κ_L because $\kappa_S'(\kappa_L) \in [-1, 0)$, which implies that $\kappa_L + \kappa_S(\kappa_L)$ is increasing in κ_L. Therefore, there is at most one $\kappa_L < \kappa_L^*$ such that $g'(\kappa_L) = 0$. Finally, the second and third properties of $g(\cdot)$ in the theorem imply that there is at least one $\kappa_L < \kappa_L^*$ such that $g'(\kappa_L) = 0$.

A.3 Proof of Proposition 2

Suppose $X_L(\cdot)$ and $X_A(\cdot)$ are strictly convex and for all κ_L, $\epsilon_L(\kappa_L)(\epsilon_S(\kappa_S(\kappa_L)) + 1) \geq 1$. I follow a similar argument as in the proof of the second part of Proposition 1. I show that we can write the expected discounted profits a small firm earns from a good, Π, as a continuous function of the marginal value of a good to the large firm $\bar{V}$ that is continuously differentiable everywhere except at a single point. I then use Π to write the surplus from an acquisition S as a continuous function of $\bar{V}$ and τ_A that is continuously differentiable everywhere except at a single point. Next, I show that we can write the acquisition rate A as a continuous function of S that is continuously differentiable everywhere except at a single point. I use these functions, along with $\kappa_S(\cdot)$ from Proposition 1, $\kappa_L(\cdot)$ from the proof of Proposition 1, and equation (8) for $\bar{V}$ to show that the equilibrium value of $\bar{V}$ is a function of τ_A with the right properties so that the equilibrium values of A, κ_L, and g are functions of τ_A with the desired properties.

As in the proof of the second part of Proposition 1, equation (8) for $\bar{V}$ and Assumption 1 imply that $\bar{V} > 0$ in any equilibrium, which I impose throughout the proof. The functions $\kappa_S(\kappa_L)$ from Proposition 1 and $\kappa_L\left(\bar{V}/\beta\right)$ from the proof of the second part of Proposition 1, and equation (6) for the expected discounted profits a small firm earns from a good define a continuous function $\Pi\left(\bar{V}\right)$ on $\bar{V} > 0$, which is continuously differentiable everywhere except at the single point where $\kappa_L\left(\bar{V}/\beta\right) = \kappa_L^*$. Moreover, where it exists, $\Pi'\left(\bar{V}\right) \leq 0$ because Π depends inversely on $\kappa_L + \kappa_S$, and the proof of Proposition 1 shows that $\kappa_L'\left(\bar{V}/\beta\right) \geq 0$ and $\kappa_S'(\kappa_L) \geq -1$.

The function $\Pi\left(\bar{V}\right)$ defines a function for $\bar{V} > 0$ and $\tau_A > -1$ for the surplus from an acquisition, $S\left(\bar{V}, \tau_A\right) \equiv \bar{V} - (1+\tau_A)\Pi\left(\bar{V}\right)$, which is continuously differentiable with respect to τ_A, continuous everywhere in $\bar{V}$, and continuously differentiable with respect to $\bar{V}$ everywhere except at the single point where $\kappa_L\left(\bar{V}/\beta\right) = \kappa_L^*$.

The First Order Condition for the acquisition rate A (inequality (10)) implicitly defines a function of the surplus from an acquisition, $A(S)$, because due to the strict convexity of $X_A(\cdot)$, the left-hand side is strictly increasing in A and goes to infinity as A goes to infinity. Moreover, $A(\cdot)$ is continuous because $X_A'(\cdot)$ is continuous and strictly increasing, and so has a continuous inverse. Finally, I show that the derivative $A'(S)$ exists and is continuous everywhere except where $S = WX_A'(0)$. If $S < WX_A'(0)$, then the First Order Condition holds with a strict inequality, which implies that $A(S) = A'(S) = 0$. If $S > WX_A'(0) \geq 0$, then $A(S) > 0$ and the First Order Condition holds with equality, which given $\epsilon_A(\cdot) > 0$, implies that

$$WX_A''(A(S)) = \epsilon_A(A(S))\frac{S}{A(S)} > 0.$$

Thus, differentiating both sides of the First Order Condition yields the continuous derivative:

$$A'(S) = \frac{1}{\epsilon_A(A(S))}\frac{A(S)}{S}. \tag{30}$$

Now, define $T\left(\bar{V}\right)$ to be the terms in equation (8) not including A:

$$T\left(\bar{V}\right) \equiv \left(r + \kappa_L\left(\bar{V}/\beta\right) + \kappa_S\left(\kappa_L\left(\bar{V}/\beta\right)\right)\right)\bar{V} - W\beta X_L\left(\kappa_L\left(\bar{V}/\beta\right)\right) - (1-\sigma^{-1}-\tau_R)R + f_L W, \tag{31}$$

which, from the proof of the second part of Proposition 1, is strictly negative as $\bar{V}$ goes to 0, is continuous, and is continuously differentiable with a derivative greater than r everywhere except where $\kappa_L\left(\bar{V}/\beta\right) = \kappa_L^*$. We can write equation (8) for $\bar{V}$ using $T(\cdot)$, $\Pi(\cdot)$, $A(\cdot)$, and $S(\cdot,\cdot)$:

$$T\left(\bar{V}\right) + A\left(S\left(\bar{V}, \tau_A\right)\right)\bar{V} - WX_A\left(A\left(S\left(\bar{V}, \tau_A\right)\right)\right) - A\left(S\left(\bar{V}, \tau_A\right)\right)(1+\tau_A)\Pi\left(\bar{V}\right) = 0. \tag{32}$$

