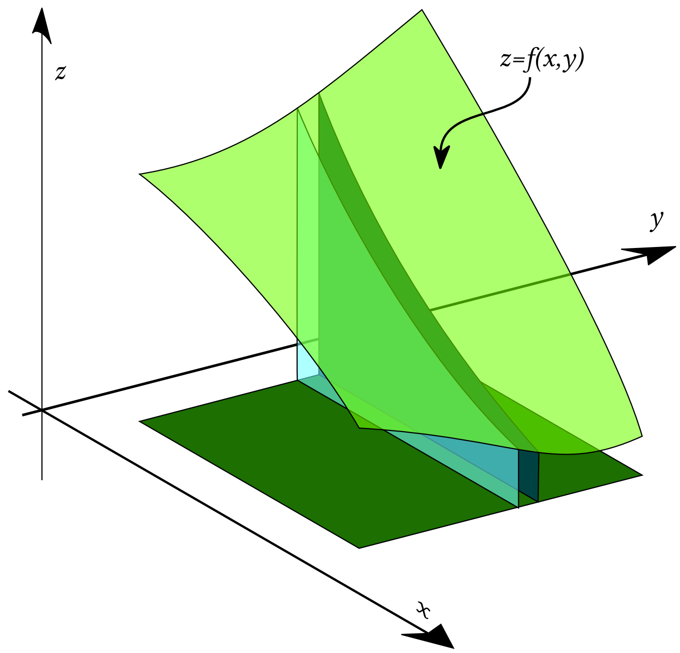


MATH 234
THIRD SEMESTER CALCULUS

Fall 2015

Math 234 – 3rd Semester Calculus
Lecture notes version 0.9.1(Fall 2015)

This is a self contained set of lecture notes for Math 234. The notes were written by Sigurd Angenent, some problems were taken from Guichard's open calculus text which is available at http://www.whitman.edu/mathematics/multivariable/src/

The LaTeX files, as well as the PYTHON and INKSCAPE-SVG files that were used to produce the notes before you can be obtained from the following web site:

http://www.math.wisc.edu/~angenent/Free-Lecture-Notes

They are meant to be freely available for non-commercial use, in the sense that "free software" is free. More precisely:

Contents

CHAPTER 1

Vector Geometry in Three dimensional space

1. Three dimensional space

The world according to our first and second semester calculus courses is flat: except for a brief digression about surfaces of revolution, everything that we discussed in Math 221 and 222 took place in the (x, y)-plane. All curves were curves in the plane and all functions had graphs that were curves in the plane. This semester we leave two dimensions behind and enter the three dimensional world. In order to understand the objects we will be dealing with, such as curves that are free to loop around in space, or functions whose graphs are themselves two dimensional curved surfaces, we will first review some three dimensional geometry. In particular, we will review the use of vectors in three dimensional geometry.

2. Geometric description of vectors

2.1. Points and their coordinates. We are used to describing the location of any point in the plane by choosing two perpendicular "coordinate axes" (the x and y axes), and specifying the corresponding (x, y)-coordinates of any given point. In the same way we can describe where points are in three dimensional space by choosing three mutually perpendicular axes, which we call the x, y, and z axes. To say where some given point P is, we travel from the origin to P, first along the x axis, then parallel to the y-axis, and finally parallel to the z-axis. The distances we had to go in the x, y, and z directions are the x, y, and z coordinates of our point P.

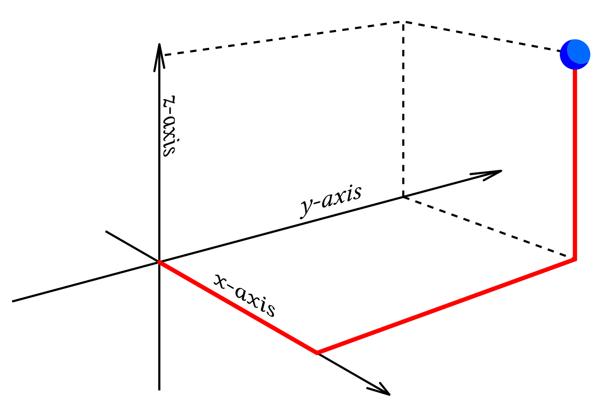


Figure 1. To determine the location of points in three dimensional space (such as the center of the blue sphere in this drawing), we should choose three coordinate axes, and specify three numbers: the x, y, and z coordinates of the point.

2.2. Vectors. While points and their coordinates are used to described locations in space, vectors are used to describe ***displacements***, i.e. how to go from one point to another. Such a displacement has a size (how far we have to go), and a direction (which way do we go). Vectors also get used in non-geometric situations to describe objects that have size and direction, e.g. velocities and forces in physics are typical examples of vector-like objects.

Informal definition of "vectors". We will think of a vector as an arrow connecting two points. If the points are A and B then we call the vector $\overrightarrow{AB}$. If we translate a vector $\overrightarrow{AB}$ ***without turning it*** then we say that the resulting vector $\overrightarrow{CD}$ is the same vector as the original vector $\overrightarrow{AB}$. A more precise way of saying that we should be able to move $\overrightarrow{AB}$ "without turning," is to insist that the line segments AB and CD should be parallel, and have the same length and orientation.

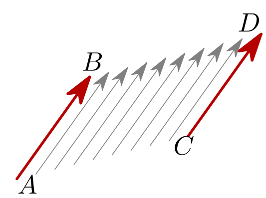


Figure 2. This figure contains ***four*** points (A, B, C, D), ***two*** line segments (AB and CD), but ***only one*** vector since $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector: $\overrightarrow{AB} = \overrightarrow{CD}$.

We say that the arrows $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ ***both represent the same vector.*** Since both $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are the same vector we will often want to use a notation for vectors that does not emphasize any particular choice of initial- and endpoint. The notation we will use in this course is

$$\vec{a} = \overrightarrow{AB} = \overrightarrow{PQ},$$

i.e., a single letter with an arrow on top will always stand for a vector in this course.

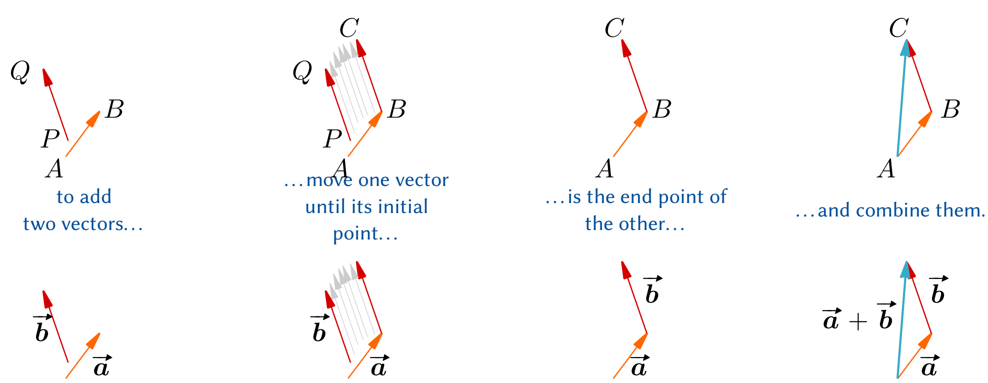


Figure 3. Adding vectors

3. Arithmetic of vectors

To add two vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ we first translate the vector $\overrightarrow{PQ}$ so that its initial point becomes B; let the result of this translation be the vector $\overrightarrow{BC}$. Then, by definition, the sum of $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ is $\overrightarrow{AC}$: in a formula,

$$\overrightarrow{AB} + \overrightarrow{PQ} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

An equivalent way of adding two vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ is to move the vectors around until they have the same initial point. Two vectors with a common initial point form two sides of a parallelogram (see Figure 4) and the sum of the two vectors is the diagonal of that parallelogram.

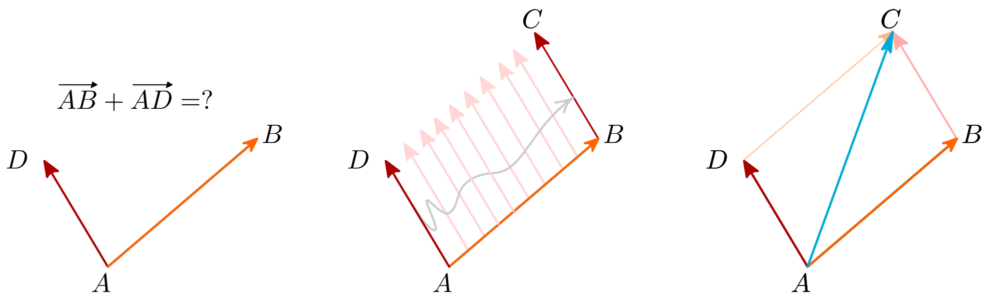


Figure 4. Using a parallelogram to add vectors. To find $\overrightarrow{AB}+\overrightarrow{AD}$ we move the vector $\overrightarrow{AD}$ so that its initial point is at B, i.e. the endpoint of $\overrightarrow{AB}$. This gives us a parallelogram $ABCD$, where $\overrightarrow{AD} = \overrightarrow{BC}$. Therefore $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

One can also multiply vectors with numbers. To multiply a vector $\vec{a}$ with a positive real number $t > 0$, we multiply the length of the vector by a factor t, without changing the direction of the vector.

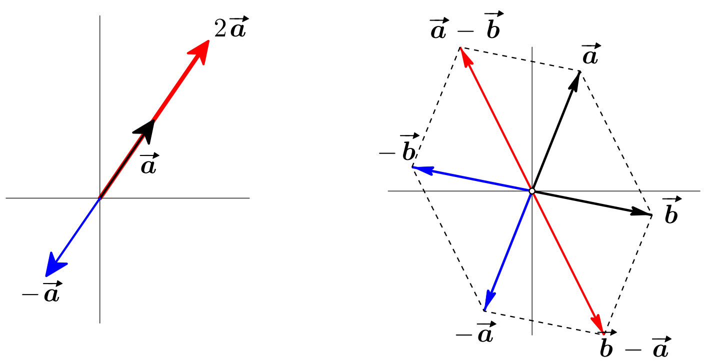


Figure 5. Multiplying and subtracting vectors

4. Vector algebra

The addition and multiplication of vectors and numbers satisfy a number of algebraic properties that should look familiar, as they are very similar to the usual algebraic properties for adding and multiplying numbers. Here they are:

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$		commutative law
$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$	$t \cdot (s \cdot \vec{a}) = (ts) \cdot \vec{a}$	associative laws
$t \cdot (\vec{a} + \vec{b}) = t\vec{a} + t\vec{b}$	$(t+s)\vec{a} = t\vec{a} + s\vec{a}$	distributive laws

5. Component representation of vectors

5.1. Components of a vector in two dimensional space. There is a way to represent a vector by specifying a list of numbers instead of by giving a geometric description of the vector. To do this for vectors in the plane, we must choose two perpendicular coordinate axes (the "x" and "y" axes). We define

$$\begin{aligned}\vec{\boldsymbol{e}}_1 &= \text{vector with length } 1\text{, in the direction of the } x \text{ axis}\\ \vec{\boldsymbol{e}}_2 &= \text{vector with length } 1\text{, in the direction of the } y \text{ axis}\end{aligned}$$

Then any other vector can be written as the sum of a multiple of $\vec{\boldsymbol{e}}_1$ and another multiple of $\vec{\boldsymbol{e}}_2$:

$$\vec{\boldsymbol{a}} = a_1\vec{\boldsymbol{e}}_1 + a_2\vec{\boldsymbol{e}}_2. \tag{1}$$

See Figure 6. The numbers a_1 and a_2 are called the ***components of the vector $\vec{\boldsymbol{a}}$.*** If we know the components a_1 and a_2 of a vector, and if we know the two vectors $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}_2$, then we can reconstruct the vector $\vec{\boldsymbol{a}}$ by using the formula (1).

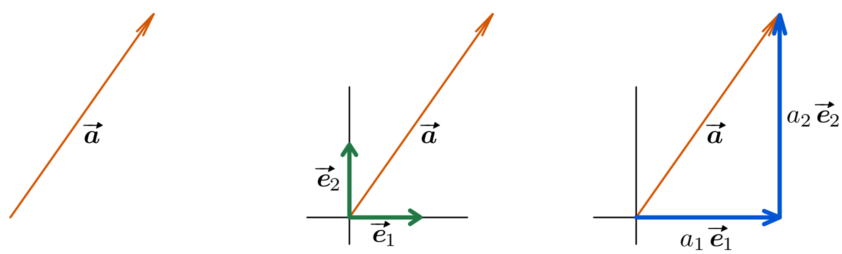


Figure 6. Describing a vector in terms of its components.

Instead of using the notation (1), one very often writes

$$\vec{\boldsymbol{a}} = \begin{pmatrix}a_1\\a_2\end{pmatrix}, \text{ or } \vec{\boldsymbol{a}} = \begin{bmatrix}a_1\\a_2\end{bmatrix}, \text{ or } \vec{\boldsymbol{a}} = \langle a_1, a_2\rangle. \tag{2}$$

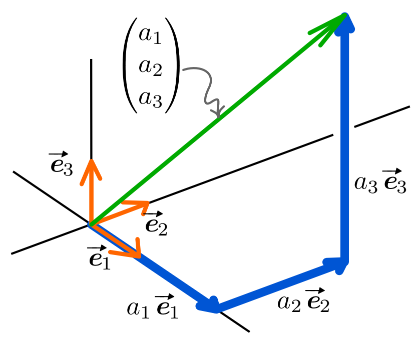
This notation says that $\vec{\boldsymbol{a}}$ is the vector whose components are a_1 and a_2. Since the two vectors $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}_2$ depend on our choice of coordinate axes, we can only use the component notation if it is clear to everyone how we chose the coordinate axes.

The first way of writing the vector, in which the components a_1 and a_2 are listed in a column enclosed in either parentheses or square brackets, is the standard way of writing "column vectors," and is used in linear algebra courses (math 320, 340, 341, etc.), as well as by most computational software (Matlab$^{\text{TM}}$, Octave, etc.). The other way of writing the components, i.e. as $\langle a_1, a_2\rangle$, also gets used, especially when one has to type the equations rather than write them by hand.

5.2. Components of a vector in three dimensional space. The preceding also applies to vectors in three dimensional space: instead of choosing two coordinate axes we choose three axes, and call them the x, y, and z axes (or, the x_1, x_2, and x_3 axes). Then we define $\vec{\boldsymbol{\imath}}$, $\vec{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{k}}$ (or $\vec{\boldsymbol{e}}_1$, $\vec{\boldsymbol{e}}_2$, and $\vec{\boldsymbol{e}}_3$) to be vectors of length one in the direction of the three coordinate axes. A vector $\vec{a}$ in space can then be written as a combination of the three vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$, namely,

$$\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}, \text{ or } \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

The $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ notation is more systematic, but the $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ notation, which was intro-



The vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is

$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$

Figure 7. Components of a vector in three dimensional space

duced into vector geometry and vector calculus by J.W.Gibbs, is also very common.

Josiah Willard Gibbs
1839–1903
https://en.wikipedia.org/wiki/Josiah_Willard_Gibbs

5.3. Length of a vector whose components are given. We will write

$$\|\vec{a}\|$$

for the length of a vector $\vec{a}$. If the vector is given in components,

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2, \text{ or } \vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3,$$

then the length of the vector is determined by Pythagoras' law (see Figures 6 and 7):

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}, \quad \text{or} \quad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{3}$$

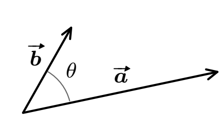
6. The dot product

There are two different descriptions of the dot product of two vectors: one geometric, and the other in terms of the components of the vectors.

6.1. Geometric description of the dot product. If $\vec{a}$ and $\vec{b}$ are two given vectors, then, by definition,

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\,\cos\theta, \tag{4}$$

where θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$.



The dot product between two vectors.

6.2. The dot product in terms of vector components. If we choose an orthonormal set of vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, and write

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

then

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3. \tag{5}$$

The fact that (4) and (5) always give the same result is not obvious (the formulas look very different), and requires a proof. A very common proof relies on the law of cosines (it was given in math 222 – see also Problem 12.17)

6.3. Algebraic properties of the dot product. The dot product has the following algebraic properties, which we will use very often throughout this course:

$$\begin{aligned} \vec{a}\cdot\vec{b} &= \vec{b}\cdot\vec{a} && \text{commutative} \\ s(\vec{a}\cdot\vec{b}) &= (s\vec{a})\cdot\vec{b} && \text{associative} \\ (\vec{a}+\vec{b})\cdot\vec{c} &= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}. && \text{distributive} \end{aligned}$$

We will not prove these properties here. Proofs can be given if one starts either from the algebraic description of the dot-product (5), or from the geometric description (4) (although the distributive property is more difficult to prove from the geometric description than from the algebraic description.)

The sign of the dot product tells us if the angle between two vectors is acute, obtuse, or if the vectors are perpendicular:

$$\vec{a} \perp \vec{b} \iff \vec{a}\cdot\vec{b} = 0 \tag{6a}$$

$$\vec{a}\cdot\vec{b} > 0 \iff \theta < \frac{\pi}{2} \tag{6b}$$

$$\vec{a}\cdot\vec{b} < 0 \iff \theta > \frac{\pi}{2}. \tag{6c}$$

7. The cross product

As with the dot product, the cross product of two vectors also has a geometric description, and a description in terms of components.

7.1. Geometric description of the cross product. Let $\vec{a}$ and $\vec{b}$ be two vectors in three dimensional space, then their ***cross product*** is the vector $\vec{a}\times\vec{b}$ that satisfies

- $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$, and also to $\vec{b}$
- the length of $\vec{a}\times\vec{b}$ is given by

  $$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\,\sin\theta,$$

  where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$,
- the three vectors $\vec{a}$, $\vec{b}$, $\vec{a}\times\vec{b}$ satisfy the ***right hand rule:*** if on your right hand $\vec{a}$ is the index finger and $\vec{b}$ is the middle finger, then your thumb points in the direction of $\vec{a}\times\vec{b}$. See Figure 8.

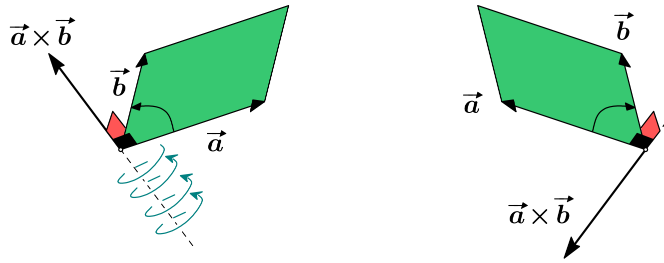
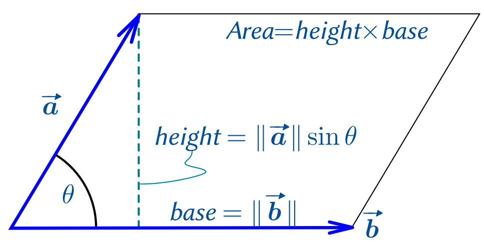


Figure 8. The cross product: $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$; its direction follows from the right-hand rule.

The length of the cross product of two vectors has a geometric interpretation. Namely, the quantity $\|\vec{a}\|\,\|\vec{b}\|\,\sin\theta$ is exactly the are of the parallelogram spanned by the vectors $\vec{a}$ and $\vec{b}$.



7.2. Algebraic description of the cross product. If $\vec{a}$ and $\vec{b}$ are given by (4), i.e. by

$$\vec{a}=a_1\vec{e}_1+a_2\vec{e}_2+a_3\vec{e}_3=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\qquad \vec{b}=b_1\vec{e}_1+b_2\vec{e}_2+b_3\vec{e}_3=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},$$

then

$$\vec{a}\times\vec{b}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}.$$

7.3. Algebraic properties of the cross product. The cross product has the distributive property, namely,

$$(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}, \tag{7}$$

holds true for any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$.

The cross product is ***not commutative***: $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are not the same thing. Instead, we have :

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}. \tag{8}$$

Because of this property the cross product is said to be "*anti-commutative.*"

The associative property fails completely for the cross product: for most vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ one has

$$(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c}) \tag{9}$$

If you need a vector that is perpendicular to two given vectors, take their cross product.

The length of the cross product $\vec{a}\times\vec{b}$ is the area of the parallelogram spanned by those vectors.

8. The triple product

Just as two vectors in the plane form a parallelogram, three vectors in space will form a shape called a parallelepiped. By definition, a parallelepiped is a solid body each of whose faces is a parallelogram.

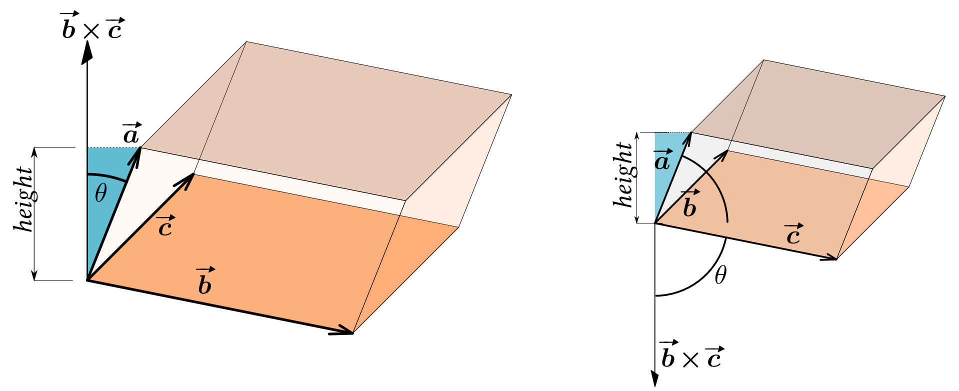


Figure 9. A parallelepiped spanned by three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. Since the base of the parallelepiped is a parallelogram with edges $\vec{b}$ and $\vec{c}$, we have

Area of base $= \|\vec{b}\times\vec{c}\|$.

The height of the parallelepiped is $\|\vec{a}\|\cos\theta$, and therefore the volume is given by

Volume = height · area of base $= \|\vec{a}\|\,\|\vec{b}\times\vec{c}\|\cos\theta = \vec{a}\cdot(\vec{b}\times\vec{c})$.

This derivation applies to the situation on the left, where the vector $\vec{a}$ and the cross product $\vec{b}\times\vec{c}$ point in the same direction. If these vectors form an obtuse angle, as is the case on the right, then $\cos\theta < 0$, and the height is $-\|\vec{a}\|\cos\theta$. In that case one has

Volume = height · area of base $= -\|\vec{a}\|\,\|\vec{b}\times\vec{c}\|\cos\theta = -\vec{a}\cdot(\vec{b}\times\vec{c})$.

If we are given three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, then the volume of the parallelepiped they determine is given by the formula

"Volume equals Area of base times height"

In terms of the three vectors this is

$$V = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|. \tag{10}$$

A derivation is sketched in Figure 9. The quantity $\vec{a}\cdot(\vec{b}\times\vec{c})$ (without the absolute values) is called the ***triple product*** of the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. Apart from its use in computing the volume of a parallelepiped, the triple product appears in many other contexts. At first sight the expression $\vec{a}\cdot(\vec{b}\times\vec{c})$ suggests that the order in which the vectors appear is important, but this turns out not to be true. One has

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

for any $\vec{a}$, $\vec{b}$, $\vec{c}$.

9. Determinants

For any four numbers a, b, c, d, one defines the 2×2 determinant to be

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\,. \tag{11}$$

One can also define 3×3 determinants. Namely, for any nine numbers $a_1, \dots, c_3$ one defines

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1\,. \tag{12}$$

This can be written as

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1\big(b_2c_3 - b_3c_2\big) - a_2\big(b_1c_3 - b_3c_1\big) + a_3\big(b_1c_2 - b_2c_1\big) \\ &= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_1 \\ b_2 & b_2 \end{vmatrix} \end{aligned} \tag{13}$$

where each coefficient in the first row is multiplied with the 2×2 determined that remains after one deletes the row and column containing the coefficient.

Instead of expanding along the first row one can also expand along the first column:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{14}$$

Many other mnemonic devices exist to remember how to compute a 3×3 determinant. A popular trick is "Sarrus' rule" (see Figure 10.)

One can also define larger determinants, i.e. 4×4, 5×5, etc, and generally $n\times n$ determinants. The theory, which is beyond the scope of this course, is treated in linear algebra courses such as Math 320, 340, or 341.

10. Determinants, the triple product, and the cross product

If the numbers $a_1, \dots, c_3$ in a determinant happen to be the components of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, i.e. if

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},\ \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\ \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

then the corresponding determinant is exactly the triple product:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \vec{a}\cdot(\vec{b}\times\vec{c}). \tag{15}$$

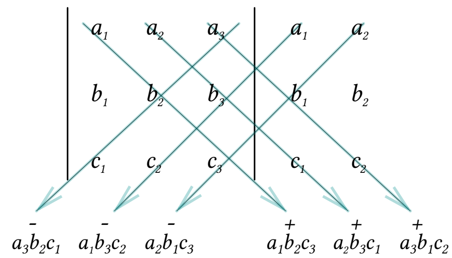


Figure 10. Computing 3×3 determinants. There are several shortcuts to remember how to compute a 3×3 determinant. Pictured here is "Sarrus' rule," which tells us to copy the first two columns of the determinant to the right of the determinant, and read off the six terms in the determinant by following the diagonals.

Related to this is the following practical trick for computing the cross product of two column vectors. Given two column vectors $\vec{b}$ and $\vec{c}$ one can write their cross product as

$$\begin{aligned}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} &= \begin{vmatrix} \vec{e}_1 & b_1 & c_1 \\ \vec{e}_2 & b_2 & c_2 \\ \vec{e}_3 & b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \vec{e}_1 - \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \vec{e}_2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \vec{e}_3.\end{aligned}$$

The 3×3 determinant in this equation is unusual in that some of its entries are vectors instead of numbers. The intention of this notation is that one expand the determinant along the first column, as in (13) and then interpret the result as a vector.

11. Defining equations for lines and planes

11.1. Lines. Let ℓ be a line in the plane, and suppose we know one point A on the line, and that we also have a vector $\vec{n}$ that is perpendicular to the line (and we exclude $\vec{n} = \vec{0}$.) Such a vector is called a ***normal vector*** to the line. Given any other point X in the plane we can form the vector $\overrightarrow{AX}$ and consider its dot-product with the normal. We have

$$\vec{n} \cdot \overrightarrow{AX} = \|\vec{n}\| \, \|\overrightarrow{AX}\| \cos\theta,$$

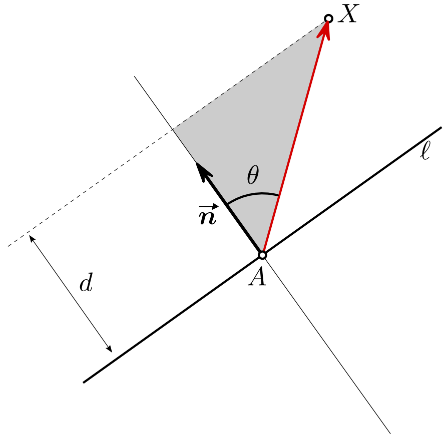
where θ is the angle between the normal vector $\vec{n}$ and $\overrightarrow{AX}$.

The combination $\|\overrightarrow{AX}\| \cos\theta$ is, up to its sign, the distance from the line ℓ to the point X: If X lies on the side of ℓ at which the normal vector points then $\vec{n} \cdot \overrightarrow{AX} > 0$; if X lies on the other side then $\vec{n} \cdot \overrightarrow{AX} < 0$. We therefore have the following formula for *the distance between a point X and the line ℓ:*

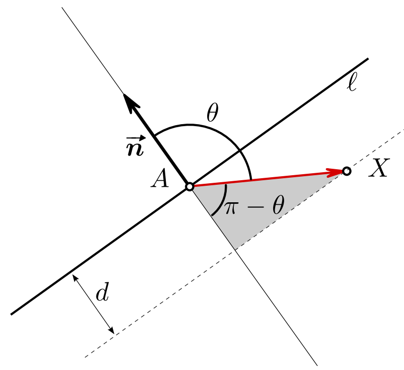
$$d = \frac{\vec{n} \cdot \overrightarrow{AX}}{\|\vec{n}\|} \tag{16}$$

When we use this equation to compute the distance from X to ℓ, it is good to recall that if $\vec{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ and $\vec{a} = \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right)$ are the position vectors of the points X and A, then

$$\overrightarrow{AX} = \vec{x} - \vec{a} = \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \end{pmatrix}.$$



$\vec{\boldsymbol{n}}\cdot\overrightarrow{AX} > 0 \qquad d = \|\overrightarrow{AX}\| \cos\theta$



$$\vec{\boldsymbol{n}}\cdot\overrightarrow{AX} < 0 \qquad \begin{aligned} d &= \|\overrightarrow{AX}\| \cos(\pi - \theta) \\ &= -\|\overrightarrow{AX}\| \cos\theta \end{aligned}$$

Moreover, the length of the normal vector is $\|\vec{\boldsymbol{n}}\| = \sqrt{n_1^2 + n_2^2}$, so we can rewrite (16) as

$$d = \frac{n_1(x_1 - a_1) + n_2(x_2 - a_2)}{\sqrt{n_1^2 + n_2^2}}.$$

This last formula is more impressive than (16), but it is better to remember (16).

The equation for the distance from any point X to a given line ℓ is also important because it gives us ***the defining equation*** for the line ℓ. The defining equation is an equation that tells us for any given point X in the plane if that point is on the line or not. Since X is on ℓ exactly when the distance from ℓ to X vanishes, it follows from (16) that X is on ℓ if and only if

$$\vec{\boldsymbol{n}}\cdot\overrightarrow{AX} = 0. \tag{17}$$

We can again rewrite this equation in a few different ways. If we want to write it in terms of the position vectors of A and X, then we get

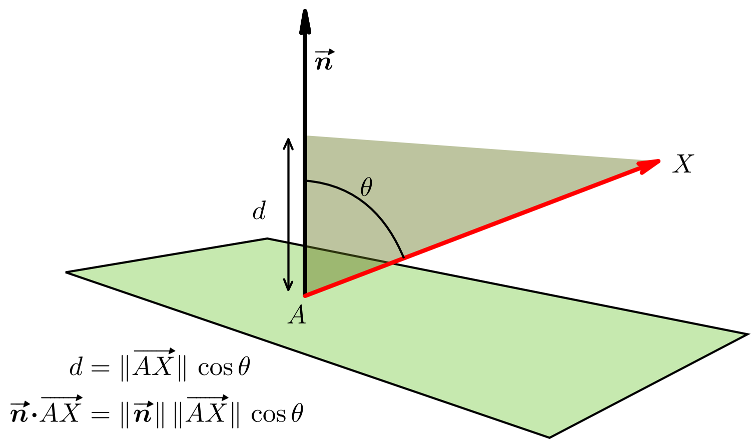
$$\vec{\boldsymbol{n}}\cdot\big(\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}\big) = 0, \qquad \text{i.e.:} \qquad \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{x}} = \vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{a}}.$$

Written without vectors, but in terms of the coordinates of the points A, X, and the components of the normal vector $\vec{\boldsymbol{n}}$, we can write this last version of our equation as

$$n_1 x_1 + n_2 x_2 = n_1 a_1 + n_2 a_2.$$

11.2. Planes. We can repeat the derivation of the distance from a point to a line in the plane and derive a formula for the distance from a point in three dimensional space to a given plane. The drawings are harder to make (at first only, practice makes perfect!), but the resulting formulas are the same.

The distance from a point X to a plane $\mathcal{P}$ is given by equation (16), where $\vec{\boldsymbol{n}}$ is a normal vector to the plane (a vector that is perpendicular to the plane), and A is some point on the plane that we happen to know.



12. Problems

1. **(a)** Simplify the following

$$\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$\vec{b} = 12 \begin{pmatrix} 1 \\ 1/3 \end{pmatrix} - 3 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\vec{c} = (1+t) \begin{pmatrix} 1 \\ 1-t \end{pmatrix} - t \begin{pmatrix} 1 \\ -t \end{pmatrix}$$

$$\vec{d} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t^2 \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) Write the vectors from part **(a)** using Gibbs' notation, i.e. write them in terms of $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$. (See § 5).

2. If $\vec{a}, \vec{b}, \vec{c}$ are as in the previous problem, then which of the following expressions mean anything? Compute those expressions that are well defined.

(a) $\vec{a} + \vec{b}$ **(b)** $\vec{b} + \vec{c}$ **(c)** $\pi\vec{a}$

(d) $\vec{b}^2$ **(e)** $\vec{b}/\vec{c}$ **(f)** $\|\vec{a}\| + \|\vec{b}\|$

(g) $\|\vec{b}\|^2$ **(h)** $\vec{b}/\|\vec{c}\|$

3. Let $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Compute:

(a) $\|\vec{a}\|$ **(b)** $2\vec{a}$ **(c)** $\|2\vec{a}\|^2$

(d) $\vec{a} + \vec{b}$ **(e)** $3\vec{a} - \vec{b}$

4. Given: points $A(2,1)$ and $B(-1,4)$. Compute the vector $\overrightarrow{AB}$. *Is $\overrightarrow{AB}$ a position vector?*

5. Given: points $A(2,1)$, $B(3,2)$, $C(4,4)$ and $D(5,2)$.
Question: *Is $ABCD$ a parallelogram?*

6. Given: points $A(0,2,1)$, $B(0,3,2)$, $C(4,1,4)$ and D.

(a) If $ABCD$ is a parallelogram, then what are the coordinates of the point D?

(b) If $ABDC$ is a parallelogram, then what are the coordinates of the point D?

7. You are given three points in the plane: A has coordinates $(2,3)$, B has coordinates $(-1,2)$ and C has coordinates $(4,-1)$.

(a) Compute the vectors $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{BC}$ and $\overrightarrow{CB}$.

(b) Find the points P, Q, R and S whose position vectors are $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$, respectively. *Make a precise drawing.*

8. Explain how you can use the dot product to find the angle between the vectors $\vec{a} = 2\vec{\imath} - 3\vec{\jmath}$, and $\vec{b} = \vec{\jmath} + \vec{k}$.

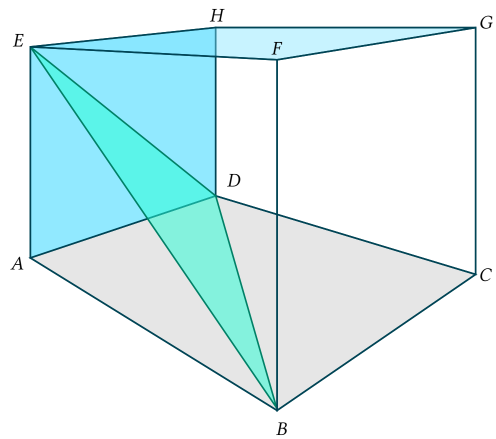


Figure 11. Figure for problem 12.10

9. For which value(s) of the number s are the vectors

$$\vec{a} = \begin{pmatrix} s \\ 1-s \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

perpendicular? For which values of s do they make an acute angle?

10. Figure 11 shows a cube whose sides have length 1.

Choose A to be the origin, and let the x, y, and z axes be along the sides AB, AD, and AE, respectively.

(a) Draw the vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ in the figure.

(b) Find a normal vector to the plane through the points B, D, and E.

(c) Draw the plane through ACH (or at least the portion of that plane that lies inside the cube). Find a normal to the plane ACH.

(d) Find the angle between the two planes BDE and ACH. (The angle between two planes is the same as the angle between their normal vectors, i.e. to find the angle between two planes find a normal vector for each of the planes and compute the angle between these two vectors.)

(e) Find the angle between the two planes BDE and HFC.

11. **(a)** Draw two vectors $\vec{a}$ and $\vec{b}$ for which $\vec{a}$ has length 3, $\vec{b}$ has length 5, and for which $\vec{a}\cdot\vec{b} = -12$. How many solutions are there?

(b) Can there be two vectors $\vec{a}$ and $\vec{b}$ whose lengths are $\|\vec{a}\| = 3$ and $\|\vec{b}\| = 5$, and whose inner product is $\vec{a}\cdot\vec{b} = 25$?

12. Compute

$$\vec{a} = (\vec{\imath}\times\vec{\jmath})\times\vec{\jmath} \quad \text{and} \quad \vec{b} = \vec{\imath}\times(\vec{\jmath}\times\vec{\jmath}).$$

What does your answer say about the associative property for the cross product? (See § 7.3.)

What about

$$\vec{c} = (\vec{\imath}\times\vec{\jmath})\times\vec{k} \quad \text{and} \quad \vec{d} = \vec{\imath}\times(\vec{\jmath}\times\vec{k})?$$

13. Which of the following vector equations are true for any pair of vectors $\vec{a}$ and $\vec{b}$? Either give a proof (using the algebraic properties or the algebraic or geometric descriptions).

(a) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \|\vec{a}\|^2 - \|\vec{b}\|^2$?

(b) If $\vec{a}\perp\vec{b}$ then

$$\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 \text{ ?}$$

(c) If $\vec{a}\perp\vec{b}$ then

$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 - \|\vec{b}\|^2 \text{ ?}$$

14. True or False:

(a) If $\vec{a} \perp \vec{b}$ and also $\vec{b} \perp \vec{c}$ then $\vec{a} \perp \vec{c}$?

(b) If $\vec{a} \perp \vec{b}$ and also $\vec{a} \perp \vec{c}$ then $\vec{a} \perp (\vec{b} + \vec{c})$?

(c) If $\vec{a} \perp \vec{b}$ and also $\vec{b} \perp \vec{c}$ then $\vec{b} \perp (\vec{a} - \vec{c})$?

(d) If $\vec{a} \perp \vec{b} + \vec{c}$ and also $\vec{a} \perp \vec{b} - \vec{c}$ then $\vec{a} \perp \vec{b}$?

15. Simplify the following expressions

(a) $(\vec{a} + \vec{b})\times(\vec{a} + \vec{b})$ •

(b) $(\vec{a} + \vec{b} + \vec{c})\times(\vec{a} + \vec{b} + \vec{c})$ •

(c) $(\vec{a} - \vec{b})\times(\vec{a} + \vec{b})$ •

(d) $(\vec{a} + \vec{b} - \vec{c})\times(\vec{a} - \vec{b} + \vec{c})$

(e) $(\vec{a} + \vec{b} - \vec{c})\cdot(\vec{a} - \vec{b} + \vec{c})$

16. This problem is about "cross division," i.e. can you solve $\vec{a}\times\vec{b} = \vec{c}$ for $\vec{b}$ if you know $\vec{a}$ and $\vec{c}$?

(a) Let

$$\vec{a} = \vec{e}_1 - \vec{e}_3, \qquad \vec{c} = \vec{e}_1 + 3\vec{e}_2 + 2\vec{e}_3.$$

Find a vector $\vec{b}$ for which $\vec{a}\times\vec{b} = \vec{c}$, if there is such a thing. (Hint: if $\vec{c} = \vec{a}\times\vec{b}$, then what do you know about $\vec{a}\cdot\vec{c}$?) •

(b) Let $\vec{a} = 2\vec{e}_1 - \vec{e}_3$, and $\vec{c} = \vec{e}_1 + 3\vec{e}_2 + 2\vec{e}_3$. Find a vector $\vec{b}$ for which $\vec{a}\times\vec{b} = \vec{c}$, if such a thing exists. •

17. The *law of cosines* says that in a triangle $\triangle ABC$ for which you know the sides AB and AC, as well as the angle $\angle A$, the length of the opposing side BC is given by

$$(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)\cos\angle A.$$

Show how you can use the dot product to (re)prove this law.

Hint: consider the vector equation $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$. You will need both the geometric description (4) of the dot product, and the algebraic properties from § 6.3.

CHAPTER 2

Parametric curves and vector functions

1. Vector functions

So far in calculus we have only considered functions $y = f(x)$ where both the independent variable x and the dependent variable y are real numbers.

A ***vector function*** is a function of one variable whose values are vectors instead of numbers. One way to specify a vector function is to say what its components are:

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = x(t)\vec{\boldsymbol{e}}_1 + y(t)\vec{\boldsymbol{e}}_2 + z(t)\vec{\boldsymbol{e}}_3.$$

2. Using vector functions to describe motion

One way to visualize a vector function $\vec{\boldsymbol{x}}(t)$ is to think of the vector $\vec{\boldsymbol{x}}(t)$ for any given value of t as the position vector of some point in space (or the plane, if $\vec{\boldsymbol{x}}(t)$ is a two-dimensional vector). In other words, we represent the vector $\vec{\boldsymbol{x}}(t)$ as an arrow starting at the origin, and ending at some point $X(t)$ whose coordinates are $(x(t), y(t), z(t))$:

$$\vec{\boldsymbol{x}}(t) = \overrightarrow{OX(t)}.$$

As t varies, the point $X(t)$ moves around and traces out a curve. Such a curve is called a ***parametrized curve,*** or a ***parametric curve***. The quantity t is called the ***parameter.***

We will now take a look at some examples of parametric curves.

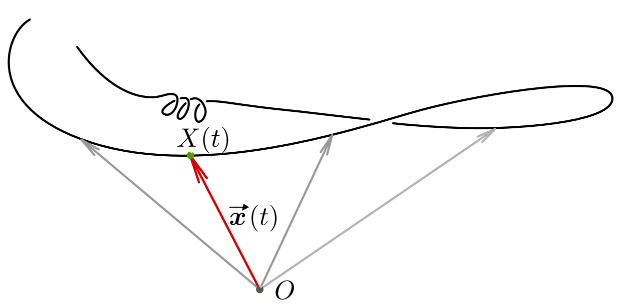


Figure 1. A parametric curve: as the parameter t changes, the vector $\vec{\boldsymbol{x}}(t)$ will also move. Keeping the initial point of the vector $\vec{\boldsymbol{x}}(t)$ at the origin O, the endpoint $X(t)$ traces out a space curve.

3. Lines

Consider the parametric curve given by

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{a}} + t\vec{\boldsymbol{v}} \tag{18}$$

where $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{v}}$ are given constant vectors. As before we let $X(t)$ be the point with $\vec{\boldsymbol{x}}(t) = \overrightarrow{OX(t)}$, i.e. $\vec{\boldsymbol{x}}(t)$ is the position vector of the point $X(t)$, and as t changes, $X(t)$ traces out the parametric curve.

To see what the parametric curve looks like, we let A be the point with $\overrightarrow{OA} = \vec{\boldsymbol{a}}$, then, since

$$\overrightarrow{OX(t)} = \overrightarrow{OA} + \overrightarrow{AX(t)},$$

it follows from (18) that $\overrightarrow{AX(t)} = t\vec{\boldsymbol{v}}$. Now consider going from the origin O to the point $X(t)$ in two steps: first move from O to the point A, then go from A to $X(t)$. The displacement in the second step is $\overrightarrow{AX(t)} = t\vec{\boldsymbol{v}}$. Changing t will then make the point $X(t)$ slide along the line through the point A in the direction of $\vec{\boldsymbol{v}}$.

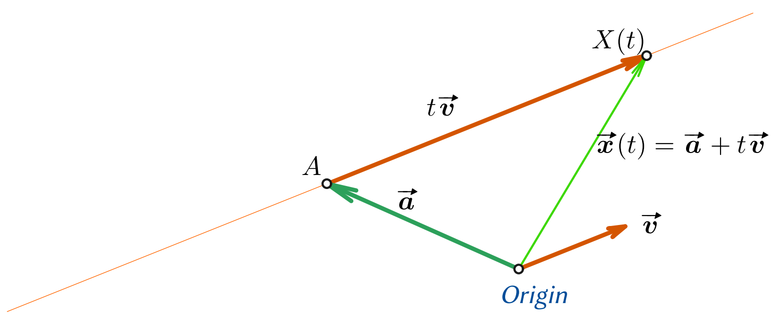


Figure 2. Vector form of linear motion given by $\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{a}} + t\vec{\boldsymbol{v}}$.

We say that $\vec{\boldsymbol{x}}(t)$ given by (18) describes motion with constant velocity, whose velocity vector is $\vec{\boldsymbol{v}}$.

4. Circular motion

For given constants $R > 0$ and ω we consider the vector function

$$\vec{\boldsymbol{x}}(t) = R\cos\omega t\,\vec{\boldsymbol{e}}_1 + R\sin\omega t\,\vec{\boldsymbol{e}}_2 = \begin{pmatrix} R\cos\omega t \\ R\sin\omega t \end{pmatrix}. \tag{19}$$

The corresponding point is $X(t) = \big(R\cos\omega t, R\sin\omega t\big)$. It lies on the circle of radius R with center at the origin, and the angle subtended by $OX(t)$ and the positive x-axis is exactly ωt.

If $\omega > 0$ then as t increases, the angle ωt increases and the point $X(t)$ goes around the circle in counter-clockwise direction. If $\omega < 0$ then $X(t)$ goes around in the clockwise direction.

The number ω is the rate of increase of the angle ωt, and is called the ***angular velocity*** of the motion.

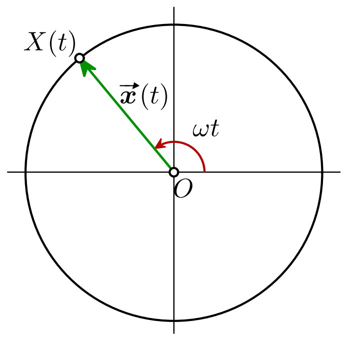


Figure 3. Circular motion with angular velocity ω.

5. The cycloid

The ***cycloid*** is the curve we get if we put a (bicycle) wheel on the ground, mark the point on the tire that touches the ground, and follow this point as we roll the wheel forward. If we call the point X, then it depends on the angle θ that the wheel has turned since X was on the ground. Figure 4 provides a derivation of the vector function $\vec{x}(\theta) = \overrightarrow{OX(\theta)}$ that describes the cycloid. The result is

$$\vec{x}(\theta) = \begin{pmatrix} R\theta - R\sin\theta \\ R - R\cos\theta \end{pmatrix}. \tag{20}$$

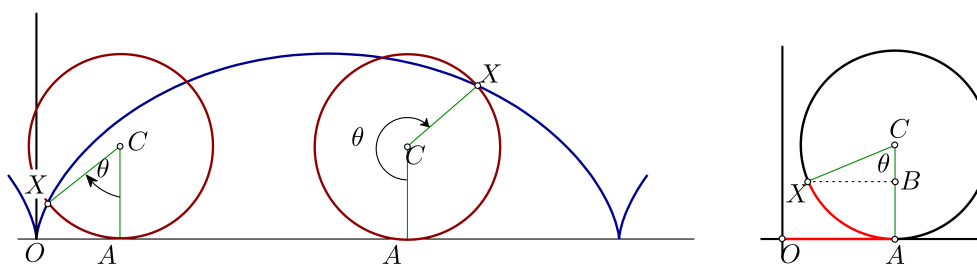


Figure 4. The cycloid. A wheel of radius R rolls over the x-axis. Initially the wheel touches the x-axis at the origin O. The cycloid is the curve traced out by a point X on the wheel.

Derivation of the cycloid motion. The arc AX and the line segment OA have the same length. Since AX has length $R\theta$, the x coordinates of the points A, B, and C are $R\theta$. The right triangle CXB has hypotenuse R, so the lengths of XB and CB are $R\sin\theta$, and $R\cos\theta$, respectively. Therefore the coordinates of the point X are $x = R\theta - R\sin\theta$, and $y = R - R\cos\theta$.

6. The helix

When we walk up a spiral staircase we are tracing out a helix: we are going around in circles, and moving upward at the same time. The parametric curve that does this (and that has the z-axis as its central axis) is given by

$$\vec{\boldsymbol{x}}(\theta) = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ a\theta \end{pmatrix} \quad \text{or:} \quad \vec{\boldsymbol{x}}(\theta) = R\cos\theta\,\vec{\boldsymbol{e}}_1 + R\sin\theta\,\vec{\boldsymbol{e}}_2 + a\theta\,\vec{\boldsymbol{e}}_3. \tag{21}$$

Here $R > 0$ is the radius of the helix, i.e. the radius of the circle on the ground above which the helix lies; the number a represents the rate at which the helix goes up.

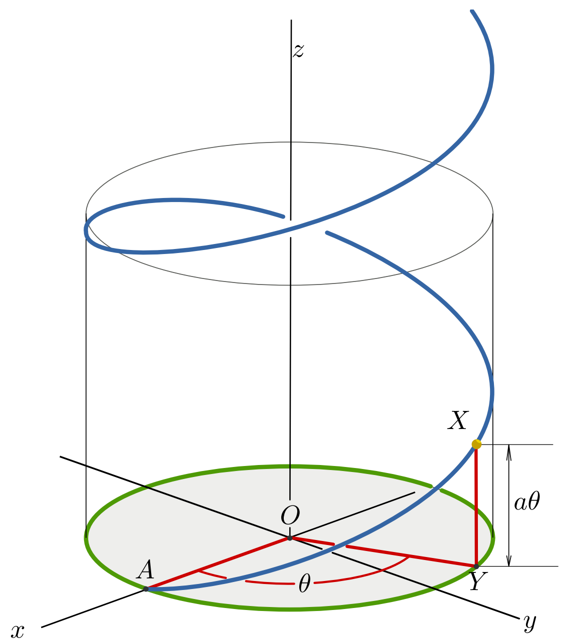


Figure 5. The Helix. The point X traces out a helix: it sits at a height $a\theta$ above the point Y, while Y runs around on a circle of radius R; here $\theta = \angle AOY$

7. The derivative of a vector function

For a function $y = f(x)$ of one variable we had two ways of describing the derivative: on one hand we had a geometric description of $f'(x)$ as "the slope of the tangent to the graph," and on the other we could describe $f'(x)$ in terms of a difference quotient, i.e.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

For vector functions we can imitate both descriptions. We begin with the formal definition in terms of limits and then proceed to the geometric description, in which we interpret the derivative as the "instantaneous velocity vector."