The left-hand side is differentiable with respect to $\bar{V}$ and τ_A everywhere except where $\kappa_L\left(\bar{V}/\beta\right) = \kappa_L^*$ and $S\left(\bar{V}, \tau_A\right) = WX_A'(0)$. Taking the derivatives and plugging in that if $A'(S) \neq 0$, then $S = WX_A'(A(S))$ yields

$$\begin{aligned}
\frac{\partial LHS}{\partial \bar{V}} &= T'\left(\bar{V}\right) + A\left(S\left(\bar{V}, \tau_A\right)\right)\left(1 - (1+\tau_A)\Pi'\left(\bar{V}\right)\right) \\
\frac{\partial LHS}{\partial \tau_A} &= -A\left(S\left(\bar{V}, \tau_A\right)\right)\Pi\left(\bar{V}\right).
\end{aligned} \tag{33}$$

The first derivative is greater then $r > 0$ because $T'\left(\bar{V}\right) \geq r$ and $\Pi'\left(\bar{V}\right) \leq 0$.

Equation (32) defines $\bar{V}$ as a continuous function of τ_A because the left-hand side is continuous in $\bar{V}$ and τ_A, and for a fixed τ_A, the left-hand side is strictly increasing in $\bar{V}$, goes to $T\left(\bar{V}\right)$, which goes to a strictly negative number as $\bar{V}$ goes to 0, and goes to infinity as $\bar{V}$ goes to infinity. Thus, there is a unique equilibrium value $\bar{V}$, which is continuous in τ_A, and using $\kappa_L(\cdot)$, $\kappa_S(\cdot)$, $\Pi(\cdot)$, $S(\cdot,\cdot)$, and $A(\cdot)$, there are unique equilibrium values κ_L, κ_S, Π, S, and A, which are also continuous in τ_A. The equilibrium value of $\bar{V}$ is increasing in τ_A because it is continuous in τ_A and differentiable everywhere except at finitely many points with derivative

$$\frac{\partial \bar{V}}{\partial \tau_A} = \frac{A\Pi}{T'\left(\bar{V}\right) + A\left(1 - (1+\tau_A)\Pi'\left(\bar{V}\right)\right)},$$

which is positive because $T'(\cdot) > 0$ and $\Pi'(\cdot) \leq 0$. Moreover, $\bar{V}$ is strictly increasing in τ_A if $A > 0$ because in that case, the derivative is strictly positive. The equilibrium value of S is strictly decreasing in τ_A because it is continuous in τ_A and differentiable everywhere except at finitely many points with derivative

$$\frac{\partial S}{\partial \tau_A} = \frac{\partial \bar{V}}{\partial \tau_A}\left(1 - (1+\tau_A)\Pi'\left(\bar{V}\right)\right) - \Pi = \frac{-T'\left(\bar{V}\right)\Pi}{T'\left(\bar{V}\right) + A\left(1 - (1+\tau_A)\Pi'\left(\bar{V}\right)\right)}, \tag{34}$$

which is strictly negative.

It follows from $\kappa_L(\cdot)$ that the equilibrium value of κ_L is increasing in τ_A, and strictly so if $\kappa_L, A > 0$ because in that case, $\bar{V}$ is strictly increasing in τ_A and $\kappa_L\left(\bar{V}\right)$ is strictly increasing. It follows from $A(\cdot)$ that the equilibrium value of A is decreasing in τ_A, and strictly so if $A > 0$ because in that case, $A(S)$ is strictly increasing.

Finally, suppose $\kappa_S, \kappa_L, A > 0$. The equilibrium values of $\bar{V}$ and S are continuously differentiable with respect to τ_A because in that case, $T(\cdot)$ and $\Pi(\cdot)$ are continuously differentiable. Moreover, the equilibrium values of κ_L and A are continuously differentiable with respect to τ_A because in that case, $\kappa_L(\cdot)$ and $A(\cdot)$ are continuously differentiable.

Finally, the results concerning the long-run growth rate g follow from equation (2) for the growth rate, evaluated on a balanced growth path, and equation (3) for the long-run large firm revenue share $\mathcal{L}$, which imply that, as long as $\kappa_L + A + \kappa_S > 0$ (always the case in equilibrium), g is a continuously differentiable function of κ_S, κ_L, and A that does not depend otherwise on τ_A.