Definition. If $\vec{\boldsymbol{x}}(t)$ is a vector function, then we set

$$\vec{\boldsymbol{x}}'(t) \stackrel{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\vec{\boldsymbol{x}}(t + \Delta t) - \vec{\boldsymbol{x}}(t)}{\Delta t}. \tag{22}$$

For (22) to make sense we would have to define what the limit of a vector function is. This can be done, but we will not go into the precise definitions in this course. More important for our use is that if the components of a vector function $\vec{x}(t)$ are given, then the derivative can be computed by just differentiating those components:

$$\vec{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}, \quad \text{or} \quad \vec{x}'(t) = x'(t)\vec{e}_1 + y'(t)\vec{e}_2 + z'(t)\vec{e}_3. \tag{23}$$

As with ordinary functions of one variable we will use Leibniz' notation for the derivative whenever it seems convenient. Thus the following are equivalent ways of expressing the same derivative:

$$\vec{a}'(t) = \frac{d\vec{a}(t)}{dt} = \frac{d}{dt}\vec{a}(t).$$

Example. For instance,

$$\vec{x}(\theta) = \begin{pmatrix} \cos\theta \\ 0 \\ \theta \end{pmatrix} = \cos\theta\,\vec{e}_1 + \theta\,\vec{e}_3$$

defines a vector function. Here we have called the independent variable θ instead of t. The derivative of this vector function is

$$\frac{d\vec{x}}{d\theta} = \frac{d}{d\theta}\begin{pmatrix} \cos\theta \\ 0 \\ \theta \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ 0 \\ 1 \end{pmatrix} = -\sin\theta\,\vec{e}_1 + \vec{e}_3.$$

8. The derivative as velocity vector

Suppose the motion of some point $X(t)$ in space is described by its position vector function $\vec{x}(t)$. Let us try to define the instantaneous velocity of the point. This velocity should have magnitude ("how fast the point is moving") and also direction ("which way

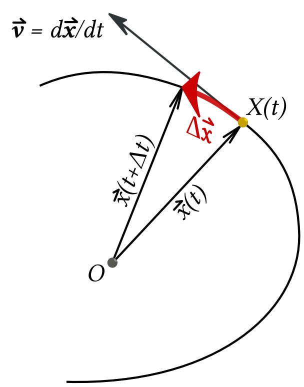


Figure 6. The vector function $\vec{x}(t)$ traces out a curve in space. The vector $\vec{x}(t)$ is the position vector of a point $X(t)$ on this curve. As we increase time from t to $t + \Delta t$, the point $X(t)$ moves. The displacement of the point $X(t)$ is given by $\Delta\vec{x} = \vec{x}(t + \Delta t) - \vec{x}(t)$. The average velocity vector during this displacement is "displacement/time", i.e. $\Delta\vec{x}/\Delta t$.

If we let $\Delta t \to 0$, then the average velocity becomes the instantaneous velocity at time t: $\vec{v} = \lim_{\Delta t\to 0} \Delta\vec{x}/\Delta t = \vec{x}'(t)$. This vector is tangent to the curve traced out by the vector function $\vec{x}(t)$. We call it a ***tangent vector***.

is the point going?"). The velocity should therefore be a vector. To see which vector, we go back to the notion that "velocity" is always "displacement divided by time."

We consider two instances in time, say, time t and time $t+\Delta t$. Then the position vectors of the point X at these two different times are $\vec{\boldsymbol{x}}(t)$ and $\vec{\boldsymbol{x}}(t+\Delta t)$. The displacement of the point X between these two times is then

$$\Delta\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(t+\Delta t) - \vec{\boldsymbol{x}}(t)$$

(see Figure 6.) We say that the average velocity over the time interval from t to $t+\Delta t$ is "the displacement divided by Δt," i.e.

$$\vec{\boldsymbol{v}}_{\text{average}} = \frac{\vec{\boldsymbol{x}}(t+\Delta t) - \vec{\boldsymbol{x}}(t)}{\Delta t}.$$

Note that the average velocity is a vector. If we write it out in components, we get a much larger formula:

$$\vec{\boldsymbol{v}}_{\text{average}} = \begin{pmatrix} \dfrac{x(t+\Delta t)-x(t)}{\Delta t} \\ \dfrac{y(t+\Delta t)-y(t)}{\Delta t} \\ \dfrac{z(t+\Delta t)-z(t)}{\Delta t} \end{pmatrix}.$$

One big advantage of using vector notation is that many formulas simplify considerably when written in terms of vectors.

To get the instantaneous velocity, we do the same thing as in one variable calculus: we take the limit as $\Delta t \to 0$ of the average velocity over the time interval from t to $t+\Delta t$. Thus we get

$$\vec{\boldsymbol{v}}(t) = \lim_{\Delta t\to 0} \frac{\vec{\boldsymbol{x}}(t+\Delta t) - \vec{\boldsymbol{x}}(t)}{\Delta t} \stackrel{\text{def}}{=} \frac{d\vec{\boldsymbol{x}}}{dt}. \tag{24}$$

In terms of components this derivative is

$$\vec{\boldsymbol{x}}'(t) = \frac{d\vec{\boldsymbol{x}}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}.$$

Thus the velocity vector of any given vector function $\vec{\boldsymbol{x}}(t)$ is the same as the derivative of this vector function.

9. Acceleration

Having found the velocity vector of a point $X(t)$ whose position vector is a given vector function $\overrightarrow{OX(t)} = \vec{\boldsymbol{x}}(t)$, we can also define the ***acceleration vector*** of the moving point. By definition, the acceleration vector is the derivative of the velocity vector, i.e.

$$\vec{\boldsymbol{a}}(t) = \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d^2\vec{\boldsymbol{x}}}{dt^2} = \begin{pmatrix} x''(t) \\ y''(t) \\ z''(t) \end{pmatrix}. \tag{25}$$

This definition is entirely analogous to the definition of acceleration ("$a = \frac{dv}{dt}$") from first semester calculus. The only difference is that, here, the position, velocity, and acceleration all have directions in addition to magnitudes: they are vectors.

Newton's famous law relating forces and acceleration continues to hold. If a point $X(t)$ moves according to some vector function $\vec{x}(t)$, then some force must be acting on this point. This force is a vector (it has magnitude and direction), and, according to Newton, it is given by

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{x}}{dt^2}, \tag{26}$$

where m is the mass of the object at the point $X(t)$ whose motion we are considering. It is always assumed to be a positive number.

Note that according to this law, the absence of forces, i.e. $\vec{F} = \vec{0}$, is the same as $\frac{d\vec{v}}{dt} = \vec{0}$, i.e. no force acts on the point if and only if its velocity vector is constant. Here "constant" means constant magnitude ***and*** constant direction.

10. The differentiation rules

Just as with ordinary derivatives, the derivatives of vector functions satisfy certain rules, such as the product rule. The purpose of these rules is not the same as in one variable calculus. There we used sum, product, quotient and chain rules to compute derivatives of given functions without having to fall back on the definition of a derivative all the time. For vector functions we do not need such rules, because we can differentiate them by simply differentiating each of their components (see the above example). Instead, the differentiation rules for vector functions are mostly used to gain insight and establish general facts about vector functions, a number of which we will see shortly.

10.1. The sum rule. The analog of the sum rule ("derivative of the sum is the sum of the derivatives") looks exactly like the ordinary sum rule. It says that for any two vector functions $\vec{a}(t)$ and $\vec{b}(t)$ one has

$$\frac{d}{dt}\big(\vec{a}(t) \pm \vec{b}(t)\big) = \frac{d\vec{a}(t)}{dt} \pm \frac{d\vec{b}(t)}{dt}.$$

10.2. The many product rules. There is no quotient rule for vector functions, simply because we have no way of dividing vectors. On the other hand we have two ways of multiplying vectors, and we can also multiply vectors and numbers, so there are ***three*** different product rules. Fortunately they all look like the product rule from first semester calculus.

If $\vec{a}(t)$ and $\vec{b}(t)$ are vector functions, and if $f(t)$ is a function, then

$$\frac{d\vec{a}(t)\cdot\vec{b}(t)}{dt} = \frac{d\vec{a}(t)}{dt}\cdot\vec{b}(t) + \vec{a}(t)\cdot\frac{d\vec{b}(t)}{dt}$$

$$\frac{d\vec{a}(t)\times\vec{b}(t)}{dt} = \frac{d\vec{a}(t)}{dt}\times\vec{b}(t) + \vec{a}(t)\times\frac{d\vec{b}(t)}{dt}$$

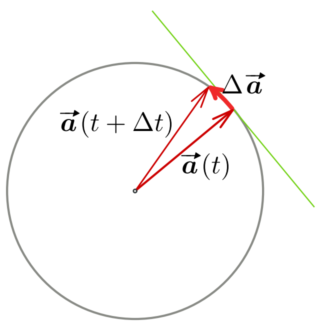
$$\frac{d\,f(t)\vec{a}(t)}{dt} = \frac{df(t)}{dt}\vec{a}(t) + f(t)\frac{d\vec{a}(t)}{dt}$$

In spite of the fact that these rules "look right," they could still be wrong, so to be sure we would have to prove them. The proofs are very straightforward. Here is a short proof for the product rule involving the dot product. To shorten the formulas we omit the "(t)" from all functions:

$$
\begin{aligned}
\frac{d\vec{a}\cdot\vec{b}}{dt} &= \frac{d}{dt}\big(a_1b_1 + a_2b_2\big) \\
&= \frac{da_1b_1}{dt} + \frac{da_2b_2}{dt} \\
&= \frac{da_1}{dt}b_1 + a_1\frac{db_1}{dt} + \frac{da_2}{dt}b_2 + a_2\frac{db_2}{dt} && \text{ordinary product rule} \\
&= \frac{da_1}{dt}b_1 + \frac{da_2}{dt}b_2 + a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} && \text{switch terms around} \\
&= \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}. && \text{recognize the dot-products}
\end{aligned}
$$

11. Vector functions of constant length

As an immediate application of the product rule for the dot-product we prove the following fact about vector functions whose length does not change, i.e. vector functions $\vec{a}(t)$ that change their direction, but not their length.



If a vector function $\vec{a}(t)$ has constant length, then, when the parameter t undergoes a small change Δt, the corresponding small change $\Delta\vec{a}$ in the vector function will be almost perpendicular to $\vec{a}(t)$ itself.

Theorem. Let $\vec{a}(t)$ be a vector function. Then a necessary and sufficient condition for the length $\|\vec{a}(t)\|$ to be constant is that $\vec{a}(t)$ and $\vec{a}'(t)$ be perpendicular for all t.

PROOF. Differentiating both sides of the equation

$$\|\vec{a}(t)\|^2 = \vec{a}(t)\cdot\vec{a}(t)$$

we get

$$\frac{d}{dt}\|\vec{a}(t)\|^2 = \vec{a}'(t)\cdot\vec{a}(t) + \vec{a}(t)\cdot\vec{a}'(t) = 2\vec{a}(t)\cdot\vec{a}'(t). \tag{27}$$

If $\vec{a}(t)$ has constant length, then $\|\vec{a}(t)\|^2$ is also constant, and thus $\frac{d}{dt}\|\vec{a}(t)\|^2 = 0$. Therefore, for a vector function $\vec{a}(t)$ whose length is constant, $\vec{a}(t)\cdot\vec{a}'(t) = 0$, i.e. $\vec{a}(t) \perp \vec{a}'(t)$.

Conversely, if $\vec{a}(t)$ is a vector function for which $\vec{a}(t) \perp \vec{a}'(t)$ holds for all t, then $\vec{a}(t)\cdot\vec{a}'(t) = 0$, and (27) implies that $\frac{d}{dt}\|\vec{a}(t)\|^2 = 0$, i.e. that $\|\vec{a}(t)\|^2$ and hence $\|\vec{a}(t)\|$ are constant.

$\square$

12. Two examples

12.1. Motion on a straight line. We return to the motion given by (18), i.e.

$$\vec{x}(t) = \vec{a} + t\vec{v}. \tag{28}$$

The velocity and acceleration are easy to compute:

$$\frac{d\vec{x}(t)}{dt} = \vec{v}, \qquad \frac{d^2\vec{x}(t)}{dt} = \frac{d\vec{v}}{dt} = \vec{\mathbf{0}},$$

since $\vec{v}$ is a constant vector in this case.

We see that if a point $X(t)$ moves according to the parametrization (18), then its velocity is constant, and its acceleration is zero. According to Newton's law, no force is exerted on an object undergoing this motion.

12.2. Circular motion. For the point $X(t)$ moving on a circle of radius R with angular velocity ω we have (19), i.e.

$$\vec{x}(t) = R\cos\omega t\,\vec{e}_1 + R\sin\omega t\,\vec{e}_2$$

so that the velocity and acceleration are easy to compute:

$$\begin{aligned}\vec{v}(t) = \vec{x}'(t) &= -\omega R\sin\omega t\ \vec{e}_1 + \ \omega R\cos\omega t\ \vec{e}_2,\\ \vec{a}(t) = \vec{v}'(t) &= -\omega^2 R\cos\omega t\ \vec{e}_1 - \omega^2 R\sin\omega t\ \vec{e}_2.\end{aligned}$$

Note that the velocity vector $\vec{v}(t)$ is perpendicular to the position vector $\vec{x}(t)$, as predicted in § 11. Our expression for the velocity vector $\vec{v}(t)$ contains the familiar relation between angular velocity and velocity: the velocity $v = \|\vec{v}(t)\|$ with which the point $X(t)$ is moving is

$$\begin{aligned} v(t) &= \|-\omega R\sin\omega t\ \vec{e}_1 + \ \omega R\cos\omega t\ \vec{e}_2\| \\ &= \sqrt{\omega^2R^2\sin^2\omega t + \omega^2R^2\cos^2\omega t} \\ &= \omega R. \end{aligned} \tag{29}$$

Hence the angular velocity of an object undergoing circular motion is

$$\omega = \frac{v}{R}. \tag{30}$$

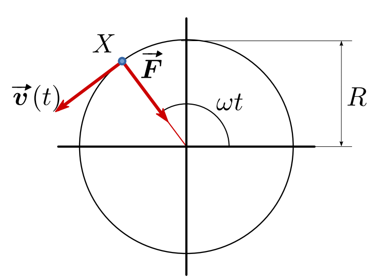


Figure 7. If an object moves along a circle with constant angular velocity, then the force $\vec{F}$ required to make the object follow that motion is $\vec{F} = -\omega^2\vec{x}$. In particular it is parallel to the position vector $\vec{x}$ but in the opposite direction.

We also note that the acceleration is a multiple of the position vector:

$$\vec{\boldsymbol{a}}(t) = -\omega^2 \vec{\boldsymbol{x}}(t).$$

According to Newton the force acting on the object at $X(t)$ is $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} = -m\omega^2\vec{\boldsymbol{x}}$, and its magnitude is

$$F = \|\vec{\boldsymbol{F}}\| = \|m\omega^2\vec{\boldsymbol{x}}(t)\| = m\omega^2 R, \tag{31}$$

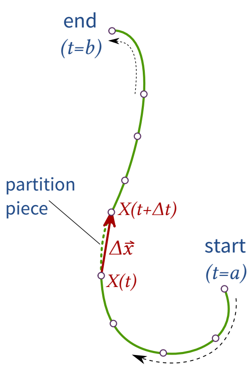
because $\|\vec{\boldsymbol{x}}(t)\| = R$ at all times.

Using (30) we can replace the angular velocity ω by the actual velocity, which leads to the classical formula for the centrifugal force

$$F = \frac{mv^2}{R}. \tag{32}$$

13. Arc length

For any given vector function there is a simple formula for the length of the curve it traces out. The formula is essentially the same as the formula for the length of a parametric curve (or, to a lesser extent, of the graph of a function) that was described in Math 221. Here we repeat the intuitive derivation of the formula, written in terms of vectors this time.



Let $\vec{\boldsymbol{x}}(t)$ $(a \le t \le b)$ be a vector function. To determine the length of the arc traced out by $X(t)$ as t varies from $t = a$ to b, we divide the interval $a \le t \le b$ into many very short subintervals. The corresponding points $X(t)$ on the curve split the curve into many short segments, each of which will be "close to a line segment." We approximate the length of the curve by adding the lengths of all these short segments. Finally we take the limit in which the number of partition points becomes infinite and our sum of lengths of short segments becomes an integral. To see which integral we get, we need to find an expression for the length of a short segment between two adjacent partition points on the curve.

Suppose we have two points on the curve, with parameter values t and $t + \Delta t$, respectively. The points are $X(t)$ and $X(t + \Delta t)$, and the distance between them is the length of the vector $\Delta\vec{\boldsymbol{x}}$ from one point to the next. This vector is

$$\Delta x = \vec{\boldsymbol{x}}(t + \Delta t) - \vec{\boldsymbol{x}}(t) = \frac{\vec{\boldsymbol{x}}(t + \Delta t) - \vec{\boldsymbol{x}}(t)}{\Delta t}\,\Delta t \approx \vec{\boldsymbol{x}}'(t)\Delta t,$$

so that its length is $\approx \|\vec{\boldsymbol{x}}'(t)\|\,\Delta t$. Adding the lengths of the short segments together, we find that the length is approximately $\sum \|\vec{\boldsymbol{x}}'(t)\|\,\Delta t$ (where the summation is over all short pieces of the curve). Taking the limit we arrive at this formula for the length of the curve traced out by $\vec{\boldsymbol{x}}(t), a \le t \le b$:

$$\text{Length} = \int_{t=a}^{b} \|\vec{\boldsymbol{x}}'(t)\|\,dt. \tag{33}$$

This integral looks simple, but that appearance turns out to be deceptive as we find out when we write it in terms of the components of the vector function $\vec{\boldsymbol{x}}(t)$. Suppose $\vec{\boldsymbol{x}}(t) = x(t)\vec{\boldsymbol{e}}_1 + y(t)\vec{\boldsymbol{e}}_2 + z(t)\vec{\boldsymbol{e}}_3$. Then

$$\vec{\boldsymbol{x}}'(t) = x'(t)\vec{\boldsymbol{e}}_1 + y'(t)\vec{\boldsymbol{e}}_2 + z'(t)\vec{\boldsymbol{e}}_3,$$

so that

$$\|\vec{\boldsymbol{x}}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

Therefore the length formula (33) of the curve is equivalent to

$$\text{Length} = \int_{t=a}^{b} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt. \tag{34}$$

The square root makes this formula a reliable source of very difficult integrals. In fact the list of curves whose length one can actually compute by doing the integral is rather short (see Problem ...).

14. Arc length derivative

Let $\vec{x}(t)$ be some vector function that describes the motion through space of some point $X(t)$, and let $f(t)$ be some other function. In what follows it will help to think of the parameter t as "time." Typical examples of functions f that we might want to consider are $f(t) = \|\vec{x}(t)\|$ (the distance to the origin of the point $X(t)$) or $f(t) = \|\vec{x}'(t)\|$ (the speed at which the point is moving.)

To describe the rate with which $f(t)$ is changing we could compute its derivative,

$$\frac{df}{dt}$$

which tells us what the ratio between the change Δf of f, and the change Δt in the parameter t is (at least approximately, if Δt is small). If we interpret t as "time" then this derivative tells us how fast $f(t)$ changes per second. But sometimes it is more useful to know how much f changes after we have travelled a small *distance* along the curve, rather than after a short amount of time has passed. In other words, for two nearby points $X(t)$ and $X(t+\Delta t)$ on the curve we would like to know the ratio

$$\frac{\text{change in } f}{\text{distance travelled}} = \frac{f(t+\Delta t) - f(t)}{\text{distance from } X(t) \text{ to } X(t+\Delta t)} \tag{35}$$

We can work this out by observing that the distance from $X(t)$ to $X(t+\Delta t)$ is the length of the vector from $X(t)$ to $X(t+\Delta t)$, i.e.

$$\text{distance from } X(t) \text{ to } X(t+\Delta t) = \|\vec{x}(t+\Delta t) - \vec{x}(t)\|.$$

Assuming Δt is small, we have

$$\|\vec{x}(t+\Delta t) - \vec{x}(t)\| = \left\| \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} \right\| \Delta t \approx \left\|\vec{x}'(t)\right\| \Delta t.$$

We substitute this in (35), and get

$$\frac{\text{change in } f}{\text{distance travelled}} \approx \frac{f(t+\Delta t) - f(t)}{\|\vec{x}'(t)\|\Delta t}.$$

Now let $\Delta t \to 0$: the quantity on the left becomes what is called the ***arc length derivative*** of the function f along the curve $vx(t)$, and which is commonly denoted by $\frac{df}{ds}$ In the quantity on the right we recognize the derivative of f with respect to t (time), which leads to

$$\frac{df}{ds} = \frac{1}{\|\vec{x}'(t)\|} \frac{df}{dt}. \tag{36}$$

Here $\frac{df}{dt} = f'(t)$ is the usual derivative of f with respect to t.

If we want to emphasize the distinction between these two derivatives, then we can call $\frac{df}{dt}$ the "time derivative of f."

15. Unit Tangent and Curvature

A vector with length 1 is called a ***unit vector***

15.1. Unit tangent. We have seen that we can find a tangent vector to the curve traced out by some vector function $\vec{x}(t)$, simply by differentiating the vector function: $\vec{x}'(t)$ always provides a tangent vector (if $\vec{x}'(t) \neq \vec{0}$). In fact any multiple $\lambda\vec{x}'(t)$ of this vector will also be a tangent vector (provided $\lambda \neq 0$.) We can single out one special tangent vector, by choosing $\lambda > 0$ so that $\lambda\vec{x}'(t)$ has length 1. Since for $\lambda > 0$ we have $\|\lambda\vec{x}'(t)\| = \lambda\|\vec{x}'(t)\|$ the value of λ that will make $\lambda\vec{x}'(t)$ a unit vector is $\lambda = 1/\|\vec{x}'(t)\|$.

For this reason the vector

$$\vec{T}(t) = \frac{d\vec{x}}{ds} = \frac{\vec{x}'(t)}{\|\vec{x}'(t)\|} \tag{37}$$

is called the ***unit tangent vector*** to the curve corresponding to the vector function $\vec{x}(t)$.

15.2. Example. For our constant velocity parametrization (18) of a straight line from § 3 we have

$$\vec{x}(t) = \vec{a} + t\vec{v},$$

so that $\vec{x}'(t) = \vec{v}$ and hence

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}.$$

We see that the unit tangent vector is constant.

15.3. Curvature and normal. If the curve described by a vector function $\vec{x}(t)$ is not a straight line, then the tangent to the curve will turn as one moves along the curve. The ***curvature vector*** $\vec{\kappa}$ measures how much the curve is curved. It is defined to be the rate of change of the unit tangent, but with respect to arc length instead of with respect to the given parameter t. Thus

$$\vec{\kappa} \stackrel{\text{def}}{=} \frac{d\vec{T}}{ds}. \tag{38}$$

According to our definition of "derivative with respect to arc length" the right hand side stands for

$$\frac{d\vec{T}}{ds} = \frac{1}{\|\vec{x}'(t)\|}\frac{d\vec{T}}{dt}. \tag{39}$$

To write this completely in terms of the original vector function $\vec{x}(t)$ we use (37)

$$\vec{\kappa} = \frac{1}{\|\vec{x}'(t)\|}\frac{d}{dt}\left\{\frac{1}{\|\vec{x}'(t)\|}\frac{d\vec{x}}{dt}\right\} \tag{40}$$

This formula is not as short as the original definition (38), but it does show that the curvature vector comes about by differentiating the vector function $\vec{x}(t)$ twice (and dividing by $\|\vec{x}'(t)\|$ at the right moments.)

Theorem. The curvature vector $\vec{\boldsymbol{\kappa}}$ is perpendicular to the tangent, i.e. $\vec{\boldsymbol{\kappa}} \perp \vec{\boldsymbol{T}}$.

Proof. We have to show that $\vec{\boldsymbol{\kappa}} \cdot \vec{\boldsymbol{T}} = 0$. From the second form (39) of the definition of $\vec{\boldsymbol{\kappa}}$ we see

$$\vec{\boldsymbol{\kappa}} \cdot \vec{\boldsymbol{T}} = \Big(\frac{1}{\|\vec{\boldsymbol{x}}'(t)\|} \frac{d\vec{\boldsymbol{T}}}{dt}\Big) \cdot \vec{\boldsymbol{T}} = \frac{1}{\|\vec{\boldsymbol{x}}'(t)\|} \frac{d\vec{\boldsymbol{T}}}{dt} \cdot \vec{\boldsymbol{T}} .$$

Remember that $\vec{\boldsymbol{T}}(t)$ is always a unit vector, i.e. $\vec{\boldsymbol{T}}(t)$ has constant length: by § 11 this implies that $\frac{d\vec{\boldsymbol{T}}}{dt} \perp \vec{\boldsymbol{T}}(t)$ and thus $\frac{d\vec{\boldsymbol{T}}}{dt} \cdot \vec{\boldsymbol{T}} = 0$, so we are done. □

There are two concepts that are derived from the curvature vector: the ***curvature*** κ is by definition the length of the curvature vector $\vec{\boldsymbol{\kappa}}$,

$$\kappa = \|\vec{\boldsymbol{\kappa}}\| = \left\| \frac{d\vec{\boldsymbol{T}}}{ds} \right\|, \tag{41}$$

and the ***normal vector*** to the curve is

$$\vec{\boldsymbol{N}} = \frac{\vec{\boldsymbol{\kappa}}}{\|\vec{\boldsymbol{\kappa}}\|} = \frac{\frac{d\vec{\boldsymbol{T}}}{ds}}{\left\|\frac{d\vec{\boldsymbol{T}}}{ds}\right\|}. \tag{42}$$

The normal vector is undefined when $\vec{\boldsymbol{\kappa}} = \vec{\boldsymbol{0}}$, because it would require division by zero.

Since $\vec{\boldsymbol{\kappa}}$ is perpendicular to $\vec{\boldsymbol{T}}$, the normal vector $\vec{\boldsymbol{N}}$ is also perpendicular to $\vec{\boldsymbol{T}}$ (hence its name).

$$\frac{d\vec{\boldsymbol{T}}}{ds} = \kappa\, \vec{\boldsymbol{N}} \tag{43}$$

16. Osculating plane

At any point $X(t)$ on a space curve given by $\vec{\boldsymbol{x}}(t)$ one defines the ***osculating plane*** to be the plane that contains the point $X(t)$ and that is parallel to both the tangent $\vec{\boldsymbol{T}}(t)$ and normal $\vec{\boldsymbol{N}}(t)$ of the curve.

If we want to write a defining equation for the osculating plane as in § 11.2 then we need a vector perpendicular to the osculating plane. Since this plane is defined to be parallel to both $\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{N}}$, we can find a normal vector to the osculating plane by taking the cross product of $\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{N}}$. This vector is called the ***binormal*** to the curve. In a formula, it is defined to be

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{T}} \times \vec{\boldsymbol{N}}. \tag{44}$$

17. Problems

1. Let ℓ be the line given by

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

(a) Find the unit tangent vector, the curvature, and the tangent line to the line ℓ at the point where $t = 2$.

(b) Find the unit tangent vector, the curvature, and the tangent line to the line ℓ at any point on the line.

2. What sign does ω have in Figure 7 ? How would the figure change if we change the sign of ω? Does the force $\vec{F}$ on the object change if we change the sign of ω?

3. Suppose a point P is rotating around a line ℓ, keeping its distance to the line fixed at r, and moving in a plane perpendicular to the line. Suppose the point has angular velocity ω: this means that during a time interval of length t the angle swept out by the line segment connecting P to ℓ is exactly ωt.

In a previous math or physics class it was shown that the velocity of the point P is ωr, where r is the distance from P to the line ℓ.

The ***angular velocity vector*** is defined to be the vector $\vec{\omega}$ whose length is ω, and that is parallel to the line ℓ. There are two such vectors ($\pm\vec{\omega}$). By definition $\vec{\omega}$ points in the direction in which a screw would move if it were turning in the same direction as the point P.

(a) Assuming the line ℓ passes through the origin show from the drawing that the velocity vector of the point P is $\vec{v}$ is given by $\vec{\omega}\times\vec{x}$. You can do this in two steps, namely:

— *show that $\vec{\omega}\times\vec{x}$ has the same direction as $\vec{v}$,*
— *show that $\vec{\omega}\times\vec{x}$ has the same length as $\vec{v}$.*

(b) Show that the acceleration vector is given by $\vec{a} = \vec{\omega}\times(\vec{\omega}\times\vec{x})$. (hint: don't use the drawing, but combine the definitions of $\vec{v}$ and $\vec{a}$, in (24) and (25) and also the product rule; finally, keep in mind that you have just found that $\vec{v} = \vec{\omega}\times\vec{x}$.)

(c) If someone told you they had computed the acceleration vector and found

$$\vec{a} = (\vec{\omega}\times\vec{\omega})\times\vec{x},$$

could they be right? Explain! What if they told you they got $\vec{a} = \vec{\omega}\times\vec{\omega}\times\vec{x}$?

(d) True or False (explain your answers):

(a) $\vec{v} \perp \vec{x}$? (b) $\vec{a} \perp \vec{v}$? (c) $\vec{a}$ and $\vec{x}$ are parallel?

(e) Include the acceleration vector $\vec{a}$ in the above drawing.

4. Consider the "twisted cubic," i.e. the curve given by $\vec{x}(t) = t\vec{e}_1 + t^2\vec{e}_2 + t^3\vec{e}_3$.

(a) Find a parametrization for the tangent to the curve at the point where $t = 1$. Where does this point intersect the xy-plane?

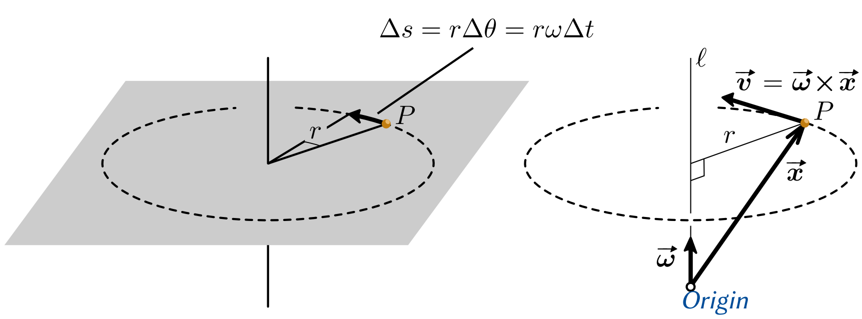
(b) For any given t find the tangent line to the curve at the point $X(t)$, and find where this curve intersects the xy-plane.

(c) If you call that intersection point $P(t)$, then which curve is traced out by the point $P(t)$ as t varies?

5. Compute the length of one full turn of the helix by taking the parametrization given in (21) and computing the length of the segment with $0 \le \theta \le 2\pi$.

After computing the length, consider this: let P be the perimeter of the circle underneath the helix, and let H be the height achieved by one full turn of the helix. Show that the length L of the helix satisfies $L^2 = P^2 + H^2$.

6. There is a multistory parking ramp where the way out is a path in the shape of a helix that is wound around the outside of the building. As a car drives down this path at night its headlights shine a spot on the ground. Which curve is traced out by this light spot as the car drives all the way down?



Make a good drawing. Assume for simplicity that the center of the Parking ramp is the z-axis.

7. Compute the tangent, curvature, normal and binormal for the following curves

(a) The parabola: $\vec{x}(t) = \begin{pmatrix} t^2 \\ t \end{pmatrix}$. At which point on the curve is the curvature the largest?

(b) Neil's parabola: $\vec{x}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$. At which point on the curve is the curvature the largest?

(c) The helix: $\vec{x}(\theta) = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ a\theta \end{pmatrix}$ (see § 6 for an explanation of the constants R and a). At which point on the curve is the curvature the largest?

(d) The graph of $y = e^x$ by using the parametrization $\vec{x}(t) = \begin{pmatrix} t \\ e^t \end{pmatrix}$. Where on the graph is the curvature the largest?

CHAPTER 3

Functions of more than one variable

1. Functions of two variables and their graphs

1.1. Definition. A function of two variables has two ingredients: a *domain* and *a rule*. The domain of the function is a collection of points in the xy-plane. For each point (x, y) from the domain of the function, the rule should tell us how to find the function value $f(x, y)$.

Just as with functions of one variable, the "rule" that gives us the function value is often specified by some formula, e.g. $f(x, y) = x + y$. The domain of a function is the set of points at which we define the function. This can in principle be any set of points in the plane. Typically the domain will be a rectangle, or a disc, or it could be the entire xy-plane, possibly with some points and lines removed.

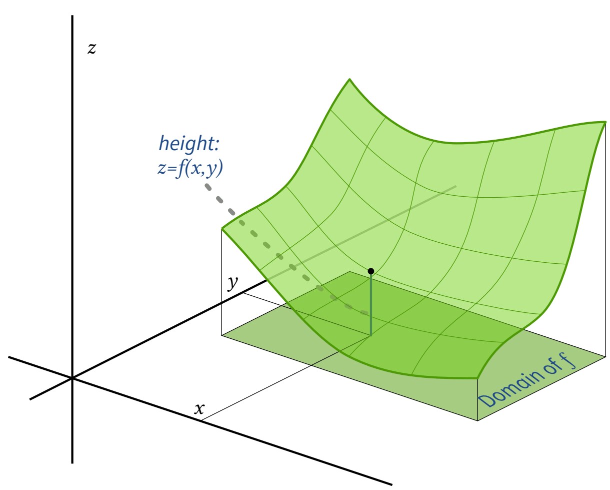


Figure 1. The graph of some function, and its domain (a rectangle in this example).

1.2. Graphs. By definition, the graph of a function $z = f(x, y)$ is the collection of all points (x, y, z) in three dimensional space that satisfy the equation $z = f(x, y)$.

The graph is usually a surface that floats above (or below) the domain of the function (see Figure 2).

1.3. Level sets. The graph of a function of two variables is a surface sitting in three dimensional space, which can be difficult to draw or visualize. Instead of looking at the graph we can also consider its level sets. If c is any real number, then, by definition, the ***level set at level*** c of the function is the set of all points (x, y) in the plane that satisfy $f(x, y) = c$.

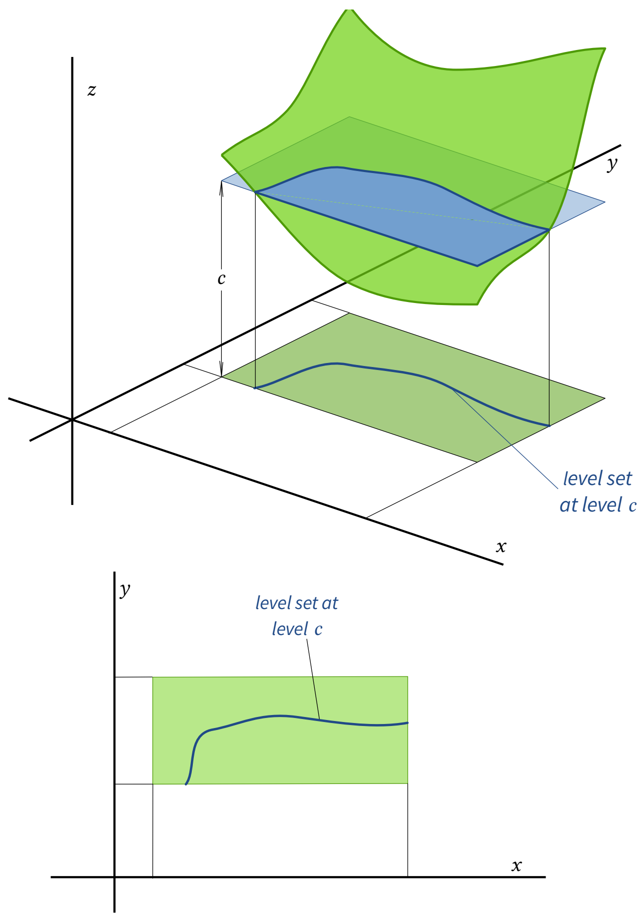


Figure 2. The graph of some function (*top*), and a construction of one of its level sets (*bottom*). Note that *by definition* the level set ("at level c") is the curve in the xy-plane under the graph: it is obtained by intersecting the graph of the function with a horizontal plane at height c, and then projecting this curve of intersection onto the xy-plane.

Since the level set is the set of all solutions to the equation $f(x, y) = c$, one often uses the notation $f^{-1}(c)$ ("f-inverse of c") for the level set. We can summarize the definition in an equation:

$$f^{-1}(c) = \{(x, y) : f(x, y) = c\}.$$

Note that the definition says that $f^{-1}(c)$ ***is not a number, but a set of points!***

Level sets tend to be curves in the xy-plane, although in general level sets can have any shape (see Problem 5.13 for an example.) They are usually easier to draw than the graphs of the corresponding functions.

1.4. An example from the "real" world. Here is a function of local interest. The domain of the function is the water surface of Lake Mendota (let's pretend this is a plane domain), and the function, which we will call d instead of f, is given by $d(x, y) =$ the depth of the lake at location (x, y). There is no formula for this function, but the Wisconsin Department of Natural Resources has measured the depth and presented the results in terms of the level sets of the function d.

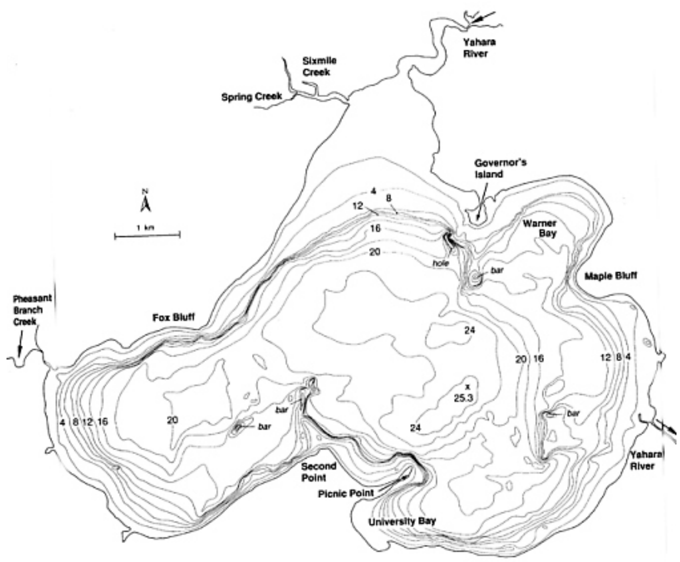


Figure 3. The level curves of a function $z = d(x, y)$. The domain of this function is the lake surface, and $d(x, y)$ is the depth in meters of Lake Mendota at (x, y). To see the graph of the function we could try to drain the lake.
See `http://limnology.wisc.edu/lake_information/mendota/mendota.html`

1.5. A comment about language and set-theoretic notation. We will often say "consider a function $z = f(x, y)\ldots$", but there is a sense in which this is incorrect. It is convenient to say "consider a function $z = f(x, y)\ldots$" since it not only names the function, but it also gives the independent variables x, y, and the dependent variable z a name. Nevertheless, the symbol in the equation $z = f(x, y)$ that actually represents the function is "f". The correct way of introducing the function[1] would be to say "consider a function f."

In fact, in the notation that is used in modern mathematics one would write "Consider the function $f : D \to \mathbb{R}\ldots$" Here f is the name of the function we are introducing, D is

[1]Saying "consider the function $z = f(x, y)\ldots$" to introduce the function f is like saying "Please meet my brother Joe, Bill, and Sue" when you want to introduce your brother Joe, who happens to be standing next to Bill and Sue. To introduce your brother, you would of course say "Please meet my brother Joe." and to introduce the function you should really say "Consider the function f."

the domain of that function (so D is a set of points in the plane), and $\mathbb{R}$ stands for the set of real numbers, indicating that computing f always results in a real number.

1.6. Vector notation. If $\vec{\boldsymbol{x}}$ is the position vector of the point (x, y) in the plane, i.e. if $\vec{\boldsymbol{x}} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, then one sometimes writes

$$f(x, y) = f(\vec{\boldsymbol{x}}).$$

Physicists have a preference for $\vec{\boldsymbol{r}}$ instead of $\vec{\boldsymbol{x}}$ (because they call the position vector the "radius vector"), and will write $f(x, y) = f(\vec{\boldsymbol{r}})$.

2. Linear functions

The simplest function of one variable are those of the form $f(x) = ax + b$. Their graphs are lines, and we called them linear functions.

A linear function of two variables is a function f of the form

$$z = f(x, y) = ax + by + c, \tag{45}$$

where a, b, c are constants.

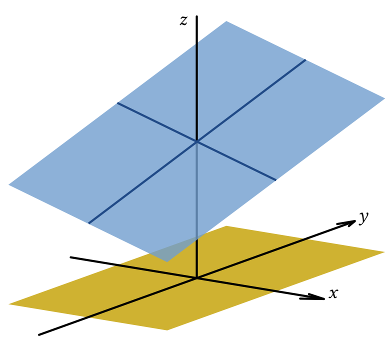


Figure 4. The graph of a linear function $z = ax + by + c$.

The graph of a linear function is always a plane. Indeed, the graph consists of all points (x, y, z) that satisfy the equation

$$-ax - by + z = c,$$

which we can write as

$$\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{x}} = \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{p}},$$

where

$$\vec{\boldsymbol{n}} = \begin{pmatrix} -a \\ -b \\ 1 \end{pmatrix}, \quad \text{and} \quad \vec{\boldsymbol{p}} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}.$$

3. Quadratic forms

After learning about linear functions in pre-calculus one usually goes on to quadratic functions. We will do the same for functions of two variables and study *Quadratic Forms*. Just as in the one variable case where quadratic functions can have a maximum or minimum, quadratic forms provide examples of functions of two variables that can have a maximum or a minimum, or, it turns out, a third kind of "min-max" or "saddle shape." They provide the basic profile of what we will run into when we look for local minima and maxima of functions of two variables. In particular, the technique of classifying quadratic forms by completing the square, which we will see in this section, is the key to the second derivative test for functions of more than one variable.

3.1. Definition.

The general quadratic form in two variables is

$$f(x, y) = Ax^2 + Bxy + Cy^2, \tag{46}$$

where A, B, and C are constants. Depending on the values of these constants the graphs of the functions can have a number of different shapes.

In addition to these quadratic forms one can also consider the more general class of quadratic functions,

$$f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F,$$

which also have terms of degree 1 and 0. We will restrict ourselves to quadratic forms (for now).

The prototypical examples. There are several important special cases that are representative of what the graphs of quadratic forms can look like. These special cases are

$$f(x, y) = x^2 + y^2, \text{ and } g(x, y) = -x^2 - y^2, \tag{47a}$$

$$h(x, y) = x^2, \text{ and } \tilde{h}(x, y) = -x^2, \tag{47b}$$

$$k(x, y) = xy \tag{47c}$$

Their graphs are discussed in Figure 5.

3.2. Classifying quadratic forms – the general procedure.

All quadratic forms have graphs that look like one of the examples shown above – but how can we tell which it is? In other words, if $Q(x, y)$ is a given quadratic form how can we tell if it is definite, indefinite, or semidefinite? How do we know for which (x, y) the form $Q(x, y)$ is positive or negative? It turns out that we can always find out by using the trick of "completing the square."

The general procedure for a given quadratic form $Q(x, y) = Ax^2 + Bxy + Cy^2$ is as follows:

(1) If $A = 0$, then we really have $Q = Bxy + Cy^2$ and we can factor Q as

$$Q(x, y) = (Bx + Cy)y.$$

(2) Assume $A \neq 0$. We factor out A, and complete the square for the first two terms:

$$\begin{aligned} Q(x,y) &= A\Big\{x^2 + \frac{B}{A}xy + \frac{C}{A}y^2\Big\} \\ &= A\Big\{\big(x + \frac{B}{2A}y\big)^2 - \big(\frac{B}{2A}y\big)^2 + \frac{C}{A}y^2\Big\} \\ &= A\Big\{\underbrace{\big(x + \frac{B}{2A}y\big)^2}_{u^2} + \underbrace{\frac{4AC - B^2}{4A^2}y^2}_{\pm v^2}\Big\}. \end{aligned}$$

(3) If $4AC - B^2 > 0$, then the expression in braces is positive, and we can write

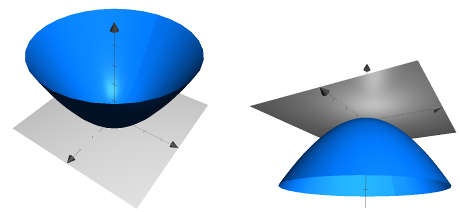
$$Q(x,y) = A(u^2 + v^2), \quad \text{where} \quad u = x + \frac{B}{2A}y, \text{ and } v = \frac{\sqrt{4AC - B^2}}{2A}y.$$

Depending on the sign of A our function is always positive or always negative, and we say the form is ***positive definite*** or ***negative definite.***

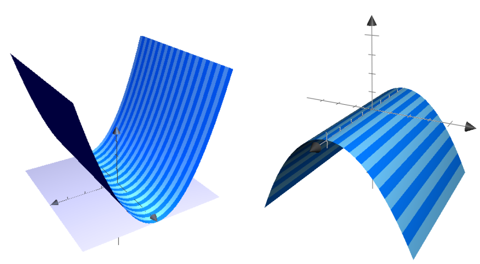
The two forms f and g from (47a) are called ***definite***, since they cannot change sign:

$$f(x,y) = x^2 + y^2$$

is the sum of two squares, and therefore is always positive, unless both x and y vanish. Similarly, $g(x,y) = -f(x,y)$ is always negative, except at $(x,y) = (0,0)$.



The form $h(x,y) = x^2$ is called ***semidefinite*** because it too cannot change its sign. Clearly, $h(x,y) = x^2$ is never negative, but for $h(x,y)$ to be positive, we need $x \neq 0$. So, the function $h(x,y)$ is positive, except on the line $x = 0$ (the y axis). The graph of the function $\tilde{h}(x,y) = -y^2$ is similar, but upside down.



The form $k(x,y) = xy$ is called ***indefinite,*** because it can be both positive and negative: if x and y have the same sign, then $xy > 0$, but if they have opposite signs, then $xy < 0$. Thus the graph of $z = xy$ lies above the xy-plane in the first and third quadrants, and below the xy-plane in the second and fourth quadrants.

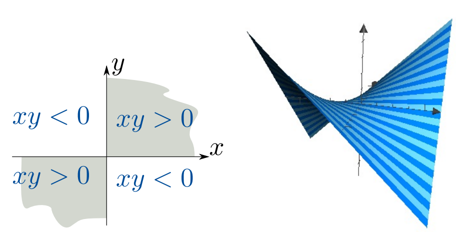


Figure 5. Graphs of some representative quadratic forms.

(4) If $4AC - B^2 < 0$, then we have

$$Q(x,y) = A(u^2 - v^2), \quad \text{where} \quad u = x + \frac{B}{2A}y, \text{ and } v = \frac{\sqrt{B^2 - 4AC}}{2A}y.$$

When this happens we can factor the quadratic form, i.e. we have

$$Q(x,y) = A(u+v)(u-v).$$

The form is ***indefinite.***

(5) in the only remaining case we have $4AC - B^2 = 0$, so that

$$Q(x,y) = A\Big(x + \frac{B}{2A}y\Big)^2.$$

In this case the form is a perfect square (times A). The form is ***semi-definite.***

To understand this procedure it is perhaps best to look at how it works in some examples.

3.3. Classifying quadratic forms – two examples.

3.3.1. *An indefinite quadratic form.* Consider the form $Q(x,y) = -3x^2 + 9xy + 6y^2$. We rewrite this as follows:

$$\begin{aligned}
Q &= -3x^2 + 6xy + 9y^2 \\
&= -3\big(x^2 - 2xy - 3y^2\big) \\
&= -3\big[\underbrace{x^2 - 2xy + y^2} - 4y^2\big] && \textit{complete the square} \\
&= -3\big[(x-y)^2 - 4y^2\big] && \textit{in this case we get the difference of two squares, so use } a^2 - b^2 = (a-b)(a+b) \\
&= -3(x - y - 2y)(x - y + 2y) \\
&= -3(x - 3y)(x + y).
\end{aligned}$$

This shows that $Q(x,y) > 0$ when $y > \frac{1}{3}x$ or $y < -x$, and $Q(x,y) < 0$ when $-x < y < \frac{1}{3}x$.

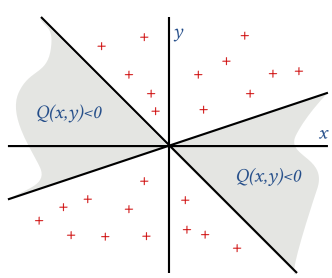


Figure 6. The signs of the quadratic form in example 3.3.1.

3.3.2. *A positive definite quadratic form.* To see a different example, consider the quadratic form $Q(x,y) = 2x^2 - 4xy + 6y^2$. By completing the square we can write it as

$$\begin{aligned} Q(x,y) &= 2\left\{x^2 - 2xy + 3y^2\right\} \\ &= 2\left\{x^2 - 2xy + y^2 + 2y^2\right\} \qquad \textit{the square is complete} \\ &= 2\left\{(x-y)^2 + 2y^2\right\} \\ &= 2(x-y)^2 + 4y^2. \end{aligned}$$

We see that this particular quadratic form is positive definite.

4. Functions in polar coordinates r, θ

Recall that instead of using Cartesian coordinates (x,y) to specify the location points in the plane, we can also use polar coordinates. In many cases it is much easier to describe a function using polar coordinates than in Cartesian coordinates.

To go back and forth between Cartesian and Polar Coordinates we can use the following relations

$$x = r\cos\theta \tag{48a}$$

$$y = r\sin\theta \tag{48b}$$

$$r = \sqrt{x^2 + y^2} \tag{48c}$$

$$\theta = \arctan\frac{y}{x} \tag{48d}$$

The equation for θ is only valid for $x > 0$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. In other regions of the plane there are other expressions relating θ to (x,y). See problem 5.8.

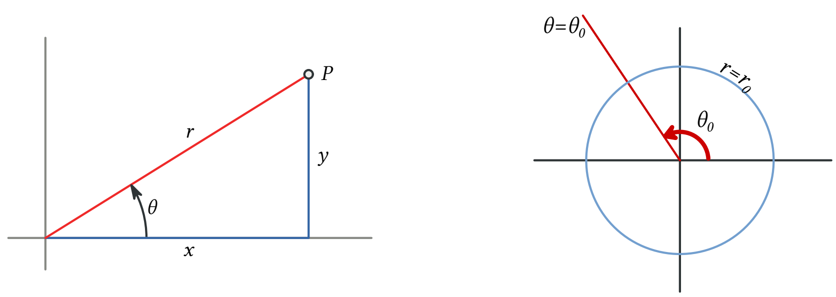


Figure 7. Polar coordinates are defined in the picture on the right (see also equations (48)). On the left: the set of points at which θ has one given value θ_0 form a half line emanating from the origin that makes an angle θ_0 with the positive x-axis. The set of points at which r has a given value r_0 form a circle centered at the origin, with radius r_0.

The simplest kinds of functions one can consider in polar coordinates are those that only depend on one of those coordinates, i.e. functions that only depend on the radius r, and functions that only depend on the polar angle θ. Let's look at some examples of such functions.

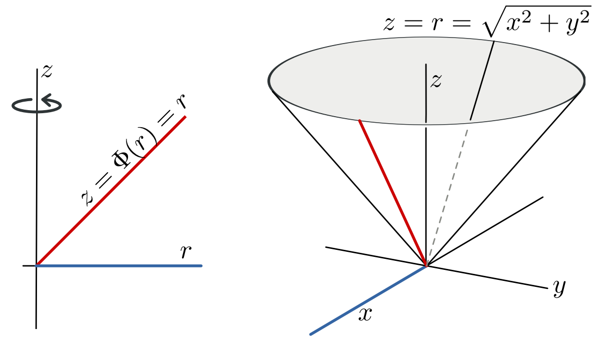


Figure 8. Radially symmetric functions. The graph of $z = r$.

4.1. Radially symmetric functions. The functions

$$f(x, y) = x^2 + y^2, \quad g(x, y) = \sqrt{x^2 + y^2}, \quad h(x, y) = \ln\bigl(x^2 + y^2\bigr),$$

all can be expressed in terms of the radius r only. Namely, using $r^2 = x^2 + y^2$, we have

$$f(x, y) = r^2, \quad g(x, y) = r, \quad h(x, y) = \ln r^2 (= 2 \ln r).$$

In general, a function $z = f(x, y)$ that can be written in terms of the radius r only, i.e. a function for which there is some function Φ of one variable with

$$f(x, y) = \Phi(r), \quad \text{i.e. } f(x, y) = \Phi\bigl(\sqrt{x^2 + y^2}\bigr),$$

is called a ***radially symmetric function.***

Since a radially symmetric function only depends on the radius r, its level sets consist of circles centered at the origin (one exception: the origin, $r = 0$ can also be a level set, and this is obviously not a circle but a point.)

As an example, we consider the function $g(x, y) = \sqrt{x^2 + y^2} = r$ in more detail. The function Φ of one variable here is $\Phi(r) = r$. We can try to visualize the graph of g by first looking at the positive x-axis only. There we have $f(x, 0) = \sqrt{x^2} = x$. We get the graph of g by revolving the graph of $z = x$ around the z-axis. See Figure 8.

4.2. Functions of θ only. Here are two functions that happen to depend on the polar angle θ only:

$$f(x, y) = \sin \theta, \qquad h(x, y) = \theta.$$

We can rewrite these functions in terms of x and y by using the relations between Cartesian and Polar coordinates (48). We get

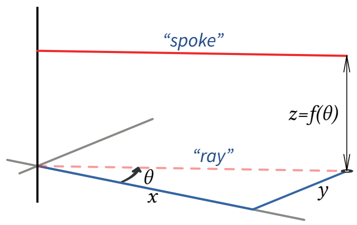
$$f(x, y) = \sin \theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

for f, and

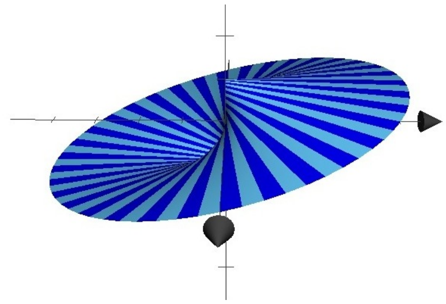
$$h(x, y) = \theta = \arctan \frac{y}{x}$$

for h, at least in the right half plane where $x > 0$.

A function that only depends on θ is constant on rays emanating from the origin because the polar angle θ is constant on such rays. The level sets of such a function therefore consist of half-lines ("rays") starting at the origin. Its graph consists of "spokes" attached to the z-axis. Each spoke lies above a ray in the xy-plane with some polar angle θ, and is attached to the z-axis at a height given by the function value. As we vary θ, the spoke rotates around the vertical axis and moves up or down, as dictated by the function. Figure 9 shows what happens for $f(x, y) = \sin\theta$.



The graph of a function of θ only consists of horizontal spokes attached to the z-axis.



The graph of $z = \sin\theta$ (the x-axis is coming right at us.)

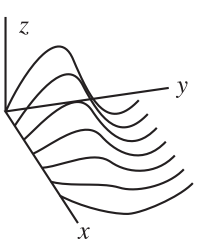
Figure 9

The function $z = \theta$ has a simpler formula in polar coordinates but actually has a more complicated graph. Let us try to visualize its graph: the spokes that make up the graph are horizontal, attached to the z-axis, and are at height θ. If we increase the angle θ the spokes go up at a steady rate in a way that should remind us of a helix (see § 6 and Figure 5). Based on this description its graph should look like the surface drawn in Figure 10. The surface is called the ***helicoid***, and it is ***not*** the graph of a function (it fails the “vertical line test.”) We could have known this from the beginning , because when we described our function as $f(x, y) = \theta$, we should have immediately asked *which θ?* The polar angle θ of any given point is only determined up to a multiple of 2π. The “graph” that we have drawn of the “function” $z = \theta$ reflects this. To make $h(x, y) = \theta$ into an honest function we have to say which of the many possible angles θ we choose when we are given a point. One possible choice is to always require the polar angle θ to lie between 0 and 2π (radians). More precisely, we can insist on

$$0 \leq \theta < 2\pi.$$

If we do this then there is a unique angle θ for each point (x, y) in the plane. The graph of this function is shown on the right in Figure 10.

5. Methods of visualizing the graph of a function



5.1. Freezing a variable. If a function is not familiar, then a good strategy for drawing its graph is to “***freeze a variable.***” In other words, to analyze a function $z = f(x, y)$ we pretend y is a constant: then x is the only independent variable, and we can try to draw the graph of the function $z = f(x, y)$, now thinking of this as a function of only one variable. This graph is a curve in the xz plane. We get one such curve for each choice of y. Piecing these graphs together then gives us the graph of the two-variable function $z = f(x, y)$.

We could apply the same procedure with the roles of x and y switched: i.e. for each fixed x you try to graph $z = f(x, y)$ as a function of the variable y only, after which we try to fit all the graphs we get for different values of x together.

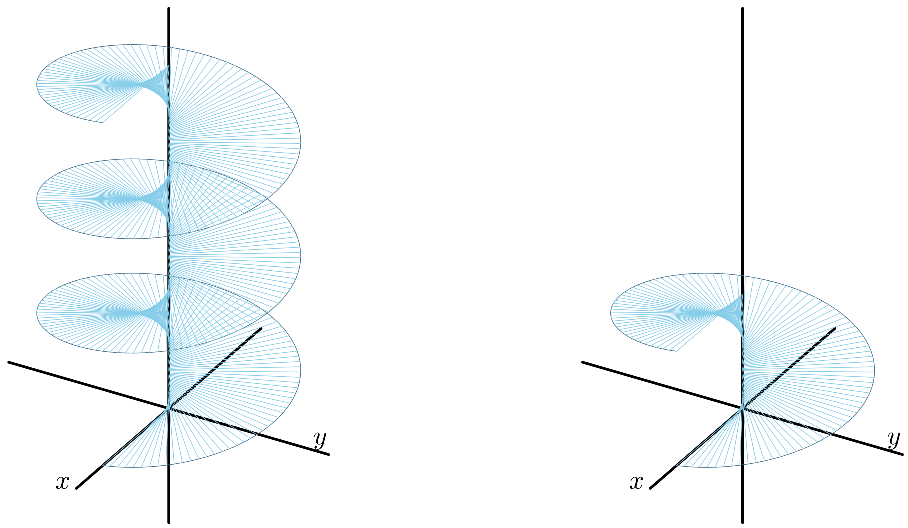


Figure 10. The graph of $z = \theta$ is the helicoid. It is not the graph of a function, but one can extract a function by choosing a "branch" of the function. One possible choice, drawn here on the right, is to restrict the polar angle θ to the interval $0 \leq \theta < 2\pi$. There are many other possible choices.

5.2. Moving graphs. There is another way of visualizing a function $z = f(x, y)$ of two variables in which we think of one of the independent variables (e.g. y) as "time." The final picture is not one static image of a three dimensional surface, but rather a movie of a graph that is moving around in the xz plane.

If we have a function $z = f(x, y)$, then let us think of y as time, and let us relabel it as t, so that we are looking at the function $z = f(x, t)$. Now at each moment in time t we can think of $z = f(x, t)$ as a function of one variable x whose graph we can try to draw, regarding it as a still-image. Then, as we let time t vary, putting the still images in a sequence, you get a movie of a graph of a changing function of one variable.

For instance, if the function is (once again) the saddle surface function $z = xy$, then we would be considering the function $z = xt$. At each moment t the graph of $z = xt$ is

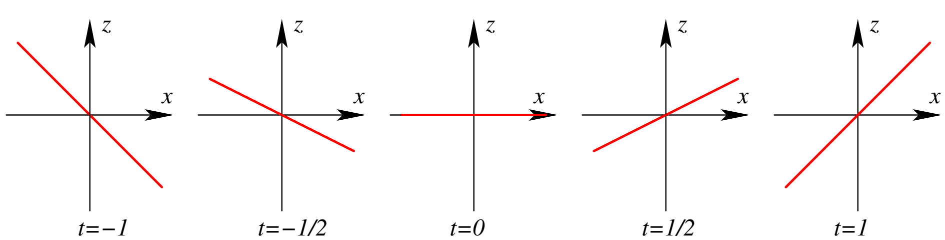


Figure 11. The saddle movie. It's about a line segment whose slope changes, even though it is otherwise stuck to the origin.

a line with slope t. Putting these graphs together gives a movie which begins with a line of rather negative slope; during the movie the slope increases, and in the middle of the movie our line has achieved horizontality; finally, the closing shot presents us with a line with a very positive slope. Figure 11 shows some stills from the movie.

This interpretation is not very different from the procedure of "freezing the y variable." The only real difference lies in what we do with all the separate graphs we get after we freeze a variable. In one case we try to piece them together to make a bigger drawing of a three-dimensional object, in the other we put them together to make a motion picture.

Problems

In the problems in this stage of the course, you will be asked to "sketch the graph of a function." From math 221 you remember that this meant you had to find minima, maxima, inflection points, and other features of the graph. In 234 you will learn to do the same for functions of two (and more) variables, but for now you should try to use the method of "freezing a variable" or other similar tricks to get an idea of what the graph of f looks like.

You can use a graphing program (such as `Grapher.app` on the Mac, `GraphCalc` on Windows, or one of the many websites such as `http://www.graphycalc.com/`) to check your answer.

> Note: very often students try to fit their drawings into a region the size of a post-it. In this course, whenever you make a drawing, especially if it's a three-dimensional drawing, ***make it large!*** Use half a page for a drawing. Make sure you have enough paper, try to find lots of cheap scrap paper.

1. If we were to drain Lake Mendota, as suggested in § 1.4, would the lake bottom give us the graph of $d(x, y)$ or of $-d(x, y)$? (where d is the depth of the lake)? •

2. What are the signs of the coefficients a, b, and c for the linear function whose graph is drawn in Figure 4? •

3. *About planes and their intersections with the coordinate axes.*

(a) Where does the plane $z = 3x - y + 6$ intersect the three coordinate axes? •

(b) Find the equation for the plane that intersects the x-axis at $x = 4$, the y-axis at $y = 2$, and the z-axis at $z = 3$. •

(c) Find the equation for the plane that intersects the x-axis at $x = a$, the y-axis at $y = b$, and the z-axis at $z = c$. (Write the equation as nice as possible.) •

4. Find a formula for the distance to the origin of the graph of (45). •

5. Classify the following quadratic forms as definite, indefinite, or other, by completing the square. Determine the zero set for each of these quadratic forms.

(a) $f(x, y) = x^2 + 2y^2$ •

(b) $Q(x, y) = x^2 - y^2$ •

(c) $g(x, y) = x^2 - 4xy + 3y^2$ •

(d) $Q(s, t) = 9s^2 - 36st + 81t^2$ •

(e) $M(\alpha, \beta) = \frac{1}{2}\alpha^2 - \alpha\beta + \beta^2$. •

(f) $Q(x, y) = xy + y^2$ •

(g) $Q(x, y) = x^2 + 2xy$ •

6. For which values of the constant k is the quadratic form

$$Q(x, y) = x^2 + 2kxy + y^2$$

positive definite? •

7. Which functions of two variables $z = f(x, y)$ are defined by the following formulae?

▷ Find draw the domain of each function (the largest domain on which the definition would make sense).

▷ Try to sketch their graphs.

▷ Draw the level sets for each function.

(a) $z = xy$

(b) $z - x^2 = 0$

(c) $z^2 - x = 0$

(d) $z - x^2 - y^2 = 0$

(e) $z^2 - x^2 - y^2 = 0$

(f) $xyz = 1$

(g) $xy/z^2 = 1$

(h) $x + y + z^2 = 0$

(i) $x + y + z^2 = 1$

8. The following expressions are all equal to the polar angle θ in some region of the xy-plane. Explain why the expression gives θ, and identify in which region this holds.

(a) $\theta = \arctan \dfrac{y}{x}$

(b) $\theta = \pi + \arctan \dfrac{y}{x}$

(c) $\theta = 2\pi + \arctan \dfrac{y}{x}$

(d) $\theta = \dfrac{\pi}{2} - \arctan \dfrac{x}{y}$

(e) $\theta = \arcsin \frac{y}{\sqrt{x^2+y^2}}$.

9. *"The level set is always a curve..." — not!* If $d(x, y)$ is the depth function of Lake Mendota (see §1.4), then what are the level sets $d^{-1}(c)$ for $c = 0$, $c = +24$ and for $c = -24$ (meters)? What is the level set $d^{-1}(400)$ (meter)?

10. Describe and explain the relation between the graph of the function $y = g(x)$ of one variable, and the corresponding function $f(x, y) = g\big(\sqrt{x^2 + y^2}\big)$ of two variables.

What do the level sets of $f(x, y)$ look like?

For instance, if $g(x) = x$, then $f(x, y) = \sqrt{x^2 + y^2}$: what is the relation between the graphs of g and f?

11. Find the largest domain on which the following functions of two (or occasionally three) variables can be defined:

(a) $f(x, y) = \sqrt{9 - x^2} + \sqrt{y^2 - 4}$

(b) $f(x, y) = \arcsin(x^2 + y^2 - 2)$

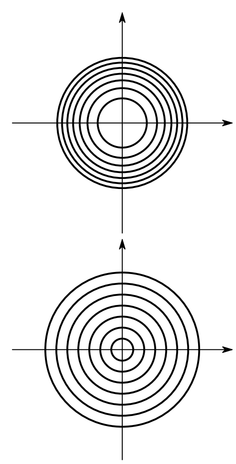
(c) $f(x, y) = \sqrt{x} \cdot \sqrt{y}$

(d) $f(x, y) = \sqrt{xy}$

(e) $f(x, y, z) = 1/\sqrt{xyz}$

(f) $f(x, y) = \sqrt{16 - x^2 - 4y^2}$

12. Here are two sets of level curves with levels $z = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4$. One is for a function whose graph is a cone ($z = \sqrt{x^2 + y^2}$), the other is for a paraboloid ($z = x^2 + y^2$). Which is which? Explain.



13. Let Q be the square in the plane consisting of all points (x, y) with $|x| \leq 1$, $|y| \leq 1$. This problem is about the so-called ***distance function*** to Q. This function is defined as follows: $f(x, y)$ is the distance from the point (x, y) to the point in Q nearest to (x, y).

(a) Which point in Q is nearest to $(0, \frac{1}{2})$? Which is closest to $(0, 2)$? Which is closest to $(3, 4)$?

(b) Compute $f(0, \frac{1}{2})$, $f(0, 2)$ and $f(3, 4)$.

(c) What is the zero set of f?

(d) Draw the level sets of f at levels -1, 1, 2, and 3. Describe the general level set

$f(x, y) = c$ where c is an arbitrary number. •

(e) Give a formula for $f(x, y)$. (It turns out to be too hard to capture the distance function in one formula. You will have to split the plane into different regions and describe $f(x, y)$ by different formulas, according to which region (x, y) belongs to.) •

14. Describe the "movie" that goes with each of the following functions.

(a) $f(x, t) = x \sin t$ •

(b) $f(x, t) = x \sin 2t$ •

(c) $f(x, t) = t \sin x$ •

(d) $f(x, t) = 2t \sin x$ •

(e) $f(x, t) = t \sin 2x$ •

(f) $f(x, t) = (x - t)^2$ •

(g) $f(x, t) = (x - \sin t)^2$ •

(h) $f(x, t) = (x - t^2)^2$

(i) $f(x, t) = \dfrac{t^2}{1 + x^2}$

(j) $f(x, t) = \dfrac{1}{(1 + x^2)(1 + t^2)}$ •

15. Describe the movie that goes with the function

$$f(x, t) = \arctan \frac{x}{t},$$

for $t > 0$. The function is not defined at $t = 0$, but can you describe the limit of this function as $t \to 0$? (Hint: the sign of x matters).

16. If $y = g(x)$ is any function of one variable, then a function of the form $f(x, t) = g(x - ct)$ is often called a ***traveling wave*** with wave speed c and profile g. Let g be any non constant function of your choice and describe the movie presented by the function $f(x, t) = g(x - ct)$ (can't choose? Then try "Agnesi's witch" $g(x) = \frac{1}{1+x^2}$.)

The number c is called the wave speed. If $c > 0$ is the motion to the left or to the right? Explain. •

17. If $y = g(x)$ is any function of one variable, then a function of the form

$$f(x, t) = \cos(\omega t) g(x)$$

is often called a ***standing wave.*** Let g be any non constant function of your choice and describe the movie presented by the function $f(x, t) = \cos(\omega t) g(x)$ (can't choose? Then try "Agnesi's witch" $g(x) = \frac{1}{1+x^2}$ again, or for this example, try $g(x) = \sin x$.)

The number $\frac{\omega}{2\pi}$ is called the frequency of the standing wave. The function $g(x)$ is called its profile. How long does it take before the standing wave returns to its original position, i.e. what is the smallest $T > 0$ for which $f(x, T) = f(x, 0)$ for all x? Explain. •

CHAPTER 4

Derivatives

1. Interior points and continuous functions

Before diving into the calculus of partial derivatives we need to discuss certain assumptions that we shall always implicitly make about the functions in this course. The first concerns the domains of our functions. Namely:

(49) *We only consider functions at interior points of their domain*

Here, by definition, a point (a, b) in the domain of a function is called an *interior point* if the function is also defined at all points (x, y) that lie within some small disc centered at (a, b).

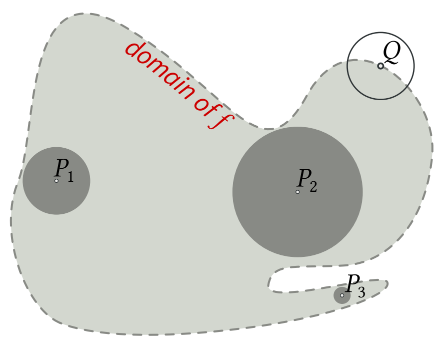


Figure 1. Interior and boundary points in the domain of f: P_1, P_2, or P_3 are interior points in the domain. Each of these points is the center of a sufficiently small disc that is still contained in the domain. For points such as Q, that lie on the edge of the domain, any disc centered at Q will "stick out of the domain," no matter how small the disc is chosen. If we talk about the derivative of a function at some point in its domain, then, *in this course*, we will always assume that we are not at an edge-point like Q.

The other standing assumption we make in this course is that

(50) *all functions we consider are continuous.*

We have seen the concept of continuity for functions of one variable. For functions of more variables "continuity" has a similar definition. In this course we will aim for an intuitive understanding of the concept, which can be formulated as follows.

> *The function $z = f(x, y)$ is continuous at some point (a, b) if the function value $f(x, y)$ at any point (x, y) is close to $f(a, b)$ when (x, y) is close to (a, b).*

There are many other ways of describing continuity, e.g. one can say that f is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

To make this precise we would have to define what "$\lim_{(x,y)\to(a,b)} \ldots$" means.

A precise definition of "f is continuous at (a, b)" invokes ε's and δ's:

> *The function $z = f(x, y)$ is continuous at some point (a, b) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every point (x, y) that lies in the disc of radius δ centered at (a, b) one has $|f(x, y) - f(a, b)| < \varepsilon$.*

In this course we will not use the definition much, but we will occasionally appeal to the intuitive notion of "continuity." The problems show some examples of how a function of two variables can fail to be continuous (e.g. Problem 3.1).

Now that we have dispensed with these preliminary issues, we can go on to the central topic in the first half of the semester: partial derivatives and the chain rule.

2. Partial Derivatives

The derivative $f'(x)$ of a function of one variable, $y = f(x)$, measures a rate of change: if we increase x by a small amount Δx then $y = f(x)$ also increases by a small amount Δy. The ratio between these two changes is the derivative: $f'(x) \approx \frac{\Delta y}{\Delta x}$.

For a function $z = f(x, y)$ of two variables there is a similar concept: if we change x and/or y by a small amount then z will also change by a small amount, and there are formulas relating the changes Δx, Δy and Δz. Because there are many different ways in which we can change x and y there are a few different formulas. We will encounter the following versions of "the derivative of $f(x, y)$":

▶ Change only one of the variables but not the other: this leads to the so-called ***partial derivatives***.

▶ Simultaneously vary both x and y: the resulting change turns out to be the sum of the changes we would get if we were to vary only x or only y, respectively. This will follow from the ***chain rule***, and the resulting formula is called the ***total derivative***.

We begin with the partial derivatives.

2.1. Definition of Partial Derivatives.

If $z = f(x, y)$ is a function of two variables then the partial derivatives of f with respect to x and with respect to y are

$$\frac{\partial f}{\partial x}(x, y) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{51}$$

and

$$\frac{\partial f}{\partial y}(x, y) = \lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{52}$$

The following more convenient notation is used very often (because it's so much shorter):

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y), \qquad f_y(x, y) = \frac{\partial f}{\partial y}(x, y). \tag{53}$$

When we are in a hurry we can also drop the "(x, y)" from our notation for derivatives and just write f_x and f_y.

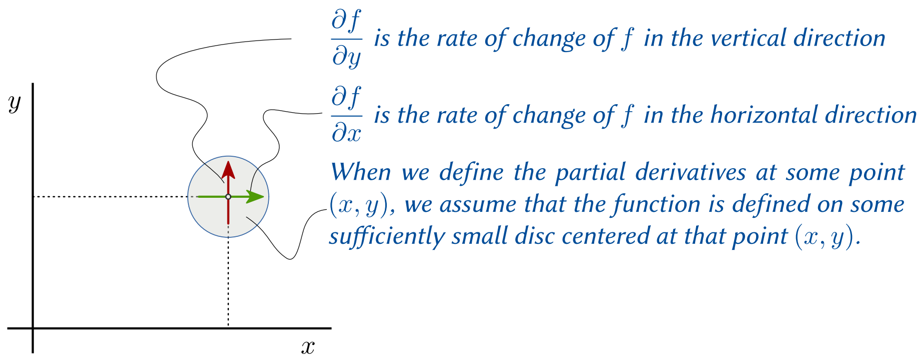


Figure 2. The partial derivatives of a function at some point (x, y) measure how fast the function $f(x, y)$ changes if we move the point either horizontally (the x direction) or vertically (the y direction).

2.2. Partial derivatives of functions of three or more variables. If a function depends on three or more variables then one can define its partial derivatives in the same way as for functions of two variables. For instance, if $w = f(x, y, z)$ is a function of three variables, then its partial derivative with respect to x is defined to be

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}.$$

The derivatives of f with respect to y and z have very similar definitions.

2.3. Examples. Computing partial derivatives is not harder than computing ordinary derivatives. To find the partial derivative of a function with respect to x we just pretend all other variables are constants and differentiate. Or, in other words, we could think of the partial derivative of $f(x, y)$ with respect to x as the ordinary derivative of the function f in which we have frozen the variable y at some particular value.

For instance, the partial derivatives of the function $f(x, y, z) = x^2 \sin \pi y + z$ of ***three*** variables x, y, and z, are

$$f_x = 2x \sin \pi y, \quad f_y = \pi x^2 \cos \pi y \text{ and } f_z = 1.$$

3. Problems

1. For each of the following functions sketch the graph (use a graphing program, if necessary) and decide if you think the function has a limit as (x, y) approaches $(0, 0)$.

(a) $f(x, y) = \dfrac{xy}{x^2 + y^2}$

(b) $g(x, y) = \dfrac{1}{x^2 + y^2}$

(c) $h(x, y) = \dfrac{x}{x^2 + y^2}$.

(d) $p(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}$.

(e) $q(x, y) = \dfrac{x^2}{\sqrt{x^2 + y^2}}$.

2. Find the partial derivatives of the following functions:

(a) $f(x, y) = x^2 y^3 - x^3 y^2$.

(b) $f(x, y) = \cos(x^2 y) + y^3$.

(c) $f(x, y) = \dfrac{xy}{x^2 + y}$.

(d) $f(x, t) = (x + t)^4$.

(e) $f(x, t) = (x - t)^4$.

(f) $f(x, t) = \sin \omega t \cos \dfrac{2\pi x}{L}$.

(g) $f(x, y) = e^{x^2+y^2}$.

(h) $f(x, y) = xy \ln(xy)$.

(i) $f(x, y) = \sqrt{1 - x^2 - y^2}$.

(j) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

(k) $f(u, v) = e^{u+v}$

(l) $f(x, y) = x \tan(y)$.

(m) $f(x, y) = \dfrac{1}{xy}$.

3. Let r be the radius in polar coordinates, as defined in § 4 of Chapter III.

(a) Compute the partial derivatives of r.

(b) Show that the partial derivatives of r can be written as

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

4. Let θ be the polar angle function, defined in § 4.2 of Chapter III.

(a) In the left half plane the function θ is defined by

$$\theta(x, y) = \arctan \frac{y}{x}.$$

Use this expression to find its partial derivatives, $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$.

(b) Check that the angle function also satisfies

$$x \sin\theta = y \cos\theta$$

at all points in the plane. Use implicit differentiation to find the partial derivatives $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$.

5. Let $f(x, y) =$ the distance from (x, y) to the origin. Find a formula for f, and compute

$$f_x, \quad f_y, \text{ and } \sqrt{f_x^2 + f_y^2}.$$

(Hint: compare this problem with problem 3.3.)

6. Suppose $f(t)$ and $g(t)$ are single variable differentiable functions. Find $\partial z/\partial x$ and $\partial z/\partial y$ for each of the following two variable functions.

(a) $z = f(x)g(y)$

(b) $z = f(xy)$

(c) $z = f(x/y)$

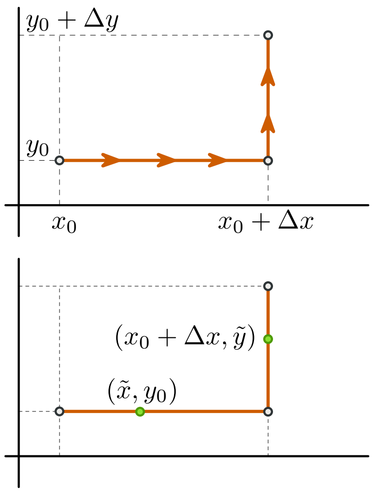
7. Let f be the ***distance to the square*** Q ***function*** from problem 5.13. Find the partial derivatives f_x and f_y of f. (You will need your answer to problem 5.13, in particular the description of f as a "piecewise defined function".)

4. The linear approximation to a function

4.1. The Chain Rule and friends. When we compute the partial derivative of a function with respect to a variable x we pretend all other variables are constants, and just differentiate with respect to x, just as we would in first semester calculus. There is therefore no need to state a product rule or quotient rule, because these are *exactly the same* as for functions of one variable. The chain rule on the other hand is different: there is a chain rule for functions of several variables, but it has more terms than the chain rule from one-variable calculus. There are several related topics that fit together in a discussion of the chain rule, namely ***Linear Approximation***, ***Tangent Planes to a Graph***, and ***The Total Derivative***. We will go through these one at a time in the next few sections.

4.2. The linear approximation formula. The key to the chain rule is the linear approximation formula. This formula tells us approximately how much a function $z = f(x, y)$ of two variables changes if both variables are subjected to a small change.

More precisely, if we have a function $z = f(x, y)$, and we know its value $f(x_0, y_0)$ at some point (x_0, y_0), then how much does the function value change if x is increased from x_0 to $x_0 + \Delta x$, and if y is similarly increased from y_0 to $y_0 + \Delta y$?



We can change (x_0, y_0) to $(x_0 + \Delta x, y_0 + \Delta y)$ in two steps:
first keep y fixed and increase x by Δx,
then keep x fixed and increase y by Δy

To express the change in function values in terms of derivatives, we can use the Mean Value Theorem. We get two intermediate points:
one at $x = \tilde{x}$ for the increase in f when x changes, and
one at $y = \tilde{y}$ for the increase in f when y changes.

Figure 3. Computation of the linear approximation (54)

The basic idea in the computation of the change in $f(x, y)$ is to go from (x_0, y_0) to $(x_0 + \Delta x, y_0 + \Delta y)$ in two steps:

$$\begin{aligned} \Delta f &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\ &= \underbrace{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)}_{\text{only } y \text{ changes}} + \underbrace{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}_{\text{only } x \text{ changes}} \end{aligned} \tag{54}$$

We have written the total change in f as the sum of two changes, one of them caused by the change in x, and the other due to the change in y. See Figure 3.

In the second difference only x changes while y remains the same, so we can use the one variable Mean Value Theorem to conclude that there is some number $\tilde{x}$ between x_0 and $x_0 + \Delta x$ with

$$\frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = f_x(\tilde{x}, y_0),$$

i.e.

$$f(x_0 + \Delta x, y_0) - f(x_0, y_0) = f_x(\tilde{x}, y_0) \cdot \Delta x. \tag{55}$$

Likewise, in the difference in (54) where only y changes we can use the Mean Value Theorem to conclude that there is some $\tilde{y}$ between y_0 and $y_0 + \Delta y$ such that

$$\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)}{\Delta y} = f_y(x_0 + \Delta x, \tilde{y}),$$

and hence

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) = f_x(x_0 + \Delta x, \tilde{y}) \cdot \Delta x. \tag{56}$$

If we now combine (55) and (56) with (54) then we get

$$\Delta f = f_x(\tilde{x}, y_0) \cdot \Delta x + f_y(x_0 + \Delta x, \tilde{y}) \cdot \Delta y.$$

This equation is exactly true, i.e. we have not made any approximations, and we have not ignored any kind of "error terms." However, the equation does contain the numbers $\tilde{x}$ and $\tilde{y}$, which are provided by the Mean Value Theorem, and of which we therefore do not know anything besides the fact that $\tilde{x}$ lies between x_0 and $x_0 + \Delta x$, and $\tilde{y}$ lies between y_0 and $y_0 + \Delta y$. We can get rid of this uncertainty by settling for an approximation for Δf instead of the exact expression we have just found. To do this we assume that Δx and Δy are "small." Then, since $\tilde{x}$ lies between x_0 and $x_0 + \Delta x$, we know that $\tilde{x} \approx x_0$. We also know that $y_0 + \Delta y \approx y_0$, so, if the function f_x is continuous, then it seems reasonable to assume that

$$f_x(\tilde{x}, y_0 + \Delta y) \approx f_x(x_0, y_0). \tag{57}$$

Similarly, we will assume that

$$f_y(x_0, \tilde{y}) \approx f_y(x_0, y_0). \tag{58}$$

Substituting this in (54) we find

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \tag{59}$$

Keeping in mind that $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$, we conclude

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \tag{60}$$

The linear approximation formula (60) is often written using Leibniz-style notation for the derivatives, where one writes $\frac{\partial f}{\partial x}$ for f_x, and $\frac{\partial f}{\partial y}$ for f_y. In this notation the approximation formula takes these forms:

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \Delta y,$$

or, shorter,

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y. \tag{61}$$

The approximation (60) can also be written without Δx and Δy by a change of notation. To do this we introduce

$$x = x_0 + \Delta x \text{ and } y = y_0 + \Delta y, \tag{62}$$

and interpret (60) as a formula that tells us approximately what the function value at (x, y) is, provided (x, y) is close enought to (x_0, y_0). Written in terms of x and y, (60) says

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0). \tag{63}$$

4.3. Linear approximation – infinitesimal version. We expect the approximation in (61) to improve as we decrease Δx and Δy (and we will try to make this statement more precise in the next section, § 4.4). We could then say, as is commonly done, that there is an exact equation when Δx and Δy are "infinitely small," and write this equation as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{64}$$

The meaning of this equation is that infinitesimally small changes in x and y, of magnitudes dx and dy, respectively, lead to an infinitesimally small change in f of magnitude df, and that df, dx, and dy are related by (64). Even though it is very difficult to make sense of the "infinitely small" quantities dx, dy, df, in (64), this notation is widely used, because the make-belief it entails allows one to ignore the more awkward error terms that we will now discuss.

4.4. The linear approximation formula with error term. In our computation of the change Δf of the function we approximated $f_x(\tilde{x}, y_0)$ by $f_x(x_0, y_0)$, and $f_y(x_0 + \Delta x, \tilde{y})$ by $f_y(x_0, y_0)$. As a result our linear approximation formula (60) is not an exact equation, but only says that one thing is "approximately equal" to another.

We can make this a bit more precise by including error terms, i.e. by saying that there are small numbers e_x and e_y such that

$$f_x(\tilde{x}, y_0) = f_x(x_0, y_0) + e_x, \text{ and } f_y(x_0 + \Delta x, \tilde{y}) = f_y(x_0, y_0) + e_y.$$

Here e_x and e_y depend on Δx and Δy, and as both Δx and Δy go to zero, the errors e_x and e_y will also go to zero.

Putting this in (54) we get the linear approximation formula with error terms:

$$f(x_0 + \Delta x, y_0 + \Delta y) = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}_{\text{linear approximation}} + \underbrace{e_x \Delta x + e_y \Delta y}_{\text{error}} \tag{65}$$

in which e_x and e_y depend on Δx, Δy, and satisfy

$$\lim_{\Delta x, \Delta y \to 0} e_x = \lim_{\Delta x, \Delta y \to 0} e_y = 0.$$

If we ignore the "error term" then we recover the linear approximation formula (60). Our more precise linear approximation formula (65) tells us that the error in (60) (difference between left and right hand sides) is given by $e_x \Delta x + e_y \Delta y$, and that this error is "small " compared to Δx and Δy. We could write this as

$$\text{Error in the approximation} = e_x \Delta x + e_y \Delta y = o(\Delta x) + o(\Delta y).$$

5. The tangent plane to a graph

5.1. The tangent plane. For a function $z = f(x, y)$ and a point (x_0, y_0) the linear approximation (63) gives us an approximation for the function f at any other point (x, y) near (x_0, y_0). It says

$$z \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

If we replace "$\approx$" by equality, then we get a new function of (x, y):

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{66}$$

Keeping in mind that $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ are constants, while only (x, y) are variables here, we see that this is the equation for a plane which we call the ***tangent plane*** to the graph of f at the point $(x_0, y_0, f(x_0, y_0))$.

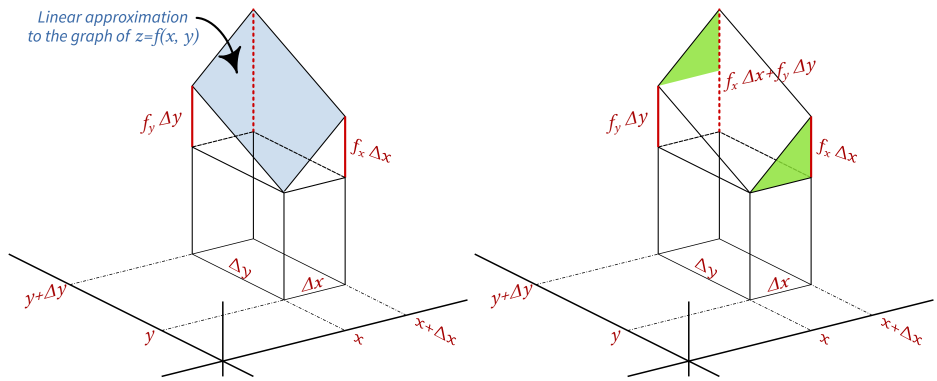
5.2. Example: tangent plane to the saddle surface at the origin. *Find the equation for the tangent plane to the saddle surface $z = xy$ at the origin.*

Solution: The saddle surface is the graph of the function $f(x, y) = xy$ whose partial derivatives are $f_x(x, y) = y$ and $f_y(x, y) = x$. To find the tangent plane at $x_0 = 0$, $y_0 = 0$, we compute the partial derivatives,

$$f_x(x, y) = \frac{\partial xy}{\partial x} = y, \text{ so at } (x_0, y_0) = (0, 0) \text{ we have } f_x(0, 0) = 0,$$

and

$$f_y(x, y) = \frac{\partial xy}{\partial y} = x, \text{ so at } (x_0, y_0) = (0, 0) \text{ we have } f_y(0, 0) = 0,$$



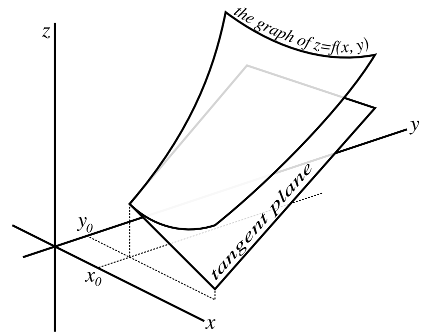


Figure 4. Top: The graph of the linear approximation of f (graph of f itself is not shown – see the bottom figure). If we increase x by Δx, then f will increase by approximately $f_x\Delta x$, and if we increase y by Δy, then f increases by approximately $f_y\Delta y$. If we increase x and y by Δx and Δy at the same time, then f increases by roughly $f_x\Delta x + f_y\Delta y$. The vertical dotted line behind the parallelogram represents this increase in f.

Bottom: The graph of a function, and of its tangent plane at some point (x_0, y_0, z_0). The tangent plane is the graph of the linear approximation to f.

Moreover, we also have $f(x_0, y_0) = f(0, 0) = 0$, so that the equation for the tangent plane is

$$z = 0 + 0 \cdot (x - 0) + 0 \cdot (y - 0) = 0,$$

i.e.,

$$z = 0.$$

The tangent plane at the origin is just the xy-plane.

5.3. Example: another tangent plane to the saddle surface. *Find the equation for the tangent plane to the saddle surface $z = xy$ at the point $(2, 1, 2)$. Where does this plane intersect the coordinate axes?*

Solution: This is almost the same problem as before. The only difference is that we are trying to find the tangent plane at a point other than the origin. To get the tangent plane at the point $(x_0, y_0) = (2, 1)$ we compute the derivatives

$$f_x(x, y) = y \implies f_x(2, 1) = 1,$$

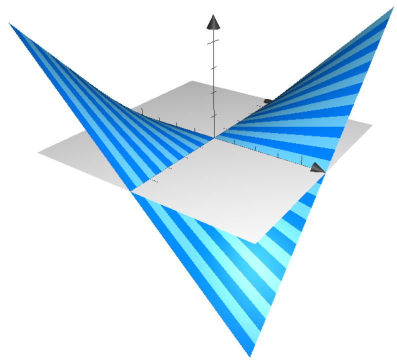


Figure 5. The graph of $z = xy$ and the tangent plane at the origin.

and

$$f_y(x, y) = x \implies f_y(2, 1) = 2.$$

The equation for the tangent plane is therefore

$$\begin{aligned} z &= x_0y_0 + y_0(x - x_0) + x_0(y - y_0) \\ &= 2 + 1 \cdot (x - 2) + 2 \cdot (y - 1) \\ &= -2 + x + 2y \end{aligned} \tag{67}$$

The intersections with the x, y and z axes are, respectively, $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, -2)$.

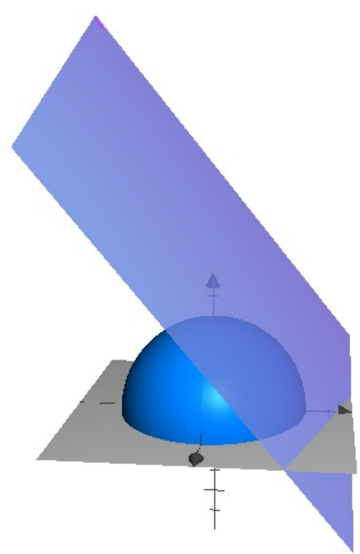
5.4. Example: tangent plane to a sphere. *The point (x_0, y_0, z_0) lies on the upper half of the sphere with radius 4 centered at the origin. Find an equation for the tangent plane to the sphere at that point, if $x_0 = 1$ and $y_0 = 3$.*

Solution: The equation for the sphere is $x^2 + y^2 + z^2 = 4^2 = 16$, so the upper half is the graph of the function

$$f(x, y) = \sqrt{16 - x^2 - y^2}.$$

The z coordinate of the given point is therefore $z_0 = \sqrt{16 - 1^2 - 3^2} = \sqrt{6}$. The partial derivatives of f at $(x_0, y_0) = (1, 3)$ are

$$\frac{\partial f}{\partial x} = \frac{-x_0}{\sqrt{16 - x_0^2 - y_0^2}} = -\frac{1}{\sqrt{6}},$$
$$\frac{\partial f}{\partial y} = \frac{-y_0}{\sqrt{16 - x_0^2 - y_0^2}} = -\frac{3}{\sqrt{6}}.$$



The equation for the tangent plane is then

$$\begin{aligned} z &= \sqrt{6} - \frac{1}{\sqrt{6}}(x - 1) - \frac{3}{\sqrt{6}}(y - 3) \\ &= \frac{16}{\sqrt{6}} - \frac{x}{\sqrt{6}} - \frac{3y}{\sqrt{6}}. \end{aligned}$$

6. The Two Variable Chain Rule

6.1. The chain rule. Given two functions $x = x(t)$, $y = y(t)$ of one variable, and a function $z = f(x, y)$ of two variables, then what is the derivative of the function

$$g(t) = f(x(t), y(t))?$$

We can find a general formula for $g'(t)$ by using the linear approximation (§ 4) in the following way.

To find $g'(t_0)$ for some t_0, we must compute

$$\frac{g(t_0 + \Delta t) - g(t_0)}{\Delta t}$$

and let $\Delta t \to 0$.

If t increases by an amount Δt from t_0 to $t_0 + \Delta t$, then x and y will also change. We write Δx and Δy for the changes in x and y, i.e.

$$\Delta x = x(t_0 + \Delta t) - x_0, \qquad \Delta y = y(t_0 + \Delta t) - y_0,$$

where $x_0 = x(t_0)$ and $y_0 = y(t_0)$. The resulting change in g is thus

$$\begin{aligned} \Delta g &= g(t_0 + \Delta t) - g(t_0) \\ &= f\big(x(t_0 + \Delta t), y(t_0 + \Delta t)\big) - f\big(x(t_0), y(t_0)\big) \\ &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0). \end{aligned}$$

By the linear approximation formula (65) one then has

$$\frac{\Delta f}{\Delta t} = f_x(x_0, y_0)\frac{\Delta x}{\Delta t} + f_y(x_0, y_0)\frac{\Delta y}{\Delta t} + e_x\frac{\Delta x}{\Delta t} + e_y\frac{\Delta x}{\Delta t}$$

As we let $\Delta t \to 0$ the quotients $\Delta x/\Delta t$ and $\Delta y/\Delta t$ converge to $x'(t_0)$ and $y'(t_0)$, while the errors e_x and e_y converge to zero, so we get ***the two-variable chain rule:***

$$\frac{df(x(t), y(t))}{dt} = f_x(x_0, y_0) \cdot x'(t_0) + f_y(x_0, y_0) \cdot y'(t_0). \tag{68}$$

The chain rule is often also written as

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{69}$$

This form becomes easy to remember if we interpret the first term as "the change in f caused by the change in x" and the second term as "the change in f caused by the change in y."

In the way (69) is written a number of details are swept under the rug: the two derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are ordinary (Math 221) derivatives of the two functions $x(t)$ and $y(t)$; the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the partial derivatives of f ***in which one has substituted*** $x(t)$ ***and*** $y(t)$. A more correct way of writing the equation would be

$$\frac{df(x(t), y(t))}{dt} = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t). \tag{70}$$

Many people find (69) easier on the eyes, so that is what we will usually write.

6.2. The difference between d and ∂. Compare (69) with the linear approximation formula (64) with infinitesimal small quantities. Equation (69) is just (64) in which one has divided both sides by dt. In contrast to equation (64) which contains the strange "infinitely small quantities" dx, dy, df, equation (69) contains the derivatives $\frac{dx}{dt}$, etc. which ***are*** well-defined.

Note that we have a breakdown of Leibniz's notation: if we ignore the distinction between "d" and "∂", and just cancel dx and ∂x, and also dy and ∂y on the right then we end up with

$$\frac{df}{dt} = \frac{\partial f}{\cancel{\partial x}}\frac{\cancel{dx}}{dt} + \frac{\partial f}{\cancel{\partial y}}\frac{\cancel{dy}}{dt} = \frac{\partial f}{dt} + \frac{\partial f}{dt} = 2\frac{\partial f}{dt},$$

which doesn't make a lot of sense. The moral: don't cancel dx against ∂x!

6.3. An example. Suppose $x(t) = \cos\omega t$ and $y(t) = \sin\omega t$, so that $\vec{\boldsymbol{x}}(t) = x(t)\vec{\boldsymbol{e}}_1 + y(t)\vec{\boldsymbol{e}}_2$ traces out the unit circle.

How fast does $S(t) = 2x(t) + 3y(t)$ change along this motion?

In other words, what can we say about $\frac{dS}{dt}$?

The quantity $S(t)$ is the composition of a function of two variables with the functions $x(t)$ and $y(t)$, i.e. it is the result of substituting $x(t)$ and $y(t)$ in the function $f(x, y) = 2x + 3y$.

Answer 1 – without using the chain rule. We can simply compute $S(t) = \cos\omega t + \sin\omega t$ and differentiate:

$$\frac{dS}{dt} = \frac{d}{dt}\Big\{2\cos\omega t + 3\sin\omega t\Big\} = -2\omega\sin\omega t + 3\omega\cos\omega t. \tag{71}$$

Note that we did not use our new two-variable chain rule here. This answer shows that the point of the two-variable chain rule is not to compute $\frac{d}{dt}f(x(t), y(t))$ in situations where we have formulas for the functions $f(x, y)$, $x(t)$, and $y(t)$. In such a situation we can always substitute $x(t)$ and $y(t)$ in the function $f(x, y)$ after which we get a function $S(t) = f(x(t), y(t))$ of one variable. We learned how to differentiate those in our first calculus course.

Answer 2 – using the chain rule. The quantity we want to differentiate is

$$S(t) = f\big(x(t), y(t)\big),$$

where

$$f(x, y) = 2x + 3y, \quad \text{and} \quad x(t) = \cos\omega t, \quad y(t) = \sin\omega t.$$

The chain rule tells us that

$$\frac{dS}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{72}$$

Here the first term stands for the change in S that is caused by the change in x. To compute it we first find

$$\frac{\partial f}{\partial x} = \frac{\partial\{2x + 3y\}}{\partial x} = 2,$$

so that

$$\frac{\partial f}{\partial x}\frac{dx}{dt} = 2\cdot\frac{dx}{dt}.$$

Similarly, the second term in (72) represents the change in $S(t)$ due to the fact that y is changing:

$$\frac{\partial f}{\partial y} = \frac{\partial\{2x + 3y\}}{\partial y} = 3 \implies \frac{\partial f}{\partial y}\frac{dy}{dt} = 3\cdot\frac{dy}{dt}.$$

To get the rate of change of S we add both the x and y contributions to this rate of change, which leads us to

$$\frac{dS}{dt} = 2 \cdot \frac{dx}{dt} + 3 \cdot \frac{dy}{dt}. \tag{73}$$

So far we have not used what we know about $x(t)$ and $y(t)$. This expression we have just derived for dS/dt is true no matter which $x(t)$, $y(t)$ we are given. In our case we have

$$x(t) = \cos \omega t \implies \frac{dx}{dt} = -\omega \sin \omega t,$$
$$y(t) = \sin \omega t \implies \frac{dy}{dt} = +\omega \cos \omega t.$$

Substitute this in (73):

$$\frac{dS}{dt} = -2\omega \sin \omega t + 3 \cos \omega t,$$

as before.

The moral: In this example the answer using the chain rule was longer, much more verbose, and perhaps more complicated than the straightforward computation that led to our first answer (71). Indeed, if the derivative of S is all we want then our first computation is the most efficient way of getting dS/dt. However, the computation using the chain rule did give us some useful intermediate results, such as the general expression (73) for dS/dt. This expression remains valid if we change the path $(x(t), y(t))$ and can therefore be useful in situations where, for example, we are allowed to choose the path and we would like to choose a path for which dS/dt has some prescribed value (e.g. suppose we want to keep S constant, how do we choose the path?)

6.4. Another example. Suppose the temperature at the point (x, y) in the plane is given by $T(x, y)$, and suppose that an ant is walking along the parametrized curve

$$x(t) = R \cos \omega t, \quad y(t) = R \sin \omega t.$$

Thus the ant is walking on a circle with radius R, and with angular velocity ω.