A.4 Derivation of $\partial g / \partial \tau_A$

If κ_S, κ_L, and A are strictly positive, then using $\kappa_L' \left(\bar{V}/\beta\right)$ given by equation (27), the derivative of the equilibrium value of κ_L with respect to τ_A is

$$\frac{\partial \kappa_L}{\partial \tau_A} = \frac{1}{\epsilon_L(\kappa_L)} \frac{\kappa_L}{\bar{V}} \frac{\partial \bar{V}}{\partial \tau_A}. \tag{35}$$

Using $A'(S)$ given by equation (30), the derivative of the equilibrium value of $S = \bar{V} - (1+\tau_A)\Pi$ given by equation (34), $T'\left(\bar{V}\right)$ given by equation (29), $\kappa_S'(\kappa_L)$ given by equation (26), and $\kappa_L'\left(\bar{V}/\beta\right)$ given by equation (27), the derivative of the equilibrium value of A with respect to τ_A is

$$\frac{\partial A}{\partial \tau_A} = -\frac{1}{\epsilon_A(A)} \frac{1}{\bar{V} - (1+\tau_A)\Pi} \left(r + \kappa_L + \kappa_S - \frac{\kappa_S}{\epsilon_S(\kappa_S)(r + \kappa_L + \kappa_S) + \kappa_S} \frac{1}{\epsilon_L(\kappa_L)} \kappa_L \right) \frac{\partial \bar{V}}{\partial \tau_A}. \tag{36}$$

Using equation (2), evaluated on a balanced growth path, for the growth rate g, and equation (3) for the long-run large firm revenue share $\mathcal{L}$, the partial derivative of the growth rate with respect to the acquisition rate is

$$\frac{\partial g}{\partial A} = -\mathcal{L}(1-\mathcal{L}) \frac{\kappa_L}{\kappa_L + A}, \tag{37}$$

which is strictly negative. The partial derivative of the growth rate with respect to the large firm innovation rate, taking into account small firm optimization through $\kappa_S(\kappa_L)$, is

$$\frac{\partial g}{\partial \kappa_L} = (1-\mathcal{L}) \left(1 - \mathcal{L}\frac{\kappa_L}{\kappa_L + A}\right) - \left(\mathcal{L}^2 \frac{\kappa_L}{\kappa_L + A} + 1\right) \frac{\kappa_S}{\epsilon_S(\kappa_S)(r + \kappa_L + \kappa_S) + \kappa_S}. \tag{38}$$

Combining equations (35), (36), (37), and (38), the derivative of the equilibrium value of g with respect to τ_A is the product of

$$\mathcal{L}(1-\mathcal{L}) \frac{\kappa_L}{\kappa_L + A} \frac{1}{\bar{V} - (1+\tau_A)\Pi} \left(r + \kappa_L + \kappa_S - \frac{\kappa_S}{\epsilon_S(\kappa_S)(r + \kappa_L + \kappa_S) + \kappa_S} \frac{1}{\epsilon_L(\kappa_L)} \kappa_L \right) \frac{\partial \bar{V}}{\partial \tau_A},$$

which is strictly positive because $\epsilon_L(\kappa_L)(\epsilon_S(\kappa_S) + 1) \geq 1$, and

$$\frac{(1-\mathcal{L})((1-\mathcal{L})\kappa_L + A)(\epsilon_S(\kappa_S)(r + \kappa_L + \kappa_S) + \kappa_S) - ((1+\mathcal{L}^2)\,\kappa_L + A)\,\kappa_S}{\mathcal{L}(1-\mathcal{L})[(r + \kappa_L + \kappa_S)\epsilon_L(\kappa_L)(\epsilon_S(\kappa_S)(r + \kappa_L + \kappa_S) + \kappa_S) - \kappa_L \kappa_S]} \frac{\bar{V} - (1+\tau_A)\Pi}{\bar{V}} + \frac{1}{\epsilon_A(A)}. \tag{39}$$

The numerator in the first fraction in expression (39) is strictly negative if and only if, holding other variables constant, $\epsilon_S(\kappa_S)$ is sufficiently small, and the denominator is strictly positive because $\epsilon_L(\kappa_L)(\epsilon_S(\kappa_S) + 1) \geq 1$. If the numerator is negative, then the first fraction is strictly increasing in $\epsilon_S(\kappa_S)$ because an increase in $\epsilon_S(\kappa_S)$ makes the numerator less negative and increases the denominator, and is strictly increasing in $\epsilon_L(\kappa_L)$ because an increase in $\epsilon_L(\kappa_L)$ increases the denominator. Thus, if $\epsilon_S(\kappa_S)$ is sufficiently large, then the first fraction is positive, and the derivative of g with respect to τ_A is positive. If $\epsilon_S(\kappa_S)$ is sufficiently small, then the first fraction is strictly negative, and the derivative of g with respect to τ_A is strictly negative if and only if $\epsilon_A(A)$ is greater than a threshold. The threshold is strictly increasing in $\epsilon_S(\kappa_S)$ and $\epsilon_L(\kappa_L)$. Finally, expression (13) follows from setting $\epsilon_S(\kappa_S) = 0$.