How fast is the temperature of the ant changing?
i.e. compute $\frac{dT}{dt}$.

Here we are not given an explicit formula for the function $T(x, y)$, so we cannot substitute $x(t)$ and $y(t)$ in T and differentiate using only our first semester calculus skills. The approach in Answer 1 of our previous example does not apply here; we must use the chain rule.

In § 6.1 we have seen several equivalent ways of writing the chain rule. Let us look at two of these and consider the meaning of the terms that arise.

The short form (69) of the chain rule tells us that

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}.$$

The T on the left stands for $T(x(t), y(t))$, which we can interpret as the temperature at the point $(x(t), y(t))$. That point is the location of the ant at time t, so the T on the left is the temperature the ant feels at time t. This is a function of t. In mathematical terms it is the result of substituting (composing) the functions $x(t)$ and $y(t)$ in the function $T = T(x, y)$.

The two T's on the right appear in partial derivatives. Here $\frac{\partial T}{\partial x}$ stands for the partial derivative of the function $T = T(x, y)$ with respect to the variable x. One can compute this without knowing the ant's path $(x(t), y(t))$. Similarly, $\frac{\partial T}{\partial y}$ is the partial derivative of

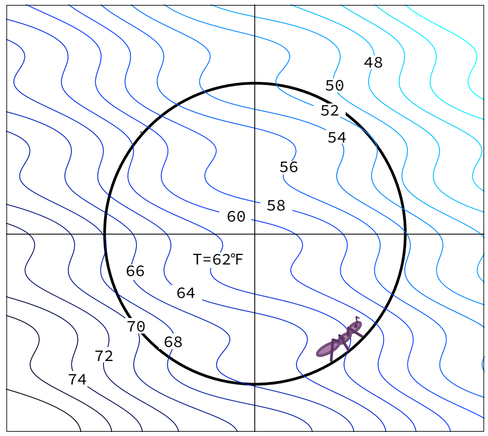


Figure 6. Ant walking in a region of varying temperature.

T with respect to y. The partial derivatives $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$ themselves are again functions of x and y. *After* computing these partials they are meant to be evaluated at the point $(x(t), y(t))$.

This leads us to the more verbose version (70) of the chain rule, which tells us

$$\frac{dT(x(t), y(t))}{dt} = \frac{\partial T}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial T}{\partial y}(x(t), y(t)) \cdot y'(t).$$

At this point the only additional information we have is about the ant's motion, namely, $x(t) = R\cos\omega t$ and $y(t) = \sin\omega t$. We can compute the derivatives of $x(t)$ and $y(t)$, which gives us the velocity of the ant in the x and y directions:

$$x'(t) = -\omega R\sin\omega t, \qquad y'(t) = \omega R\cos\omega t.$$

If we substitute everything we know in the chain rule we find that the rate at which the ant's temperature changes is

$$\frac{dT}{dt} = -\frac{\partial T}{\partial x}(R\cos\omega t, R\sin\omega t) \cdot \omega R\sin\omega t + \frac{\partial T}{\partial y}(R\cos\omega t, R\sin\omega t) \cdot \omega R\cos\omega t.$$

To make the equation more readable one can leave out the $(R\cos\omega t, R\sin\omega t)$, which results in

$$\frac{dT}{dt} = -\omega R\sin\omega t\frac{\partial T}{\partial x} + \omega R\cos\omega t\frac{\partial T}{\partial y}.$$

The disadvantage of this shorter version is that the reader has to figure out where we intended to evaluate the two partial derivatives $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$.

7. Problems

1. Find the linear approximation to $f(x, y)$ at the point (a, b) in the following cases:

(a) $f(x, y) = xy^2$, $(a, b) = (3, 1)$.

(b) $f(x, y) = x/y^2$, $(a, b) = (3, 1)$.

(c) $f(x, y) = \sin x + \cos y$, $(a, b) = (\pi, \pi)$.

(d) $f(x, y) = xy/(x + y)$, $(a, b) = (3, 1)$.

2. Find an equation for the plane tangent to the graph of $f(x, y) = \sin(xy)$ at $(\pi, 1/2, 1)$.

3. Find an equation for the plane tangent to the graph of $f(x, y) = x^2 + y^3$ at $(3, 1, 10)$.

4. Find an equation for the plane tangent to the graph of $f(x,y) = x\ln(xy)$ at $(2,1/2,0)$.

5. **(a)** Find an equation for the plane tangent to the surface defined by $2x^2+3y^2-z^2=4$ at $(1,1,-1)$. (Hint: first write the surface as a graph $z=f(x,y)$).

(b) The same question at the point $(1,1,+1)$.

6. **(a)** Suppose you have computed the two partial derivatives of a function $z = f(x_0,y_0)$, and you found $f_x(x_0,y_0) = A$ and $f_y(x_0,y_0) = B$. Find a normal vector to the tangent plane of the graph of $z = f(x,y)$ at (x_0,y_0,z_0).

(Hint: If you know the equation for a plane, then how do you find a normal vector to this plane?)

(b) Find an equation in vector form for the tangent plane to $x^2+4y^2=2z$ at $(2,1,4)$. Also find an equation for the normal line to the graph at $(2,1,4)$. (The normal line to the graph of a function at some point P, is the line through P that is perpendicular to the tangent plane to the graph at P.)

7. Imagine a differentiable function, $f(x,y)$. Make a good drawing of the function f and show how $f_x(a,b)$ and $f_y(a,b)$ are the slopes of two lines which are tangent to the graph at (a,b). Indicate clearly which two lines you mean, and describe how they are defined.

(Can't think of a nice graph? Take something like the bottom drawing in Figure 4.)

8. Let f be as in problem 7.4. Use linear approximation to approximate $f(1.98,0.4)$ by hand. Compare your answer with the actual value of $f(1.98,0.4)$ (you'll need a calculator).

9. **(a)** The tangent plane to the saddle surface $z = xy$ at the origin intersects the graph of the saddle surface in two lines. Which lines are they?

(b) Consider the tangent plane to the saddle surface at $x=2, y=1$ that was computed in §5.3. Let (x,y,z) be a point on the saddle surface, and let (x,y,z_*) be the point on the tangent plane with the same x and y coordinates. What is the difference in heights of these two points?

(c) Show that the saddle surface and its tangent plane intersect when $x=2$ or $y=1$.

10. **(a)** Find an equation for the tangent) plane to the graph of $f(x,y) = xy$ at the point (a,b,ab). Here a and b are constants which will appear in your answer.

(b) Show that the intersection of the tangent plane and the graph consists of two straight lines.

8. Gradients

8.1. The gradient vector of a function. The right hand side in the chain rule (68) can be written as a dot-product of two vectors, namely

$$\frac{df}{dt} = f_x(x_0,y_0)\cdot x'(t_0) + f_y(x_0,y_0)\cdot y'(t_0) = \begin{pmatrix} f_x(x_0,y_0) \\ f_y(x_0,y_0) \end{pmatrix} \cdot \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix} \tag{74}$$

This turns out to be so useful that the vector containing the derivatives of f has been given a name. It is called the ***gradient of*** f, and it is written as

$$\vec{\boldsymbol{\nabla}} f(x,y) \stackrel{\text{def}}{=} \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} \tag{75}$$

The symbol $\vec{\boldsymbol{\nabla}}$ is pronounced "nabla."

The chain rule, written in vector form, looks like this:

$$\frac{df(\vec{\boldsymbol{x}}(t))}{dt} = \vec{\boldsymbol{\nabla}} f(x(t))\cdot \vec{\boldsymbol{x}}'(t) \tag{76}$$

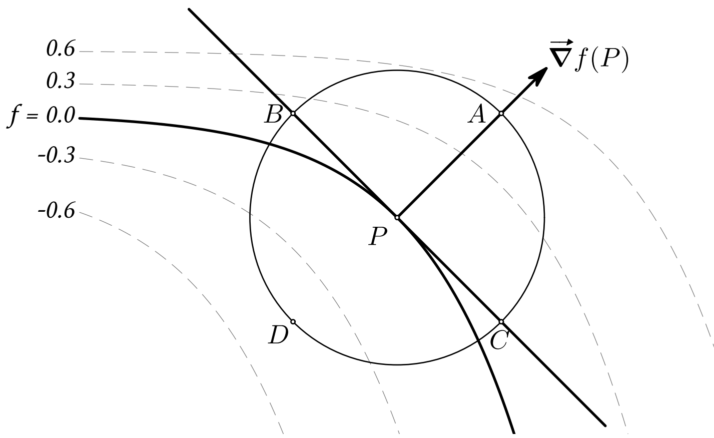


Figure 7. The gradient as direction of fastest increase: if we are at a point P, and we are allowed to jump to any point at a given fixed distance from P, and if we only know $\vec{\nabla} f(P)$, then the linear approximation formula tells us that
to maximize f we follow the gradient (choose A);
to minimize f we go in the direction opposite to $\vec{\nabla} f(P)$ (choose D);
to keep f fixed we move perpendicular to the gradient (choose B or C).

The linear approximation formula (60) can also be rewritten more compactly using the gradient vector:

$$f(\vec{x}_0 + \Delta\vec{x}) \approx f(\vec{x}_0) + \vec{\nabla} f(\vec{x}_0) \cdot \Delta\vec{x}. \tag{77}$$

8.2. The gradient as the "direction of greatest increase" for a function f.

When we apply the formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\, \|\vec{b}\| \cos \angle(\vec{a}, \vec{b}) \tag{78}$$

for the dot product to the vector form (77) of the linear approximation equation, we find a very useful interpretation of the gradient. If we are at a point with position vector $\vec{x}_0$ (P in figure 7) and we are allowed to make a small step $\Delta\vec{x}$ in any direction we like, but of prescribed length, then which way should we go if we want to increase f as much as possible? And where should we go if, instead, we want to decrease f as much as possible? What if we want to keep f the same?

From (77) we see that the change in f is (approximately) given by

$$\Delta f \stackrel{\text{def}}{=} f(\vec{x} + \Delta\vec{x}) - f(\vec{x}) \stackrel{(77)}{\approx} \vec{\nabla} f \cdot \Delta\vec{x} \stackrel{(78)}{=} \|\vec{\nabla} f\|\, \|\Delta\vec{x}\|\, \cos\theta$$

where θ is the angle between the gradient $\vec{\nabla} f$ and the vector $\Delta\vec{x}$ which represents the step we take. In this formula the lengths $\vec{\nabla} f$ and $\|\Delta\vec{x}\|$ are fixed, and the angle θ is the only thing we can change. Therefore the largest change in f results if $\cos\theta = +1$, the smallest when $\cos\theta = -1$, and no change will result if $\cos\theta = 0$. So we conclude

- To increase f as much as possible choose $\Delta\vec{x}$ in the direction of the gradient $\vec{\nabla} f$,
- To decrease f as much as possible choose $\Delta\vec{x}$ in the direction opposite to the gradient $\vec{\nabla} f$, i.e. in the direction of $-\vec{\nabla} f$,
- To keep f constant choose $\Delta\vec{x}$ perpendicular to the gradient.

This is sometimes summarized by saying that ***the gradient $\vec{\nabla} f$ points in the direction of fastest increase for the function f.***

8.3. The gradient is perpendicular to the level curve. Suppose that for some function $z = f(x,y)$ the level set at level C is a curve, and suppose that we have a parametric representation $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of this curve. This means that $x(t)$ and $y(t)$ satisfy

$$f(x(t), y(t)) = C.$$

By the chain rule we then get

$$0 = \frac{df(\vec{x}(t))}{dt} = \vec{\nabla} f(\vec{x}(t)) \cdot \vec{x}'(t),$$

which tells us that the tangent vector $\vec{x}'(t)$ to the level set is perpendicular to the gradient $\vec{\nabla} f(\vec{x}(t))$ of the function. Therefore,

if $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $\vec{\nabla} f(x_0, y_0)$ is a normal vector to the tangent to the level curve of f at (x_0, y_0).

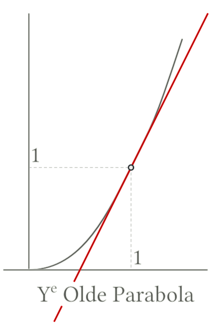
We now have the necessary ingredients to write the equation for the tangent, namely we know a point (x_0, y_0) on the line, and we know a normal vector to the line (the gradient). Thus the equation for the tangent is

$$\vec{\nabla} f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0,$$

or, equivalently,

$$\frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0) = 0.$$

8.4. The tangent to the parabola $y = x^2$, again. The very first example anyone sees in their first calculus course must surely be the computation of the tangent to the parabola $y = x^2$ at the point $(x,y) = (1,1)$. We know the answer: it is a line with slope 2, through the point $(1,1)$.



Y^{e} Olde Parabola

We can interpret the parabola as the zero set of the function of two variables given by $f(x,y) = y - x^2$, and therefore we should be able to find the same tangent at $(1,1)$ by computing the gradient of f. The computation goes like this:

$$f(x,y) = y - x^2 \implies \vec{\nabla} f(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} -2x \\ y \end{pmatrix}.$$

At $(x,y) = (1,1)$ we have

$$\vec{\nabla} f(1,1) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

This vector is perpendicular to the tangent to its zero set. If we let $\vec{x}_0 = \binom{1}{1}$ be the position vector of our point on the parabola, then the equation for the tangent to the parabola at this point is

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0,$$

i.e.

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} = 0.$$

Simplifying this we get

$$-2 \cdot (x-1) + 1 \cdot (y-1) = 0, \text{ and thus } y = 2x - 1.$$

This is the same line that we found in our first calculus course.

8.5. Example: the tangent to the zero set of $x^2 - y^2 + y^3$. Consider the zero set of the function

$$f(x,y) = x^2 - y^2 + y^3.$$

The resulting curve is not as familiar as the parabola from the previous example, and drawing the curve takes some effort[1].

We will not try to draw the whole zero set in this example, but instead we will see what happens when we try to find the tangent to the zero set at two different points on the zero set, namely, at $(0,1)$ and at the origin.

The tangent at $(0,1)$. To find the tangent at any point on the zero set of f we use that the normal to the tangent is given by the gradient of f

$$\vec{\nabla} f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 2x \\ -2y + 3y^2 \end{pmatrix}.$$

The normal to the tangent at the point $(0,1)$ is therefore

$$\vec{n} = \vec{\nabla} f(0,1) = \begin{pmatrix} 0 \\ -2+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In other words, the normal to the tangent at $(0,1)$ is the vertical unit vector $\vec{e}_2$, and therefore the tangent is a horizontal line through $(0,1)$. Its equation is $y = 1$. We could also find this equation by working out the general equation $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$ for a line with a given normal and point. Here we have

$$\vec{n} = \vec{\nabla} f(0,1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so the equation for the tangent is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x - 0 \\ y - 1 \end{pmatrix} = 0,$$

which simplifies to

$$y - 1 = 0.$$

The tangent at the origin. When we repeat the previous calculation at $(x_0, y_0) = (0,0)$ we run into problems. These problems begin when we compute the gradient $\vec{\nabla} f$ at the origin:

$$\vec{\nabla} f(0,0) = \begin{pmatrix} 2x \\ -2y + 3y^2 \end{pmatrix}_{x=0, y=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The gradient at the origin turns out to be the zero vector. This is problematic because the zero vector has no direction, and thus is not perpendicular to any particular line. *We cannot find the tangent at the origin!*

To see what is going on one has to take a closer look at the curve near the origin – see figure 8. It turns out that near the origin the zero set of f consists of two smooth curves that cross each other.[2] The gradient has to be perpendicular to both of these curves, and the only vector that achieves this is the zero vector. Note also that there is no single line

[1] One could start by solving the equation for x, which leads to $x = \pm y\sqrt{1-y}$. This shows that $y \le 1$ on the curve. Graphing $x = y\sqrt{1-y}$ using our 1st semester calculus skills then gives us half the curve; the other half is given by its reflection in the y-axis, i.e. $x = -y\sqrt{1-y}$.

[2] One way to see this is to solve $x^2 - y^2 + y^3 = 0$ for x, which gives $x = \pm y\sqrt{1-y}$. Near the origin y is very small, so we can approximate $\sqrt{1-y} \approx \sqrt{1} = 1$. The zero set near the origin is therefore approximately described by $x = \pm y$, i.e. two crossing lines.

that is tangent to the zero set at the origin. If we had seen the drawing ahead of time then we would not have expected to find a tangent to the zero set of f at the origin.

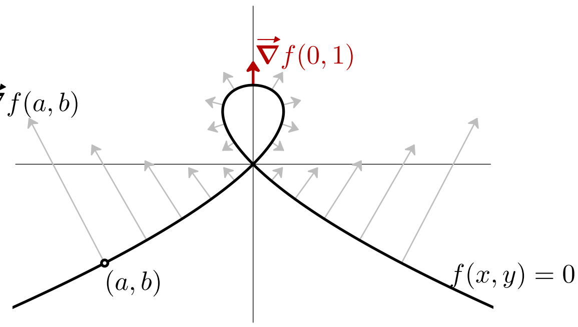


Figure 8. The zero set of the function $f(x,y) = x^2 - y^2 + y^3$, and its gradient at various points on this zero set. Since the gradient is always perpendicular to the level set of a function, a drawing of the zero set tells us the direction of the gradient. However, the drawing does not say anything about the length of the gradient.

9. The chain rule and the gradient of a function of three variables

9.1. The gradient, etc. So far we have only looked at the gradient of a function of two variables. But for a function of three variables there is a very similar definition, and the facts we have discovered have nearly identical counterparts.

If $u = f(x,y,z)$ is a function of three variables, then its gradient is defined to be the vector

$$\vec{\nabla} f(x,y,z) = \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix}.$$

The ***chain rule*** in this context says that, if $x = x(t)$, $y = y(t)$, and $z = z(t)$ are functions of one variable, then the derivative of the function we get by substituting $x(t), y(t), z(t)$ in f is given by any of the following three equivalent formulas

$$\begin{aligned}\frac{df(x(t),y(t),z(t))}{dt} &= f_x(x(t),y(t),z(t))\,x'(t) + f_y(x(t),y(t),z(t))\,y'(t) \\ &\qquad + f_z(x(t),y(t),z(t))\,z'(t) \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \vec{\nabla} f(\vec{x}(t)) \cdot \vec{x}\,'(t), \text{ where } \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.\end{aligned} \tag{79}$$

The linear approximation formula for the function f at some point (x_0, y_0, z_0), which gives us an approximation of the amount by which f increases if we go from (x_0, y_0, z_0) to $(x,y,z) = (x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$, is as follows:

$$\begin{aligned}\Delta f &= f(x,y,z) - f(x_0,y_0,z_0) \\ &\approx \frac{\partial f}{\partial x}\cdot\Delta x + \frac{\partial f}{\partial y}\cdot\Delta y + \frac{\partial f}{\partial z}\cdot\Delta z,\end{aligned} \tag{80}$$

in which the partial derivatives are to be evaluated at (x_0, y_0, z_0). Compare this with the two variable version (59). In vector form we have

$$\Delta f = f(\vec{\boldsymbol{x}}_0 + \Delta\vec{\boldsymbol{x}}) - f(\vec{\boldsymbol{x}}_0) \approx \vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{x}}_0) \cdot \Delta\vec{\boldsymbol{x}}, \tag{81}$$

$$\text{where } \vec{\boldsymbol{x}}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \ \Delta\vec{\boldsymbol{x}} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta x \end{pmatrix}.$$

This is the same formula as in the two-variable case, where we had (77). The discussion about "direction of fastest increase" applies to the three variable case without change. Thus, if we are at a point $\vec{\boldsymbol{x}}_0$, and we are allowed to change our position by a small vector $\Delta\vec{\boldsymbol{x}}$ of a prescribed length, then we should choose $\Delta\vec{\boldsymbol{x}}$ in the direction of the gradient $\vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{x}})$ if we want to increase f as much as possible; we should choose $\Delta\vec{\boldsymbol{x}}$ in the direction of $-\vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{x}})$ if we want to *decrease* f as much as possible; and we should choose $\Delta\vec{\boldsymbol{x}}$ perpendicular to $\vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{x}})$ if we want to keep f constant.

9.2. Tangent plane to a level set. If $t = f(x, y, z)$ is a function of three variables then it is hard to visualize its graph, since this involves drawing *four* mutually perpendicular axes, something we, three dimensional creatures, cannot do. However, we can try to visualize the level sets of the function. The level set at level C consists, by definition, of all points in three dimensional space whose coordinates satisfy the equation $f(x, y, z) = C$.

For instance, the unit sphere is given by the equation $x^2 + y^2 + z^2 = 1$, so it is the level set at level 1 of the function $f(x, y, z) = x^2 + y^2 + z^2$. The sphere with radius R is the level set of the same function f at level R^2.

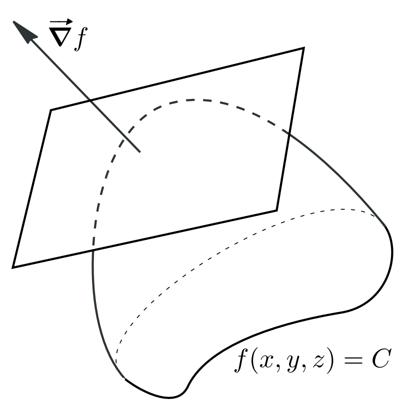
Consider a function of three variables, and let (x_0, y_0, z_0) be some point on the level set at level C (thus $f(x_0, y_0, z_0) = C$.) The equation for the level set itself is $f(x, y, z) = C$, and since (x_0, y_0, z_0) satisfies this equation we can write the equation for the level set as

$$f(x, y, z) - f(x_0, y_0, z_0) = 0.$$

Near the point (x_0, y_0, z_0) we can use the linear approximation of f to approximate the equation for the level set of f. We have

$$f(x, y, z) - f(x_0, y_0, z_0) \approx \frac{\partial f}{\partial x} \cdot (x - x_0) + \frac{\partial f}{\partial y} \cdot (y - y_0) + \frac{\partial f}{\partial z} \cdot (z - z_0),$$

where, as in (80), the partial derivatives are to be computed at the given point (x_0, y_0, z_0). They are, in particular, constants (they depend on (x_0, y_0, z_0) but not on (x, y, z).)



Thus we see that near any particular point on the level set of a function we can approximate the equation for the level set by

$$\frac{\partial f}{\partial x}\cdot(x-x_0)+\frac{\partial f}{\partial y}\cdot(y-y_0)+\frac{\partial f}{\partial z}\cdot(z-z_0)=0. \tag{82}$$

If at least one of the partial derivatives at (x_0, y_0, z_0) is non zero, then this is the equation of a plane. We call this plane the tangent plane to the level set.

In vector form the equation for the tangent plane to a level set of f at a point with position vector $\vec{\boldsymbol{x}}_0$ can be written as

$$\vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{x}}_0)\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_0)=0. \tag{83}$$

From this equation we see that, just as in the case (§8.3) of level curves of a function of two variables, ***the gradient*** $\vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{x}}_0)$ ***is perpendicular to the tangent plane of the level set of the function*** f ***at the point*** $\vec{\boldsymbol{x}}_0$**.**

9.3. Example: tangent plane to a sphere revisited. In the example in § 5.4 we found the tangent plane to the sphere at the point $(1, 3, \sqrt{6})$, where the sphere had radius 4, and was centered at the origin. There we represented the top half of the sphere as the graph of a function. We will now redo this calculation by representing the sphere as the level set of some other function.

By Pythagoras the distance d from a point (x, y, z) to the origin satisfies

$$d^2 = x^2 + y^2 + z^2.$$

The sphere with radius 4 and center at the origin therefore consists of all points (x, y, z) that satisfy

$$x^2 + y^2 + z^2 = 4^2 = 16.$$

In other words, it is the level set at level $C = 16$ of the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$

To find an equation for the tangent plane through the point $(1, 3, \sqrt{6})$ we need two ingredients: a point on the plane and a normal vector to the plane. (See Chapter I, §11.2.) We already have a point on the plane, namely our point $(1, 3, \sqrt{6})$, and the normal is given by the gradient of the function f whose level set is the sphere. This gradient is easy to compute. Since $f(x, y, z) = x^2 + y^2 + z^2$, we have

$$\frac{\partial f}{\partial x}=2x,\quad \frac{\partial f}{\partial y}=2y,\quad \frac{\partial f}{\partial z}=2z,$$

and thus

$$\vec{\boldsymbol{\nabla}} f(1,3,\sqrt{6})=\begin{pmatrix}2x\\2y\\2z\end{pmatrix}_{(x,y,z)=(1,3,\sqrt{6})}=\begin{pmatrix}2\\6\\2\sqrt{6}\end{pmatrix}.$$

The equation for the tangent plane is $\vec{\boldsymbol{n}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_0)=0$, where the normal $\vec{\boldsymbol{n}}$ to the tangent plane is the gradient $\vec{\boldsymbol{\nabla}} f$ evaluated at our given point $\vec{\boldsymbol{x}}_0$. So, the tangent plane is given by

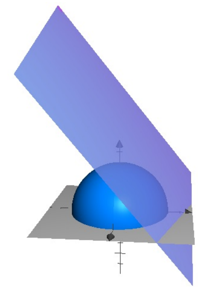
$$\vec{\boldsymbol{\nabla}} f(1,3,\sqrt{6})\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_0)=0,$$

which we can write as

$$\begin{pmatrix}2\\6\\2\sqrt{6}\end{pmatrix}\cdot\begin{pmatrix}x-1\\y-3\\z-\sqrt{6}\end{pmatrix}=0,$$

i.e.

$$2(x-1)+6(y-3)+2\sqrt{6}(z-\sqrt{6})=0.$$



After some cleaning up we get

$$x+3y+\sqrt{6}z=16.$$

This is the same answer we got in §5.4.

9.4. Example. *Find the linear approximation of $F(x,y,z) = e^{-y}(x-z)^2$ and tangent plane to its level set at $x=1, y=2, z=5$*

Solution: At the given values of x, y, z on has $F(1,2,5) = e^{-2}(1-5)^2 = 16/e^2$. The partial derivatives of F are

$$F_x = 2(x-z)e^{-y}, \quad F_y = -e^{-y}(x-z)^2, \quad F_z = -2(x-z)e^{-y},$$

which at $(x,y,z) = (1,2,5)$ reduces to $F_x = -8/e^2$, $F_y = -16/e^2$ and $F_z = +8/e^2$. If (x,y,z) is close to $(1,2,5)$, then the linear approximation formula tells us that

$$F(x,y,z) \approx F(1,2,5) - \frac{8}{e^2}(x-1) - \frac{16}{e^2}(y-2) + \frac{8}{e^2}(z-5)$$

or, in "Δx" notation,

By definition:
$\Delta x = x-1$
$\Delta y = y-2$
$\Delta z = z-5$

$$F(1+\Delta x, 2+\Delta y, 5+\Delta z) \approx F(1,2,5) - \frac{8}{e^2}\Delta x - \frac{16}{e^2}\Delta y + \frac{8}{e^2}\Delta z.$$

The equation for the tangent plane to the level set of F at the point $(1,2,5)$ is therefore

$$-\frac{8}{e^2}(x-1) - \frac{16}{e^2}(y-2) + \frac{8}{e^2}(z-5) = 0,$$

or, after cancelling e^2's and 8's: $(x-1)+2(y-2)-(z-5)=0$. Further simplification shows that the equation for the tangent plane is

$$x+2y-z=0.$$

10. Implicit Functions

In first semester calculus we learned a procedure for finding derivatives of implicitly defined functions. If some function $y = f(x)$ was not given by an explicit formula, but rather by an implicit equation

$$F(x,y)=0 \tag{84}$$

then there was a way to find the derivative of $y = f(x)$ from the above equation only. But there was no formula for $f'(x)$. The reason is that the formula for the derivative $f'(x)$ involves the partial derivatives of F.

In this section we review implicit differentiation again. The following theorem is about the zero set of the function F. One usually thinks of the zero set of a function of two variables as a curve ("an equation defines a curve") but this is not always so. The theorem below gives us a way to find out if the zero set is really a curve, at least near any given point on the zero set which we happen to know.

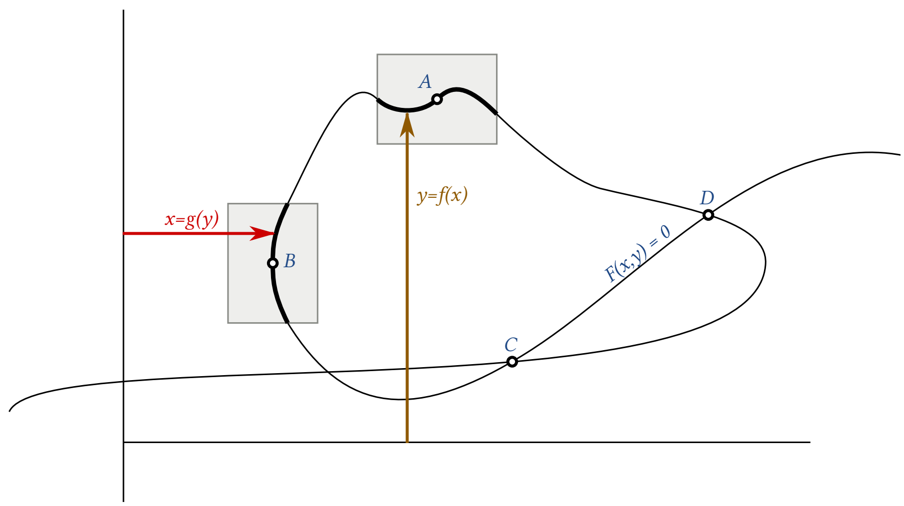


Figure 9. The Implicit Function Theorem. The zero set of a function $F(x, y)$ does not have to be the graph of a function, but if at some point (A) on the zero set we have $F_y \neq 0$, then, near that point A, the zero set is the graph of a function $y = f(x)$. If $F_x \neq 0$ at some point (B), then near B the zero set is also the graph of a function, provided we let x be a function of y: $x = g(y)$.

Exceptional points: At some points, like C and D in this figure, the level set of F cannot be represented as the graph of a function $y = f(x)$, nor can it be represented as a graph of the type $x = g(y)$. At such points the Implicit Function Theorem implies that both $F_x = 0$ and $F_y = 0$.

10.1. The Implicit Function Theorem. *Let $F(x, y)$ be a function defined on some plane domain with continuous partial derivatives in that domain, and suppose that a point (x_0, y_0) in the zero set of F is given.*

If $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ then there is a small rectangle centered at (x_0, y_0) such that within this rectangle the zero set of F is the graph of a function $y = f(x)$. The derivative of this function is

$$f'(x) = \frac{dy}{dx} = -\frac{F_x(x, f(x))}{F_y(x, f(x))}. \tag{85}$$

If $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$ then there is a small rectangle centered at (x_0, y_0) such that within this rectangle the zero set of F is the graph of a function $x = g(y)$. The derivative of this function is

$$g'(y) = \frac{dx}{dy} = -\frac{F_y(g(y), y)}{F_x(g(y), y)}. \tag{86}$$

A proof may be given in class, time permitting.

There is no need to memorize the formulas (85) and (86). We can get them by using the method of implicit differentiation from math 221. For instance, suppose that the graph of the function $y = f(x)$ gives you a piece of the zero set of F. This means that

$$F(x, f(x)) = 0 \text{ for all } x.$$

Differentiating both sides of this equation leads us via the chain rule,

$$\frac{dF(x,f(x))}{dx} = \frac{\partial F}{\partial x}(x,f(x)) \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y}(x,f(x)) \cdot \frac{df(x)}{dx},$$

to

$$0 = \frac{dF(x,f(x))}{dx} = F_x(x,f(x)) + F_y(x,f(x))f'(x). \tag{87}$$

Solve this for $f'(x)$ and we get

$$f'(x) = \frac{dy}{dx} = -\frac{F_x(x,f(x))}{F_y(x,f(x))},$$

which is what the theorem claims.

10.2. The Implicit Function Theorem with more variables. There are many variations and extensions of Theorem 10.1. The simplest is to consider the level set of a function of three rather than two variables. Suppose F is a function of three variables, with continuous partial derivatives, and consider the set of points defined by the equation

$$F(x,y,z) = C.$$

This is the level set of F at level C.

If

$$\frac{\partial F}{\partial y}(x_0,y_0,z_0) \neq 0,$$

then near (x_0,y_0,z_0) the level set of F is the graph of a function $y = g(x,z)$, meaning that the function $y = g(x,z)$ satisfies

$$G(x,g(x,z),z) = 0.$$

Hence we can find the partial derivatives of this function by implicit differentiation. The result is

$$\frac{\partial y}{\partial x} = g_x(x,z) = -\frac{F_x(x,y,z)}{F_y(x,y,z)}, \qquad \frac{\partial y}{\partial z} = g_z(x,z) = -\frac{F_z(x,y,z)}{F_y(x,y,z)}, \tag{88}$$

where $y = g(x,z)$.

10.3. Example – The saddle surface again. The saddle surface is the graph of the function $z = xy$, which we can think of as the zero set of the function

$$F(x,y,z) = z - xy.$$

The point $(2,3,6)$ lies on the saddle surface, and at this point the partial derivatives of F are

$$F_x = \frac{\partial(z-xy)}{\partial x} = -y = -3, \quad F_y = \frac{\partial(z-xy)}{\partial y} = -x = -2, \quad F_z = \frac{\partial(z-xy)}{\partial z} = 1.$$

Since $F_x(2,3,6) = -3$ is non zero, the Implicit Function Theorem tells us that near this point the zero set of F is the graph of a function $x = g(y,z)$. Solving $F = 0$ for x we see that this function is in fact

$$x = g(y,z) = \frac{z}{y}.$$

The partial derivatives of g are easy to compute in this example, but even if we couldn't find them directly, the Implicit Function Theorem would tell us that

$$g_y(3,6) = -\frac{F_y(2,3,6)}{F_x(2,3,6)} = \frac{2}{3}, \qquad g_z(3,6) = -\frac{F_z(2,3,6)}{F_x(2,3,6)} = \frac{1}{3}.$$

Problems

1. Compute the gradient of each function in Problem 3.2 of § 3.

2. Show that for any two differentiable functions f and g one has

$$\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g,$$
$$\vec{\nabla}(fg) = f\vec{\nabla} g + g\vec{\nabla} f,$$
$$\vec{\nabla}\left(\frac{f}{g}\right) = \frac{g\vec{\nabla} f - f\vec{\nabla} g}{g^2}.$$

In other words the sum-, product- and quotient rules for differentiation also apply to the gradient.

3. (a) Draw the level sets of the function $f(x, y) = x^2 + 4y^2$ at levels $0, 4, 16$.

(b) Find the points on the level set $f(x, y) = 4$ where the gradient is parallel to the vector $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$. What can you say about the tangent line to the level set at those points? Draw the gradient vectors, and the tangent lines at the points you just found.

Hint: two non-zero vectors $\vec{v}$ and $\vec{w}$ are parallel if there is a number s such that $\vec{v} = s\vec{w}$.

(c) Repeat the same two problems for the function $g(x, y) = 4xy^2$.

4. (a) Draw the zero set of the function $f(x, y, z) = x^2 + y^2 - 2z$.

(b) Find all points on the zero set of the function f where the gradient is parallel to the vector $\vec{v} = \begin{pmatrix}1\\1\\2\end{pmatrix}$.

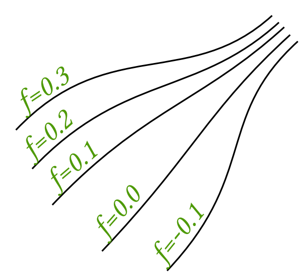
5. A bug is crawling on the surface of a hot plate, the temperature of which at the point x units to the right of the lower left corner and y units up from the lower left corner is given by $T(x, y) = 100 - x^2 - 3y^3$.

(a) If the bug is at the point $(2, 1)$, in what direction should it move to cool off the fastest?

(b) If the bug is at the point $(1, 3)$, in what direction should it move in order to maintain its temperature?

6. The level sets of a function $z = f(x, y)$ are often curves. Must they always be curves? Could the zero set of a function be a solid square (e.g. all points (x, y) with $0 \le x \le 1$ and $0 \le y \le 1$)?

7. The caption of Figure 8 says that one can only see the direction, but not the length of the gradient $\vec{\nabla} f$ of a function, from just one of its level sets. It is however possible to see where the gradient is larger from a drawing of several level sets. We can read this information from the way in which level sets are more bunched together in some regions than in others.



The picture above shows some level sets of a function. On the bottom left the level sets are further apart, on the top right they are more bunched together. Where is the gradient the larger, i.e. where is $\|\vec{\nabla} f\|$ larger: bottom-left, or top-right?

8. Have a look at Figure 8. Assume the function differentiable at the origin.

(a) What can you say about the gradient $\vec{\nabla} f$ at the origin?

(b) Where is the function positive and where is it negative (assume that the whole zero set is drawn).

9. This problem asks you to think about the Implicit Function Theorem 10.1

Consider the unit circle $\mathcal{C}$ with equation

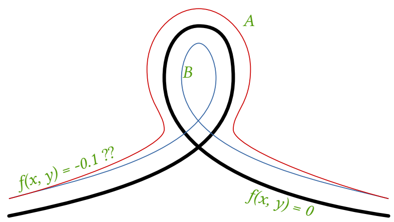
$$x^2 + y^2 = 1.$$

The unit circle $\mathcal{C}$ is a level set of the function $F(x, y) = x^2 + y^2$.

(a) Where on $\mathcal{C}$ is $F_y \neq 0$? Near which points P on $\mathcal{C}$ can one represent $\mathcal{C}$ as a graph of the form $y = f(x)$?

(b) Near which points P on $\mathcal{C}$ can one represent $\mathcal{C}$ as a graph of the form $x = g(y)$?

10. Here is the zero set of a function $z = f(x, y)$ (in bold). The function is only zero on the bold curve, it is nonzero everywhere else.

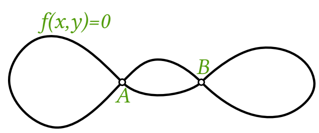


(a) One of the two other curves above is the level set $f(x, y) = -0.1$. Which one is it, A or B? As always, explain your answer.

(b) Draw a possible level set $f(x, y) = +0.1$.

(c) Draw possible gradients on the zero set (similar to Figure 8).

11. Here is the zero set of a differentiable function $z = f(x, y)$.



Explain why the Implicit Function Theorem (§10.1) implies that $\vec{\nabla} f = \vec{\mathbf{0}}$ at the two points A and B.

12. **(a)** Compute the gradient of the "distance to the square function" f from problems 5.13 and 3.7.

(b) How much is $|\vec{\nabla} f|$?

(c) Make a drawing of the level sets of f, and the gradient $\vec{\nabla} f$.

13. Let $f(x, y) = \ln(2 + 2x + e^y)$.

(a) Compute the gradient of f at the point (x_0, y_0) with position vector $\vec{x}_0 = \binom{1}{0}$.

(b) You are allowed to choose a point at a distance 0.01 from the point $(1, 0)$. Where would you choose the new point if you want f to be as large as possible? (Hint: review the linear approximation formula and subsequent discussion about the gradient as direction of greatest increase in §8.2)

(c) Is your answer to the previous the *exact* answer, or only an *approximation*? I.e., could someone else find a point at distance 0.01 from $(1, 0)$ at which f has a (slightly) higher value than at the point you found?

(d) The level set C of f through the point $(1, 0)$ happens to be the graph of a function $y = g(x)$. Find that function.

(e) Find a normal vector to the tangent line to C at the point $(1, 0)$. Find an equation for the tangent line to C at $(1, 0)$.

(f) How much is $g(1)$? Find two different ways to compute $g'(1)$ based on the work you have done so far.

14. Let (a, b, c) be a point on the sphere with radius R centered at the origin. Find an equation for the tangent plane to the sphere at (a, b, c). Simplify your answer as much as possible (a, b, c, and R will show up in your answer of course.)

11. The Chain Rule with more Independent Variables; Coordinate Transformations

The chain rule we have seen so far tells us how to differentiate expressions of the form $f(x(t), y(t))$. Such expressions are the result of substituting two functions $x(t), y(t)$ of one variable t in one function of two variables $z = f(x, y)$. What do we do if the functions x, y that get substituted in $f(x, y)$ depend on not one, but two (or more) variables? The answer is easy: ***we do exactly the same.***

For instance, suppose we want to substitute $x = x(u, v)$ and $y = y(u, v)$ in a function $z = f(x, y)$, resulting in a function $F(u, v) = f(x(u, v), y(u, v))$, and suppose we want find the partial derivatives of F with respect to u. To compute this we *keep v fixed* and regard u as the variable – then $x(u, v)$ and $y(u, v)$ are functions of one variable u and we apply the chain rule we already know. This leads to

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

The only difference with (69) is that we have written the derivatives of x and y as partial derivatives. We do this to indicate that in computing this derivative we momentarily consider x as a function of u, but later we may want to vary v again.

The same considerations lead to the partial derivative of F with respect to v:

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

11.1. An example without context. Suppose f is some function of two variables and we want to find the partial derivatives of

$$g(u, v, w) = f(2uv, u^2 + w^2).$$

By this we mean that g is the result of substituting $x = 2uv$ and $y = u^2 + w^2$ in f. Note that g is a function of three vairables, and f is a function of two variables.

The chain rule tells us that the derivatives of g are

$$\begin{aligned}
\frac{\partial g}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2v\frac{\partial f}{\partial x} + 2u\frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = 2u\frac{\partial f}{\partial x} \\
\frac{\partial g}{\partial w} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} = 2w\frac{\partial f}{\partial y}
\end{aligned}$$

11.2. Example: a rotated coordinate system. We are used to specifying the location of points in the plane by giving their x and y coordinates, but sometimes it is better to use different coordinates. For instance, two people A and B could have chosen the same origin, but their axes could be rotated with respect to each other. See Figure 10. If A's coordinates are called x, y and B's coordinates are X, Y then it should be possible to find A's coordinates of a point if we know what coordinates B assigns to this point – given

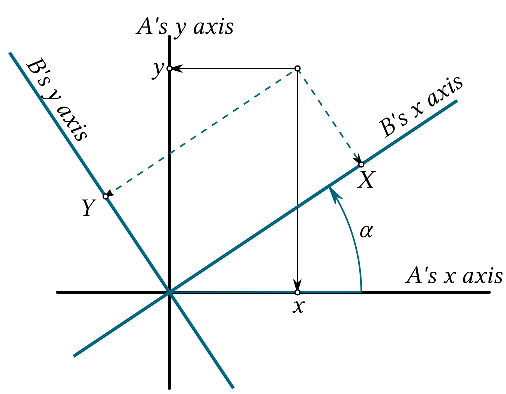


Figure 10. After choosing different x and y axes, A and B will assign different x, y coordinates to the same point in the plane. Equations (89) give the relation between these two sets of coordinates.

X, Y what are x, y? The answer to this question is[3]

$$\begin{cases} x = X\cos\alpha - Y\sin\alpha, \\ y = X\sin\alpha + Y\cos\alpha. \end{cases} \tag{89}$$

Suppose both A and B are measuring the temperature T at various points in the plane. A predicts the temperature at various points in the plane: he says that at the point with coordinates (x, y) the temperature will be $T(x, y)$. In fact he has also found the partial derivatives $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$.

Equipped with the $X, Y \to x, y$ conversion (89) B can now take A's formula for the temperature and express it in terms of her own X, Y coordinates. If we write $T_A(x, y)$ for the temperature at the point whose A-coordinates are (x, y) and $T_B(X, Y)$ for the temperature at the point whose B-coordinates are (X, Y), then we have

$$\begin{aligned} T_B(X,Y) &= T_A(x, y) \\ &= T_A(X\cos\alpha - Y\sin\alpha, X\sin\alpha + Y\cos\alpha). \end{aligned}$$

What is the relation between the partial derivatives of the temperatures as computed by A and by B? The chain rule gives the answer:

$$\begin{aligned} \frac{\partial T_B}{\partial X} &= \frac{\partial}{\partial X}\Big\{T_A\big(\underbrace{X\cos\alpha - Y\sin\alpha}_{=x}, \underbrace{X\sin\alpha + Y\cos\alpha}_{=y}\big)\Big\} \\ &= \frac{\partial T_A}{\partial x}\cos\alpha + \frac{\partial T_A}{\partial y}\sin\alpha. \end{aligned}$$

11.3. Another example – Polar coordinates. Suppose a quantity P is given in terms of Cartesian coordinates x and y: $P = f(x, y)$. How does P change if we vary the polar coordinates r and θ, i.e. what are the partial derivatives of P with respect to r and θ?

To answer this question we must write P as a function of r and θ. Recall that the relation between Cartesian coordinates and polar coordinates is

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{90}$$

Therefore $P = f(x, y) = f(r\cos\theta, r\sin\theta)$ and we get

$$\frac{\partial P}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}, \qquad \frac{\partial P}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y} \tag{91}$$

Since the function f always gives us the value of the quantity P, these relations are usually written in this way:

$$\frac{\partial P}{\partial r} = \cos\theta\frac{\partial P}{\partial x} + \sin\theta\frac{\partial P}{\partial y}, \qquad \frac{\partial P}{\partial \theta} = -r\sin\theta\frac{\partial P}{\partial x} + r\cos\theta\frac{\partial P}{\partial y} \tag{92}$$

Using the relation (90) between polar and Cartesian coordinates we can write these equations in yet another way:

$$\frac{\partial P}{\partial r} = \frac{x}{r}\frac{\partial P}{\partial x} + \frac{y}{r}\frac{\partial P}{\partial y}, \qquad \frac{\partial P}{\partial \theta} = -y\frac{\partial P}{\partial x} + x\frac{\partial P}{\partial y} \tag{93}$$

12. Problems

[3] One way of arriving at these relations is to use vectors as in the first vector work sheet of this semester.

1. Use the chain rule to compute dz/dt for $z = \sin(x^2 + y^2)$, $x = t^2 + 3$, $y = t^3$.

2. Use the chain rule to compute dz/dt for $z = x^2 y$, $x = \sin(t)$, $y = t^2 + 1$.

3. Use the chain rule to compute $\partial z/\partial s$ and $\partial z/\partial t$ for $z = x^2 y$, $x = \sin(st)$, $y = t^2 + s^2$.

4. Use the chain rule to compute $\partial z/\partial s$ and $\partial z/\partial t$ for $z = x^2 y^2$, $x = st$, $y = t^2 - s^2$.

5. **(a)** Let $x = x(u, v)$, $y = y(u, v)$ be the following set of functions of u, v:

$$x = u^2 - v^2, \quad y = 2uv.$$

If $g(u, v) = f(x(u, v), y(u, v))$ then compute $g_u(1, 0)$, $g_u(1, 1)$, $g_v(1, 0)$, and $g_v(1, 1)$, if you are given these values of the partial derivatives of f:

x	y	$f_x(x, y)$	$f_y(x, y)$
0	0	A	B
1	0	C	D
0	1	E	F
1	1	G	H
2	0	I	J
0	2	K	L

(b) Repeat the above problem if x and y are given by $x = u$, $y = v/u$.

(c) Repeat part **(a)** of this problem if x and y are given by $x = u + v$, $y = u - v$.

6. Let x, y, X, Y, T_A, and T_B be as in the example in §11.2. In that section we computed $\frac{\partial T_B}{\partial X}$.

(a) Compute $\dfrac{\partial T_B}{\partial Y}$.

(b) Show that

$$\Big(\frac{\partial T_A}{\partial x}\Big)^2 + \Big(\frac{\partial T_A}{\partial y}\Big)^2 = \Big(\frac{\partial T_B}{\partial X}\Big)^2 + \Big(\frac{\partial T_B}{\partial Y}\Big)^2.$$

In other words, A and B may measure different partial derivatives, but the temperature gradients they find have the same length. $\|\vec{\nabla} T_A\| = \|\vec{\nabla} T_B\|$.

7. (About polar coordinates). In §11.3 we saw how we can use the chain rule to find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ if we know the function f in terms of Cartesian coordinates (x, y). In this problem we turn the question around: suppose we are given a function in polar coordinates, how do we compute its gradient.

Recall that polar and Cartesian coordinates are related by

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan\frac{y}{x},$$

at least in the region where $x > 0$. (See Chapter III, § 4.)

(a) Compute $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}$. Try to simplify your answer as much as possible, by reusing the variables r and θ. For instance, the simplest way to write $\frac{\partial r}{\partial x}$ is as $\frac{\partial r}{\partial x} = \frac{x}{r}$.

(b) Suppose a quantity P is given in terms of Polar coordinates by $P = f(r, \theta)$. Express $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ in terms of $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

More precisely, compute

$$\frac{\partial P}{\partial x} \overset{\text{def}}{=} \frac{\partial\{f(r(x, y), \theta(x, y)\}}{\partial x}$$

and

$$\frac{\partial P}{\partial y} \overset{\text{def}}{=} \frac{\partial\{f(r(x, y), \theta(x, y)\}}{\partial y}$$

(c) Show that

$$\|\vec{\nabla} P\|^2 = \Big(\frac{\partial f}{\partial r}\Big)^2 + \frac{1}{r^2}\Big(\frac{\partial f}{\partial \theta}\Big)^2.$$

8. For some function f we are told that at the point with Cartesian coordinates $(4, 3)$ one has

$$\frac{\partial f}{\partial r} = 3, \quad \frac{\partial f}{\partial \theta} = 6.$$

Compute the gradient $\vec{\nabla} f$ at $(2, 1)$.

9. In physics an electric field is described by its ***potential function***, $\phi = \phi(x, y)$ (in this problem we assume the world is two-dimensional; the potential ϕ is measured in Volts). Minus the gradient of the potential function is called the ***electric field:***

$$\vec{\boldsymbol{E}} = -\vec{\nabla}\phi.$$

The electric potential of a point charge in the plane is given in Polar coordinates by $\phi = -C \ln r$, for some constant C (the physicists will tell you that C depends on the charge that was placed at the origin; for us it is just some number, and we will in fact assume that $C = 1$.)

(a) Compute the electric field $\vec{\boldsymbol{E}}$ corresponding to the potential $\phi = -\ln r$.

(b) Compute $\|\vec{\boldsymbol{E}}\|$ (this quantity measures the strength of the electric field, but not its direction.) Where is the electric field stronger?

(c) Make a drawing of the level curves of the potential ϕ, and the electric field $\vec{\boldsymbol{E}}$.

(d) In the three dimensional world the electric potential generated by a charged particle at the origin is not given by $-C \ln r$, but instead by the so-called *Coulomb potential*

$$\phi = \frac{C}{r}, \text{ where } r = \sqrt{x^2 + y^2 + z^2}.$$

Compute the corresponding electric field $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\phi$.

10. The *ideal gas law*, given by $PV = nRT$, relates the Pressure, Volume, and Temperature of n moles of gas. (R is the ideal gas constant). Thus, we can view pressure, volume, and temperature as variables, each one dependent on the other two.

(In this problem pressure is measured in Pascals, temperature in degrees Kelvin, and volume in Liters.)

Each of the following three questions can be answered by applying the chain rule to differentiate $z(t) = f(x(t), y(t))$ for suitable quantities x, y, and z. In each case state which variables play the role of x, y, z, and what the function f is.

(a) If pressure of a gas is increasing at a rate of 0.2Pa/min and temperature is increasing at a rate of 1°K/min, how fast is the volume changing?

(b) If the volume of a gas is decreasing at a rate of 0.3L/min and temperatuere is increasing at a rate of 0.5°K/min, how fast is the pressure changing?

(c) If the pressure of a gas is decreasing at a rate of 0.4Pa/min and the volume is increasing at a rate of 3L/min, how fast is the temperature changing?

11. The ideal gas law says $PV = nRT$, where P, V, T are variables, and n, R are constants. Verify the following identity:

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

12. The previous exercise was a special case of the following fact, which you are asked to verify here:

Assume that $F(x, y, z)$ is a function of 3 variables, and suppose that the relation $F(x, y, z) = 0$ defines each of the variables in terms of the other two, namely $x = f(y, z)$, $y = g(x, z)$ and $z = h(x, y)$, then

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1.$$

Hint: this is a problem about implicit differentiation.

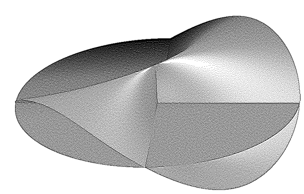
13. Four cartographers are using different coordinates to describe the same landscape. Each of them describes the landscape by specifying a the height of a point in the landscape as a function of its position above a horizontal plane.

Cartographer A uses Cartesian coordinates (x, y) in the plane, B uses Cartesian coordinates (X, Y) in the plane. The coordinates (X, Y) are rotated by 45° with respect to (x, y) (see §11.2).

Cartographer C works with A but uses polar coordinates (r, θ) (r is the distance to the origin, θ is the angle with A's x-axis).

Cartographer D works with B and uses polar coordinates (r, φ) (r is the distance to the origin, φ is the angle with B's X-axis).

Here is a picture of the landscape that A, B, C, and D are looking at:



(a) If B has found that the height is given by the function $f(X, Y) = 2XY/(X^2 + Y^2)$, then what function does A find for the height?

(b) What height function does C find?

(c) What height function does D find?

14. Brian and Ally are using different Cartesian coordinate systems in the plane: (x, y) for Ally, (X, Y) for Brian. They have the same origin, but Brian's coordinates are rotated by an angle of $\theta = \arctan \frac{4}{3}$ ($\approx 53^\circ$, although that is only an approximation. You can give exact answers in this problem, and you don't need a calculator.)

(a) What is the relation between (x, y) and (X, Y)?

(b) If Ally has found that $T_A(x, y) = 32+0.1y$, then what formula $T_B(X, Y)$ will Brian use to describe the temperature?

(c) On a different occasion Ally found that the temperature had changed. Now Ally measures the temperature and finds that at the point with $x = 1, y = 1$ one has $T_A(1, 1) = 35$, and also $\frac{\partial T_A}{\partial x} = 0.05$ and $\frac{\partial T_A}{\partial y} = 0.8$. Which coordinates does Brian assign to this point, which temperature T_B, and which derivatives $\frac{\partial T_B}{\partial X}$ and $\frac{\partial T_B}{\partial Y}$ does Brian compute at this point?

[Hint: before you compute anything, find $\sin\theta$ and $\cos\theta$; also draw a right triangle one of whose acute angles is θ.]

13. Higher Partials and Clairaut's Theorem

13.1. Higher partial derivatives. By definition

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial(\frac{\partial f}{\partial x})}{\partial x} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial(\frac{\partial f}{\partial y})}{\partial x}$$
(94)
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial(\frac{\partial f}{\partial x})}{\partial y} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial(\frac{\partial f}{\partial y})}{\partial y}$$

In subscript notation one writes these higher partial derivatives as follows:

$$f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy}(x, y) = \frac{\partial^2 f}{\partial y \partial x}$$
$$f_{yx}(x, y) = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2}\ .$$

Note the reversal in xy order in the mixed partial derivatives!

13.2. Example. If $f(x, y) = x^2 y + \cos xy$ then $f_x = 2xy - y \sin xy$, and hence

$$f_{xx} = \frac{\partial(2xy - y \sin xy)}{\partial x} = 2y - y^2 \cos xy,$$
$$f_{xy} = \frac{\partial(2xy - y \sin xy)}{\partial y} = 2x - \sin xy - xy \cos xy.$$

The other partial derivatives follow from $f_y = x^2 - x \sin xy$, and they are

$$f_{yx} = 2x - \sin xy - xy \cos xy, \quad f_{yy} = -x^2 \cos xy.$$

Every time we take a derivative, we can choose whether we differentiate with respect to x or y. Differentiating once we have two possibilities, differentiating twice we have $2 \times 2 = 4$ possibilities, etc. That is why we found four partial derivatives of second order in the above example. But if we look carefully, we also see that f_{xy} and f_{yx} are the same. This is no coincidence.

13.3. Clairaut's Theorem – mixed partials are equal. *If for a given function f of two variables the mixed partial derivative $f_{xy}(x, y)$ exists for all (x, y) in a neighborhood of a point (a, b), and if this derivative is continuous at (a, b), then the other mixed partial derivative $f_{yx}(a, b)$ also exists, and $f_{xy}(a, b) = f_{yx}(a, b)$.*

So we normally don't have to worry about the order in which we take partial derivatives.

13.4. Proof of Clairaut's theorem. With some algebra we can show that the definition of partial derivatives implies

$$\frac{\partial^2 f}{\partial x \partial y} = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x,y)}{\Delta x \Delta y} \tag{95}$$

while

$$\frac{\partial^2 f}{\partial y \partial x} = \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x,y)}{\Delta x \Delta y} \tag{96}$$

So it's a matter of showing that one can switch the two limits. We won't go into the details here, but the hypothesis that f_{xy} is continuous implies that we are indeed allowed to switch the limits.

14. Finding a function from its derivatives

We now look at integrating the partial derivatives of a function, which looks out of place here (this being a chapter on derivatives and not on integrals), but Clairaut's Theorem actually turns out to play a role.

If we have the derivative $f'(x)$ of some function of one variable then we know how to recover the function $f(x)$: we integrate, i.e.

$$f(x) = \int f'(x)dx + C.$$

Furthermore, any (continuous) function can be the derivative of a function, because, if someone gives us a continuous function $f(x)$, then

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t)dt$$

is a differentiable function whose derivative is $F'(x) = f(x)$.

What about functions of more than one variable? Suppose we know the partial derivatives

$$\frac{\partial f}{\partial x} = P(x,y) \text{ and } \frac{\partial f}{\partial y} = Q(x,y) \tag{97}$$

of a function of two variables, can you then find the function $f(x,y)$?

The answer is "yes, you can find f by integrating, ***if it exists,*** but not every pair of functions P and Q are the partial derivatives of some function."

The following two examples are typical of what can happen.

14.1. Example. Does there exist a function $f(x,y)$ of two variables such that

$$\frac{\partial f}{\partial x} = x^3 - 2xy, \text{ and } \frac{\partial f}{\partial y} = 3y^2$$

both hold? The answer is no, such a function cannot exist, and here is the reason: if there were such a function, then we could compute

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial(x^3 - 2xy)}{\partial y} = -2x, \text{ and } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial(3y^2)}{\partial x} = 0.$$

By Clairaut's Theorem both computations should give us the same answer, but they don't. Therefore the function f whose partials are as above cannot exist.

14.2. Example. Does there exist a function $f(x, y)$ of two variables whose derivatives are

$$\frac{\partial f}{\partial x} = x^3 - 2xy, \text{ and } \frac{\partial f}{\partial y} = \sin \pi y - x^2?$$

Let's check Clairaut's condition:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial (x^3 - 2xy)}{\partial y} = -2x, \text{ and } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial (\sin \pi y - x^2)}{\partial x} = -2x.$$

This time both computations gave us the same answer, so Clairaut's theorem does not rule out the existence of the function f that we are looking for. We can try to compute it by integrating both partial derivatives. There is a systematic way of doing this that usually leads to the answer.

We first integrate f_x while treating y as a constant:

$$f(x, y) = \int \{x^3 - 2xy\}\, dx = \tfrac{1}{4}x^4 - x^2 y + C(y).$$

The "constant" is only a constant in the sense that it does not depend on x. It may depend on y, and that is why we wrote it as $C(y)$. To find $C(y)$ we differentiate this result with respect to y:

$$\sin \pi y - x^2 = f_y = \frac{\partial \{\tfrac{1}{4}x^4 - x^2 y + C(y)\}}{\partial y} = -x^2 + C'(y).$$

So we see that $C'(y) = \sin \pi y$, and hence $C(y) = -\frac{1}{\pi} \cos \pi y + K$, where K is a real constant (K depends neither on x nor on y).

We find that the following function has the prescribed partial derivatives

$$f(x, y) = \tfrac{1}{4}x^4 - x^2 y - \tfrac{1}{\pi} \cos \pi y + K$$

where K is constant, i.e. where K depends on neither x nor y.

The method used in this example always works, and we summarize this fact in the following theorem.

14.3. Theorem. *Suppose $P(x, y)$ and $Q(x, y)$ are two functions that are defined on a rectangular domain $\mathcal{R} = \{(x, y) : a < x < b, c < y < d\}$, and suppose that they have continuous partial derivatives on this domain.*

If a function $f(x, y)$ exists such that (97) holds on $\mathcal{R}$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{98}$$

must hold on $\mathcal{R}$.

Conversely, if P and Q satisfy (98) then there is a function f defined on $\mathcal{R}$ that satisfies (97).

To prove this theorem we need to understand integrals of functions of several variables, and Green's theorem in particular, so this will have to wait until the end of the semester. See § VII.11.

It should be noted that the assumption above that the functions P and Q be defined on a rectangle is important: the theorem is no longer true if the domain of P and Q "has holes." See problem 15.16.

15. Problems

1. Find all first and second partial derivatives of $x^3y^2 + y^5$.

2. Find all first and second partial derivatives of $4x^3 + xy^2 + 10$.

3. Find all first and second partial derivatives of $x \sin y$.

4. Find all first and second partial derivatives of $\sin(3x)\cos(2y)$.

5. Find all first and second partial derivatives of e^{x+y^2}.

6. Find all first and second partial derivatives of $\ln \sqrt{x^3 + y^4}$.

7. Find all first and second partial derivatives of z with respect to x and y if $x^2 + 4y^2 + 16z^2 - 64 = 0$. (Hint: solve for z or use implicit differentiation…)

8. Find all first and second partial derivatives of z with respect to x and y if $xy + yz + xz = 1$. (Hint: solve for z or use implicit differentiation…)

9. How many different second partial derivatives does a function of two variables have? What about a function of three variables? How many derivatives of third degree does a function of two variables have?

10. Derive the formulas (95) and (96) from the definition of partial derivatives (51) and (52).

11. The equation which describes the *vibrating string* (as in a guitar, piano, or violin string) is

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \tag{99}$$

where $c > 0$ is some constant. The equation is called the ***wave equation***. It is an example of a ***partial differential equation.***

Note : this problem looks like a problem about differential equations, but to answer the following questions you really only have to compute partial derivatives of certain functions, and solve some (easy) algebraic equations.

(a) For which values of the constant v is a "traveling wave with velocity v and profile $F(x)$" a solution of the wave equation (99)? Does it matter which profile F is used here?

(For the terminology used here, revisit problem 5.16 in Chapter III, §5.2.)

(b) Suppose the string is clamped down at its ends, and that its length is L. For which values of the constants A and α is

$$f(x,t) = A\sin(\alpha t)\sin\frac{\pi x}{L}$$

a solution of the wave equation? (Assume $A \neq 0$).

(c) Same question for

$$g(x,t) = B\sin(\beta t)\sin\frac{2\pi x}{L}.$$

(d) Describe the movies that go with the solutions you found in **(b)** and **(c)**. Which of the two graphs moves faster?

(e) Show that $h(x,t) = f(x,t) + g(x,t)$ is again a solution of the wave equation, where f and g are as above. (Don't use the formulas for f and g: it is easier to prove a more general fact, namely, if two functions f and g satisfy (99), then so does their sum $f + g$.)

(f) Describe the movie that goes with the function $h(x,t)$ (it is probably better to use a graphing application like `grapher.app` on Mac OS X, `graphcalc.exe` on Windows or Linux).

12. Suppose $P(x,y) = x^2 - 2xy^3$ and $Q(x,y) = (xy)^2$. Does there exist a function $f(x,y)$ such that $P = f_x$ and $Q = f_y$?

13. Suppose $P(x,y) = x^2 + axy^3$ and $Q(x,y) = (xy)^2$, where a is a constant. For which a does there exist a function $f(x,y)$ such that $P = f_x$ and $Q = f_y$?

14. Suppose $P(x,y) = x^2 - 2xy^3$ and $Q(x,y) = (xy)^2$. Does there exist a function $f(x,y)$ such that $P = f_x$ and $Q = f_y$?

15. Suppose $x = u + v$, $y = u - v$, and suppose $f(x, y) = g(u, v)$. Then compute

(a) $\dfrac{\partial^2 g}{\partial u^2}$

(b) $\dfrac{\partial^2 g}{\partial v^2}$

(c) $\dfrac{\partial^2 g}{\partial u \partial v}$

(d) $\dfrac{\partial^2 g}{\partial u^2} - \dfrac{\partial^2 g}{\partial v^2}$

(e) $\dfrac{\partial^2 g}{\partial u^2} + \dfrac{\partial^2 g}{\partial v^2}$

16. [For discussion] Let

$$P(x, y) = \frac{-y}{x^2 + y^2}, \quad Q(x, y) = \frac{x}{x^2 + y^2}.$$

(a) What is the domain of P and Q?

(b) Show that

$$P = \frac{\partial \theta}{\partial x}, \qquad Q = \frac{\partial \theta}{\partial y}$$

where θ is the angle variable from polar coordinates.

(c) Show that P and Q satisfy the condition (98). (You don't have to compute the derivatives to check this, although you could.)

(d) Is there a function f such that (97) holds?

CHAPTER 5

Maxima and Minima

In first semester calculus we learned how to find the maximal and minimal values of a function $y = f(x)$ of one variable. The basic method is as follows: assuming the independent variable is restricted to some interval $a \leq x \leq b$, we first look for interior maxima and minima. These always occur at ***critical*** or ***stationary*** points of the function, i.e. solutions x of $f'(x) = 0$. We then check the function values at the endpoints a and b of the interval, to see if they might be maxima or minima.

To find out which solutions of $f'(x) = 0$ are actually local maxima or minima we can look at the sign of the derivative $f'(x)$ to see where the function is increasing or decreasing, or we can apply the ***second derivative test.***

This chapter we will see how to solve similar questions about functions of two or more variables.

1. Local and Global extrema

Let $z = f(x, y)$ be the function whose maximal or minimal values we are looking for, and let D be the domain of this function. This domain could be the largest possible domain for the given function (in case f is defined by a formula), but it could also be some smaller region which we ourselves have chosen. The question we are considering is

What are the largest and smallest values that $f(x, y)$ can have
if the point (x, y) belongs to the domain D?

1.1. Definition of global extrema. *The function f has a **global maximum** or **absolute maximum** at a point (a, b) in D if $f(x, y) \leq f(a, b)$ for all points (x, y) in D.*

Similarly, *the function f has a **global minimum** or **absolute minimum** at a point (a, b) in D if $f(x, y) \geq f(a, b)$ for all points (x, y) in D.*

1.2. Definition of local extrema. *The function f has a **local maximum** at a point (a, b) in D if there is a $r > 0$ such that $f(x, y) \leq f(a, b)$ for all points (x, y) in D which also lie in a disc of radius r centered at (a, b).*

Local minima are defined analogously.

1.3. Interior extrema. Recall that a point (a, b) in a domain D is called ***interior*** if it is not a boundary point, or, more precisely, if there is some small $r > 0$ such that the disc with radius r centered at (a, b) is entirely contained in D. We will apply this distinction to the local and global maxima and minima that we find: an ***interior local minimum*** is a local minimum that occurs at an interior point of the domain D of the function.

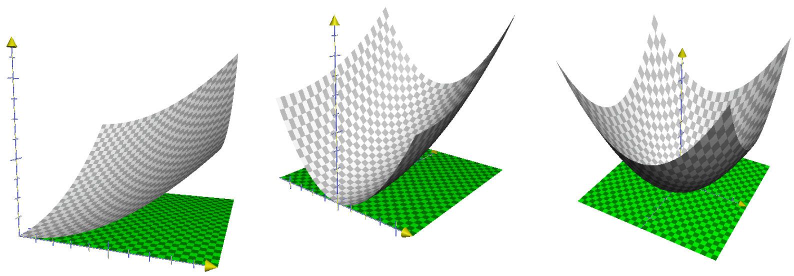


Figure 1. The graph of $f(x, y) = x^2 + y^2$ from example § 2.2 on three different rectangles Q. From left to right:

(i) $0 \le x \le 1, 0 \le y \le 1$. Both max and min are attained at a corner point of the rectangle.

(ii) $0 \le x \le 1, -1 \le y \le 1$. Two maxima, both are attained at corner points of the rectangle; the minimum is attained at an edge point.

(iii) $-1 \le x \le 1, -1 \le y \le 1$, Four maxima, all attained at corner points of the rectangle; the minimum is attained at an interior point.

2. Continuous functions on closed and bounded sets

Before we go into the details of how we can actually find the maxima and minima, it is good to know the following general fact. It tells us where to expect maxima and minima.

Let $z = f(x_1, \ldots, x_n)$ be a continuous function defined on some ***closed*** and ***bounded*** region D in $\mathbb{R}^n$. Here "closed" means that D contains all its boundary points, and "bounded" means that all points in D are not further away from the origin than some fixed radius R (D does not "stretch all the way to infinity".)

We will also assume that f is continuous on D.

2.1. Theorem about Maxima and Minima of Continuous Functions. *A continuous function defined on a closed and bounded region $D \subset \mathbb{R}^n$ has both a maximum and minimum within that region.*

The precise definitions of the concepts (continuous, closed, bounded) and the proof of this theorem all involve a fair number of ε's and δ's. This material is treated in courses like Math 421, 521 (real analysis) or 551 (point set topology) and really does not belong here in Math 234. Nevertheless it is important to have some understanding of what is meant in the above theorem. The following examples are meant to clarify this.

2.2. Example – The function $f(x, y) = x^2 + y^2$. This function is continuous, and the square $Q = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ is bounded, and it contains all boundary points (the edges of the square). Therefore Theorem 2.1 tells us that f attains both its highest and lowest values somewhere in the square. The theorem does not say where these max/min points are, but in this example they are easy to find. The function $f(x, y) = x^2 + y^2$ is at its smallest when both $x = 0$ and $y = 0$, i.e. at the bottom-left corner of the square. And $f(x, y)$ is at its largest when x and y are both as large as they can be, i.e. when $x = 1$ and $y = 1$. This happens at the top-right corner of the square.

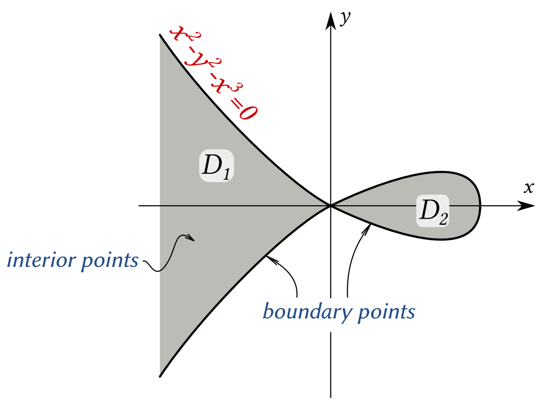
Note that the boundary of the rectangle Q has two different kinds of points: it has four corner points, and then all the other points that lie on the edges.

If we change the rectangle Q then the minimum can appear at a corner point, a point on an edge, or in an interior point. See Figure 1.

2.3. A fishy example. Consider the function $f(x,y) = x^2 - x^3 - y^2$. Its zero set is the curve $y^2 = x^2 - x^3$, which is shaped like the letter α, or like a fish – see Figure 2. The function is positive on the tail (D_1) and also on the body (D_2) of the fish, it vanishes on the curve that traces out the fish, and f is negative elsewhere.

We assume that both regions D_1 and D_2 are closed, which means that we assume that they include their boundary points. See Figure 2 below.

Theorem 2.1 does not apply to the region D_1 because D_1 is not bounded (it contains the whole negative x-axis). But the region D_2 is bounded, and our function f is continuous, so Theorem 2.1 does apply to D_2. The theorem tells us that the function f has a maximal value and a minimal value somewhere in D_2. In the interior of D_2 the function is strictly positive, and at the boundary points of D_2 we have $f = 0$. Therefore each boundary point is a minimum point of f on D_2. The point(s) in D_2 where f attains its highest value must be somewhere in the interior of D_2. In the next section we will see how to find it (and how to check that in this case there really only is ***one*** such point.)



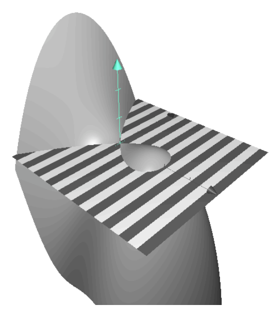


Figure 2. Left: The region where $f(x,y) = x^2 - x^3 - y^2$ is positive consists of two parts, one bounded (D_2), and the other unbounded (D_1). Theorem 2.1 does not apply to the unbounded region, but it does apply to the bounded region D_2. In that region f must attain a maximum and also a minimum. Since $f = 0$ on the boundary of the region D_2, and $f > 0$ in the interior, f achieves its lowest value in D_2 everywhere on the boundary of D_2 and its highest value somewhere in the interior. Theorem 2.1 does not tell us how to find that interior point, and allows for the possibility that there might be more interior maxima, as well as a few interior (local) minima.

Right: The graph of the function $z = x^2 - y^2 - x^3$.

3. Problems

1. Suppose you want to find the ***maximal value*** of $f(x,y) = x^2 - x^3 - y^2$ over all possible (x,y) with $x \geq 0$ (and no restriction on y – this region is called the *right half plane*).

(a) Explain why you should always choose $y = 0$ in order to maximize this particular function $f(x,y)$.

(b) Use your answer to part **(a)** to find the point (x,y) that maximizes $f(x,y)$ over the right half plane.

(c) Does our function $f(x,y)$ have a maximal value if (x,y) can be any point in the plane? (hint: what is $f(-1000,0)$?)

2. Suppose that D is a bounded and closed region in the plane (you should draw one: any region will do as long as you include the boundary points).

Where does the function $f(x, y) = x$ attain its maximum in the region that you drew? Can f attain its maximum at an interior point of the region?

What about minima?

3. Draw the region

$$R = \left\{(x, y) : y^2 \le 4(x^3 - x^4)\right\}.$$

Find the largest and smallest values that the function $f(x, y) = x$ can have on this region.

(Hint: where is $4(x^3 - x^4) = 4x^3(1 - x)$ positive? The region looks like an Onion). •

4. Critical points

For functions $y = f(x)$, $a \le x \le b$, of one variable the standard way of finding minima (and maxima) is to look for them in two different places: either the minimum is attained at one of the end points $x = a$ or $x = b$ of the interval, or else the minimum is attained at an interior point. At an interior minimum one has $f'(x) = 0$, so they can be found by solving the equation $f'(x) = 0$. The same approach works for functions of two or more variables. The basic fact that tells us that this is so, is the following theorem.

4.1. Definition (critical point). *A critical point of a function $z = f(x, y)$ of two variables is a point (a, b) at which $\vec{\nabla} f(a, b) = 0$, i.e. at which*

$$f_x(a, b) = 0 \textit{ and } f_y(a, b) = 0.$$

At a critical point of a function the tangent plane to the graph is horizontal.

4.2. Theorem. Local extrema are critical points. *If a function $z = f(x, y)$ defined on a domain D has a local minimum or local maximum at an interior point (a, b) then one has*

$$\frac{\partial f}{\partial x}(a, b) = 0, \textit{ and } \frac{\partial f}{\partial y}(a, b) = 0.$$

Picture proof. (See Figure 3.) If f has a local maximum at an interior point (a, b) then $f(x, y) \le f(a, b)$ for all (x, y) close to (a, b). This means that a small piece of the graph of f near its local maximum at $(a, b, f(a, b))$ lies below the plane $z = f(a, b)$. This plane must therefore be the tangent plane to the graph of f. Being horizontal, its slopes are zero, and these slopes are exactly the partial derivatives of f at (a, b).

Frozen variable proof. Suppose f has a local maximum at an interior point (a, b) of the domain D. Then we can freeze the y-variable at the value $y = b$ and consider the function of one variable $g(x) = f(x, b)$. This function has a maximum at $x = a$, so by first semester calculus we know that $g'(x) = 0$. By definition $g'(a) = f_x(x, b)$, so we conclude that $f_x(a, b) = 0$.

By freezing x instead of y we find that $f_y(a, b) = 0$ also must hold.

The same arguments apply in the case of a local minimum.

4.3. Three typical critical points. Let's find the critical points of the following three functions:

$$f(x, y) = x^2 + y^2, \quad g(x, y) = x^2 - y^2, \quad h(x, y) = -x^2 - y^2.$$

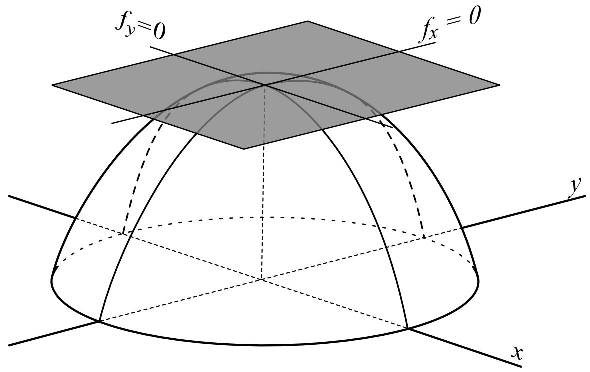


Figure 3. Theorem 4.2: at a local maximum the tangent plane to the graph is horizontal. The partial derivatives w.r.t. both x and y vanish, and in fact, the derivative along ***any*** path through (a, b) vanishes. To see a picture of a local minimum turn the page upside down.

► $f(x, y) = x^2 + y^2$. Computing the partial derivatives we find for the first function

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y.$$

If (x, y) is a critical point of f then x and y must satisfy the equations $f_x(x, y) = 0$ and $f_y(x, y) = 0$, in this case, $2x = 0$ and $2y = 0$. So we see that f has exactly one critical point, namely the origin $(x, y) = (0, 0)$.

Is this critical point perhaps a minimum or a maximum? Since squares can never be negative, $f(x, y) = x^2 + y^2$ is always non-negative, and it is at its smallest when both terms x^2 and y^2 vanish, i.e. when $x = y = 0$. So $f(x, y)$ has a global minimum at the origin.

► $h(x, y) = -x^2 - y^2$. This function is just $-f(x, y)$, and without looking at its derivatives we can tell that it has a global maximum at the origin (because $f(x, y)$ has a global minimum there). The derivatives are

$$\frac{\partial h}{\partial x} = -2x, \quad \frac{\partial h}{\partial y} = -2y$$

so that the origin is the only critical point of this function.

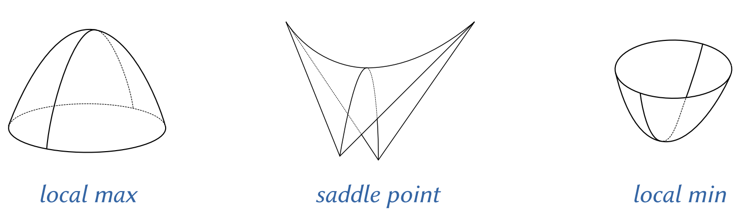


Figure 4. The three most common kinds of critical point. See the examples in §4.3 and also the second derivative test in §9.

► $g(x, y) = x^2 - y^2$. The derivatives of g are

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = -2y,$$

so, once again, the origin is the only critical point. But, unlike the previous two functions, g has neither a maximum nor a minimum at the origin. We can see this by first looking at what g does on the x-axis, and then what g does on the y-axis:

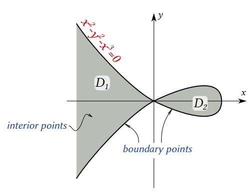
On the x-axis we have $g(x, 0) = +x^2$, so g has a ***minimum*** at the origin.

On the y-axis we have $g(0, y) = -y^2$, so g has a ***maximum*** at the origin.

So arbitrarily close to the origin we can find points (x, y) where $g(x, y)$ is larger than $g(0, 0)$, and we can find other points where $g(x, y)$ is smaller than $g(0, 0)$. Therefore g does not have a local maximum or a local minimum at the origin.

Figure 4 shows the three cases we have just discussed.

4.4. Critical points in the fishy example. *What are the critical points of the function $f(x, y) = x^2 - x^3 - y^2$ from §2.3?*



We compute the partial derivatives of the function

$$\frac{\partial f}{\partial x} = 2x - 3x^2 = (2 - 3x)x, \qquad \frac{\partial f}{\partial y} = -2y.$$

The equation $f_y = 0$ implies that $y = 0$, while $f_x = 0$ implies $x = 0$ or $x = \frac{2}{3}$. Therefore f has two critical points: one at the origin $(0, 0)$, and the other at $(\frac{2}{3}, 0)$.

In this example we could have already predicted from the shape of the zero set of f that f has at least two critical points – we don't need to compute the derivatives of f for that. Namely, the zero set of f is a curve that crosses itself at the origin, so the Implicit Function Theorem 10.1 (chapter 2) cannot hold at the origin, and hence $f_x = f_y = 0$ there. And in § 2.3 we argued that the function f must have a local maximum somewhere in the region D_2 (Figure 2), so f must have at least two critical points. On the other hand, by computing the critical points we have found that there is only one local maximum in the region D_2.

4.5. Another example – find the critical points of $f(x, y) = x - x^3 - xy^2$.
Solution: The derivatives of our function are

$$\frac{\partial f}{\partial x} = 1 - 3x^2 - y^2, \qquad \frac{\partial f}{\partial y} = -2xy.$$

The critical points are therefore the solutions of the equations

$$1 - 3x^2 - y^2 = 0, \qquad -2xy = 0.$$

This is a system of two equations, with two unknowns (that always happens when we look for critical points, since we are looking for solutions of $f_x(x, y) = 0$, $f_y(x, y) = 0$.) The second equation, $-2xy = 0$, implies that either $x = 0$ or $y = 0$ (or both). We have to treat these two cases separately:

The case $x = 0$. If $x = 0$ then we only have the first equation left, which tells us $1 - y^2 = 0$, i.e. $y = \pm 1$. We find two critical points with $x = 0$, namely, $(0, 1)$ and $(0, -1)$.

The other case, $x \neq 0$. If $x \neq 0$, then the second equation ($-2xy = 0$) implies $y = 0$. Substitute this in the first equation and we find $1 - 3x^2 = 0$, i.e. $x = \pm\frac{1}{3}\sqrt{3}$, so that we have two critical points with $x \neq 0$, namely, $(-\frac{1}{3}\sqrt{3}, 0)$ and $(\frac{1}{3}\sqrt{3}, 0)$.

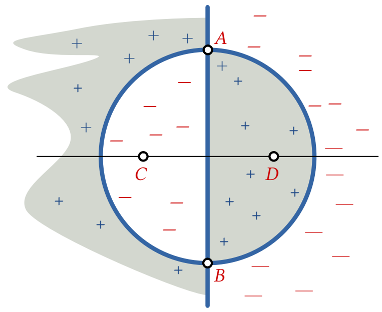


Figure 5. The zero set and signs of the function $f(x,y) = x - x^3 - xy^2$.

The conclusion is that this function has four critical points, two on the x-axis, and two on the y-axis. Without looking into this in any further detail we cannot tell if any of these points are local maxima or minima. In general the second derivative test (to be explained in § 9) will provide this information. For this example a look at the zero set of f also helps us figure out what kind of critical points we have found. Since f factors as

$$f(x,y) = x \cdot (1 - x^2 - y^2),$$

we see that its zero set consists of the line $x = 0$ and the unit circle $x^2 + y^2 = 1$. In the above picture $f > 0$ in the grey region, and $f < 0$ in the white area. Consider the right half of the unit disc. The function is positive in the interior, and zero on the boundary of this region. Just as in the "fishy example" of § 2.3, we have another case where the maximum of the function must be attained at one or more interior points of the right half of the unit disc. According to our computation f only has *one* critical point in the right half circle, and therefore this point must be a local maximum of the function. Conclusion: $D = (\frac{1}{3}\sqrt{3}, 0)$ is a local maximum.

In the same spirit you can argue that f has a local minimum at C.

The other two points A, B are neither local maxima nor minima, since *arbitrarily close to A or B* there are both points (x,y) with $f(x,y)$ positive, and points with $f(x,y)$ negative. The points A and B turn out to be "saddle points" (see §9 on the second derivative test.)

5. When there are more than two variables

The whole discussion so far has been about functions of two variables. Fortunately, not much changes when you have more variables. The concepts *local minimum* and *local maximum* are defined in the same way, and it turns out that any *interior local maximum or minimum must be a critical point of the function.* Here, by definition, a critical point of a function $w = f(x_1, \ldots, x_n)$ of n variables is a solution of the equations

$$\begin{cases} \dfrac{\partial f}{\partial x_1}(x_1, \cdots, x_n) = 0 \\ \dfrac{\partial f}{\partial x_2}(x_1, \cdots, x_n) = 0 \\ \qquad\qquad \vdots \\ \dfrac{\partial f}{\partial x_n}(x_1, \cdots, x_n) = 0. \end{cases}$$

Observe that there are n equations, and that there are also n unknowns $(x_1, \ldots, x_n)$ so that we should *in principle* be able to solve these equations. In practice the system of equations we get can be very easy, difficult, or simply impossible to solve.

6. Problems

1. Find all critical points of the following functions. Try to classify them into local/global maxima/minima, saddles, or other kind of critical points. (Write clear solutions. You will need your solutions later in problem 10.5.)

(a) $f(x, y) = x^2 + 4y^2 - 2x + 8y - 1$

(b) $f(x, y) = x^2 - y^2 + 6x - 10y + 2$

(c) $f(x, y) = x^2 + 4xy + y^2 - 6y + 1$

(d) $f(x, y) = x^2 - xy + 2y^2 - 5x + 6y - 9$

(e) $f(x, y) = y^2 - 18x^2 + x^4$

(f) $f(x, y) = y^4 - 4y^2 - 18x^2 + x^4$

(g) $f(x, y) = 9 + 4x - y - 2x^2 - 3y^2$

(h) $f(x, y) = xy(4 - x - 2y)$

(i) $f(x, y) = x(x - y)(x - 1)$

(j) $f(x, y) = (x - y)(xy - 4)$

(k) $f(x, y) = y^2 + \cos x$

(l) $f(x, y) = x^2y - \frac{1}{3}y^3$

(m) $f(x, y) = (x - y^2)(x - 1)$

(n) $f(x, y) = (x - y)(xy - 4)$

(o) $f(x, y) = x^2$

(p) $f(x, y) = x^2y$

(q) $f(x, y) = \left(1 - x^2 - y^2\right)^2$

(r) $f(x, y) = x^2y$

2.

(a) Draw the zero set of the function $f(x, y) = \sin(x)\sin(y)$.

(b) Where is the function f positive? Find as many critical points as you can without computing f_x or f_y.

(c) Find all critical points of $f(x, y)$. Which are local minima or local maxima?

3. Find the critical points of the function

$$f(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y - 2.$$

4. Draw the zero set and find the critical points of the functions

$$f(x, y, z) = x^2 + y^2 - z^2$$

and

$$g(x, y, z) = x^2 - y^2 - z^2$$

5. If we have three points A, B, and C in the plane, *which point is closest to all three of them?* The answer depends on what we mean by "closest to all three points." The following problem gives us one interpretation of this general question.

Consider the three points $(1, 4)$, $(5, 2)$, and $(3, -2)$ in the plane. The function

$$\begin{aligned} f(x, y, z) = {} & \\ & (x - 1)^2 + (y - 4)^2 + \\ & (x - 5)^2 + (y - 2)^2 + \\ & \qquad (x - 3)^2 + (y + 2)^2 \end{aligned}$$

is the sum of the squares of the distances from point (x, y) to the three points.

(a) Assuming that there is a global minimum, find x and y so that $f(x, y)$ is minimized.

(b) (For discussion) Does $f(x, y)$ have a global minimum? How can we be sure that the point we found in part **(a)** is not actually a maximum or some other critical point?

(c) Given the three points (a, b), (c, d), and (e, f), let $f(x, y)$ be the sum of the squares of the distances from point (x, y) to the three points. Find x and y so that this quantity is minimized.

6. Suppose that a function $f(x, y)$ factors, i.e. we can write it as the product of two other differentiable functions, $f(x, y) = g(x, y)h(x, y)$.

Prove: if a point (a, b) lies in the zero set of g and also in the zero set of h, then (a, b) is a critical point of f.

Hint: compute the partial derivatives of f by applying the product rule to $f = g \cdot h$.

7. Find the critical points of the functions

(a) $f(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y - 2$

(b) $f(x, y, z) = x^4 + y^2 + z^2 - 2xz + 4y$

(c) $f(x, y, z) = xyze^{-x-y-z}$

(d) $f(x, y, z) = x^2 + y^2 + z^2 - 2xyz$

7. A Minimization Problem: Linear Regression

Suppose we are measuring two quantities x and y in some experiment, and suppose that we expect that there is a linear relation of the form $y = ax + b$ between x and y. If we have a set of data points (x_k, y_k) from our experiment, then what do they tell us about a and b? ***Which choice of coefficients a and b bests fits our data?*** Because of experimental errors we would not expect our data points to lie on a straight line, but instead, we expect them to be clustered around a straight line. We could plot the data points, get a ruler, and draw a straight line by hand that looks like the best match – then we could measure a, b from our drawing. A more systematic approach is to first define what we mean by "best match" and then find the line that best matches according to our chosen criterion.

A very common criterion is the least-mean-square-fit. To describe it, imagine we have N data points, $(x_1, y_1), \ldots , (x_N, y_N)$, and consider the line with coefficients a and b. Most data points (x_k, y_k) will then probably not lie on the line $y = ax + b$, and one uses

$$E_k = \tfrac{1}{2}\bigl(ax_k + b - y_k\bigr)^2$$

as a measure for the mismatch between the data point (x_k, y_k) and the line $y = ax + b$ (the factor $\frac{1}{2}$ makes formulas later on nicer). Adding all these errors we get the total "mean square" error

$$E = E_1 + \cdots + E_N.$$

If we think of all the numbers $x_1, \ldots, x_N, y_1, \ldots, y_N$ as given constants (after all, we measured them, so we shouldn't change them any more[1]), then the total error only depends on the coefficients a and b. It is a measure for how well the line $y = ax + b$ fits our data points, and the common method of ***linear regression*** consists in choosing the coefficients a and b so as to minimize this error E.

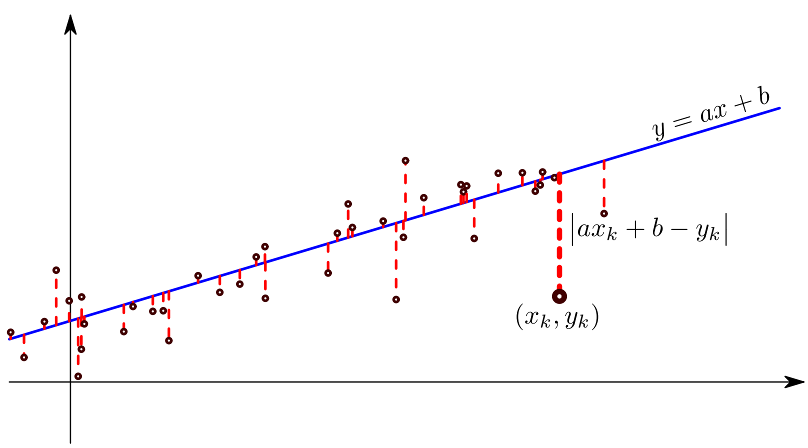


Figure 6. Which line best fits a set of data points?

This leads us to the problem of finding the critical points of the total error E as a function of a and b. We have to solve

$$\frac{\partial E}{\partial a} = 0 \qquad \frac{\partial E}{\partial b} = 0.$$

[1]This is called the "Sushi Principle": *raw data is better than cooked data.*

The total error is the sum of the individual errors $E_k(a,b)$ so we get

$$\frac{\partial E}{\partial a} = \frac{\partial E_1}{\partial a} + \cdots + \frac{\partial E_N}{\partial a}, \qquad \frac{\partial E}{\partial b} = \frac{\partial E_1}{\partial b} + \cdots + \frac{\partial E_N}{\partial b}.$$

The individual errors have the following derivatives:

$$\frac{\partial E_k}{\partial a} = x_k\big(ax_k + b - y_k\big), \qquad \frac{\partial E_k}{\partial b} = ax_k + b - y_k.$$

Adding all these derivatives then leads to

$$\begin{aligned}\frac{\partial E}{\partial a} &= \sum x_k\big(ax_k + b - y_k\big)\\ &= (\textstyle\sum x_k^2)a + (\sum x_k)b - \sum x_k y_k\end{aligned}$$

and

$$\begin{aligned}\frac{\partial E}{\partial b} &= \sum \big\{ax_k + b - y_k\big\}\\ &= (\textstyle\sum x_k)a + Nb - \sum y_k\end{aligned}$$

Here "$\sum$" represents summation over $k = 1, \cdots, N$, i.e. $\sum x_k y_k = x_1y_1 + \cdots + x_N y_N$, etc.

If (a,b) is a critical point then a and b must satisfy

$$\begin{aligned}(\textstyle\sum x_k^2)a + (\sum x_k)b &= \textstyle\sum x_k y_k\\ (\textstyle\sum x_k)a + Nb &= \textstyle\sum y_k\end{aligned}$$

These are two linear equations for the two unknowns a and b. Solving them leads to

$$a = \frac{N\sum x_k y_k \; - \; \sum x_k \sum y_k}{N\sum x_k^2 - \big(\sum x_k\big)^2}\;; \qquad b = \frac{-\sum x_k \sum x_k y_k + \sum x_k^2 \sum y_k}{N\sum x_k^2 - \big(\sum x_k\big)^2}\;.$$

These are the standard formulas for the coefficients a and b provided by the method of linear regression. Most calculators, and certainly all spreadsheets (like Excel) have these formulas preprogrammed, so we only have to enter the data points (x_k, y_k) and "push the right button" to get a and b.

8. Problems

1. We are given N measurements x_1, ..., x_N from some experiment, and, inspired by the Linear Regression example, we decide to see which number a "best fits the data." We define the error (or "measure of misfit") for each measurement to be

$$E_k(a) = \tfrac{1}{2}(a - x_k)^2$$

and we look for the number a which minimizes the total error

$$E(a) = E_1(a) + \cdots + E_N(a).$$

(a) Is this a problem about several variable calculus, or about one variable calculus? •

(b) Which number a do we find? •

2. We have a series of data points (x_k, y_k), and when we plot them we think we see a convex curve rather than a straight line. In fact it looks like a parabola, and so we set out to find a quadratic function $y = ax^2 + bx + c$ that minimizes the error

$$E(a,b,c) = E_1 + \cdots + E_N,$$

with

$$E_k(a,b,c) = \tfrac{1}{2}\big(ax_k^2 + bx_k + c - y_k\big)^2.$$

(a) How many variables are there in this problem? •

(b) If (a,b,c) is a critical point of $E(a,b,c)$ then a, b, and c satisfy three linear equations. Find these equations (don't solve them). •

3. A measurement in a certain experiment results in three numbers (x,y,z). The point of the experiment is to see if there is a linear relation of the form $z = ax + by + c$ between the three measured quantities, and to estimate the coefficients a, b, c.

After repeating the experiment N times we have N data points (x_k, y_k, z_k) ($k = 1, \ldots, N$). We decide to choose a, b, c so as to minimize the mean square error

$$E = E_1 + \cdots + E_N,$$

with

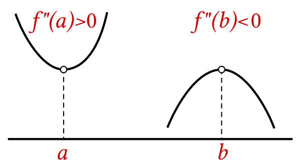
$$E_k(a, b, c) = \tfrac{1}{2}\bigl(ax_k + by_k + c - z_k\bigr)^2.$$

Which (linear) equations will we get for a, b, and c?

9. The Second Derivative Test

9.1. Review of the one-variable second derivative test and Taylor's formula.

For a function $y = f(x)$ of one variable you can tell if a critical point a is a local maximum or minimum by looking at the sign of the second derivative $f''(a)$ of the function at that point.



If $f''(a) > 0$ then the graph of f is curved upwards and f has a local minimum at a; if $f''(a) < 0$ then f has a local max. This section is about the analogous test for critical points of functions of two variables.

One way to understand the second derivative test is to look at the Taylor expansion of the function $y = f(x)$. If $x = a$ is a critical point for f, then

$$f(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2 + \cdots$$

Since a is a critical point of f we have $f'(a) = 0$, so that the Taylor expansion reduces to

$$f(x) = f(a) + \tfrac{1}{2}f''(a)(x - a)^2 + \cdots \tag{100}$$

If we ignore the remainder term (the dots), then we find that

$$f(x) \approx f(a) + \tfrac{1}{2}f''(a)(x - a)^2.$$

Near the critical point the graph of $y = f(x)$ is a approximately a parabola. It is curved upwards if $f''(a) > 0$, and downwards if $f''(a) < 0$.

To apply the same reasoning to a function of two (or more) variables we need to know the Taylor expansion of such a function.

9.2. Taylor's formula for a function of several variables.

The Taylor expansion of a function $z = f(x, y)$ should give us an approximation of $f(a+\Delta x, b+\Delta y)$ in terms involving powers of Δx and Δy. There is a general formula, but here we only need the second order terms, so we'll derive those and stop there.

The trick to finding the Taylor expansion is to consider the function

$$g(t) = f(a + t\Delta x, b + t\Delta y). \tag{101}$$

By definition

$$g(1) = f(a + \Delta x, b + \Delta y)$$

is the quantity we want to approximate, and $g(0) = f(a, b)$. Since $g(t)$ is a function of one variable, we can apply Taylor's formula from Math 222 to it. We get:

$$g(t) = g(0) + g'(0)t + g''(0)\frac{t^2}{2!} + \cdots \tag{102}$$

The dots contain the remainder term, which we will ignore. Now we set $t = 1$, and we get

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(0) + \cdots$$

The derivatives of g can be computed with the chain rule:

$$\begin{aligned} g'(t) &= \frac{df(a + t\Delta x, b + t\Delta y)}{dt} \\ &= f_x(a + t\Delta x, b + t\Delta y)\frac{d(a + t\Delta x)}{dt} + f(a + t\Delta x, b + t\Delta y)\frac{d(b + t\Delta y)}{dt} \\ &= f_x(a + t\Delta x, b + t\Delta y)\Delta x + f_y(a + t\Delta x, b + t\Delta y)\Delta y. \end{aligned} \tag{103}$$

The second derivative is

$$\begin{aligned} g''(t) = f_{xx}&(a + t\Delta x, b + t\Delta y)(\Delta x)^2 \\ &+ 2f_{xy}(a + t\Delta x, b + t\Delta y)\Delta x\Delta y \\ &\qquad + f_{yy}(a + t\Delta x, b + t\Delta y)(\Delta y)^2. \end{aligned} \tag{104}$$

In computing $g''(t)$ we run into terms involving f_{xy} and terms with f_{yx}. Because of Clairaut's theorem these are the same, and combining them leads to the coefficient "2" in front of f_{xy} above.

Setting $t = 0$ in (103) and in (104) gives you expressions for $g'(0)$ and $g''(0)$, and by substituting these in (102) we get ***the second order Taylor expansion of a function of two variables:***

$$\begin{aligned} f(a + \Delta x, b + \Delta y) &= f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y \\ &+ \frac{1}{2}\Big\{f_{xx}(a, b)(\Delta x)^2 + 2f_{xy}(a, b)\Delta x\Delta y + f_{yy}(a, b)(\Delta y)^2\Big\} + \cdots \end{aligned} \tag{105}$$

The first three terms are exactly the linear approximation (60) of the function that we

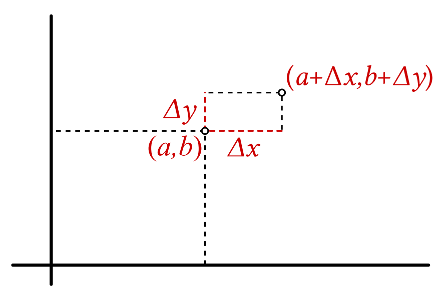


Figure 7. **Δx and Δy:** Taylor's formula lets us approximate a function $z = f(x, y)$ at points $(x, y) = (a + \Delta x, b + \Delta y)$ close to (a, b). The expansion gives us $f(x, y) = f(a + \Delta x, b + \Delta y)$ as a function of Δx and Δy.

saw in Chapter III, § 4.2. The next terms in 105 are

$$\frac{1}{2}f_{xx}(a, b)(\Delta x)^2 + f_{xy}(a, b)\Delta x\Delta y + \frac{1}{2}f_{yy}(a, b)(\Delta y)^2.$$

These terms determine a quadratic form in the variables Δx and Δy. The quantities $\frac{1}{2}f_{xx}(a,b)$, etc. are the coefficients of the form.