A.5 Proof of Theorem 2

Suppose $\epsilon_S(\cdot)$, $\epsilon_L(\cdot)$, and $\epsilon_A(\cdot)$ are constants with $\epsilon_S = 0$, $\epsilon_L \geq 1$, and $\epsilon_A > 0$. It follows that $X_A'(0) = 0$ because otherwise $\epsilon_A(0) = 0$.

I first show that if $\mathcal{L} \in (0, 1)$ and there is an increase in f_L and/or τ_R, then the equilibrium $\mathcal{L}$ is strictly lower, the equilibrium marginal value of a good to the large firm, $\bar{V}$, is strictly lower, the equilibrium creative destruction rate of a small firm's good and expected discounted profits a small firm earns from a good, $\kappa_L + \kappa_S$ and Π, are constant, the equilibrium large firm innovation rate and acquisition rate, κ_L and A, are weakly lower, and the equilibrium sum $\kappa_L + A$ is strictly lower. I next show the existence of $\mathcal{L}^*$. Finally, I show the existence of $\mathcal{L}^{**}$.

Suppose $\mathcal{L} \in (0, 1)$. It follows that one of κ_L and A is strictly positive, and $\kappa_S > 0$. The proof of Proposition 2 shows that equation (32) must hold in equilibrium. It follows from the definition of $T(\cdot)$ in equation (31) that for a fixed $\bar{V}$, the left-hand side of equation (32) is strictly decreasing in $(1 - \sigma^{-1} - \tau_R)\, R - f_L W$, and does not otherwise depend on τ_R or f_L. Moreover, expression (33) shows that the left-hand side of equation (32) is strictly increasing in $\bar{V}$. It follows that if $(1 - \sigma^{-1} - \tau_R)\, R - f_L W$ is strictly higher, then the equilibrium value of $\bar{V}$ is strictly higher. Thus, using $\kappa_L \left(\bar{V}/\beta\right)$ defined implicitly by the First Order Condition for κ_L in the proof of the second part of Proposition 1, the equilibrium value of κ_L is higher, and strictly so if $\kappa_L > 0$. Next, $\epsilon_S = 0$ and $\kappa_S > 0$ imply that $\kappa_S'(\kappa_L) = -1$. It follows that the equilibrium value of the creative destruction rate of a small firm's good, $\kappa_L + \kappa_S$, and so the equilibrium value of the expected present discounted profits a small firm earns from a good, Π, are constant. Therefore, the equilibrium value of the surplus from an acquisition, $\bar{V} - (1 + \tau_A)\Pi$, is strictly higher, which implies, using $A(S)$—the acquisition rate as a function of the surplus from an acquisition—defined implicitly by the First Order Condition for A, that the equilibrium value of A is higher, and strictly so if $A > 0$. Finally, it follows that the equilibrium value of the long-run large firm revenue share $\mathcal{L}$ is strictly higher because if $\kappa_L > 0$, then the numerator of $1 - \mathcal{L}$, κ_S, is strictly lower, and the denominator, $\kappa_L + A + \kappa_S$, is higher, or if $A > 0$, then the numerator is lower and the denominator is strictly higher.

The equilibrium variables mentioned are continuous functions of $(1-\sigma^{-1}-\tau_R)\,R-f_LW$ because the proof of Proposition 2 shows that equation (32) is continuous in $(1-\sigma^{-1}-\tau_R)\,R-f_LW$ and in $\bar{V}$, and that all the functions mentioned are continuous.

It follows that there exists an $\mathcal{L}^*$ such that $A>0$ if and only if $\mathcal{L}>\mathcal{L}^*$ because if $\mathcal{L}$ is higher, then so is $(1-\sigma^{-1}-\tau_R)\,R-f_LW$, and thus so is A.

From the derivation in Appendix A.4, the sign of $\partial g/\partial\tau_A$ is the sign of expression (13). From the above arguments, it follows that if $\mathcal{L}$ is strictly larger, then so is $(1-\sigma^{-1}-\tau_R)\,R-f_LW$. Thus, expression (13) is strictly lower because $1-\mathcal{L}$ is strictly lower, $2\kappa_L+A$ is strictly higher, $\epsilon_L(r+\kappa_L+\kappa_S)$ is constant, $-\kappa_L$ is lower, Π is constant, and $\bar{V}-(1+\tau_A)\Pi$ is strictly higher. Hence, if $\mathcal{L}^*<1$, then there exists an $\mathcal{L}^{**}$ such that $\partial g/\partial\tau_A<0$ if and only if $\mathcal{L}>\mathcal{L}^{**}$. Moreover, $\mathcal{L}^{**}>\mathcal{L}^*$ because if $\mathcal{L}^*\geq 0$ and $\mathcal{L}=\mathcal{L}^*$, then $\bar{V}-(1+\tau_A)\Pi=0$ (since $X_A'(0)=0$), which implies that expression (13) is strictly positive. Next, $\mathcal{L}^{**}>0$ because if $\mathcal{L}=0$, then $2\kappa_L+A=0$, which implies that expression (13) is strictly positive. Finally, $\mathcal{L}^{**}<1$ because if $\mathcal{L}=1$, then $1-\mathcal{L}=0$, which implies that expression (13) is strictly negative.