As always, the dots in the expansion (105) contain the remainder term. By carefully including the one-variable Lagrange remainder in the derivation we can get a formula for the remainder in (105). We will not do that, but it can be shown that the remainder is $o\big((\Delta x)^2+(\Delta y)^2\big)$, i.e. that it is small compared to the other terms in the expansion, at least when Δx and Δy are small.

9.3. Example – compute the Taylor expansion of $f(x,y)=\sin 2x\cos y$ at the point $(\frac{1}{6}\pi,\frac{1}{6}\pi)$. To find the expansion we need to compute f, f_x, f_y, f_{xx}, f_{xy}, and f_{yy} at $(\frac{1}{6}\pi,\frac{1}{6}\pi)$. Here goes:

$$\begin{aligned} f &= \sin 2x\cos y = \tfrac{3}{4} & f_{xx} &= -4\sin 2x\cos y = -3\\ f_x &= 2\cos 2x\cos y = \tfrac{1}{2}\sqrt{3} & f_{xy} &= -2\cos 2x\sin y = -\tfrac{1}{2}\\ f_y &= -\sin 2x\sin y = -\tfrac{1}{4}\sqrt{3} & f_{yy} &= -\sin 2x\cos y = -\tfrac{3}{4}. \end{aligned}$$

Substituting in the Taylor expansion we get

$$\begin{aligned} &f\big(\tfrac{1}{6}\pi+\Delta x,\tfrac{1}{6}\pi+\Delta y\big)\\ &\qquad= \tfrac{3}{4}+\tfrac{1}{2}\sqrt{3}\,\Delta x-\tfrac{1}{4}\sqrt{3}\,\Delta y+\frac{1}{2}\Big\{-3(\Delta x)^2-2\cdot\tfrac{1}{2}\Delta x\Delta y-\tfrac{3}{4}(\Delta y)^2\Big\}+\cdots\\ &\qquad= \tfrac{3}{4}+\tfrac{1}{2}\sqrt{3}\,\Delta x-\tfrac{1}{4}\sqrt{3}\,\Delta y-\tfrac{3}{2}(\Delta x)^2-\tfrac{1}{2}\Delta x\Delta y-\tfrac{3}{8}(\Delta y)^2+\cdots \end{aligned}$$

Note that the first three terms in the expansion are the linear approximation of the function:

$$f\big(\tfrac{1}{6}\pi+\Delta x,\tfrac{1}{6}\pi+\Delta y\big)=\tfrac{3}{4}+\tfrac{1}{2}\sqrt{3}\,\Delta x-\tfrac{1}{4}\sqrt{3}\,\Delta y+\cdots$$

9.4. Another example – the Taylor expansion of $f(x,y)=x^3+y^3-3xy$ at the point $(1,1)$. The function $f(x,y)=x^3+y^3-3xy$ has the following derivatives at $(1,1)$:

$$\begin{aligned} f &= x^3+y^3-3xy &&=1 & f_{xx} &= 6x &&=6\\ f_x &= 3x^2-3y &&=0 & f_{xy} &= -3 &&=-3\\ f_y &= 3y^2-3x &&=0 & f_{yy} &= 6y &&=6 \end{aligned}$$

The first derivatives vanish, so $(1,1)$ is a critical point of f. The second order Taylor expansion of f at $(1,1)$ is

$$f(1+\Delta x,1+\Delta y)=1+3(\Delta x)^2-3\Delta x\Delta y+3(\Delta y)^2+\cdots \tag{106}$$

Note that there are not first order terms in this expansion because $(1,1)$ is a critical point – the coefficients of the first order terms are both zero.

To see what kind of critical point $(1,1)$ is, we have to analyze the second order, quadratic, terms

$$3(\Delta x)^2-3\Delta x\Delta y+3(\Delta y)^2. \tag{107}$$

This expression is a quadratic form in Δx and Δy, and by completing the square (see Chapter III, § 3) we find that

$$3(\Delta x)^2-3\Delta x\Delta y+3(\Delta y)^2=3\Big[\big(\Delta x-\tfrac{1}{2}\Delta y\big)^2+\tfrac{3}{4}(\Delta y)^2\Big].$$

In particular, the quadratic terms in the Taylor expansion of f at the critical point are always positive, no matter what Δx and Δy we choose (as long as they are not both zero). If we are allowed to ignore the remainder term (the "$\cdots$"), then this implies that the function has a local minimum: after all, the Taylor expansion (106) says that for small Δx and Δy the function value $f(1+\Delta x, 1+\Delta y)$ is

$$f(1+\Delta x, 1+\Delta y) \approx f(1,1) + 3\big(\Delta x - \tfrac{1}{2}\Delta y\big)^2 + \tfrac{9}{4}(\Delta y)^2.$$

The second order terms are all positive, so the Taylor expansion tells us that

$$f(1+\Delta x, 1+\Delta y) \geq f(1,1),$$

at least for small Δx and Δy. The function therefore has a local minimum at $(1,1)$.

9.5. Example of a saddle point. The same function $f(x,y) = x^3 + y^3 - 3xy$ has another critical point, namely, the origin. By calculating the derivatives at $(0,0)$ we find that the Taylor expansion at the origin is

$$f(\Delta x, \Delta y) = -3\Delta x \Delta y + \cdots \tag{108}$$

Ignoring the remainder terms we see that near the origin $f(\Delta x, \Delta y) \approx -3\Delta x\Delta y$, which suggests that f is negative when Δx and Δy are both positive, or when they are both negative, while f is positive when Δx and Δy have opposite signs.

Arbitrarily close to the origin the function f therefore has both positive and negative values, and therefore f has neither a local maximum nor a local minimum at the origin. In fact the Taylor expansion (108) suggests that the graph of f should look like that of the "saddle function" $z = xy$.

9.6. The two-variable second derivative test. The last two examples essentially show us how the second derivative test for functions of two variables works. To explain how it works in general, let's suppose a function f has a critical point at (a,b). Then the first partial derivatives of f vanish at (a,b) and hence the Taylor expansion has no first order terms. We get

$$\begin{aligned} f(a+\Delta x, b+\Delta y) &= f(a,b) + \\ &\frac{1}{2}\Big\{ f_{xx}(a,b)(\Delta x)^2 + 2f_{xy}(a,b)\Delta x\Delta y + f_{yy}(a,b)(\Delta y)^2 \Big\} + \cdots \end{aligned} \tag{109}$$

This is the two-variable analog of equation (100). To see if (a,b) is a local maximum or minimum (or something else), we have to see if the quadratic terms in (109) are always negative, always positive, or if they can have either sign, depending on the choice of Δx, Δy.

The precise statement of the second derivative test uses the terminology introduced in Chapter I, §3 and Figure 5 in that chapter.

Theorem (second derivative test). *If (a,b) is a critical point of $f(x,y)$, and if*

$$Q(\Delta x, \Delta y) = \frac{1}{2}\Big\{ f_{xx}(a,b)(\Delta x)^2 + 2f_{xy}(a,b)\Delta x\Delta y + f_{yy}(a,b)(\Delta y)^2 \Big\}$$

is the quadratic part of the Taylor expansion of f at the critical point, then

- *If Q is **positive definite** then (a,b) is a **local minimum** of f,*
- *If Q is **negative definite** then (a,b) is a **local maximum** of f,*
- *If Q is **indefinite** then (a,b) is a **saddle point** of f*
- *If Q is **semidefinite** the second derivative test is inconclusive.*

When the form Q is indefinite, so that it can be factored as

$$Q(\Delta x, \Delta y) = (k\Delta x + l\Delta y)(m\Delta x + n\Delta y),$$

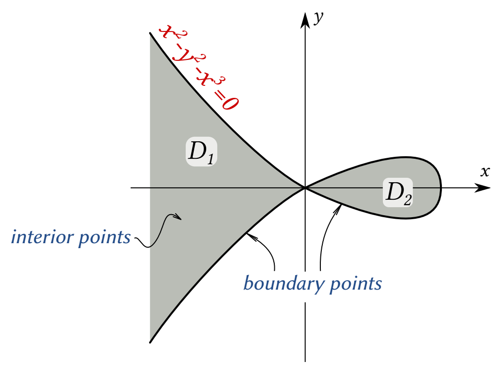
then the level set of the function f containing the critical point (a, b) consists of two curves. One of these curves is tangent to the line

$$k\Delta x + l\Delta y = 0, \text{ i.e. } k(x-a) + l(y-b) = 0$$

while the other is tangent to

$$m\Delta x + n\Delta y = 0, \text{ i.e. } m(x-a) + l(y-b) = 0.$$

9.7. Example – apply the second derivative test to the fishy example. In § 2.3 and § 4.4 we had found that the function $f(x, y) = x^2 - x^3 - y^2$ has two critical points, one at the origin, and one at the point $(\frac{2}{3}, 0)$. By carefully looking at the zero set of the



function we discovered that the origin is neither a local maximum nor a local minimum, and that the point $(\frac{2}{3}, 0)$ is a local maximum. The second derivative test provides a more systematic way of reaching these conclusions. To apply the test we need to know the second derivatives of f at the critical points. They are:

(x, y)	$f_{xx}(x, y)$	$f_{xy}(x, y)$	$f_{yy}(x, y)$
(x, y)	$2 - 6x$	0	-2
$(0, 0)$	2	0	-2
$(\frac{2}{3}, 0)$	-2	0	-2

Therefore the second order Taylor expansion of f at the origin is

$$\begin{aligned} f(\Delta x, \Delta y) &= f(0,0) + \tfrac{1}{2}\left\{2 \cdot (\Delta x)^2 + 2 \cdot 0 \cdot \Delta x \Delta y + (-2)(\Delta y)^2\right\} + \cdots \\ &= (\Delta x)^2 - (\Delta y)^2 + \cdots \\ &= (\Delta x - \Delta y)(\Delta x + \Delta y) + \cdots \end{aligned}$$

The quadratic part of the Taylor expansion can be factored, so this is the "indefinite" case. It can be both positive and negative, depending on our choice of Δx and Δy. The second derivative test implies that the origin is a saddle point. It also says that the zero set of f near the origin consists of two curves, whose tangents at the origin are given by the two equations

$$\Delta x - \Delta y = 0 \text{ and } \Delta x + \Delta y = 0. \tag{110}$$

In this case the point (a, b) is the origin, so $\Delta x = x - a = x$ and $\Delta y = y - b = y$, and the two tangents are the lines $y = \pm x$.

The second order Taylor expansion at the other critical point $(\frac{2}{3}, 0)$ is given by

$$f(\tfrac{2}{3} + \Delta x, \Delta y) = f(\tfrac{2}{3}, 0) - (\Delta x)^2 - (\Delta y)^2 + \cdots \tag{111}$$

This time we see that the second order terms of the Taylor expansion are negative definite. The second derivative test therefore says that we have a local maximum at $(\frac{2}{3}, 0)$.

10. Problems

1. [for discussion] Are Δx in § 9.4 and § 9.5 the same?

Are the Δx in the equations (110) and in (111) of the second derivative test example the same? Explain what they stand for. •

2. Compute the second order Taylor expansion of the following functions at the indicated points:

[In this problem you are asked to find Taylor expansions of functions at various points. Since these points are not necessarily critical points, the expansions you find will generally have first and second oder terms. In the expansions you will compute when you use the second derivative test later on, there will be no first order terms.]

(a) $f(x, y) = \big(1 - x + xy\big)^2$ at $(0, 0)$ •

(b) $f(x, y) = \big(1 - x + xy\big)^2$ at $(1, 1)$ •

(c) $f(x, y) = e^{x - y^2}$ at $(0, 0)$ •

(d) $f(x, y) = e^{x - y^2}$ at $(1, 1)$ •

(e) $f(x, y) = \dfrac{x}{1 - y}$ at $(0, 0)$

(f) $f(x, y) = \dfrac{x}{1 + y}$ at $(1, 0)$

3. Factor, or complete the square in the following quadratic forms, draw their zero sets, and determine if they are positive definite, negative definite, indefinite or degenerate.

(a) $Q(x, y) = x^2 + 3xy + y^2$

(b) $Q(x, y) = x^2 + xy + y^2$

(c) $Q(x, y) = 2x^2 + 3xy - 4y^2$

(d) $Q(x, y) = 2x^2 + 3xy - 5y^2$

(e) $Q(\Delta x, \Delta y) = (\Delta x)^2 + (\Delta y)^2$

(f) $Q(\Delta x, \Delta y) = (\Delta x)^2 - 3(\Delta y)^2$

(g) $Q(\Delta x, \Delta y) = \Delta x \Delta y$

(h) $Q(\Delta x, \Delta y) = \Delta x \Delta y - 2(\Delta y)^2$

4. If a is a constant, then for which values of a is the form $Q(x, y) = x^2 + 2axy + y^2$ positive/negative definite, indefinite, or degenerate? •

5. Find all critical points of the following functions (you did many of these in problem 6.1). Apply the second derivative test to all critical points you find. •

(a) $f(x, y) = x^2 + 4y^2 - 2x + 8y - 1$

(b) $f(x, y) = x^2 - y^2 + 6x - 10y + 2$

(c) $f(x, y) = x^2 + 4xy + y^2 - 6y + 1$

(d) $f(x, y) = x^2 - xy + 2y^2 - 5x + 6y - 9$

(e) $f(x, y) = y^2 - 18x^2 + x^4$

(f) $f(x, y) = y^4 - 4y^2 - 18x^2 + x^4$

(g) $f(x, y) = 9 + 4x - y - 2x^2 - 3y^2$

(h) $f(x, y) = xy(4 - x - 2y)$

(i) $f(x, y) = x(x - y)(x - 1)$

(j) $f(x, y) = (x - y)(xy - 4)$

(k) $f(x, y) = y^2 + \cos x$

(l) $f(x, y) = x^2 y - \frac{1}{3} y^3$

(m) $f(x, y) = (x - y^2)(x - 1)$

(n) $f(x, y) = (x - y)(xy - 4)$

(o) $f(x, y) = x^2$

(p) $f(x, y) = x^2 - y^4$

(q) $f(x, y) = x^2 + y^4$

(r) $f(x, y) = x^2 y$

6. **(a)** Draw the zero set of the function $f(x, y) = \sin(x)\sin(y)$. **(b)** Where is the function f positive? Find as many critical points as you can without computing f_x or f_y.

(c) Find all critical points of $f(x, y)$. Which are local minima or local maxima?

7. Find all critical points of the following functions, and apply the second derivative test to the points you find.

(a) $f(x, y) = x^2 + y^2 - \frac{1}{2}xy^2$

(b) $f(x, y) = x^2 + y^2 - x^2y^2$

(c) $f(x, y) = x + 2y - xy^2$

(d) $f(x, y) = 8x^4 + y^4 - xy^2$

8. Suppose that $f(x, y) = x^2 + y^2 + kxy$. Find and classify the critical points, and discuss how they change when k takes on different values.

9. Consider the function

$$f(x, y) = x^3 - 3xy^2.$$

The graph of this function is known as the "Monkey Saddle."

(a) Show that $(0, 0)$ is the only critical point of f.

(b) Show that the second derivative test is inconclusive for f.

(c) Draw the zero set of f, and indicate where $f > 0$ and where $f < 0$.

(d) What kind of critical point is $(0, 0)$?

10. Consider the function

$$f(x, y) = x^3 - x^2y.$$

(a) Draw the zeroset of f and indicate where $f(x, y)$ is positive, and where $f(x, y)$ is negative.

(b) Find all the critical points of the function.

(c) Does the second derivative test apply to any of the critical points of f?

(d) Use the sign-diagram you made in part **(a)** to decide which critical points are local maxima or minima.

11. Second derivative test for more than two variables

The ideas that lead to the second derivative test for functions of two variables also work when we have a function with more variables. However, the second derivative test for functions of more than two variables is beyond the scope of Math 234, and this short section tries to explain why.

11.1. The second order Taylor expansion.

If $z = f(x_1, x_2, \cdots, x_n)$ is a function of n variables, then its Taylor expansion of order two at some point $(a_1, a_2, \cdots, a_n)$ turns out to be

$$\begin{aligned} f(a_1 + \Delta x_1, \cdots, a_n + \Delta x_n) = \\ f(a_1, \cdots, a_n) + f_{x_1}\Delta x_1 + \cdots + f_{x_n}\Delta x_n + \\ \frac{1}{2}\Big\{ f_{x_1x_1}(\Delta x_1)^2 + \cdots + f_{x_1x_n}\Delta x_1\Delta x_n \\ + f_{x_2x_1}\Delta x_2\Delta x_1 + \cdots + f_{x_2x_n}\Delta x_2\Delta x_n \\ \vdots \\ + f_{x_nx_1}\Delta x_n\Delta x_1 + \cdots + f_{x_nx_n}(\Delta x_n)^2 \Big\} + \cdots \end{aligned}$$

where the partial derivatives f_{x_i} and $f_{x_ix_j}$ are to be evaluated at the point $(a_1, \cdots, a_n)$. The same trick involving the function "$g(t)$" that was used in §9.2 to derive the two-variable Taylor expansion works without modification.

If $(a_1, \cdots, a_n)$ is a critical point then $f_{x_1} = f_{x_2} = \cdots = f_{x_n} = 0$, so the linear terms are absent, and the function is described by the quadratic terms of the Taylor expansion

$$\begin{aligned} f(a_1 + \Delta x_1, \cdots, a_n + \Delta x_n) = \\ f(a_1, \cdots, a_n) + \frac{1}{2}\Big\{ & f_{x_1x_1}(\Delta x_1)^2 + \cdots + f_{x_1x_n}\Delta x_1 \Delta x_n \\ & + f_{x_2x_1}\Delta x_2 \Delta x_1 + \cdots + f_{x_2x_n}\Delta x_2 \Delta x_n \\ & \vdots \\ & + f_{x_nx_1}\Delta x_n \Delta x_1 + \cdots + f_{x_nx_n}(\Delta x_n)^2 \Big\} + \cdots \end{aligned}$$

Just as in the two-variable case we could now try to see if the quadratic terms are positive definite or negative definite by completing squares. The procedure is however much more complicated, and best understood in terms of *"eigenvalues of matrices,"* a subject which is explained in courses on linear algebra or matrix algebra (Math 320, 340, or 341). Therefore, we will only use the second derivative test for functions of two variables in this course.

12. Optimization with constraints and the method of Lagrange multipliers

In many optimization problems we want to find the maximal or minimal value of a function $f(x, y)$ where (x, y) can be any point satisfying a certain ***constraint***

$$g(x, y) = C. \tag{112}$$

Thus the domain D of the function we want to minimize consists of all points (x, y) that satisfy the equation $g(x, y) = C$: it is a level set of g.

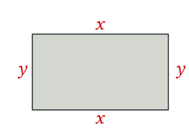
12.1. Solution by elimination or parametrization. One approach to minimization problems with a constraint is to "eliminate one variable." If we are asked to find the minimal value that $f(x, y)$ can have if (x, y) must satisfy the constraint $g(x, y) = C$, then we first try to solve the constraint equation for one of the variables, say, for y:

$$g(x, y) = C \iff y = h(x).$$

Now the only (x, y) that we have to consider are points of the form $(x, h(x))$, so the old minimization problem is equivalent to a new problem: find the minimal value of $F(x) = f(x, h(x))$, where there are no constraints on x. This new problem is a one variable problem of the kind we learned to solve in Math 221.

12.2. Example – which rectangle with perimeter 1 has the largest area? This is another problem, like finding the tangent to the parabola $y = x^2$, that appears in almost every first semester calculus course. We recall its solution.

If the sides of the rectangle are x and y, then its area is xy and its perimeter is $2(x+y)$. Hence the function we want to maximize is $f(x, y) = xy$ and the constraint is



$$g(x, y) = 2(x + y) = 1.$$

Solving the constraint for y tells you that $y = \frac{1}{2} - x$, so we want to maximize the function $F(x) = f(x, \frac{1}{2} - x) = x(\frac{1}{2} - x)$. The only remaining constraint is that x cannot be negative, and that $y = \frac{1}{2} - x$ also cannot be negative. Thus we want to maximize $F(x) = x(\frac{1}{2} - x)$ over all x in the interval $0 \le x \le \frac{1}{2}$.

12.3. Example – maximize $x + 2y$ over the unit circle. We are asked to find the maximal value of $f(x, y) = x + 2y$ where (x, y) is allowed to be any point that satisfies the constraint $g(x, y) = x^2 + y^2 = 1$. If we try to solve for y we find that there are two solutions, $y = \pm\sqrt{1 - x^2}$, and so the "function" $F(x) = x + 2y = x \pm 2\sqrt{1 - x^2}$ is not really a function at all. In this case we can still solve the problem by noting that any point on the unit circle can be written as $(x, y) = (\cos\theta, \sin\theta)$ for some angle θ, and thus we have to maximize the function

$$F(\theta) = f(\cos\theta, \sin\theta) = \cos\theta + 2\sin\theta.$$

Here there are no constraints on θ, and we again have a first semester calculus problem.

12.4. Solution by Lagrange multipliers. In both examples above we were lucky because we could either solve the constraint equation or we could parametrize all possible points that satisfy the constraint. There is a method due to Joseph-Louis Lagrange (known from the remainder term) that does not require this kind of luck. His method is based on the following observation (see Figure 8).

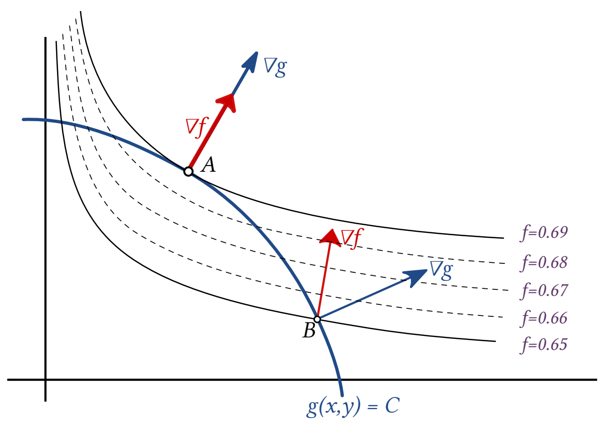


Figure 8. Lagrange multipliers: if, at some point like B on the constraint set the gradients of f and g are not parallel, then we can increase f by moving along the constraint set in the direction of $\vec{\nabla} f$. At a point (such as A) where the function f reaches a maximum, the gradients $\vec{\nabla} f$ and $\vec{\nabla} g$ must be parallel.

Let $B = (x, y)$ be a point on the constraint set as in the figure. Assume that $\vec{\nabla} g \neq \vec{0}$ at B, then near B the Implicit Function Theorem says that the constraint set $g(x, y) = C$ is a curve, and that its tangent is perpendicular to $\vec{\nabla} g(B)$.

If $\vec{\nabla} f(B)$ is not perpendicular to the constraint set at B, then it provides us a direction along the constraint set in which f will increase (see Figure 8). Therefore f does not have a maximum at B. It follows that at a maximum of f on the constraint set $g(x, y) = C$ the gradient $\vec{\nabla} f(B)$ must be perpendicular to the constraint set, and hence it must be parallel to $\vec{\nabla} g(B)$. Since one vector is parallel to another if it is a multiple of the other vector, we have found the following fact.

12.5. Theorem (Lagrange multipliers). *If the function $z = f(x, y)$ attains its largest value among all points that satisfy the constraint $g(x, y) = C$ at the point (a, b), and if*

$$\vec{\boldsymbol{\nabla}} g(a, b) \neq 0, \tag{113}$$

then the point (a, b) satisfies the Lagrange Multiplier equation,

$$\vec{\boldsymbol{\nabla}} f(a, b) = \lambda \vec{\boldsymbol{\nabla}} g(a, b) \tag{114}$$

The number λ is called the *Lagrange multiplier*, and it is one of the unknowns in the equations we must solve when we use Lagrange's method.

12.6. Example. We again try to find the largest rectangle with perimeter 1, as in example 12.2.

The problem is to maximize $f(x, y) = xy$ with constraint $g(x, y) = 2x + 2y = 1$. We compute the gradients

$$\vec{\boldsymbol{\nabla}} f = \begin{pmatrix} y \\ x \end{pmatrix}, \qquad \vec{\boldsymbol{\nabla}} g = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

The gradient of g never vanishes, i.e. $\vec{\boldsymbol{\nabla}} g(x, y) \neq \vec{\mathbf{0}}$ for all (x, y), so Lagrange tells us that at any minimum or maximum the following equations hold:

$$\begin{aligned} f_x &= \lambda g_x, \text{ i.e. } y = 2\lambda \\ f_y &= \lambda g_y, \text{ i.e. } x = 2\lambda \\ g(x, y) &= C, \text{ i.e. } 2x + 2y = 1. \end{aligned}$$

The first two equations come from $\vec{\boldsymbol{\nabla}} f = \lambda \vec{\boldsymbol{\nabla}} g$, and the last equation is the constraint. We have three equations, and we also have three unknowns: x, y and the Lagrange multiplier λ.

In this case it is easy to solve the equations: the first two say that both y and x equal 2λ, so in particular, they equal each other: $y = x$. This already tells us that the solution is a square! To complete the problem we must still solve for x, y, λ. Since $x = y$ the constraint implies $4x = 1$, so $x = y = \frac{1}{4}$. Finally, either of the first two equations provides $\lambda = \frac{1}{2}x = \frac{1}{2}y = \frac{1}{8}$.

What is the meaning of λ? In this example you see that we first found the solution (x, y), and then computed λ. The multiplier λ is the ratio between the lengths of the gradients of f and g at the maximum, and can be interpreted as *the rate at which the maximum f changes if the value of the constraint $g = C$ is varied*; the details go beyond this course (but see Problems 13.15 and 13.16). In any case, the upshot is that, when using Lagrange's method, *you must always also find λ,* or at least make sure that a λ exists for the x and y you have found.

Did we find a maximum or a minimum? Lagrange's method does not tell us if we have a maximum or a minimum, and we will have to use different methods to figure this out. There does exist a second derivative test for constrained minimization problems, but it falls outside the scope of this course.

12.7. A three variable example. *Find the largest value of $x + y + z$ on the sphere with equation $x^2 + y^2 + z^2 = 1$.*

Solution: We must maximize $f(x, y, z) = x + y + z$ with constraint $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

Lagrange's method says that the minimum and maximum either occur at a point (x_0, y_0, z_0) with $\vec{\nabla} g(x_0, y_0, z_0) = \vec{0}$, or else at a point that satisfies Lagrange's equations. The gradient of g is

$$\vec{\nabla} g(x, y, z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix},$$

and the only point where $\vec{\nabla} g = \vec{0}$ is at the origin. The origin does not satisfy the constraint $g(x, y, z) = 1$, so we can rule out the possibility of the maximum or minimum occurring at a point with $\vec{\nabla} g = \vec{0}$.

This leads us to consider the Lagrange multiplier equations, which are

$$\begin{aligned} 1 &= \lambda \cdot 2x && (f_x = \lambda g_x) \\ 1 &= \lambda \cdot 2y && (f_y = \lambda g_y) \\ 1 &= \lambda \cdot 2z && (f_z = \lambda g_z) \\ x^2 + y^2 + z^2 &= 1 && (g(x, y, z) = C) \end{aligned}$$

Solve the first three equations for x, y, z and substitute the result in the constraint, and we find

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1 \implies \frac{3}{4\lambda^2} = 1 \implies \lambda = \pm\tfrac{1}{2}\sqrt{3}.$$

We therefore find two points on the sphere,

$$(x, y, z) = \left(\tfrac{1}{3}\sqrt{3}, \tfrac{1}{3}\sqrt{3}, \tfrac{1}{3}\sqrt{3}\right) \text{ and } (x, y, z) = \left(-\tfrac{1}{3}\sqrt{3}, -\tfrac{1}{3}\sqrt{3}, -\tfrac{1}{3}\sqrt{3}\right)$$

By computing the function values we find that the first point maximizes $x + y + z$, and the second minimizes $x + y + z$.

13. Problems

1. Minimize xy subject to the constraint

$$x^2 + \tfrac{1}{4}y^2 = 1.$$

Draw the constraint set.

2. A six-sided rectangular box is to hold $1/2$ cubic meter. Which shape should the box be to minimize surface area?

(a) Find the solution without using Lagrange's method.

(b) Use Lagrange multipliers to solve this problem.

3. Using the methods of this section, find the shortest distance from the origin to the plane $x+y+z = 10$. (suggestion: instead of minimizing the distance, you can also minimize the square of the distance.)

4. Use Lagrange multipliers to find the largest and smallest values of $f(x, y) = x$ under the constraint $g(x, y) = y^2 - x^3 + x^4 = 0$.

5. (a) Using Lagrange multipliers, find the shortest distance from the point $(2, 1, 4)$ to the plane $2x - y + 3z = 1$.

(b) Using Lagrange multipliers, find the shortest distance from the point (x_0, y_0, z_0) to the plane $ax + by + cz = d$.

6. (a) Find the shortest distance from the point $(0, b)$ to the parabola $y = x^2$, using Lagrange multipliers.

(b) Find the shortest distance from the point $(0, 0, b)$ to the paraboloid $z = x^2 + y^2$.

(c) Find the shortest distance from the point $(0, 0, b)$ to the paraboloid $z = x^2 + \frac{1}{4}y^2$.

7. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid

$$2x^2 + 72y^2 + 18z^2 = 288.$$

8. A six-sided rectangular box is to hold $1/2$ cubic meter; what shape should the box be to minimize surface area?

9. A circular cone has height H, and its base has radius R. If the volume of the cone is fixed, then which ratio of radius to height ($R : H$) minimizes the surface area of the cone? (The area of the cone is $A = \pi R\sqrt{R^2 + H^2}$, its volume is $V = \frac{1}{3}\pi R^2 H$, and instead of minmizing the area you could also minimize the square of the area.)

10. The post office will accept packages whose combined length and girth are at most 130 inches (girth is the maximum distance around the package perpendicular to the length). What is the largest volume that can be sent in a rectangular box? •

11. The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost. •

12. Find all points on the surface

$$xy - z^2 + 1 = 0$$

that are closest to the origin. •

13. The material for the bottom of an aquarium costs half as much as the high strength glass for the four sides. Find the shape of the cheapest aquarium that hold a given volume V. •

14. The plane $x - y + z = 2$ intersects the cylinder $x^2 + y^2 = 4$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin. (Hint: on the plane you always have $z = 2 - x + y$, so you can eliminate z and make this a problem about functions of (x, y) only.) •

15. (Interpretation of the Lagrange multiplier–general case.) Suppose that for all values of the constraint parameter C we have a solution $\big(x(C), y(C)\big)$ to the Lagrange multplier equations $\vec{\nabla} f(x, y) = \lambda\vec{\nabla} g(x, y)$, $g(x, y) = C$.

Show that the derivative of $f(x(C), y(C))$ with respect to C is exactly λ.

16. (Interpretation of the Lagrange multiplier–example.)

(a) Use Lagrange multpliers to find the rectangle with sides x and y and enclosed area A whose perimeter is as small as possible. Find the x and y coordinates of the solution, the Lagrange multiplier λ, as well as the smallest perimeter L, and write all of them as functions of the prescribed area A.

(b) Compute the derivative

$$\frac{dL}{dA}.$$

Describe in words what this derivative represents ("the rate of change of …"), and verify that in this example $\frac{dL}{dA} = \lambda$.

CHAPTER 6

Integrals

1. Ways of Integrating

In this chapter we will see several different ways of integrating functions of several variables. Before introducing them one by one, we spend this section reviewing how integration was defined in first semester calculus and outlining the general features that all different ways of integrating have in common.

1.1. The one variable integral. To begin, let us quickly recall how the integral of a function of one variable is defined. Given a function $y = f(x)$ and an interval $[a, b]$, we choose a ***partition*** of the interval $[a, b]$, which means that

- we split the interval $[a, b]$ into shorter intervals $[x_0, x_1], [x_1, x_2], \dots, [x_{N-1}, x_N]$, where $a = x_0 < x_1 < \cdots < x_N = b$,
- and we choose one ***sample point*** ξ_k from each interval $[x_{k-1}, x_k]$.

From these ingredients we compute the ***Riemann sum***

$$R = f(\xi_1)\Delta x_1 + \cdots + f(\xi_N)\Delta x_N = \sum_{k=1}^{N} f(\xi_k)\Delta x_k$$

where $\Delta x_k = x_k - x_{k-1}$ is the length of the k^{th} interval.

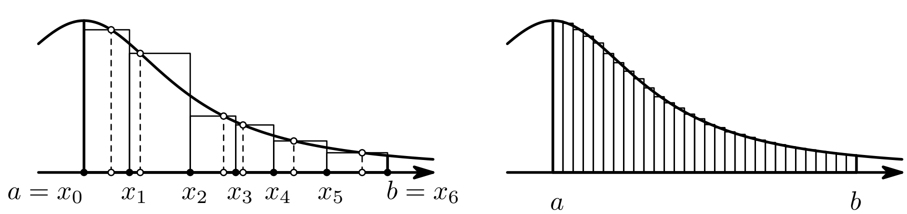


Figure 1. Riemann sums for $\int_a^b f(x)dx$ with one partition on the left, and a finer partition on the right. The dashed lines in the figure on the left indicate where the sample points ξ_k were chosen.

For most functions $y = f(x)$ it is true that upon making the intervals $[x_{k-1}, x_k]$ shorter (and hence choosing more partition intervals), the resulting Riemann sums approach a limiting value. When this happens we call the limiting value of the Riemann sums *the integral of the function* $f(x)$ *over the interval* $[a, b]$:

$$\int_a^b f(x)dx = \lim_{\substack{\text{"as the partition}\\ \text{gets finer"}}} f(\xi_1)\Delta x_1 + \cdots + f(\xi_N)\Delta x_N.$$

The individual terms in the Riemann sum are areas of the narrow rectangles in the figure. Added together they approximate the area of the region under the graph, so that the integral is the area between the graph of $y = f(x)$ and the x-axis (at least in the case that f is a positive function, so that its graph lies above the x-axis.)

A note about rigor. Our quick description of the single variable integral is lacking in mathematical precision. It is based on a belief that we know what "area" is. In the late 19^{th} and early 20^{th} centuries many examples of geometric figures were found in which area computations give unexpected and counterintuitive results. Therefore one cannot base a theory on our intuitive idea of "area," and instead the integral, defined as limit of Riemann sums is used a way of giving a rigorous definition of the notion of "area." For a proper treatment of these issues the student is referred to a more advanced course on Real Analysis (e.g. Math 421 or 521).

1.2. Generalizing the one variable integral. While there is essentially only one kind of integral in single variable calculus, there are many different ways of integrating functions of several variables. All these different notions of "integral" fit the following broad description.

In any kind of integral we have these ingredients:

- a ***domain***. Depending on the kind of integral, this can be a region in the plane, a region in space, a plane curve, a space curve, or even some surface in three dimensional space.
- a ***function*** that is defined on the domain
- a way of measuring the "***size***" of pieces of the domain

To define the integral we "partition" the region, i.e. we divide it into lots of little pieces. Given any such partition of the region into smaller pieces, we then form the following "Riemann sum"

$$\sum_{\substack{\text{pieces in the}\\ \text{partition}}} \begin{pmatrix} f \text{ at sample point} \\ \text{in piece \#k} \end{pmatrix} \times \{\text{Size of piece \#k}\}$$

This gives us a number for each way of partitioning the region. As we make the partition finer, i.e. as we choose more, smaller, pieces, the Riemann sums tend to get closer to one particular number, which is called the integral of the function. In short, the integral is the limit of the Riemann sums we find as we take finer and finer partitions:

$$\int_{\text{some region}} f(x)\,dx = \lim_{\substack{\text{as the}\\ \text{partition}\\ \text{gets finer}}} \sum_{\substack{\text{pieces in the}\\ \text{partition}}} \begin{pmatrix} f \text{ at sample point} \\ \text{in piece \#k} \end{pmatrix} \times \{\text{Size of piece \#k}\}$$

Depending on what kind of function we have, and what kind of region the function is defined on, and also how we decide to measure the size of the small pieces in the partition, this process can lead to many different kinds of integrals. The integrals we will meet in this chapter are ***double integrals*** and ***triple integrals***; in the next chapter on vector calculus we will also see ***line integrals*** and ***surface integrals***. See Table 1.

2. Double Integrals

Let $z = f(x, y)$ be a function of two variables defined on some region D in the plane. The ***double integral of*** f ***over*** D is defined in terms of Riemann sums, following the general scheme described in the previous section. To form a Riemann sum we first need a

Kind of integral	Domain	Typical piece of partition	Size of piece
"Good old 221 Integral" $\int_a^b f(x)\,dx$	interval $a \le x \le b$	small subinterval (x_{k-1}, x_k)	length of subinterval $\Delta x_k = x_k - x_{k-1}$
Multiple integral $\iint_D f(x,y)dA$	region in the plane	tiny sub domain	area ΔA of tiny sub domain
Multiple integral $\iiint_D f(x,y,z)dV$	region in space	tiny sub domain	volume ΔV of tiny sub domain
Line integral $\int_{\mathcal{C}} f(x,y)\,ds$	curve in the plane	short sub arc of the curve	length Δs of the sub arc
Line integral $\int_{\mathcal{C}} f(x,y,z)\,ds$	curve in space	short sub arc of curve	length Δs of the sub arc
Surface integral $\iint_{\mathcal{S}} f(x,y,z)\,dA$	surface in space	small patch on the surface	area ΔA of the patch

Table 1. A list of the different kinds of integrals that we will encounter in math 234.

partition of the region D into smaller regions $D_1, \ldots, D_N$, and we need to choose a sample point (x_k, y_k) from each region D_k. If ΔA_k is the area of region D_k, then the Riemann sum corresponding to the partition $D_1, \cdots, D_N$ and the choice of sample points (x_1, y_1), $\ldots$, (x_N, y_N) is

$$R = f(x_1, y_1)\Delta A_1 + \cdots + f(x_N, y_N)\Delta A_N = \sum_{k=1}^{N} f(x_k, y_k)\,\Delta A_k. \tag{115}$$

If the partition is "sufficiently fine" then this Riemann sum will in many cases be close to one particular number, which we will call the integral of the function f over the region

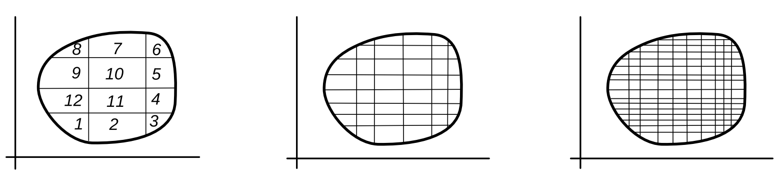


Figure 2. On the left: a region in the plane with some partition. Many pieces of the partition are rectangles. This is a common choice, but the pieces don't have to be rectangles: here the pieces that touch the boundary of the domain have at least one curved edge. **On the right:** the same region with two finer partitions.

D. Thus

$$\iint_D f(x,y)\, dA = \lim_{\substack{\text{as the partition}\\ \text{“gets finer\&finer”}}} \sum_{k=1}^{N} f(x_k, y_k)\, \Delta A_k. \tag{116}$$

To make this more precise one has to resort to ε's and δ's, which results in the following definition.

2.1. Definition. *If for every $\varepsilon > 0$ there is a $\delta > 0$ such that the Riemann sum corresponding to any partition of the region D into smaller pieces $D_1, \dots, D_N$, whose pieces have diameter no more than δ satisfies*

$$\left| I - \sum_{k=1}^{N} f(x_k, y_k)\, \Delta A_k \right| < \varepsilon$$

then we say that

$$\iint_D f(x,y)\, dA = I.$$

On one hand it can be shown in many cases that that the integral of a function exists according to the above definition. On the other hand the ε-δ definition is neither a practical method of computing such integrals, nor does it provide an easy intuitive understanding of the properties of the integral. Therefore, we will stick to the less precise definition (116) in this course.

2.2. The integral is the volume under the graph, when $f \geq 0$. If the function f is positive, then its graph lies above the xy-plane, and there is a simple interpretation of the integral, namely

$$\iint_D f(x,y)\, dA = \textit{Volume of}\ \mathcal{R},$$

where $\mathcal{R}$ is “the region under the graph of f above the domain D” – in symbols,

$$\mathcal{R} = \big\{(x,y,z) : (x,y) \text{ lies in } D, \text{ and } 0 \leq z \leq f(x,y)\big\}. \tag{117}$$

To see why this is so, imagine that we have a positive function $z = f(x,y)$ defined on some region D in the xy-plane, and let us try to compute the integral $\iint_D f(x,y)dA$ “geometrically.” To compute the integral we begin by finely partitioning the region D into smaller regions $D_1, D_2, \dots, D_N$ (see Figure 3 on the left where the small pieces were themselves chosen to be rectangles). We also choose one “sample point” (x_k, y_k) in each region D_k. The Riemann sum we get this way is

$$R = f(x_1, y_1)\Delta A_1 + \cdots + f(x_N, y_N)\Delta A_N$$

where ΔA_k is the area of D_k. The k^{th} term, $f(x_k, y_k)\Delta A_k$, is the volume of a block whose base is D_k and whose top is some point on the graph of the function above the region D_k. This volume is almost, but usually not exactly the same as the volume of the region between the graph of the function and the small region D_k in the xy-plane. The volume $f(x_k, y_k)\Delta A_k$ of the block above D_k is not exactly the same as the volume of the region under the graph because the top of the block is a piece of a horizontal plane while the graph of f will usually have a slope (see Figure 3).

The total Riemann sum is therefore the sum of the volumes of such blocks, (see Figure 4) and this will approximate the volume between the graph of f and the domain of

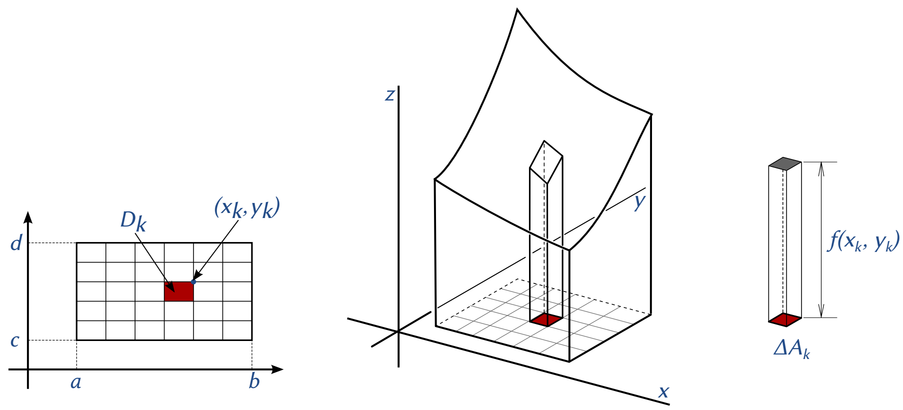


Figure 3. On the left: the domain of the function f partitioned into 6×5 pieces, each with the same width Δx and height Δy. To form a Riemann sum we have to choose one sample point (x_k, y_k) in each piece D_k of the partition. Below we will always choose the upper-right-hand corner of the rectangle to be the sample point. **On the right:** Any piece in the partition corresponds to a term in the Riemann sum of the form $f(x_k, y_k)\Delta A_k$. This is the volume of a block of height $f(x_k, y_k)$, and base D_k, which is approximately the volume of the region under the graph of f and above the piece D_k. Adding all these volumes together we see that a Riemann sum approximates the total volume between the graph and the region D.

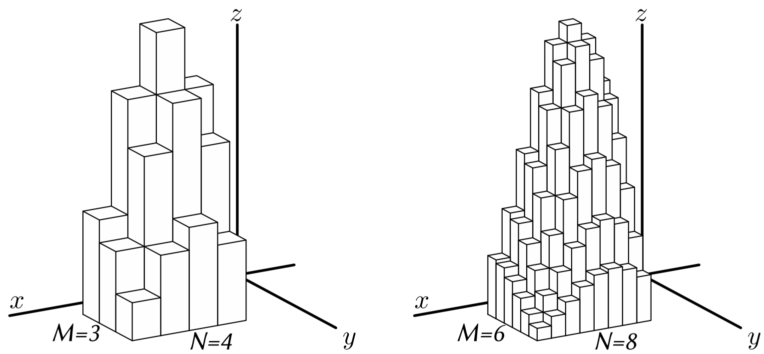


Figure 4. Approximating the region under the graph of $z = f(x, y)$ from Figure 3 by vertical blocks. The base of each block is a rectangle in a partition of the domain of f. As we choose finer and finer partitions, the region occupied by the vertical blocks gets closer to the region under the graph of f.

integration D. The finer the partition, the better the approximation and so we can conclude[1] that the limit of the Riemann sums is the volume under the graph, i.e. the volume of the region $\mathcal{R}$ defined in (117).

[1]As promised before, this is not a very precise "proof," a proof that the limit of Riemann sums exists quickly lead us to $\varepsilon\&\delta$ arguments.

2.3. How to compute a double integral. So far, we have a definition for the double integral $\iint_D f(x,y)dA$, and an interpretation of the integral as "volume under the graph of f." What is missing is a method of actually computing the integral. In this section we'll see how one can compute a double integral by doing two one-variable integrals.

Let us take another look at the integral of the function f over the rectangle

$$D = \{(x,y) : a \le x \le b,\ c \le y \le d\},$$

from the previous section.

We again partition D into smaller rectangles, as in Figure 3, but instead of just counting them and arbitrarily numbering the pieces $1, 2, \dots, N$, we can use the fact that the smaller rectangles appear in rows and columns. If we take N rectangles in the x direction, and M in the y direction, then the smaller rectangles will measure Δx by Δy, where

$$\Delta x = \frac{b-a}{N}, \qquad \Delta y = \frac{d-c}{M}.$$

We let (x_k, y_l) be the upper-right-hand corner of the rectangle in the k^{th} column from the left, and the l^{th} row from below. Then

$$x_k = a + k\Delta x, \qquad y_l = c + l\Delta y. \tag{118}$$

The Riemann sum corresponding to this partition and choice of sample points (x_k, y_l) is

$$\begin{aligned} R &= \sum f(x_k, y_l)\Delta x\Delta y \\ &= f(x_1,y_1)\Delta x\Delta y + \cdots + f(x_N,y_1)\Delta x\Delta y + \\ &\quad f(x_1,y_2)\Delta x\Delta y + \cdots + f(x_N,y_2)\Delta x\Delta y + \\ &\quad\quad \vdots \\ &\quad f(x_1,y_M)\Delta x\Delta y + \cdots + f(x_N,y_M)\Delta x\Delta y \end{aligned} \tag{119}$$

Since we are choosing the upper-right-hand corner of each rectangle as sample point in that rectangle, the sample point for the rectangle at the top-right is (x_N, y_M). (See Figure 3 on the left.) Therefore, in this summation k can have any value with $1 \le k \le N$ and l can be any integer with $1 \le l \le M$.

The term corresponding to rectangle (k, l) represents the volume of a block whose height is $f(x_k, y_l)$ and whose base is a $\Delta x \times \Delta y$ rectangle. Together these blocks approximate the region between the graph of the function and the xy-plane.

Consider the terms on the k^{th} row in equation (119); after factoring out Δy we get

$$\text{row } \#k \text{ of } (119) = \Delta y \left\{ f(x_1,y_k)\Delta x + f(x_2,y_k)\Delta x + \cdots + f(x_N,y_k)\Delta x \right\}.$$

Note that in this sum the function is always evaluated at the same value of y, namely y_k. The sum between braces $\{\cdots\}$ is actually a Riemann sum for the one-variable integral

$$I = \int_a^b f(x, y_k)dx$$

in which we treat $f(x, y_k)$ as a function of x only and consider the variable y to be frozen at $y = y_k$. The value of this integral depends on the value at which y is frozen, so it is better to write

$$I(y) = \int_a^b f(x,y)dx.$$

With this notation we find that

$$\text{row } \#k \text{ of } (119) \approx \Delta y \times \{I(y_k)\} = I(y_k)\Delta y.$$

To find the value of the Riemann sum that approximates the double integral $\iint_D f(x,y)dA$ we add the rows in (119) and find

$$R \approx I(y_1)\Delta y + I(y_2)\Delta y + \cdots + I(y_M)\Delta y.$$

The sum on the right is again a Riemann sum for a one variable integral, namely, $\int_c^d I(y)dy$. Therefore we find that

$$R \approx \int_c^d I(y)dy$$

If we now take the limit in which we let the size of the pieces in the partition go to zero, then it can be shown (with quite a bit of effort) that the approximation above gets better, and that one has

$$\iint\limits_D f(x,y)dA = \int_c^d I(y)dy.$$

Therefore, remembering the definition of $I(y)$, we have found the following method of computing a double integral.

2.4. Theorem. *If $f(x,y)$ is a function defined on a rectangle*

$$D = \{(x,y) : a \le x \le b, c \le y \le d\},$$

then the double integral of f over D is given by

$$\iint\limits_D f(x,y)dA = \int_c^d \Big\{\int_a^b f(x,y)dx\Big\}dy. \tag{120}$$

One can also first integrate with respect to y and then x, so that

$$\iint\limits_D f(x,y)dA = \int_a^b \Big\{\int_c^d f(x,y)dy\Big\}dx. \tag{121}$$

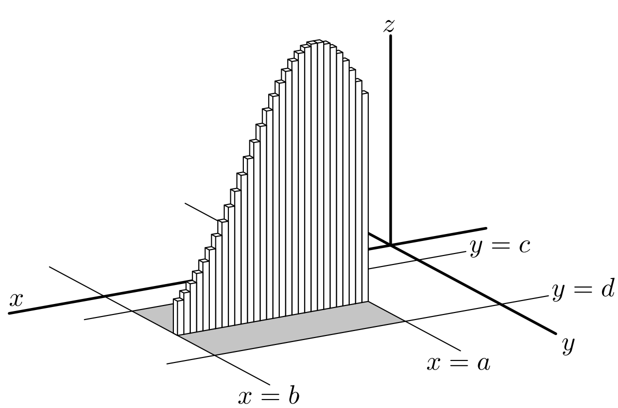


Figure 5. This picture shows the blocks corresponding to all those terms in the Riemann sum R from equation (119) in which $y = y_k$. These terms $\{f(x_1,y_k)\Delta x + \cdots + f(x_N,y_k)\Delta x\}\Delta y$ give you the total volume of one row of "matchsticks" from Figure 4. In this sum y is frozen at the value $y = y_k$, so we can think of $f(x_1,y_k)\Delta x + \cdots + f(x_N,y_k)\Delta x$ as a Riemann sum for the integral $\int_a^b f(x,y_k)\,dx$.

The second way of computing the double integral $\iint_D f(x,y)\ dA$, i.e. equation (121), follows by the same reasoning that led us to (120), except in (119) one groups the terms by columns rather than rows.

To compute the right hand side in this equation we have to compute two one-variable integrals. The expression

$$\int_c^d \Big\{\int_a^b f(x,y)dx\Big\}dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

is called an ***iterated integral***.

The two integrals that appear in an iterated integral are often called “inner” and “outer” integral:

$$\underbrace{\int_c^d \Big\{\underbrace{\int_a^b f(x,y)dx}_{\text{inner integral}}\Big\}\ dy}_{\text{outer integral}}\,.$$

2.5. Example: the volume under the graph of the paraboloid $z = x^2 + y^2$ above the square $Q = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. The double integral we have to compute is

$$\text{Volume} = \iint_Q (x^2+y^2)dA$$

and to compute it we write it as an iterated integral

$$\iint_Q (x^2+y^2)dA = \int_0^1 \Big\{\int_0^1 (x^2+y^2)dx\Big\}dy.$$

In the inner integral the variable y is frozen, so to compute the inner integral, we simply treat y as a constant, and integrate with respect to x. We get

$$\int_0^1 (x^2+y^2)dx = \big[\tfrac13 x^3 + y^2x\big]_{x=0}^1 = \tfrac13 + y^2.$$

(This is $I(y)$ in the notation of the previous section.)

To get the double integral we must still do the outer integral:

$$\begin{aligned}\iint_Q (x^2+y^2)dA &= \int_0^1 \Big\{\int_0^1 (x^2+y^2)dx\Big\}dy\\ &= \int_0^1 \big(\tfrac13 + y^2\big)dy\\ &= \big[\tfrac13 y + \tfrac13 y^3\big]_0^1\\ &= \tfrac13 + \tfrac13 = \tfrac23.\end{aligned}$$

Since the surrounding block (Figure 6) is a $1\times1\times2$ block, its volume is 2, and the region under the graph occupies exactly one third of the whole block.

To compute the volume of the region under the graph of the same function above the rectangle $\{(x,y) : 0 \le x \le a, 0 \le y \le b\}$ one can compute either of the iterated integrals

$$\int_0^a \int_0^b (x^2+y^2)\ dy\ dx \text{ or } \int_0^b \int_0^a (x^2+y^2)\ dx\ dy.$$

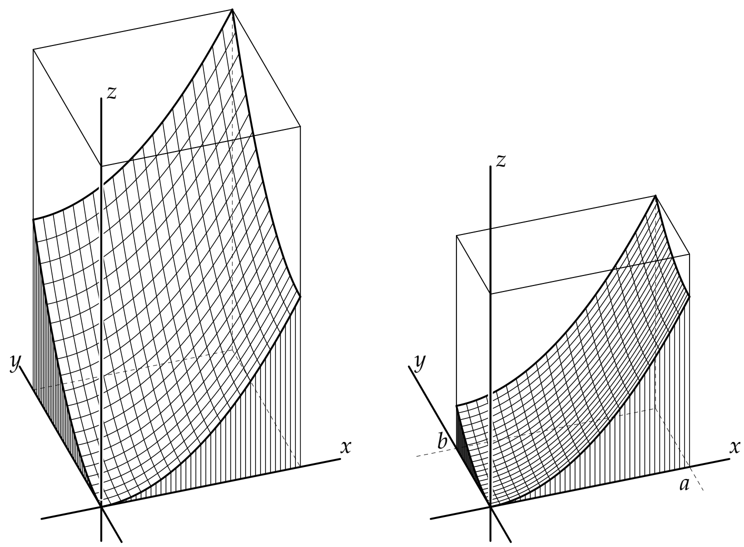


Figure 6. The graph of $z = x^2 + y^2$ above the unit square Q on the left, and rectangle $\{(x, y) : 0 \le x \le a \text{ and } 0 \le y \le b\}$, on the right, together with the surrounding block. *What fraction of the volume of the block lies below the graph?*

2.6. Double integrals when the domain is not a rectangle. We have seen how to compute a double integral when the domain is a rectangle. The reasoning that led us from a double integral to an iterated integral also works for non rectangular domains, provided they are not too complicated. Suppose we want to compute $\iint_D f(x, y)dA$ where the domain D is the region caught between the graphs of two functions:

$$D = \bigl\{(x, y) : a \le x \le b,\ f(x) \le y \le g(x)\bigr\}.$$

We again partition the region by cutting it along many vertical lines $x = x_1$, $x = x_2$, ..., $x = x_N$, and many horizontal lines $y = y_1$, ..., $y = y_M$. Most of the pieces of the partition will be rectangles, but those that overlap with the boundary of the region D may have curved edges. See Figures 7 and 8.

This time, all the terms in a Riemann sum corresponding to one particular strip $x_{k-1} \le x \le x_k$ add up to a Riemann sum for an integral over the y variable,

$$\int_{c(x)}^{d(x)} f(x_k, y)\, dy \ \times\ \Delta x,$$

and adding all these we get the iterated integral

$$\iint_D f(x, y)\, dA = \int_a^b \left\{ \int_{c(x)}^{d(x)} f(x, y)\, dy \right\} dx. \tag{122}$$

2.7. An example–the parabolic office building. Consider the region under the graph of $f(x, y) = x + y$, above the domain

$$D = \bigl\{(x, y) : 0 \le x \le 1, (1 - x)^2 \le y \le 1\bigr\}.$$

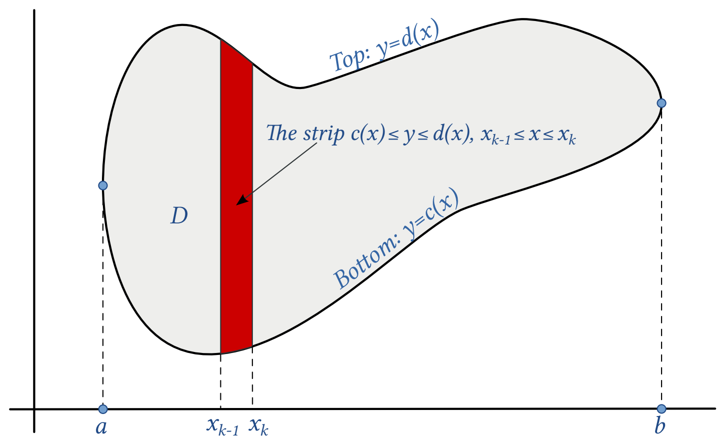


Figure 7. The region between the graphs of $y = f(x)$ and $y = g(x)$.

The volume of this region is given by

$$V = \iint_D (x+y)dA.$$

We can compute this volume by finding the following iterated integral

$$V = \int_{x=0}^{1} \int_{(1-x)^2}^{1} (x+y)\; dy\; dx. \tag{123}$$

Alternatively, the region D can also be described as

$$D = \left\{(x, y) : 0 \le y \le 1,\ 1-\sqrt{y} \le x \le 1\right\}.$$

This leads to the following iterated integral for the volume

$$V = \int_{y=0}^{1} \int_{1-\sqrt{y}}^{1} (x+y)\; dx\; dy. \tag{124}$$

Both iterated integrals should give the same answer. Let's compute the first one:

$$\begin{aligned}
V &= \int_0^1 \int_{(1-x)^2}^{1} (x+y)\; dy\; dx \\
&= \int_0^1 \left[\tfrac{1}{2}xy + \tfrac{1}{2}y^2\right]_{(1-x)^2}^{1}\; dx \\
&= \int_0^1 \left[x\big(1-(1-x)^2\big) + \tfrac{1}{2}\big(1^2-(1-x)^4\big)\right]\; dx \\
&= \int_0^1 \left[2x^2 - x^3 + \tfrac{1}{2}\big(4x - 6x^2 + 4x^3 - x^4\big)\right]\; dx \\
&= \int_0^1 \left[2x^2 - x^3 + 2x - 3x^2 + 2x^3 - \tfrac{1}{2}x^4\right]\; dx \\
&= \tfrac{2}{3} - \tfrac{1}{4} + 1 - 1 + 2 \times \tfrac{1}{4} - \tfrac{1}{2} \times \tfrac{1}{5} \\
&= \tfrac{16}{15}.
\end{aligned}$$

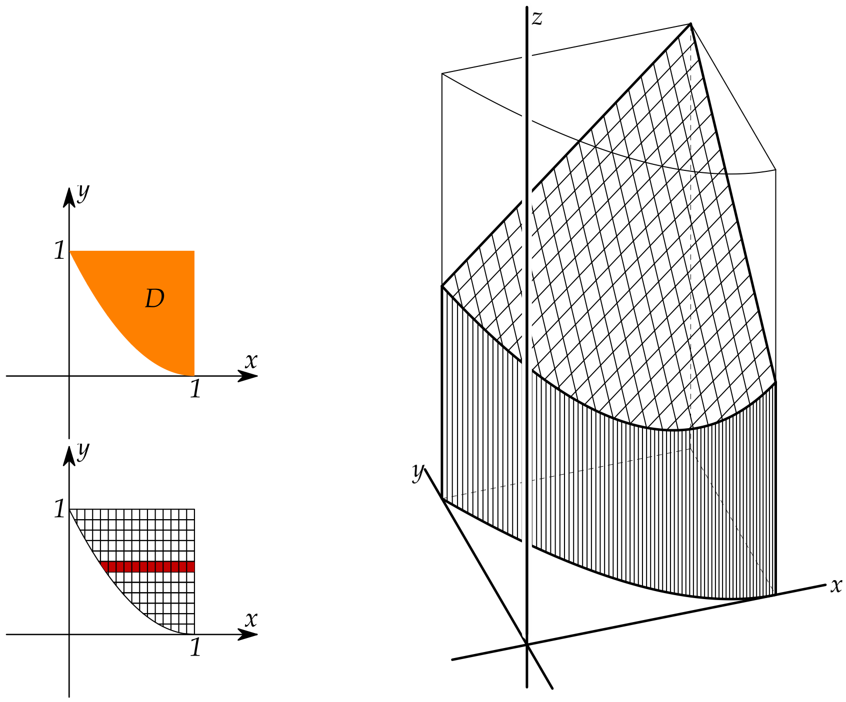


Figure 8. On the left: the domain of integration, a partition, and all pieces in the partition corresponding to one value of y. **On the right:** The "parabolic office building," being the region whose volume is computed in example 2.7

Note that even though the function we integrated is very simple (it's just $x + y$) the integral can still become complicated because of the shape of the domain D over which we are integrating.

2.8. Double integrals in Polar Coordinates. Sometimes Cartesian coordinates are just not the best choice. For instance, a disc or radius R, centered at the origin, is very easy to describe in polar coordinates as "all points with $r \le R$." In Cartesian coordinates we need Pythagoras, and we have to say "all points with $x^2 + y^2 \le R^2$." In the same spirit a "polar rectangle" is a domain of the form

$$R = \{\text{all points with } \theta_0 \le \theta \le \theta_1, r_0 \le r \le r_1\}.$$

See Figure 9 (on the left). There is a very natural way of partitioning such a region into many smaller regions, by cutting the region along curves of constant r (arcs centered at the origin) or constant θ (rays emanating from the origin). If the partition is sufficiently fine, then the pieces in the partition will almost be real Cartesian rectangles, with sides $r\Delta\theta$ and Δr ($\Delta\theta$ being the angle between adjacent rays, and Δr being the difference in radius between two consecutive arcs). The area of such a small partition piece is therefore $\Delta A \approx r\Delta\theta \times \Delta r$, and one arrives at the following formula for the integral of a function of a polar rectangle

$$\iint_R f(x, y)\, dA = \int_{r_0}^{r_1} \int_{\theta_0}^{\theta_1} F(r, \theta)\, r d\theta\, dr = \int_{\theta_0}^{\theta_1} \int_{r_0}^{r_1} F(r, \theta)\, r dr\, d\theta. \tag{125}$$

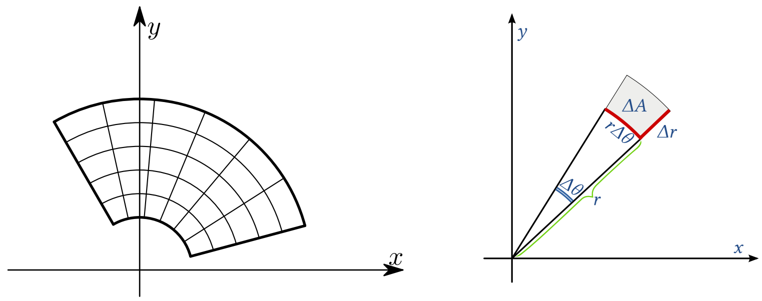


Figure 9. **Left:** A "polar rectangle" and a partition by lines of constant θ (the spokes) and curves of constant r (the arcs). **Right:** The area of a small piece of such a partition is approximately $\Delta A \approx \Delta r \times r\Delta\theta$.

Here $F(r,\theta) = f(r\cos\theta, r\sin\theta)$ is the function $f(x,y)$ written in polar coordinates.[2]

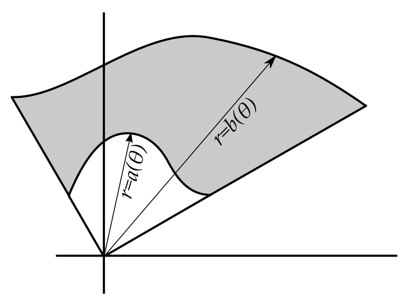


Figure 10. The gray region is the region between the polar graphs $r = a(\theta)$ and $r = b(\theta)$.

There is a similar formula for more complicated domains. If a domain can be described in polar coordinates by

$$D = \{\text{all points with } \alpha \le \theta \le \beta, a(x) \le r \le b(x)\}$$

and if we want to integrate a function $z = f(x,y)$ of this domain, then we can again partition the domain D into many small pieces that are bounded by circular arcs centered at the origin, and rays emanating from the origin. The area of a small piece in the partition is once again given by $\Delta A \approx \Delta r \times r\Delta\theta$, and therefore the integral of f over D is

$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{a(\theta)}^{b(\theta)} F(r,\theta)\, r\, dr\, d\theta. \tag{126}$$

[2]It is very common to use the same letter f for both functions, i.e. to write $f(x,y)$ for f as a function of Cartesian coordinates, and also $f(r,\theta)$ for the same function but written in Polar coordinates. This begs the question of what $f(0.3, 1.24)$ means – are $(0.3, 1.24)$ the polar or the Cartesian coordinates of the point at which f is to be evaluated? To avoid this kind of ambiguity we will try to use different letters for the same quantity regarded as a function of Cartesian coordinates, and of Polar coordinates.

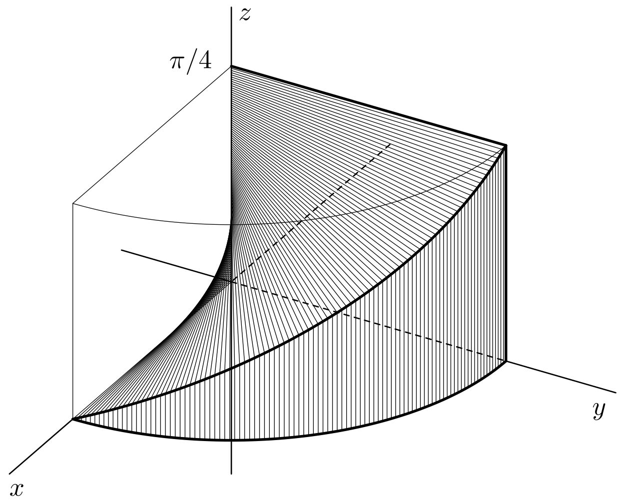


Figure 11. The graph of the function $z = a\theta$ in polar coordinates is called the *helicoid*. Here we see one quarter turn of a helicoid with $a = \frac{1}{2}$. The volume under the helicoid is given by a double integral which is best computed using polar coordinates. *Which fraction of the volume in the surrounding quarter cylinder lies beneath the helicoid?*

2.9. Example: the volume under a quarter turn of a helicoid. A ***helicoid*** is the surface that in polar coordinates is given by

$$z = a\theta$$

where $a > 0$ is some constant. (See Chapter III, § 4.2)

If we choose the constant $a = \frac{1}{2}$, and take the first quarter turn of this surface, on which $0 \le \theta \le \frac{1}{2}\pi$, then we get the picture in Figure 11. In that drawing we have only included the part with $0 \le r \le 1$. To compute the volume of the region under the quarter helicoid using Cartesian coordinates, we would have to compute this integral

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} \tfrac{1}{2} \arctan \frac{y}{x}\, dy\, dx.$$

(Try to set up this integral yourself!)

In Polar coordinates things are easier. The domain is a polar rectangle,

$$0 \le r \le 1, \quad 0 \le \theta \le \tfrac{1}{2}\pi,$$

and the function is very simple,

$$F(r, \theta) = \tfrac{1}{2}\theta.$$

The double integral that represents the volume is therefore

$$V = \iint_D \tfrac{1}{2}\theta\, dA = \int_0^1 \int_0^{\pi/2} \tfrac{1}{2}\theta r\, d\theta\, dr = \frac{\pi^2}{32}.$$

3. Problems

1. Compute these iterated integrals:

(a) $\displaystyle\int_0^1 \int_0^4 x \; dy \; dx$

(b) $\displaystyle\int_0^1 \int_0^4 x \; dx \; dy$

(c) $\displaystyle\int_{-1}^1 \int_0^{x^2} \; dy \; dx$

(d) $\displaystyle\int_0^\pi \int_0^y \frac{\sin y}{y} \; dx \; dy$

(e) $\displaystyle\int_0^\pi \int_0^\theta \frac{\sin \theta}{\theta} \; dr \; d\theta$

(f) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \; dy \; dx$

2. What is wrong with the iterated integral

$$\int_x^1 \Big\{ \int_0^1 \sin(\pi x) dx \Big\} dy \quad ?$$

Is the answer a number – does it depend on x or y?

3. **(a)** Is the following true or false? *For any two functions $f(x)$ and $g(y)$ one has*

$$\int_0^1 \int_0^2 f(x)g(y) \; dx \; dy = \left(\int_0^1 f(x) \; dx \right) \cdot \left(\int_0^2 g(y) \; dy \right) .$$

Explain your answer (if you claim "true" give a proof, if you claim "false" give a counterexample.)

(b) Is the following true or false? *For any two functions $f(x)$ and $g(y)$ one has*

$$\int_0^2 \int_0^1 f(x)g(y) \; dy \; dx = \left(\int_0^1 f(x) \; dx \right) \cdot \left(\int_0^2 g(y) \; dy \right) .$$

Explain your answer (no, this is not the same question as before. Look at the integration bounds.)

(c) Suppose D is the unit disc, $D = \{(x, y) : x^2 + y^2 < 1\}$. True or False: *For any two functions $f(x)$ and $g(y)$ one has*

$$\iint_D f(x)g(y) \; dx \; dy = \left(\int_{-1}^1 f(x) \; dx \right) \cdot \left(\int_{-1}^1 g(y) \; dy \right) .$$

Again, explain your answer.

4. Answer the question posed in Figure 6.

5. Compute the following double integrals. In each case sketch the domain of integration and show which iterated integral you must compute to find the given double integral.

(a) $\displaystyle\iint_D (1 + x) \, dA$ $\quad D = \{(x, y) : 0 \le x \le 2, 0 \le y \le 4\}$.

(b) $\displaystyle\iint_D (x + y) \, dA$ $\quad D = \{(x, y) : |x| \le 1, 0 \le y \le 4\}$.

(c) $\displaystyle\iint_D xy \, dA$ $\quad D = \{(x, y) : 0 \le x \le y, 1 \le y \le 2\}$.

(d) $\displaystyle\iint_D dA$ $\quad D = \big\{(x, y) : \tfrac{1}{2}y^2 \le x \le \sqrt{y}, 0 \le y \le 1\big\}$.

(e) $\displaystyle\iint_D \frac{x^2}{y^2} \, dA$ $\quad D = \{(x, y) : 1 \le x \le 2, 1 \le y \le x\}$.

(f) $\displaystyle\iint_D \frac{y}{e^x} \, dA$ $\quad D = \big\{(x, y) : 0 \le x \le 1, 0 \le y \le x^2\big\}$.

(g) $\displaystyle\iint_D x \cos y \, dA$ $\quad D = \Big\{(x, y) : 0 \le x \le \sqrt{\pi/2}, 0 \le y \le x^2\Big\}$.

(h) $\displaystyle\iint_D \sqrt{x^3 + 1} \, dA$ $\quad D = \{(x, y) : 0 \le y \le 1, \sqrt{y} \le x \le 1\}$.

(i) $\displaystyle\iint_D y\sin(x^2)\,dA$ $\qquad D = \{(x,y) : 0 \le y \le 1, y^2 \le x \le 1\}$.

(j) $\displaystyle\iint_D x\sqrt{1+y^2}\,dA$ $\qquad D = \{(x,y) : 0 \le x \le 1, x^2 \le y \le 1\}$.

(k) $\displaystyle\iint_D \frac{2}{\sqrt{1-x^2}}\,dA$ D is the triangle bounded by the y axis, the line $y = 1$ and the line $y = x$.

6. Find the volumes of the following regions by computing a double integral.

(a) the region bounded by $z = x^2 + y^2$ and $z = 4$.

(b) the region in the first octant bounded by $y^2 = 4 - x$ and $y = 2z$.

(c) the region in the first octant bounded by $y^2 = 4x$, $2x + y = 4$, $z = y$, and $y = 0$.

(d) the region in the first octant bounded by $x + y + z = 9$, $2x + 3y = 18$, and $x + 3y = 9$.

(e) the region in the first octant bounded by $x^2 + y^2 = a^2$ and $z = x + y$.

(f) the region bounded by $x^2 + y^2 = 4z$ and $z = 2$.

(g) the region bounded by $z = x^2 + y^2$ and $z = y$.

7. The average value of a function $f(x,y)$ over a domain D is by definition

$$\text{average } f \text{ over } D = \frac{\iint_D f(x,y)\,dA}{\text{area of } D}$$

Find the average value of $f(x,y) = e^y\sqrt{x+e^y}$ on the rectangle with vertices $(0,0)$, $(4,0)$, $(4,1)$ and $(0,1)$.

8. Suppose $f(x)$ is a positive function defined on an interval $a \le x \le b$. Let A be the area under the graph of $y = f(x)$, $(a \le c \le b)$, and let B be the area under the graph of $y = f(x)^2$ $(a \le c \le b)$

(a) Compute $\int_a^b \int_0^{f(x)} dydx$.

(b) Compute $\int_a^b \int_0^{f(x)} ydydx$.

9. Let V be the volume under the graph of the function

$$z = \frac{2xy}{x^2+y^2},$$

above the region

$$D = \{(x,y) : x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 1\}.$$

(a) Write an iterated integral for the volume V, using Cartesian coordinates. (You don't have to compute the integral you get.)

(b) Compute V using polar coordinates.

10. Let V be the volume under the graph of $z = xy$ above the domain

$$D = \{(x,y) : x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 4\}.$$

Try to draw the region D, and the graph of $z = xy$ above D.

(a) Use Cartesian coordinates to compute V. (Hint: this is similar to part (i) of the previous problem, but the integral in this problem isn't as bad.)

(b) Use Polar Coordinates to compute V.

4. Triple integrals

Instead of integrating over two-dimensional regions in the plane, we can also integrate over three-dimensional regions in space. In this section we will see the definition, how to compute triple integrals using iterated integrals, and some examples of how triple integrals come up in the real world.

4.1. Definition, and how to compute triple integrals. The definition of triple integrals follows the same pattern as that of double integrals. Let D be some three dimensional region in three dimensional space: D could be a cube, a "block," a cylinder, a sphere, or in general, the region enclosed by some surface. A particular case is that of a ***rectangular block,*** which is a region defined by the inequalities

$$a_x \le x \le b_x, \quad a_y \le y \le b_y, \quad a_z \le z \le b_z. \tag{127}$$

To define the triple integral of a function $w = f(x, y, z)$ over such a region we consider a partition of D into many smaller pieces. We number the pieces $1, 2, \cdots, N$ and for each j we choose a sample point (x_j, y_j, z_j) from the j^{th} partition piece. Let ΔV_j be the volume of the j^{th} partition piece and consider the Riemann sum

$$f(x_1, y_1, z_1)\Delta V_1 + \cdots + f(x_N, y_N, z_N)\Delta V_N = \sum_{j=1}^{N} f(x_j, y_j, z_j)\Delta V_j.$$

If these Riemann sums converge to some number as we choose finer and finer partitions, then we call this limit is called the ***triple integral, or volume integral, of*** f ***over*** D. The notation we use is

$$\iiint\limits_D f(x, y, z)\, dV = \lim_{\substack{\text{as the}\\ \text{partition}\\ \text{gets finer}}} \sum_{j=1}^{N} f(x_j, y_j, z_j)\Delta V_j. \tag{128}$$

If the domain D is a rectangular block, defined by the inequalities (127), then the triple integral can be computed by an iterated integral

$$\iiint\limits_D f(x, y, z)\, dV = \int_{a_z}^{b_z}\int_{a_y}^{b_y}\int_{a_x}^{b_x} f(x, y, z)\, dx\, dy\, dz. \tag{129}$$

This follows from the same kind of arguments that allowed us to turn a double integral into an iterated integral in § 2.3.

We can use (129) to compute a triple integral over any three dimensional block. To compute triple integrals over more general domains we can use the same slicing method as in § 2.6. If the domain D is given by inequalities of the type

$$a_x(y, z) \le x \le b_x(y, z), \quad a_y(z) \le y \le b_y(z), \quad a_z \le z \le b_z. \tag{130}$$

where $a_y(z), b_y(z), a_z(y, z)$, and $b_z(y, z)$ now are functions rather than constants, then the triple integral of a function $f(x, y, z)$ over D is given by

$$\iiint\limits_D f(x, y, z)\, dV = \int_{a_z}^{b_z}\int_{a_y(z)}^{b_y(z)}\int_{a_x(y,z)}^{b_x(y,z)} f(x, y, z)\, dx\, dy\, dz.$$

4.2. Example – the integral of $f(x, y, z) = x^2 + y^2$ over a rectangular block. Let's compute the integral of $f(x, y, z) = x^2 + y^2$ over the domain

$$D = \{(x, y, z) : 0 \le x \le A,\ 0 \le y \le B,\ 0 \le z \le C\},$$

where A, B, and C are the sides of the block.

The integral of f over D is

$$\iiint\limits_D \left(x^2 + y^2\right)\, dV = \int_0^C\int_0^B\int_0^A \left(x^2 + y^2\right)\, dx\, dy\, dz$$

It is a good idea to write such an integral as

$$\iiint\limits_D \left(x^2+y^2\right)\,dV = \int\limits_{z=0}^{C}\int\limits_{y=0}^{B}\int\limits_{x=0}^{A}\left(x^2+y^2\right)\,dx\,dy\,dz = \tfrac{1}{3}ABC\left(A^2+B^2\right),$$

to emphasize which integral goes with which variable.

The computation goes in three steps (there are three integrals). The innermost integral is

$$\int_0^A (x^2+y^2)\,dx = \tfrac{1}{3}A^3 + y^2A.$$

Next we integrate this with respect to y:

$$\int_0^B\int_0^A \left(x^2+y^2\right)\,dx\,dy = \int_0^B \left(\tfrac{1}{3}A^3+y^2A\right)\,dy = \tfrac{1}{3}A^3B + \tfrac{1}{3}AB^3.$$

finally, we integrate with respect to z:

$$\begin{aligned}\int_0^C\int_0^B\int_0^A \left(x^2+y^2\right)\,dx\,dy\,dz &= \int_0^C \left(\tfrac{1}{3}A^3B+\tfrac{1}{3}AB^3\right)\,dz\\ &= \tfrac{1}{3}A^3BC + \tfrac{1}{3}AB^3C\\ &= \tfrac{1}{3}ABC\left(A^2+B^2\right).\end{aligned}$$

4.3. Example of setting up a triple iterated integral– the integral of e^x over the unit sphere. Suppose we needed to know the integral

$$\iiint\limits_D e^x\,dV,$$

where the domain

$$D = \left\{(x,y,z) : x^2+y^2+z^2 \le 1\right\}$$

is the unit sphere. By slicing the domain D in the x, y, and z directions we can describe following the general template in (130):

- z can take any value between -1 and $+1$,
- for given z the coordinate y can be anything between $-\sqrt{1-z^2}$ and $+\sqrt{1-z^2}$,
- for given y and z the remaining coordinate x can have all values from $-\sqrt{1-y^2-z^2}$ to $+\sqrt{1-y^2-z^2}$.

(See Figure 12.)

This lets us write the triple integral as an iterated integral:

$$\iiint\limits_D e^x\,dV = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} e^z\,dx\,dy\,dz.$$

Even though it can be computed this is not an easy integral – the point of this example was to find the integration bounds in the iterated integral.

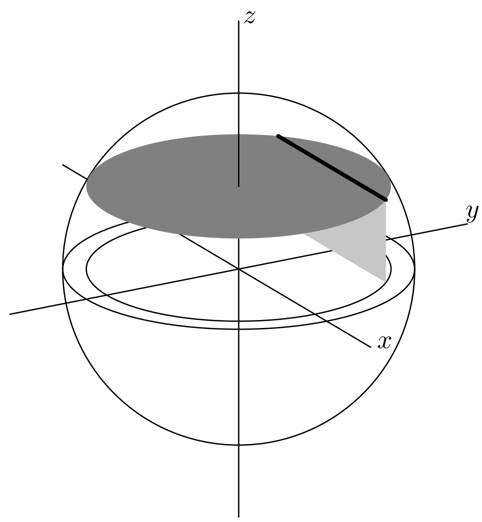
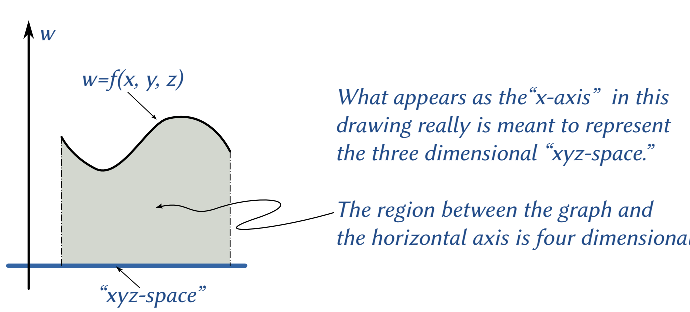


Figure 12. Turning a triple integral over the unit sphere into an iterated integral. The horizontal gray disc contains all points with a given fixed value of z; the solid line in that disc contains all points at height z whose y coordinate is also fixed at a particular value. From this drawing we can see that z runs between -1 and $+1$; for any given z, the y coordinate runs between $-\sqrt{1-z^2}$ and $+\sqrt{1-z^2}$; for fixed y and z, the x coordinate can take any value between $\sqrt{1-y^2-z^2}$ and $\sqrt{1-y^2-z^2}$.

5. Why compute a Triple Integral?

5.1. The 4D-volume under a graph. Just as $\int_a^b f(x)\ dx$ is the area between the graph of the function $y = f(x)$ and the interval $[a, b]$ on the x-axis, and $\iint_D f(x, y)\ dA$ is the volume caught between the graph of $z = f(x, y)$ and the domain D in the xy plane, there should be a similar description of $\iiint_D f(x, y, z)\ dV$. There is, but it requires some imagination: the graph of f is the set of points in *four dimensional space* whose coordinates (x, y, z, w) satisfy $w = f(x, y, z)$, and the triple integral $\iiint_D f(x, y, z)dV$ is the "four dimensional volume" of the four dimensional region caught between the graph of f and the domain D in xyz-space. Of course, even though people will draw cartoon like representations of the situation like this,



we cannot really visualize four dimensional volumes. Rather than telling us what the triple integral is, the interpretation "integral=volume" gives a definition of what "four dimensional volume" should be.

5.2. The average of a function over a domain D. There is a formula for the "average value of a function on a region." The only rigorous definition for the "average" is just that formula, so we could simply state the formula be done with it. Here it is: the average of a function $w = f(x, y, z)$ over a region D is defined to be

$$\begin{array}{c}\text{Average of } f\\ \text{over } D\end{array} = \frac{1}{V_D} \iiint_D f(x, y, z)\, dV. \tag{131}$$

There is however an intuitive derivation (a story) that justifies why we call this particular quantity the average. Understanding this derivation is at least as important as just knowing the formula (131).

Why (131) *deserves to be called the average.* What is an average? If we have finitely many numbers $a_1, \ldots, a_N$ then their average is just

$$\text{Average} = \frac{a_1 + \cdots + a_N}{N}.$$

If we only have finitely many points $(x_1, y_1, z_1), \ldots, (x_N, y_N, z_N)$ in the region D then the average function value at these points is

$$\begin{array}{c}\text{Average function value}\\ \text{at given points}\end{array} = \frac{f(x_1, y_1, z_1) + \cdots + f(x_N, y_N, z_N)}{N}.$$

To define the average of a function over a region D, we cannot simply add all the function values of f at all the points in D because there are infinitely many such points. Instead, we sprinkle the region D with a very large but finite number of points, and calculate the average value of the function at all these points. If the points are evenly distributed, and if there are enough of them, then the average value of the function at the dots should be a good approximation for the average value of the function on the region. E.g. the average of our function over the region on the left should be approximately the average of the

function at the dots drawn in that region.

To approximate the average at the dots we partition the region into many small pieces, which we label $D_1, \ldots, D_n$. We write ΔV_j for the volume of the j^{th} piece D_j, and V_D for the volume of the whole region D. We assume that the pieces are so small that we may assume that the function is practically constant in each piece.

Since the dots are evenly distributed over D, the number of dots in the j^{th} partition piece is proportional to the volume of that piece, so

$$\frac{N_j}{N} \approx \frac{\Delta V_j}{V_D} \tag{132}$$

where N_j is the number of dots in the j^{th} piece, and N is the total number of dots.

To compute the average value of f at all the dots we begin with

$$\text{sum of } f \text{ at all dots} = \sum_j \text{sum of } f \text{ at all dots in } j^{\text{th}} \text{ piece} .$$

If we pick a sample point (x_j, y_j, z_j) in each piece D_j, then, since the pieces are assumed to be small, we may approximate the function value at every dot in D_j by the value of the function at the sample point. There are N_j dots in D_j, so we find that

$$\text{sum of } f \text{ at all dots} \approx \sum_j N_j f(x_j, y_j, z_j)$$

Using (132) we therefore find that the average function value at all the dots is

$$\begin{aligned}\frac{\text{sum of } f \text{ at all dots}}{\text{number of dots in } D} &\approx \frac{1}{N}\sum_j N_j f(x_j, y_j, z_j)\\ &= \sum_j \frac{\Delta V_j}{V_D} f(x_j, y_j, z_j)\\ &= \frac{1}{V_D}\sum_j f(x_j, y_j, z_j)\,\Delta V_j\\ &\approx \frac{1}{V_D}\iiint_D f(x, y, z)\, dV.\end{aligned}$$

This is exactly how we had defined the average of f over the region D.

Keep in mind that the above is not a proof of the equation (131), but rather an intuitive justification for taking (131) as definition of the average.

5.3. Example 4.2 continued. In §4.2 we computed the volume integral of

$$f(x, y, z) = x^2 + y^2$$

over the rectangular block D given by $0 \le x \le A$, $0 \le y \le B$, $0 \le z \le C$ and we found

$$\iiint_D \left(x^2 + y^2\right)\, dV = \tfrac{1}{3}ABC\left(A^2 + B^2\right).$$

Since the volume of the block is ABC, the average value of $f(x, y, z) = x^2 + y^2$ over the block D is

$$\begin{matrix}\text{Average of } x^2 + y^2\\ \text{over } D\end{matrix} = \frac{\frac{1}{3}ABC\left(A^2 + B^2\right)}{ABC} = \tfrac{1}{3}\left(A^2 + B^2\right).$$

5.4. Densities. If a substance (for an example, think of a gas in a cylinder) occupies a certain region D in space, then its ***density*** μ is defined to be

$$\mu = \text{density} = \frac{\text{mass in } D}{\text{volume of } D}.$$

If the substance is evenly distributed throughout the region D, then the mass-to-volume ratio will be the same for any subregion D'. Thus the mass contained in any smaller region D' will be proportional to the volume of that region:

$$\text{mass in } D' = \mu \times \text{volume of } D'.$$

When the substance is not distributed evenly this proportionality will no longer hold, and we say that "the density varies from point to point." If we now want to give a precise definition of ***the density at any point*** P, we run into the same kind of problem we had in first semester calculus when we tried to define the slope of a tangent, or the velocity at one moment in time. Namely, the "density at P" should be the mass of the substance at P divided by the volume of the point P – but there is no mass at one point, and the volume of one point is zero, so this leads to density$=\frac{0}{0}=$?? The way out of this is to calculate the average density for very small regions D' surrounding the point P, and to declare those as approximations of the density at P. To get a better approximation we should choose a smaller region D'.

This is summarized in the following formula,

$$\mu(x,y,z) = \lim_{D' \searrow P} \frac{\text{mass in } D'}{\text{volume of } D'} \tag{133}$$

where "$D' \searrow P$" means that we are taking the limit as the region D' shrinks to the point P.

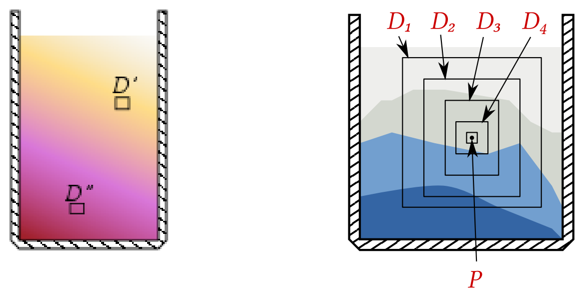


Figure 13. Density of gas in a container; in these drawings most of the gas concentrates in the bottom of the container. **Left:** The total mass in two regions, D' and D'', depends on their location, even though they have the same shape and volume. **Right:** To define the density at a point P, we compute the average density over smaller and smaller regions $D_1, D_2, \ldots$ which shrink to the given point P. If the average densities converge to some number, then we call that limit the density at P.

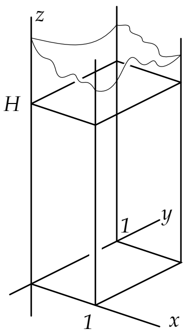
5.5. Mass as integral of the density. Suppose the density of a substance is given to us as a function $\mu(x,y,z)$, how do we find the total mass of the substance present in a particular region D? The answer is in terms of a triple integral, and the way this integral comes about is typical for a large number of applications of double and triple integrals.

To find the total mass present in a region D we partition it into many small pieces, and compute the mass in each small piece. Consider one such piece. If it is small enough, then we assume that the density $\mu(x,y,z)$ is nearly constant in that small piece, and hence the total mass in one small piece will be

$$\text{mass in a piece of the partition} = \mu(x,y,z) \times \Delta V.$$

Here (x,y,z) is a sample point in the partition piece, and ΔV is the volume of the piece. So when we compute the total mass by adding all the masses of the partition pieces, each piece in the partition contributes one term of the form $f(x,y,z)\Delta V$. Our formula for the total mass is therefore a Riemann sum for the following triple integral

$$\text{total mass} = \iiint_D \mu(x,y,z)\, dV. \tag{134}$$



5.6. Example: air in the atmosphere. *How much air is there in the atmosphere in a vertical column of height H above one square meter?*

According to one model of the atmosphere, the density of the atmosphere decays exponentially with height, so that

$$\mu(x, y, z) = Ce^{-z/L} \quad (\mathrm{kg/m^3}) \tag{135}$$

where z is the height above sea level, and x, y are horizontal coordinates. The constant C is the density of air at sea level, and L is another constant (L must have the units of length).

We adapt our coordinates to the 1×1 square which is the base of the air column whose mass we are to compute, namely, we let the origin be one of the corners of the square, and we let the sides at this corner be the x and y axes. The region occupied by the air column is then a rectangular block

$$D = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le H\}$$

and the mass of the air in this block is

$$\begin{aligned} M &= \iiint_D \mu(x, y, z)\, dV \\ &= \int_0^H \int_{y=0}^1 \int_{x=0}^1 Ce^{-z/L}\, dx\, dy\, dz \\ &= LC\big(1 - e^{-H/L}\big). \end{aligned}$$

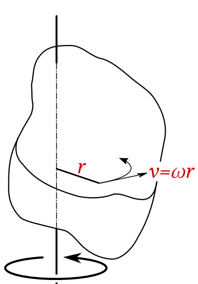
To get the mass of *all* the air above our 1×1 square, we let $H \to \infty$ which leads to

$$\text{Total Mass} = LC.$$

5.7. The moment of inertia of a solid about an axis of rotation. An object of mass m that moves with velocity v has kinetic energy given by

$$K = \frac{1}{2}mv^2. \tag{136}$$

If a solid object is rotating about an axis, then it also has kinetic energy, but the formula (136) does not apply, because different parts of the solid will be moving with different velocities. The problem is that v is not a constant: it varies from place to place, and thus it is a function of where we measure the velocity.



the kinetic energy of a whirling potato

To compute the kinetic energy of a rotating solid we break it up into small pieces: if each of the pieces is small enough, then all the particles in that small piece will have nearly the same velocity. A well known formula from trigonometry says that if the object is rotating with angular velocity ω about an axis, then the velocity of a particle in the object is given by $v = \omega r$ where r is the distance from the particle to the axis of rotation. On the other hand the mass of such a small piece will be $\mu \cdot \Delta V$, where μ is the density of the material (which we assume to be constant here), and ΔV is the volume of the small piece. Therefore if we break the object into many small pieces (partition the object), the kinetic energy of any one of the small pieces is

$$\text{K.E. of one piece} = \frac{1}{2}\mu(\omega r)^2 \Delta V.$$

Adding the kinetic energies of all the small pieces again gives us a Riemann sum for an integral, and this leads us to the formula

$$K = \iiint_D \tfrac{1}{2}\mu\omega^2 r^2 \, dV = \frac{1}{2}M\omega^2, \tag{137}$$

where

$$M \stackrel{\text{def}}{=} \mu \iiint_D r^2 \, dV. \tag{138}$$

is called the ***moment of inertia*** of the given object about the given axis of rotation.

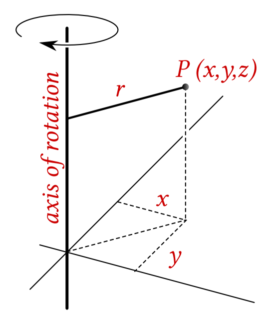
5.8. Example. *Compute the moment of inertia of a wooden rectangular block*

$$D = \big\{(x, y, z) : 0 \le x \le A, 0 \le y \le B, 0 \le z \le C\big\}.$$

around the z axis. The density of the wood is μ.

The integral we have to calculate is

$$M = \mu \iiint_D r^2 \, dV.$$



To compute this we have to figure out what r is: since r is the distance from the point (x, y, z) to the axis of rotation, and since this axis is the z-axis, we get, by Pythagoras, $r^2 = x^2 + y^2$. Therefore we have to compute

$$M = \mu \iiint_D \big(x^2 + y^2\big) \, dV.$$

We have already computed this integral in §4.2, where we found that

$$M = \tfrac{1}{3}\mu ABC(A^2 + B^2).$$

6. Integration in special coordinate systems

Many volume integrals arise in situations where there is a lot of symmetry. When this happens Cartesian "x, y, z" coordinates are usually not the best choice to compute the integral. There are many different coordinates besides Cartesian. In this section we will look at the two most commonly used coordinate systems. They can both be thought of as three-dimensional variations on polar coordinates in the plane.

6.1. Cylindrical coordinates. Let P be some point in three dimensional space. If we provide the z coordinate as well as the polar coordinates (r, θ) of the projection of P on the xy plane, then the location of P is completely determined. See the drawing on the left in Figure 14. From this drawing it is easy to derive the relation between cylindrical Cartesian coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \tag{139}$$

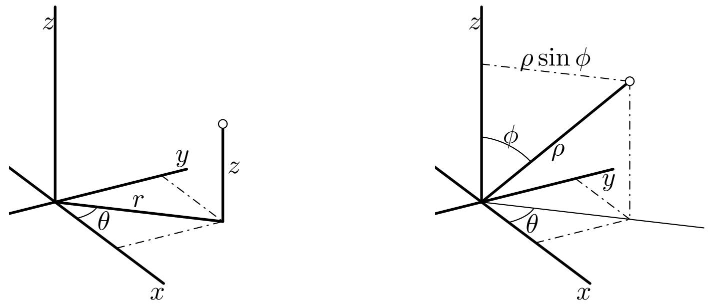


Figure 14. Left: In ***cylindrical coordinates*** we specify the location of a point by its height z above the xy-plane, and the polar coordinates (r, θ) of its projection on the xy-plane. **Right:** In ***spherical coordinates*** we specify the location of a point by its distance ρ to the origin, the polar angle θ of its projection on the xy-plane, and the angle ϕ between the z-axis and the line segment from the point to the origin.

6.2. Spherical coordinates. We can also specify the location of a point P by providing these three numbers:

– the distance ρ from P to the origin
– the angle ϕ between the positive z-axis and the line from the origin to the point P
– the polar angle θ of the projection of P onto the xy-plane.

DO NOT MEMORIZE. What if the north pole happens to be on the x-axis? Can you still relate spherical and Cartesian coordinates?

See the drawing on the right in Figure 14, from which we can derive the following relation between the spherical coordinates (ρ, ϕ, θ) and the Cartesian coordinates (x, y, z) of a point:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned} \tag{140}$$

The angle ϕ takes values between 0 and $+\pi$, with $\phi = 0$ on the north pole, and $\phi = \pi$ on the south pole. The polar angle θ can take all values from 0 to 2π, or more generally any value in some interval of length 2π (like $-\pi < \theta < \pi$).

6.3. Triple integral in cylindrical coordinates. Suppose we wanted to find a triple integral

$$\iiint_D f(x, y, z)\, dV$$

over a domain D which is a "rectangular block" in cylindrical coordinates, i.e. suppose D is given by the inequalities

$$r_0 \le r \le r_1,\ z_0 \le z \le z_1,\ \theta_0 \le \theta \le \theta_1.$$

Let's try to write it as an iterated integral. To do this we partition the region D into many small pieces by dividing the interval $r_0 \le r \le r_1$ into pieces of length Δr, the interval $z_0 \le z \le z_1$ into pieces of length Δz, and interval $\theta_0 \le \theta \le \theta_1$ into pieces of length $\Delta\theta$. The whole region D then gets broken up into small regions in which the radius is constrained to lie in the interval $(r, r+\Delta r)$, the height to the interval $(z, z+\Delta z)$ and the polar angle to $(\theta, \Delta\theta)$. See Figure 15. Such a small region is approximately a rectangular

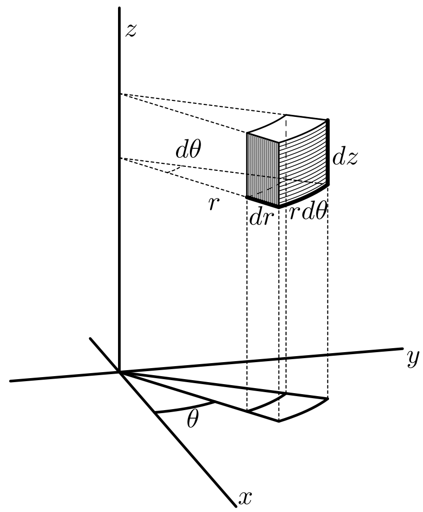


Figure 15. The cylindrical volume element.

block, so that we can approximate its volume by multiplying the lengths of its sides, which leads to

$$\Delta V \approx \Delta r \times r\Delta\theta \times \Delta z.$$

Arguing as in the case of polar coordinates (see §2.8) we get the following iterated integral formula for a triple integral over a rectangular block in cylindrical coordinates:

$$\iiint_D f(x,y,z)dV = \int_{r_0}^{r_1}\int_{z_0}^{z_1}\int_{\theta_0}^{\theta_1} f(x,y,z)r\,d\theta\,dz\,dr \tag{141}$$

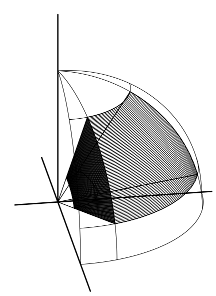
If the function to be integrated is given in terms of the Cartesian coordinates x, y, z, then we first have to rewrite it in terms of cylindrical coordinates using (139).