A.6 Quantitative Model Proofs

To make the proof of Theorem 3 more readable, I split it into a few steps. I begin with the following proposition.

Proposition 3. *On a balanced growth path equilibrium, the long-run industry revenue share of the large firm in an industry is a sufficient statistic for that industry's growth rate in the sense that two industries with the same $\mathcal{L}_n$ have the same g_n. Moreover, we can write the industry growth rate as a continuously differentiable function $g(\mathcal{L}_n)$ defined on $[0,1)$ such that $g(0)>0$ and $g'(0)>0$.*

Proof. Throughout the proof, I drop time t subscripts when possible because the theorem is concerned with a balanced growth path. To prove the theorem, I decompose the industry growth rate into total innovation and the growth share of total innovation, i.e., the composition of innovation: $(\gamma-1)g_n=I_n(\gamma-1)g_n/I_n$, where total innovation is

$$I_n\equiv N\kappa_{S,n}+\kappa_{L,n}(S)+(\gamma-1)g_n. \tag{40}$$

I first show that total innovation and the growth share of innovation are differentiable functions of the large firm industry revenue share, $\mathcal{L}_n$. In particular, they are not functions of the large firm cost of innovation, $\chi_{L,n}$, or the large firm fixed cost, $\chi_{F,L,n}$. It follows that the growth rate is as well, and its derivative with respect to the large firm industry revenue share is

$$\frac{\partial(\gamma-1)g_n}{\partial\mathcal{L}_n} = \frac{(\gamma-1)g_n}{I_n}\frac{\partial I_n}{\partial\mathcal{L}_n} + I_n\frac{\partial(\gamma-1)g_n/I_n}{\partial\mathcal{L}_n}. \tag{41}$$

I use small firm optimal innovation to show that total innovation is a continuously differentiable function of $\mathcal{L}_n$ for all $\mathcal{L}_n \in [0,1)$. Small firm innovation is a function of the present discounted profits a small firm earns from a good with relative productivity 1 divided by the wage. Present discounted profits at time t are given by the HJB equation (20). On a balanced growth path, the interest rate is $r = \rho + g$, aggregate output is $C_{t'} = Z_t e^{g(t'-t)} L^p$ for all $t' \geq t$, and the first term on the second line is zero because the industry state is constant over time. We can thus guess and verify that present discount profits are

$$\pi_{S,n,t} = (1-\sigma^{-1})C_t/(\rho + I_n) \tag{42}$$

because then $\dot{\pi}_{S,n,t} = g\pi_{S,n,t}$. Since the wage is $W_t = Z_t/\sigma$, it follows that present discounted profits over the wage is

$$\pi_{S,n,t}/W_t = (\sigma-1)L^p/(\rho + I_n), \tag{43}$$

which is not a function of t. Equation (17) for the evolution of the industry state over time, evaluated on a balanced growth path, shows that we can write total innovation in terms of only small firm innovation:

$$I_n = \lambda^{\gamma-1}N(\kappa_{S,n} + \delta_{S,n})/(1-\mathcal{L}_n).$$

Thus, using equation (21) for optimal small firm innovation, it follows that

$$I_n = \frac{\lambda^{\gamma-1}N}{1-\mathcal{L}_n}\left(\chi_C^{\frac{-1}{\epsilon}} + 1\right)\left(\frac{\lambda^{\gamma-1}(\sigma-1)L^p}{\rho + I_n}\right)^{\frac{1}{\epsilon}}. \tag{44}$$

Thus, I_n is a strictly increasing continuously differentiable function of $\mathcal{L}_n$ on $[0,1)$.

To see that the growth share of innovation is a differentiable function of the large firm industry revenue share, use equation (17) for the evolution of the industry state over time, evaluated on a balanced growth path, to write total innovation as $I_n = I_{L,n}/\mathcal{L}_n = I_{S,n}/(1-\mathcal{L}_n)$, where $I_{L,n}$ and $I_{S,n}$ are total innovation from the large firm and small firms, respectively:

$$I_{L,n} \equiv \kappa_{L,n}(S) + (\gamma-1)g_{L,n} \qquad\qquad I_{S,n} \equiv N(\kappa_{S,n} + (\gamma-1)g_{S,n}).$$

It follows that the growth share of innovation is

$$\frac{(\gamma-1)g_n}{I_n} = \mathcal{L}_n\frac{(\gamma-1)g_{L,n}}{I_{L,n}} + (1-\mathcal{L}_n)\frac{N(\gamma-1)g_{S,n}}{I_{S,n}}. \tag{45}$$

Equation (21) shows that $(\gamma-1)g_{S,n}/I_{S,n}$ is a strictly positive constant, and equation (23) shows that $(\gamma-1)g_{L,n}/I_{L,n}$ is a strictly decreasing continuously differentiable function of $\mathcal{L}_n$ on $[0,1)$ that is equal to $(\gamma-1)g_{S,n}/I_{S,n}$ at $\mathcal{L}_n=0$. Thus, the growth share of innovation is a strictly decreasing continuously differentiable function of $\mathcal{L}_n$ on $[0,1)$.

Now, I show that as $\mathcal{L}_n$ goes to 0, the industry growth rate converges to a strictly positive number. Equation (44) shows that as $\mathcal{L}_n$ goes to 0, total innovation I_n converges to a strictly positive number. Equation (45) then shows that the industry growth rate converges to the product of that strictly positive number and $N(\gamma-1)g_{S,n}/I_{S,n}$, which is strictly positive.

Next, I show that as $\mathcal{L}_n$ goes to 0, the derivative of the industry growth rate with respect to $\mathcal{L}_n$ converges to a strictly positive number. Since total innovation I_n is strictly increasing, and the growth share of innovation at $\mathcal{L}_n=0$ is $N(\gamma-1)g_{S,n}/I_{S,n}$, which is strictly positive, it follows that the first term in the derivative in (41) is strictly positive at $\mathcal{L}_n=0$. Equation (23) shows that the derivative of $(\gamma-1)g_{L,n}/I_{L,n}$ with respect to $\mathcal{L}_n$ is a finite number at $\mathcal{L}_n=0$. Thus, (45) shows that the derivative of $(\gamma-1)g_n/I_n$ is equal to 0 at $\mathcal{L}_n=0$. It follows from (41) that the derivative of the industry growth rate is strictly positive at $\mathcal{L}_n=0$. ■

To proceed in the proof of Theorem 3, I first show that if ϵ (the curvature of the innovation cost function) is sufficiently small (including 1), then the industry growth rate goes to 0 as the large firm industry revenue share goes to 1. I then show that if $\epsilon=1$, then a function with the same sign as the derivative of the industry growth rate with respect to the large firm industry revenue share is strictly decreasing. I finally prove that Theorem 3 follows.

Proposition 4. *If* $\epsilon \in (0,(3+\sqrt{5})/2-1)$*, then* $\lim_{z\to 1}(g(z))=0$*. If* $\epsilon > (3+\sqrt{5})/2-1$*, then* $\lim_{z\to 1}(g(z))=\infty$*.*

Proof. First, I show that as $\mathcal{L}_n$ goes to 1, total innovation I_n converges to the product of a strictly positive finite number and $(1-\mathcal{L}_n)^{\frac{-\epsilon}{\epsilon+1}}$. Equation (44) shows that as $\mathcal{L}_n$ goes to 1, I_n diverges to positive infinity. Multiplying each side of (44) by $(\rho+I_n)^{\frac{1}{\epsilon}}(1-\mathcal{L}_n)$ then shows that as $\mathcal{L}_n$ goes to 1, $I_n^{\frac{\epsilon+1}{\epsilon}}(1-\mathcal{L}_n)$ converges to a strictly positive finite number. The result follows.

Next, equations (45) and (23) show that as $\mathcal{L}_n$ goes to 1, the growth share of innovation $(\gamma-1)g_n/I_n$ converges to the product of a strictly positive finite number and $(1-\mathcal{L}_n)^{\frac{1}{\epsilon}}$ because that is the lowest power of $1-\mathcal{L}_n$ contained in any term.

It follows that the industry growth rate g_n converges to the product of a strictly positive finite number and $(1-\mathcal{L}_n)^{\frac{1}{\epsilon}+\frac{-\epsilon}{\epsilon+1}}$. The exponent on $1-\mathcal{L}_n$ is

$$\frac{1}{\epsilon}+\frac{-\epsilon}{\epsilon+1}=\frac{(\epsilon-\epsilon_1^*)(\epsilon_2^*-\epsilon)}{(\epsilon)(\epsilon+1)},$$

where

$$\epsilon_1^*=(3-\sqrt{5})/2-1 \qquad\qquad \epsilon_2^*=(3+\sqrt{5})/2-1.$$

Since $(3-\sqrt{5})/2<1$, it follows that for all $\epsilon\in(0,\epsilon_2^*)$, the industry growth rate converges to 0 as $\mathcal{L}_n$ goes to 1, and for all $\epsilon>\epsilon_2^*$, the industry growth rate diverges to infinity as $\mathcal{L}_n$ goes to 1. ■

Proposition 5. *If $\epsilon=1$, then there exists a strictly decreasing differentiable function $G(\cdot)$ defined on $[0,1)$ and a strictly positive differentiable function $h(\cdot)$ defined on $[0,1)$ such that for all $z\in[0,1)$, $g'(z)=h(z)G(z)$.*

Proof. Given $\epsilon=1$, equations (23) simplify to

$$\frac{(\gamma-1)g_{L,n}}{\kappa_{L,n}}=\left(\lambda^{\gamma-1}\chi_C+\lambda^{\gamma-1}-1\right)\frac{\lambda^{\gamma-1}(1-\mathcal{L}_n}{\lambda^{\gamma-1}(1-\mathcal{L}_n)+\mathcal{L}_n}. \tag{46}$$

From (44), total innovation is given by the quadratic equation

$$I_n^2+\rho I_n-A/(1-\mathcal{L}_n)=0,$$

where $A\equiv\lambda^{\gamma-1}(1/\chi_C+1)\lambda^{\gamma-1}(\sigma-1)L^p$. The quadratic equation has a unique positive solution, which must therefore be I_n:

$$I_n=-\rho/2+\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}.$$

Next, let B denote small firm growth relative to creative destruction from (21):

$$B\equiv(\gamma-1)g_{S,n}/\kappa_{S,n}=\lambda^{\gamma-1}\chi_C+\lambda^{\gamma-1}-1.$$

Then, from (45) and (46), the growth share of total innovation is

$$\frac{(\gamma-1)g_n}{I_n}=\mathcal{L}_n\left(B^{-1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}(1-\mathcal{L}_n)}+B^{-1}+1\right)^{-1}+(1-\mathcal{L}_n)(B^{-1}+1)^{-1}.$$

Differentiating total innovation and the growth share of total innovation with respect to the large firm industry revenue share yields

$$\frac{\partial I_n}{\partial\mathcal{L}_n}=\frac{1}{1-\mathcal{L}_n}\frac{A/(1-\mathcal{L}_n)}{2\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}}$$

and

$$\frac{\partial(\gamma-1)g_n/I_n}{\partial\mathcal{L}_n} = \left(B^{-1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}(1-\mathcal{L}_n)}+B^{-1}+1\right)^{-1} - (B^{-1}+1)^{-1}$$
$$-\frac{1}{1-\mathcal{L}_n}\left(B^{-1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}(1-\mathcal{L}_n)}+B^{-1}+1\right)^{-2} B^{-1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}(1-\mathcal{L}_n)}.$$

Thus, writing the derivative as

$$\frac{\partial(\gamma-1)g_n}{\partial\mathcal{L}_n} = \frac{(\gamma-1)g_n}{I_n}\frac{\partial I_n}{\partial\mathcal{L}_n} + I_n\frac{\partial(\gamma-1)g_n/I_n}{\partial\mathcal{L}_n}$$

and multiplying by $(1-\mathcal{L}_n)\left(B^{-1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}(1-\mathcal{L}_n)}+B^{-1}+1\right)\left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+1-\mathcal{L}_n\right)$, which is strictly positive for all $\mathcal{L}_n \in [0,1)$, yields a function $G(\mathcal{L}_n)$ that has the same sign as the derivative of growth with respect to $\mathcal{L}_n$:

$$G(\mathcal{L}_n) = \left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+1-\mathcal{L}_n\right)\left(1+\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}\right)\frac{A/(1-\mathcal{L}_n)}{2\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}}$$
$$-\left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+1-\mathcal{L}_n+1\right)\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}\left(-\rho/2+\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}\right).$$

The derivative of $G(\mathcal{L}_n)$ is

$$G'(\mathcal{L}_n) = \left(-1+2\left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+1-\mathcal{L}_n\right)\frac{1}{B+1}\frac{1}{\lambda^{\gamma-1}}\right)\frac{A/(1-\mathcal{L}_n)}{2\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}}$$
$$+\left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+1-\mathcal{L}_n\right)\left(1+\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}\right)\frac{A/(1-\mathcal{L}_n)}{2\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}}\frac{1-\frac{1}{2}\frac{A/(1-\mathcal{L}_n)}{(\rho/2)^2+A/(1-\mathcal{L}_n)}}{1-\mathcal{L}_n}$$
$$-2\left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+1-\mathcal{L}_n\right)\frac{1}{B+1}\frac{1}{\lambda^{\gamma-1}}\left(-\rho/2+\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}\right)$$
$$-\left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+2-\mathcal{L}_n\right)\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}\frac{A/(1-\mathcal{L}_n)}{2\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}}\frac{1}{1-\mathcal{L}_n}.$$

I show that $G'(\mathcal{L}_n) < 0$, and so complete the proof, in two steps. First, the positive term on the first line is outweighed by the third line because

$$-\rho/2+\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)} > \frac{A/(1-\mathcal{L}_n)}{2\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}}.$$

To see that the inequality holds, add $\rho/2$ to each side, multiply each side by $2\sqrt{(\rho/2)^2+A/(1-\mathcal{L}_n)}$, subtract $A/(1-\mathcal{L}_n)$ from each side, square each side, and subtract $\rho^4/4+\rho^2 A/(1-\mathcal{L}_n)$ from each side, to get that the inequality is equivalent to $(A/(1-\mathcal{L}_n))^2 > 0$. Second, we can write the beginning of the second line of $G'(\mathcal{L}_n)$ as

$$\left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+1-\mathcal{L}_n\right)\left(1+\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}\right) = 1-\mathcal{L}_n+\left(\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}+2-\mathcal{L}_n\right)\frac{1}{B+1}\frac{\mathcal{L}_n}{\lambda^{\gamma-1}}.$$

The $1-\mathcal{L}_n$ term is outweighed by the negative term in the first line of $G'(\mathcal{L}_n)$ because the final term on the second line is strictly less than $1/(1-\mathcal{L}_n)$. The remaining term is outweighed by the fourth line of $G'(\mathcal{L}_n)$ for the same reason. ■

Finally, I complete the proof of Theorem 3.

Proof. Since $g'(0) > 0$ and $g(0) > \lim_{z \to 1} g(z)$, it is sufficient to show that $g(\cdot)$ is single peaked, i.e., there exists a z^* such that $g'(z) > 0$ if and only if $z < z^*$. Since $g(\cdot)$ is continuously differentiable it is sufficient to show that if $g'(z) = 0$, then $g''(z) < 0$. If $g'(z) = 0$, then $g'\prime(z) = h'(z)g'(z)/h(z) + h(z)G'(z)$. Since $h(z) > 0$ and $G(\cdot)$ is strictly decreasing, it follows that the first term is zero and the second term is strictly negative. Thus if $g'(z) = 0$, then $g'\prime(z) < 0$, completing the proof of part of Theorem 3. ■

I now use the proof of Proposition 3 to prove Theorem 3.

Proof. Throughout the proof, I drop time and industry subscripts when possible because the theorem is concerned with a balanced growth path in which all industries are identical.

In the proof of Proposition 3, I decompose the growth rate on a balanced growth path into total innovation I, defined in (40), and the growth share of total innovation $(\gamma - 1)g/I$. I show that the latter is a strictly decreasing continuously differentiable function of the large firm industry revenue share $\tilde{Z}_L$ on $[0, 1)$, and does not depend separately on the large firm cost of innovation or fixed cost. It is therefore sufficient to show that on a balanced growth path with identical industries, the common value of total innovation I is not a function of the common large firm industry revenue share $\tilde{Z}_L$ (or the large firm cost of innovation or fixed cost). To do so, I use the small firm free entry condition.

The cost of entry at time t is $\chi_E W_t$, and the value of entering is the value of being a small firm with zero goods at time t, $V_{S,t}$. Hence, the value of being a small firm with zero goods relative to the wage on a balanced growth path is given by $V_{S,t}/W_t = \chi_E$, which is a constant.

The value of being a small firm with zero goods at time t is given by the HJB equation (22). Optimal small firm creative destruction and new good development are given by (21). Plugging in (43) for the present discounted value of profits relative to the wage on a balanced growth path, we have that small firm innovation rates are not a function of time t:

$$\delta_S = \left(\lambda^{\gamma-1} \frac{(\sigma - 1)L^p}{\rho + I} \right)^{\frac{1}{\epsilon}}$$
$$\kappa_S = \chi_C^{\frac{-1}{\epsilon}} \left(\lambda^{\gamma-1} \frac{(\sigma - 1)L^p}{\rho + I} \right)^{\frac{1}{\epsilon}}.$$

Equation (42) for small firm profits on a balanced growth path shows that for all $t' \geq t$,

$$\pi_{S,t'} = \frac{(1-\sigma^{-1})C_t}{\rho + I} e^{g(t'-t)}$$

because aggregate output on a balanced growth path is $C'_t = Z_t e^{g(t'-t)} L^p$. We can thus guess and verify that the value of being a small firm with zero goods at time t is given by

$$(\rho+\eta)V_{S,t} = (1 - 1/(\epsilon+1)) \left(\chi_C^{\frac{-1}{\epsilon}} + 1 \right) \left(\lambda^{\gamma-1} \frac{(\sigma-1)L^p}{\rho+I} \right)^{\frac{1}{\epsilon}} \lambda^{\gamma-1} \frac{(1-\sigma^{-1})C_t}{\rho+I}$$

because then $\dot{V}_{S,t} = gV_{S,t}$ (recall that $r = \rho + g$). Since the wage is $W_t = Z_t/\sigma$, and using that $V_{S,t}/W_t = \chi_E$, it follows that

$$(\rho+\eta)\chi_E = (1 - 1/(\epsilon+1)) \left(\chi_C^{\frac{-1}{\epsilon}} + 1 \right) \left(\lambda^{\gamma-1} \frac{(\sigma-1)L^p}{\rho+I} \right)^{\frac{(\epsilon+1)}{\epsilon}},$$

which determines total innovation I as a function of exogenous parameters not including the large firm cost of innovation or fixed cost. ■