6.4. Triple integral in spherical coordinates. A spherical block is a region D which in spherical coordinates is given by the inequalities

$$\rho_0 \le \rho \le \rho_1,\ \theta_0 \le \theta \le \theta_1,\ \phi_0 \le \phi \le \phi_1$$

To integrate a function over such a block we divide into many small spherical blocks. In each of these blocks ρ increases by $\Delta\rho$, θ by $\Delta\theta$, and ϕ by $\Delta\phi$. See Figure 16. Any sufficiently small spherical block is approximately rectangular, and we can therefore compute its volume by multiplying the lengths of its sides. If we carefully look at the drawing on the right in Figure 16, then we find that

$$\Delta V \approx \rho\Delta\phi \times \rho\sin\phi\Delta\theta \times \Delta\rho.$$

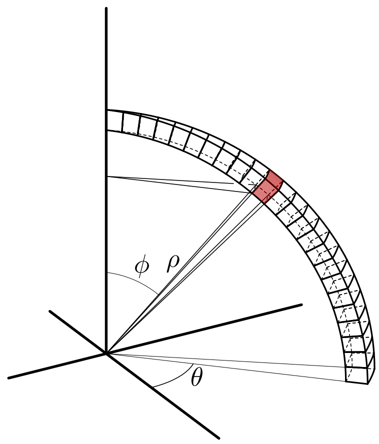


A spherical block

This leads us to the formula for integration in spherical coordinates:

$$\iiint_D f(x,y,z)dV = \int_{\rho_0}^{\rho_1}\int_{\theta_0}^{\theta_1}\int_{\phi_0}^{\phi_1} f(x,y,z)\rho^2\sin\phi\,d\phi\,d\theta\,d\rho \tag{142}$$

As in the cases of polar coordinates and cylindrical coordinates we first have to express the function $f(x,y,z)$ in terms of the variables ρ, ϕ, and θ, using (140).



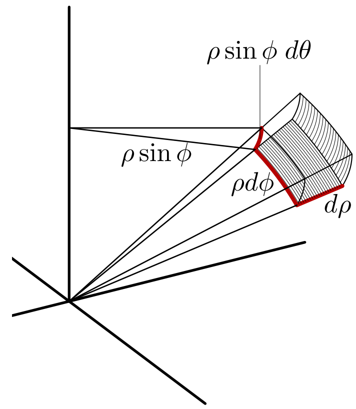


Figure 16. Left: a number of small spherical blocks with varying ϕ but the same θ and ρ stacked together. **Right:** the volume of a small spherical block is approximately the product of the lengths of its sides, so $\Delta V \approx \rho^2 \sin\phi \Delta\rho\, \Delta\theta\, \Delta\phi$.

6.5. Example – Rotational Kinetic energy of the Earth. The earth is roughly a sphere with radius $a \approx 6400\mathrm{km}$, which rotates around its axis with angular velocity $\omega = 2\pi \mathrm{rad}/\mathrm{day}$. Let's assume that the density of the earth is constant, say, $\mu \mathrm{kg}/\mathrm{m}^3$.

To compute the total kinetic energy of the earth we can use formula (137), which tells us that we have to find

$$\iiint\limits_{\text{Earth}} r^2 \, dV,$$

where r is the distance to the earth's axis of rotation.

This integral is best computed using spherical coordinates, in which

$$r = \rho \sin\phi \quad \text{(see Figure 14, right).}$$

Thus the kinetic energy is

$$\begin{aligned} K &= \tfrac{1}{2}\mu\omega^2 \iiint\limits_{\text{Earth}} \rho^2 \sin^2\phi \, dV \\ &= \tfrac{1}{2}\mu\omega^2 \int_{\phi=0}^{\pi} \int_{\rho=0}^{a} \int_{\theta=0}^{2\pi} \rho^2 \sin^2\phi \; \underbrace{\rho^2 \sin\phi \, d\theta \, d\rho \, d\phi}_{dV}. \end{aligned}$$

After doing the θ and ρ integrals, we get

$$K = \pi\mu\omega^2 \frac{a^5}{5} \int_0^{\pi} \sin^3\phi \, d\phi.$$

This last integral can be done several ways (integrate by parts and find a reduction formula, or substitute $u = -\cos\phi$). The result is

$$K = \tfrac{4}{15}\pi\mu\omega^2 a^5.$$

7. Problems

1. Describe the following sets (given in spherical coordinates):

(a) All points with $\phi = \pi/6$.

(b) All points with $\phi = \pi$.

(c) All points with $\phi = \pi/2$.

(d) All points with $\theta = \pi/2$

2. Let E be the part of the sphere with radius a, centered at the origin, and contained in the first octant

(a) Describe E in terms of spherical coordinates.

(b) Describe E in terms of cylindrical coordinates. There are two possible answers, find both.

3. Draw the volume elements in cylindrical and in spherical coordinates and show how these lead to $dV = r dr d\theta dz$, and $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$, respectively.

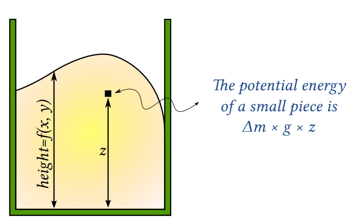
4. Look at Figure 12. Suppose the grey disc has height z, and suppose all points on the line segment drawn in this disc have the same y-coordinate (y).

(a) What are the radii of the two circles drawn in the xy plane?

(b) What are the coordinates of the two endpoints of the drawn line segment?

5. *The potential energy in a pile of honey.* If you lift an object to height h above the ground, then the potential energy you give it is mgh, where m is the mass of the object, and g is the acceleration of gravitation ($g \approx 9.8\text{m}/\text{sec}^2$).

Suppose that a certain substance occupies a three dimensional region D (think of honey that has just been poured into a jar, see the drawing which gives a two dimensional side view of the situation).



Assuming the base of the jar is an $A \times B$ rectangle, the honey occupies the region

$$D = \{(x, y, z) : \\ 0 \le x \le A,\ 0 \le y \le B, \\ 0 \le z \le f(x, y)\}.$$

Here $f(x, y)$ is the height of the honey above the point (x, y) in the base of the jar.

(a) What is the potential energy of a small piece of the honey at (x, y, z) (assume the density of the honey is μ, and that this is constant.) Is your formula an exact formula?

(b) Write a volume integral for the total (gravitational) potential energy contained in the honey.

(c) Write your triple integral as an iterated integral, and show that you can do the integration in the z direction even if you don't know the height function $f(x, y)$.

6. *The kinetic energy in a tornado.*

Assume an airmass is whirling around the z-axis, and assume that the wind velocity $v(r)$ only depends on the distance from the z-axis.

Assume furthermore that the air has constant density μ.

(a) Derive a volume integral for the total kinetic energy of the airmass in a given region D. (The kinetic energy of an object of mass m and velocity v is $\frac{1}{2}mv^2$. See the derivation of the moment of inertia in §5.7).

(b) Suppose the velocity is actually given by $v(r) = 1/\sqrt{1+r^2}$, the density is $\mu = 1$. Let D be the cylinder of height H and radius R, with the z-axis as its central axis. How much kinetic energy does the airmass in D have? (Hint: which coordinates should you use?)

7. For each of the following iterated integrals, describe and draw the domain of integration. Then compute the integral.

(a) $\displaystyle\int_0^2 \int_{-1}^{x^2} \int_1^y xyz\,dz\,dy\,dx.$

(b) $\displaystyle\int_0^1 \int_0^x \int_0^{\ln y} e^{x+y+z}\,dz\,dy\,dx.$

(c) $\displaystyle\int_0^{\pi/2} \int_0^{\sin\theta} \int_0^{r\cos\theta} r^2\,dz\,dr\,d\theta.$

(d) $\displaystyle\int_0^{\pi} \int_0^{\sin\theta} \int_0^{r\sin\theta} r\cos^2\theta\,dz\,dr\,d\theta.$

(e) $\displaystyle\int_0^1 \int_0^{y^2} \int_0^{x+y} x\,dz\,dx\,dy.$

(f) $\displaystyle\int_1^2 \int_y^{y^2} \int_0^{\ln(y+z)} e^x\,dx\,dz\,dy.$

8. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.

9. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.

◇

If a mass is distributed throughout a region D with density $\mu(x, y, z)$, then, by definition the coordinates (X, Y, Z) of the ***center of mass***

$$X \stackrel{\text{def}}{=} \frac{\iiint_D x\mu(x, y, z))\,dV}{\text{Mass of } D},$$

and similarly for Y and Z.

10. An object occupies the volume of the upper hemisphere of $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the center of mass.

11. An object occupies the volume of the pyramid with corners at $(1, 1, 0)$, $(1, -1, 0)$, $(-1, -1, 0)$, $(-1, 1, 0)$, and $(0, 0, 2)$ and has density $x^2 + y^2$ at (x, y, z). Find the center of mass.

12. Let $z = f(x, y)$ be a function on some domain D, and assume that D is split into two parts: D_+, on which $f \geq 0$, and D_-, on which $f(x, y) < 0$.

Let V_+ be the volume of the region beneath the graph of f and above the domain D_+ in the xy-plane, and, similarly, let V_- be the volume of the region above the graph of f and beneath the region D_- in the xy-plane.

Reminder: ***volumes are never negative***, so both $V_+ \geq 0$ and $V_- \geq 0$.

(a) Express the following integrals in terms of V_+ and V_-:

$$I = \iint_{D_+} f(x, y)\,dA,$$
$$J = \iint_{D_-} f(x, y)\,dA$$
$$K = \iint_{D} f(x, y)\,dA$$
$$L = \iint_{D} |f(x, y)|\,dA.$$

(b) Find the region E in three dimensional space for which

$$\iiint_E (1 - x^2 - y^2 - z^2)\,\mathrm{d}V$$

is a maximum. [Hint: Suppose E is some region; consider then what happens to the integral if you make E larger by adding on a piece.]

13. Evaluate

$$\int_0^1 \int_0^x \int_0^{\sqrt{x^2+y^2}} \frac{(x^2+y^2)^{3/2}}{x^2+y^2+z^2}\,dz\,dy\,dx.$$

14. Evaluate $\iiint x^2\,\mathrm{d}V$ over the interior of the cylinder $x^2+y^2 = 1$ between $z = 0$ and $z = 5$.

15. Evaluate $\iiint xy\,\mathrm{d}V$ over the interior of the cylinder $x^2+y^2 = 1$ between $z = 0$ and $z = 5$.

16. Evaluate $\iiint z\,\mathrm{d}V$ over the region above the x-y plane, inside $x^2+y^2-2x = 0$ and under $x^2 + y^2 + z^2 = 4$.

17. Evaluate $\iiint yz\,\mathrm{d}V$ over the region in the first octant, inside $x^2+y^2-2x = 0$ and under $x^2 + y^2 + z^2 = 4$.

18. Evaluate $\iiint x^2+y^2\,\mathrm{d}V$ over the interior of $x^2+y^2+z^2=4$.

19. Evaluate $\iiint \sqrt{x^2+y^2}\,\mathrm{d}V$ over the interior of $x^2+y^2+z^2=4$.

20. Find the mass of a right circular cone of height h and base radius a if the density is proportional to the distance from the base.

21. Find the mass of a right circular cone of height h and base radius a if the density is proportional to the distance from its axis of symmetry.

22. An object occupies the region inside the unit sphere at the origin, and has density equal to the distance from the x-axis. Find the mass.

23. An object occupies the region inside the unit sphere at the origin, and has density equal to the square of the distance from the origin. Find the mass.

24. An object occupies the region between the unit sphere at the origin and a sphere of radius 2 with center at the origin, and has density equal to the distance from the origin. Find the mass.

CHAPTER 7

Vector Calculus

1. Vector Fields

So far we have been studying the calculus of *functions* of several variables. Functions are used to describe things that have different values at different locations, e.g. quantities like temperature, or density. Many other physical phenomena are described by ***vector fields,*** i.e. by vectors whose direction and magnitude can vary from place to place. Vector calculus is the theory of integration and differentiation of vector fields.

By definition, a vector field in the plane is a vector valued function of two variables: whereas an ordinary function of two variables gives us a number for each (x, y) in its domain, a vector field gives us a vector in the plane for each point (x, y) in its domain. Such a vector is determined by its two components, both of which are ordinary functions of (x, y). The notation we will use in this course is as follows:

$$\vec{\boldsymbol{v}}(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix} = P(x,y)\,\vec{\boldsymbol{\imath}} + Q(x,y)\,\vec{\boldsymbol{\jmath}}. \tag{143}$$

For a vector field in three dimensional space we must specify a vector $\vec{\boldsymbol{v}}(x, y, z)$ at each point (x, y, z) in a three dimensional domain :

$$\vec{\boldsymbol{v}}(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \\ R(x,y) \end{pmatrix} = P(x,y)\,\vec{\boldsymbol{\imath}} + Q(x,y)\,\vec{\boldsymbol{\jmath}} + R(x,y)\,\vec{\boldsymbol{k}}.$$

To draw a vector field in the plane we would have compute $\vec{\boldsymbol{v}}(x, y)$ at lots of points and simply plot them. The more points we pick, the busier the picture gets. See for example Figure 1, in which the vector field

$$\vec{\boldsymbol{v}}(x,y) = \begin{pmatrix} -y/(x^2+y^2) \\ x/(x^2+y^2) \end{pmatrix} \tag{144}$$

is drawn.

2. Examples of vector fields

2.1. Gradients as vector fields. We have already seen examples of vector fields before, namely *the gradient of any function $f(x, y)$ is a vector field:*

$$\vec{\boldsymbol{\nabla}} f(x,y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix}.$$

In fact, the example (144) is such a vector field: it is the gradient of the polar angle θ. In §III.4.2 we saw that for $x > 0$ this angle is given by $\theta(x, y)$, and we checked in problem III.15.16 that $\vec{\boldsymbol{v}}$ given by (144) and shown below in Figure 1 satisfies $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\nabla}}\theta$.

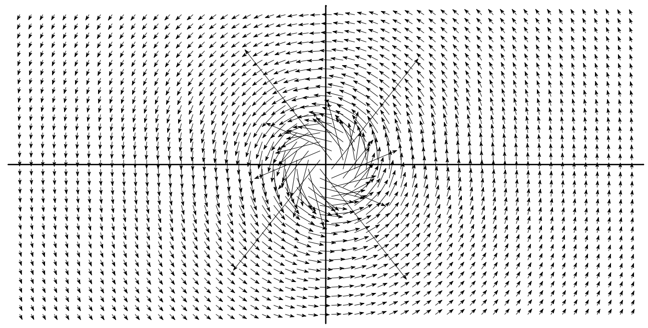


Figure 1. A vector field in the plane. This vector field is

$$\vec{v}(x,y) = \vec{\nabla}\theta(x,y) = \frac{-y}{x^2+y^2}\vec{\imath} + \frac{x}{x^2+y^2}\vec{\jmath}.$$

In general, drawings of vector fields become messy in the region where the vectors are long, because they tend to overlap. Drawing a three dimensional vector field is challenging.

2.2. Fluid flow. Vector fields appear in various ways in physics. The easiest way to visualize a vector field is by thinking of it as the velocity field of a fluid flow. Suppose a fluid is flowing through a certain region in space. The velocities of the fluid particles will generally vary from place to place, and also with time. A fluid flow is called *steady* if the velocity of a fluid particle only depends on its location. This means that the velocity vector $\vec{v}$ of a fluid particle is a function of its coordinates (x, y, z) only, and does not depend on time.

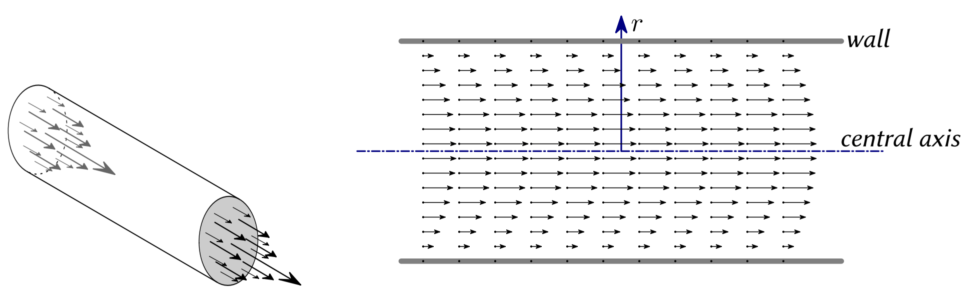


Figure 2. Fluid flow in a cylindrical pipe. Left: as a viscous fluid flows through a pipe it sticks to the walls, and so its velocity will be highest at the center of the pipe. **Right:** a drawing of a cross section the flow on the left. We see the vector field corresponding to so-called *Poiseuille flow*, given by Equation (145).

For instance, if a viscous fluid flows through a cylindrical pipe, the velocity of the fluid will only depend on the distance to the central axis of the pipe. On the walls the velocity will vanish (the fluid sticks to the wall of the pipe), and in the center the fluid will move fastest. Under certain circumstances it follows from the laws of fluid mechanics that the velocity field

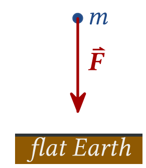
- is always parallel to the central axis, and
- depends quadratically on the distance to the central axis.

It is given by

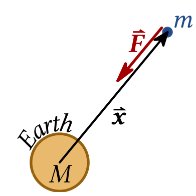
$$\vec{v}(x,y,z) = v_c\left(1 - \frac{r^2}{R^2}\right)\vec{\imath} = \begin{pmatrix} v_c\left(1-(r/R)^2\right) \\ 0 \\ 0 \end{pmatrix}, \tag{145}$$

where R is the radius of the pipe, r is the distance to its central axis, and v_c is the velocity at the center of the pipe.

This example describes the motion of a fluid, but a vector field can be the velocity field of anything that moves, in particular, a gas flow has a velocity field, and the velocities in a moving elastic solid (think "Jello") must also be described by a vector field.



2.3. Force fields. If we assume the Earth is flat, then the gravitational force it exerts on a mass m is always the vector $\vec{F} = \left(\begin{smallmatrix} 0 \\ -mg \end{smallmatrix}\right)$. We can think of this as a constant vector field: its magnitude and direction are the same everywhere.



But the Earth is not flat, and according to Newton the gravitational force $\vec{F}$ is a vector pointing towards the center of the earth, whose magnitude is inversely proportional to the distance to the center of the Earth. If we choose the Earth's center to be the origin, then Newton's law looks like this:

$$\vec{F}(x,y,z) = -C\frac{\vec{x}}{\|\vec{x}\|^3}, \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{146}$$

Here C is a constant that depends on the mass m of the object, and the mass M of the Earth (physics tells us that $C = GMm$, where G is called the "universal gravitational constant.")

Other prominent examples of vector fields appear in the theory of electromagnetism. The electric currents and charges around us create an electric field and a magnetic field, which at each point in space are given by vectors $\vec{E}$ and $\vec{B}$. These vectors change from place to place, and so they define vector fields

$$\vec{E} = \vec{E}(x,y,z), \qquad \vec{B} = \vec{B}(x,y,z).$$

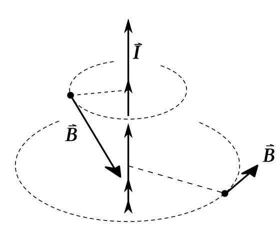
For example, Coulomb's law states that the electric field generated by a charged particle at the origin is given by

$$\vec{E}(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{\vec{x}}{\|\vec{x}\|^3}, \tag{147}$$

which is almost the same as Newton's law (146) for the gravitational field. Here ϵ_0 is some constant, and Q is the electric charge of the particle.

If an electric current of strength I runs upward through the z-axis, then this current will create a magnetic field which is given by

$$\vec{B}(x,y,z) = \frac{\mu_0 I}{2\pi}\begin{pmatrix} -y/(x^2+y^2) \\ x/(x^2+y^2) \\ 0 \end{pmatrix}. \tag{148}$$



Again, a constant (μ_0) appears. If we compare (148) with (144), then we see that, except for the constant factor $\mu_0 I/2\pi$ this vector field is a three dimensional version of the one drawn in Figure 1: we can regard Figure 1 as a "top view" of the magnetic field $\vec{B}$ of an electric current.

3. Line integrals

3.1. Line integrals of functions. Instead of integrating over plane domains, or regions in space, it often turns out to be useful to integrate over a curve in the plane, or a curve in space.

If $\mathcal{C}$ is a curve in the plane, or in space, (think of a line segment, a circular arc, or a fancier curve), and if $w = f(x, y, z)$ is a function then the basic pattern for defining the integral of f over the curve $\mathcal{C}$ is the same as for all the other integrals we have defined in the previous chapter.

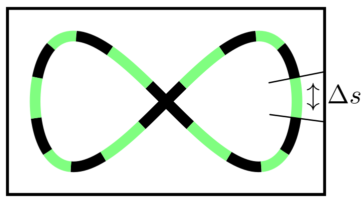


Figure 3. Partitioning a curve.

To define the integral we divide the curve $\mathcal{C}$ into many short arcs, and label them $\mathcal{C}_1$, …, $\mathcal{C}_n$; we choose one sample point (x_k, y_k, z_k) on arc $\mathcal{C}_k$ for every $k = 1, \cdots, n$; and we compute the length Δs_k of each arc $\mathcal{C}_k$. With these data we form the Riemann sum

$$\mathcal{R} = f(x_1, y_1, z_1)\Delta s_1 + \cdots + f(x_n, y_n, z_n)\Delta s_n, \tag{149}$$

and if these Riemann sums converge as one makes the partition arbitrarily fine, then we call the limit the line integral of f with respect to arc length over the curve $\mathcal{C}$:

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \lim_{\substack{\text{"as the partition} \\ \text{gets finer"}}} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k. \tag{150}$$

The length of the curve $\mathcal{C}$ can be expressed as a line integral

$$\text{Length of } \mathcal{C} = \int_{\mathcal{C}} ds.$$

3.2. How to calculate a line integral. Recall that a curve $\mathcal{C}$ is usually given by a parametrization

$$\vec{x} = \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \ (a \le t \le b)$$

also written as $\vec{x}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$.

Given such a parametrization it is easy to make partitions by just partitioning the parameter interval $a \le t \le b$ into many short sub intervals, $a = t_0 < t_1 < \cdots < t_n = b$. We could choose the k^{th} sample point to be the point $\vec{x}(t_k)$. The length of the arc from $\vec{x}(t_{k-1})$ to $\vec{x}(t_k)$ is approximately the same as the distance between these two points (for as one makes the partition finer, the arcs become more and more like short line segments). Thus we find

$$\Delta s_k \approx \|\vec{x}(t_k) - \vec{x}(t_{k-1})\| = \left\| \frac{\vec{x}(t_k) - \vec{x}(t_{k-1})}{\Delta t_k} \right\| \Delta t_k \approx \|\vec{x}'(t_k)\|\, \Delta t_k,$$

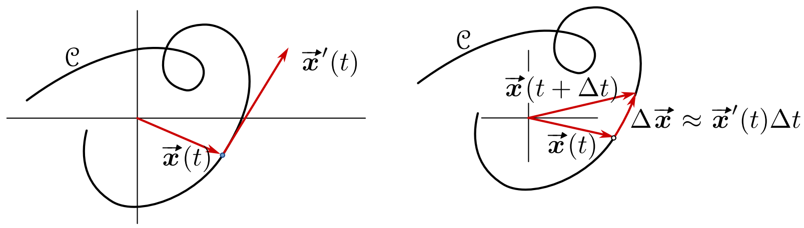


Figure 4. A parametrized curve: Left: The vector $\vec{\boldsymbol{x}}'(t)$ is tangent to the curve at the point $\vec{\boldsymbol{x}}(t)$. The vector $\vec{\boldsymbol{x}}(t)$ is the position vector of a point on the curve. **Right:** Increasing the parameter t by a small amount Δt changes the position vector to $\vec{\boldsymbol{x}}(t+\Delta t)$, causing the corresponding point on the curve to move by $\vec{\boldsymbol{x}}(t+\Delta t) - \vec{\boldsymbol{x}}(t) \approx \vec{\boldsymbol{x}}'(t)\Delta t$.

where $\Delta t_k = t_k - t_{k-1}$. The Riemann sum for the line integral is

$$\mathcal{R} \approx \sum_{k=1}^{n} f(\vec{\boldsymbol{x}}(t_k))\, \|\vec{\boldsymbol{x}}'(t_k)\|\, \Delta t_k.$$

As the partition is made finer the approximation gets better, and in the limit we get

$$\int_{\mathcal{C}} f(\vec{\boldsymbol{x}})\, ds = \int_a^b f(\vec{\boldsymbol{x}}(t))\|\vec{\boldsymbol{x}}'(t)\|\, dt. \tag{151}$$

3.3. Example – What is the average of $f(x,y) = x$ over the quarter unit circle in the first quadrant? Just as with double and triple integrals, the average of a function over a curve $\mathcal{C}$ is defined to be

$$\text{Average of } f(\vec{\boldsymbol{x}}) = \frac{\int_{\mathcal{C}} f(\vec{\boldsymbol{x}})\, ds}{\int_{\mathcal{C}}\, ds},$$

where $\int_{\mathcal{C}}\, ds$ is the length of $\mathcal{C}$.

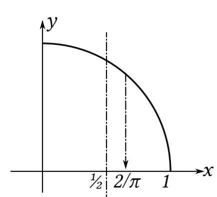


Figure 5. The average x-coordinate on a quarter circle

To compute these integrals we must first parametrize the curve. Since the curve is the unit circle, we can parametrize points on the curve by their polar coordinate θ, which gives us:

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad \text{and thus} \quad \|\vec{\boldsymbol{x}}'(t)\| = \left\|\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right\| = 1.$$

Therefore

$$\int_{\mathcal{C}} x\, ds = \int_0^{\pi/2} \cos\theta\, d\theta = 1.$$

The length of the curve is $\pi/2$, so the average value of x on the quarter circle is

$$\frac{1}{\pi/2} = \frac{2}{\pi} \approx 0.636\,619\,8\ldots.$$

4. Problems

1. If $\mathcal{C}$ is the quarter of the unit circle that lies in the first quadrant, then…

(a) What is the average distance to the origin on $\mathcal{C}$?

(b) what is the average polar angle θ?

2. **(a)** Compute the average x and y coordinates of the polygon from $A(1,0)$ to $B(1,a)$ to $C(0,a)$ ($a>0$ is a constant; the polygon has the shape of an upside-down "L").

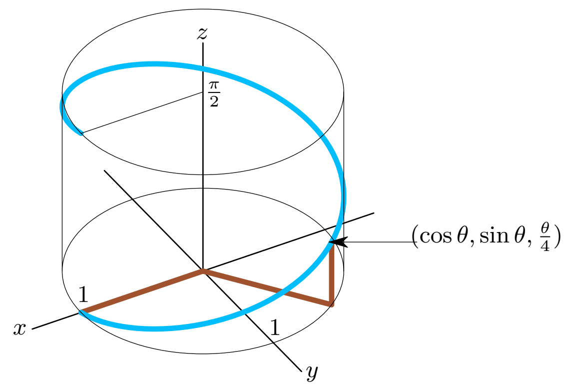
(b) Compute the average polar angle θ on the same polygon ABC.

3. Find the average x and y-coordinates on the part that lies above the x-axis of the circle with radius R and center at the origin.

4. Compute $\int_{\mathcal{C}} x\, ds$ where $\mathcal{C}$ is the parabola $y=x^2$, with $0\le x\le 1$.

5. A wire is made in the shape of a helix, of radius a and height H, with parametrization

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ Ht/2\pi \end{pmatrix} \qquad (0\le t\le 2\pi).$$



Suppose the temperature at (x,y,z) is $T = T_0 e^{-z/L}$ for constants L and T_0.

(a) What are the units of a, H, T_0, and L?

(b) What values do a and H have for the helix in the drawing?

(c) What is the average temperature on the wire? (Check that your answer has the right units.)

5. Line integrals of vector fields

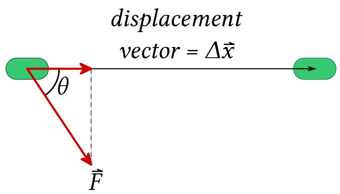
5.1. Definition. *If $\mathcal{C}$ is a curve in three dimensional space, and $\vec{\boldsymbol{F}}(x,y,z)$ is a vector field, then the* ***line integral of $\vec{\boldsymbol{F}}$ over*** *$\mathcal{C}$ is defined to be*

$$\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} = \lim_{\substack{\text{"as the partition}\\ \text{gets finer"}}} \sum_{k=1}^{n} \vec{\boldsymbol{F}}(x_k,y_k,z_k)\cdot\Delta\vec{\boldsymbol{x}}_k \tag{152}$$

To define the Riemann sum we have partitioned the curve into n pieces; (x_k,y_k,z_k) is a sample point on the k^{th} short arc in the partition, and $\Delta\vec{\boldsymbol{x}}_k$ is the vector connecting the initial and final points of the k^{th} partition arc. See Figure 6 (right).

5.2. Integrals over closed curves. A curve $\mathcal{C}$ is ***closed*** if its initial and final points coincide. If we are integrating a vector field over a closed curve, and if we want to emphasize this in the notation, then we can write

$$\oint_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}}, \text{ or } \oint_{\mathcal{C}} P dx + Q dy + R dz.$$



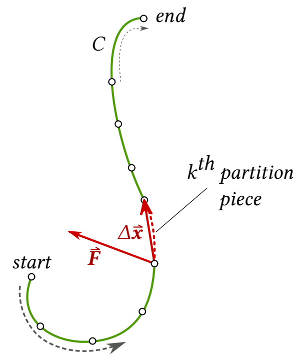


Figure 6. Left: The work done by a force $\vec{\boldsymbol{F}}$ acting on an object is equal to the product of the length of the displacement $\Delta\vec{\boldsymbol{x}}$ and the magnitude of the force in the direction of the displacement. If the angle between force and displacement is θ, then this is $\mathcal{W} = \|\vec{\boldsymbol{F}}\|\|\Delta\vec{\boldsymbol{x}}\|\cos\theta = \vec{\boldsymbol{F}}\cdot\Delta\vec{\boldsymbol{x}}$. **Right:** To define $\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}}$ we partition the curve into small pieces, and add the work done by the force $\vec{\boldsymbol{F}}$ over all partition pieces.

5.3. Differential form notation for line integrals. The $d\vec{\boldsymbol{x}}$ that appears in line integrals is often interpreted as an "infinitesimally short vector" connecting two adjacent points on the curve $\mathcal{C}$. Its components give us the amounts by which the coordinates x, y, and z change as we go "from one point to the next" on the curve, and therefore one often writes

$$d\vec{\boldsymbol{x}} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

If the vector field $\vec{\boldsymbol{F}}$ has components $\vec{\boldsymbol{F}} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$, where P, Q, and R are functions of (x, y, z), then the expression $\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}}$ can be written as

$$\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz.$$

Because of this the following notation for line integrals is often used:

$$\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} = \int_{\mathcal{C}} Pdx + Qdy + Rdz.$$

For instance, the integral

$$\int_{\mathcal{C}} xdx + zdy - xydz$$

stands for the line integral of the vector field $\vec{\boldsymbol{F}} = \begin{pmatrix} x \\ z \\ -xy \end{pmatrix}$ over the curve $\mathcal{C}$.

Expressions of the form $Pdx + Qdy + Rdz$, such as $x\,dx + z\,dy - xy\,dz$ above, are called ***differential forms.***

5.4. The orientation of a curve. The Riemann sum (152) contains the vectors $\Delta\vec{\boldsymbol{x}}_k$, which connect two adjacent points in our partition of the curve $\mathcal{C}$. Whenever we have two points A and B, there are *two* vectors connecting them, namely $\overrightarrow{AB}$ and $\overrightarrow{BA} = -\overrightarrow{AB}$. To make sure that the direction of the vector $\Delta\vec{\boldsymbol{x}}_k$ in the Riemann sum (152) is unambiguous, we have to agree on a direction in which the curve $\mathcal{C}$ is traversed. Such a direction is called ***an orientation*** of the curve. A curve can have exactly two orientations, and to distinguish between a curve and the same curve with the opposite orientation, one writes

$$-\mathcal{C} = \text{the curve } \mathcal{C} \text{ with its orientation reversed.}$$

If one reverses the orientation of a curve (e.g. by switching its begin and end points, see Figure 6), then each vector $\Delta\vec{\boldsymbol{x}}_k$ in the Riemann sum in (152) changes its sign, and as a result the whole Riemann sum changes its sign. In the limit the integral changes its sign. Thus we have

$$\int_{-\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} = -\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}}. \tag{153}$$

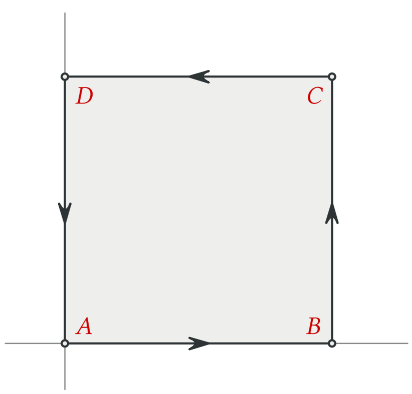
It is important to realize that the integral changes its sign here because it is the line integral of a vector field. If $w = f(x, y, z)$ is a function of three variables then

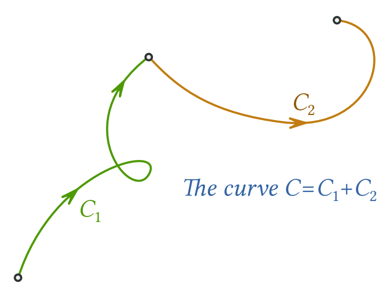
$$\int_{-\mathcal{C}} f(x, y, z)\, ds = +\int_{\mathcal{C}} f(x, y, z)\, ds.$$

For instance, if $f = 1$ is constant then $\int_{-\mathcal{C}} ds$ and $\int_{\mathcal{C}} ds$ are the length of $-\mathcal{C}$ and $\mathcal{C}$. Since the length of a curve is always a positive number and does not depend on its orientation, we have

$$\int_{\mathcal{C}} ds = \text{length of } \mathcal{C} = \text{length of } -\mathcal{C} = \int_{-\mathcal{C}} ds.$$

5.5. Integrating over piecewise defined curves. To compute a line integral it is often best to start with a parametrization of the curve and use (151). In practice it can be very difficult to find such a parametrization of the whole curve, even though the curve can be broken into a few pieces, each of which does have a simple parametrizations. For instance, the edges of a square together form a closed curve. It is difficult to find one parametrization for all four edges at once, but each edge of the square is a simple line segment for which one can easily find a parametrization. In this situation one can write





the line integral over the whole curve as a sum of line integrals over the separate pieces. Going back to the example of the square, we have

$$\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} = \int_{AB} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} + \int_{BC} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} + \int_{CD} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} + \int_{DA} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}}.$$

In general, if a curve $\mathcal{C}$ consists of two parts, $\mathcal{C}_1$ and $\mathcal{C}_2$, then we express this by writing

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2.$$

A line integral over the whole curve is the sum of the line integrals over the separate pieces:

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{x} = \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{x} + \int_{\mathcal{C}_2} \vec{F}\cdot d\vec{x}. \tag{154}$$

5.6. The line integral as the integral of the tangential component of a vector field. In the Riemann-sum (152) the k^{th} term contains the vector $\Delta\vec{x}_k$, which connects two adjacent points in the partition of the curve (see figure 6, on the right). We can write this vector as the product of a unit vector and a positive number:

$$\Delta\vec{x}_k = \frac{\Delta\vec{x}_k}{\|\Delta\vec{x}_k\|}\ \|\Delta\vec{x}_k\|.$$

The vector $\Delta\vec{x}_k$ will be almost tangent to the curve, and the finer one makes the partition, the smaller the angle between $\Delta\vec{x}_k$ and the tangent to the curve will be. If the partition is sufficiently fine, then we will have

$$\frac{\Delta\vec{x}_k}{\|\Delta\vec{x}_k\|} \approx \vec{T}_k,$$

where $\vec{T}_k$ is the unit tangent vector to the curve at the point (x_k, y_k, z_k). Furthermore, $\Delta\vec{x}_k \approx \Delta s_k$ approximates the length of the k^{th} arc in the partition, and hence we can write the Riemann sum as

$$\sum_{k=1}^{n} \vec{F}(x_k, y_k, z_k)\cdot\Delta\vec{x}_k \approx \sum_{k=1}^{n} \vec{F}(x_k, y_k, z_k)\cdot\vec{T}_k\ \Delta s_k.$$

The sum on the right is a Riemann sum for the line integral $\int_{\mathcal{C}} \vec{F}(\vec{x})\cdot\vec{T}\ ds$. Taking the limit of arbitrarily fine partitions, we conclude that

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{x} = \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\ ds. \tag{155}$$

Since $\vec{T}$ is a unit vector, the quantity $\vec{F}\cdot\vec{T}$ is the length of component of the vector field $\vec{F}$ tangential to the curve. Equation (155) therefore says that the line integral of the vector field $\vec{F}$ along a curve $\mathcal{C}$ is the same as the line integral (in the sense of §3.1) of the tangential component of $\vec{F}$ along $\mathcal{C}$.

5.7. Example – work around a circle. The formula (155) for the line integral is useful if we know the angle between the force $\vec{F}$ and the curve, and the magnitude of the force. For instance, consider this problem:

Compute the work done by the vector field $\vec{F}(x, y) = x\vec{\imath} + y\vec{\jmath} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ along the curve $\mathcal{C}$, where $\mathcal{C}$ is some piece of the unit circle in the plane.

We are asked to compute $\int_{\mathcal{C}} \vec{F}\cdot d\vec{x}$. The vector field $\vec{F} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ always points away from the origin, and thus it is always perpendicular to the tangent $\vec{T}$ to the unit circle. (See Figure 7.) Hence $\vec{F}\cdot\vec{T} = 0$, and we find that

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{x} = \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\ ds = \int 0\ ds = 0.$$

Those who prefer the differential form notation (§ 5.3) can write this as

$$\int_{\mathcal{C}} x\ dx + y\ dy = \int_{\mathcal{C}} \vec{F}\cdot d\vec{x} = 0.$$

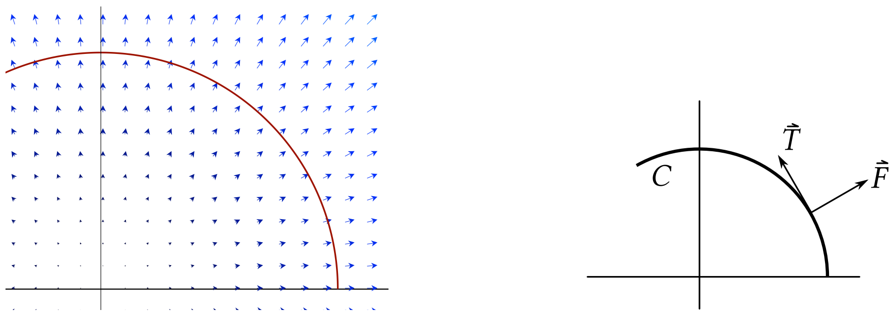


Figure 7. Left: the vector field $\vec{F}(x,y) = x\vec{\imath} + y\vec{\jmath}$ from § 5.7. **Right:** the vector field $\vec{F}$ is perpendicular to the path $\mathcal{C}$ and hence does no work.

5.8. How to compute a line integral. If a parametrization of a curve $\mathcal{C}$ is given, so that the curve $\mathcal{C}$ is the image of

$$\vec{x} = \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \quad a \le t \le b,$$

then we can partition the curve $\mathcal{C}$ by partitioning the parameter interval $a \le t \le b$ by choosing partition points $a = t_0 < t_1 < \ldots < t_n = b$, just as in §3.2. The k^{th} term in the Riemann sum (152) defining $\int_{\mathcal{C}} \vec{F} \cdot d\vec{x}$ is $\vec{F}(x_k, y_k, z_k) \cdot \Delta\vec{x}_k$, with

$$\Delta\vec{x}_k = \vec{x}(t_k) - \vec{x}(t_{k-1}) \approx \vec{x}\,'(t_k)\Delta t_k$$

(again as in § 3.2). The Riemann sum for $\int_{\mathcal{C}} \vec{F} \cdot d\vec{x}$ is therefore

$$\sum_{k=1}^{n} \vec{F}(x_k, y_k, z_k) \cdot \Delta\vec{x}_k \approx \sum_{k=1}^{n} \vec{F}(x_k, y_k, z_k) \cdot \vec{x}\,'(t_k)\Delta t_k.$$

The sum on the right converges to the integral $\int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\, dt$, and thus we have found

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{x} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{x}\,'(t) dt. \tag{156}$$

We can think of this as a substitution formula for integrals, in which we substitute $\vec{x} = \vec{x}(t)$ in the integral $\int_{\mathcal{C}} \vec{F}(\vec{x}) \cdot d\vec{x}$, using the rule

$$d\vec{x} = \vec{x}\,'(t)dt.$$

5.9. Three examples. Let $\mathcal{C}_1$ be the line segment from the origin to the point $(1, 1)$, and let $\mathcal{C}_2$ be the piece of the parabola $y = x^2$ between the origin and the point $(1, 1)$. Compute the work done by the vector field $\vec{F}(x, y) = \left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)$ along each of these two paths.

The two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ together bound a region $\mathcal{R}$. Let $\mathcal{C}_3$ be the boundary of this region, traversed in clockwise direction, and compute the work done by $\vec{F}$ along the closed curve $\mathcal{C}_3$.

To find these integrals we need parametrizations of the curves. For $\mathcal{C}_1$ we can use

$$\vec{x}_1(t) = \begin{pmatrix} t \\ t \end{pmatrix}, \quad 0 \le t \le 1,$$

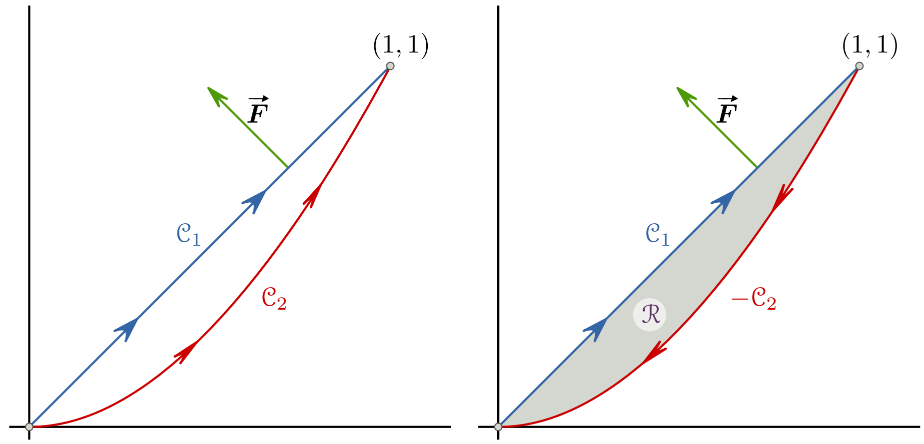


Figure 8. Left: Two different paths from the origin to the point $(1, 1)$, and the vector field $\vec{F}$. **Right:** By reversing the orientation of the second path $\mathcal{C}_2$ we can create a closed path that starts and ends at the origin. This path ($\mathcal{C}_1$ combined with $-\mathcal{C}_2$) is the boundary of the shaded region $\mathcal{R}$, traversed in the *clockwise* sense.

and for $\mathcal{C}_2$ we can use

$$\vec{x}_2(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, \quad 0 \leq t \leq 1.$$

To show how both notations work, we will do the first integral using vector notation, and the second using the differential form notation.

Integral over $\mathcal{C}_1$. The first integral is computed as follows:

$$\begin{aligned} \int_{\mathcal{C}_1} \vec{F} \cdot d\vec{x} &= \int_{\mathcal{C}_1} \begin{pmatrix} -y \\ x \end{pmatrix} \cdot d\vec{x} && \text{substitute } \vec{x} = \vec{x}_1(t) = \begin{pmatrix} t \\ t \end{pmatrix} \\ &= \int_{t=0}^{1} \underbrace{\begin{pmatrix} -t \\ t \end{pmatrix}}_{\vec{F}} \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix} dt}_{d\vec{x}} && \text{since } d\vec{x} = \vec{x}_1'(t)dt = \begin{pmatrix} 1 \\ 1 \end{pmatrix} dt \\ &= \int_0^1 0 \, dt \\ &= 0. \end{aligned}$$

Integral over $\mathcal{C}_2$. The second integral written using differential forms is

$$\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{x} = \int_{\mathcal{C}_2} -y \, dx + x \, dy.$$

Here we substitute the parametrization of the path

$$x = x_2(t) = t \text{ and } y = y_2(t) = t^2,$$

with

$$dx = dt, \qquad dy = dt^2 = 2t \, dt,$$

and we find

$$\int_{\mathcal{C}_2} \vec{F}\cdot d\vec{x} = \int_{t=0}^{1} \underbrace{-t^2\,dt}_{-y\,dx} + \underbrace{t\,2t\,dt}_{x\,dy} = \int_0^1 t^2\,dt = \frac{1}{3}.$$

Integral over $\mathcal{C}_3$. We had defined $\mathcal{C}_3$ to be the combination of the curves $\mathcal{C}_1$ and $-\mathcal{C}_2$ (which is $\mathcal{C}_2$ with its orientation reversed). Therefore

$$\begin{aligned}\int_{\mathcal{C}_3} \vec{F}\cdot d\vec{x} &= \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{x} + \int_{-\mathcal{C}_2} \vec{F}\cdot d\vec{x}\\ &= \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{x} - \int_{\mathcal{C}_2} \vec{F}\cdot d\vec{x}.\end{aligned}$$

We have already computed these two integrals so there is no need to do a new integration. The result we are looking for is

$$\int_{\mathcal{C}_3} \vec{F}\cdot d\vec{x} = 0 - \frac{1}{3} = -\frac{1}{3}.$$

6. Another Fundamental Theorem of Calculus

If we know the derivative $f'(x)$ of a function $y = f(x)$ of one variable then the Fundamental Theorem of Calculus tells us that we can recover the function by integrating the derivative:

$$f(b) = f(a) + \int_a^b f'(x)\,dx \tag{157}$$

This semester we saw in chapter IV, § 14 that one can do the same for functions of several variables, i.e. following a somewhat complicated procedure one can recover a function of two or more variables if one knows it s partial derivatives. In this section we show that the procedure has a much shorter description in terms of a line integral.

6.1. Theorem. *For any path $\mathcal{C}$ and any differentiable function f one has*

$$f(B) - f(A) = \int_{\mathcal{C}} \vec{\nabla} f(\vec{x})\cdot d\vec{x}, \tag{158}$$

where A and B are the initial and final points, respectively, of the path $\mathcal{C}$.

In differential form notation the same statement is written as

$$\int_{\mathcal{C}} \left\{ \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \right\} = f(B) - f(A).$$

6.2. Line integral of a gradient does not depend on the path. The examples in § 5.9 show that the line integral $\int_{\mathcal{C}} \vec{F}\cdot d\vec{x}$ of some vector field $\vec{F}$ normally depends on the path $\mathcal{C}$. However, it follows from Theorem 6.1 that if the vector field $\vec{F}$ happens to be the gradient of a function, $\vec{F} = \vec{\nabla} f$, then the line integral $\int_{\mathcal{C}} \vec{F}\cdot d\vec{x}$ only depends on the initial and final points, A and B, of the path $\mathcal{C}$, but not on the way that $\mathcal{C}$ gets from A to B.

6.3. Line integral of a gradient around a closed curve vanishes. An important special case of Theorem 6.1 is that in which the curve $\mathcal{C}$ is closed. If $\mathcal{C}$ is a closed curve, then its initial and final points coincide, so that one always has

$$\oint_{\mathcal{C}} \vec{\nabla} f(\vec{x})\cdot d\vec{x} = 0. \tag{159}$$

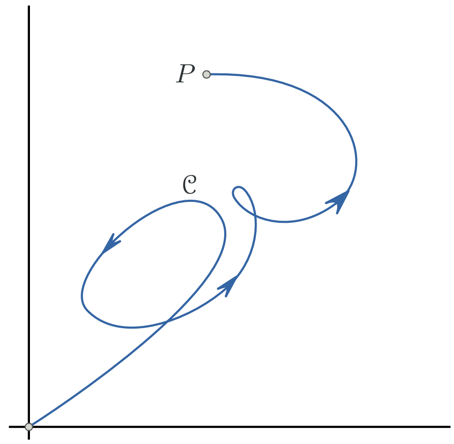


Figure 9. If we know the gradient of a function, and its value at one point (say, the origin), then we can compute $f(P)$ at any other point P by choosing a path $\mathcal{C}$ from the origin to P, and computing the line integral of the gradient. We have $f(P) = f(0,0) + \int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{x}$. It does not matter which path we choose.

6.4. Proof of the Fundamental Theorem. Suppose $\vec{F} = \vec{\nabla} f(x,y,z)$, and let the curve $\mathcal{C}$ be parametrized by $\vec{x} = \vec{x}(t)$, $a \le t \le b$. Then

$$\vec{F} = \vec{\nabla} f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix},$$

and hence

$$\begin{aligned}\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{x} &= \int_{\mathcal{C}} \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\ &= \int_a^b \Big\{ \frac{\partial f}{\partial x}(x(t), y(t), z(t)) \cdot x'(t) \\ &\qquad + \frac{\partial f}{\partial y}(x(t), y(t), z(t)) \cdot y'(t) \\ &\qquad\qquad + \frac{\partial f}{\partial z}(x(t), y(t), z(t)) \cdot z'(t) \Big\}\, dt\end{aligned}$$

The expression between $\{\cdots\}$ is what the Chain Rule would give us if we tried to differentiate $f(x(t), y(t), z(t))$ with respect to t. So we get

$$\begin{aligned}\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{x} &= \int_a^b \frac{df(x(t), y(t), z(t))}{dt}\, dt \\ &= f\big(x(b), y(b), z(b)\big) - f\big(x(a), y(a), z(a)\big).\end{aligned}$$

The point $B = (x(b), y(b), z(b))$ is the end point of the curve $\mathcal{C}$, and $A = (x(a), y(a), z(a))$ is its initial point, so we have found the fundamental theorem (158).

7. Conservative vector fields

7.1. Definition. *A vector field $\vec{F}$ is called **conservative** if one has*

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{x} = 0 \tag{160}$$

for every closed curve $\mathcal{C}$.

The name "conservative" derives from the interpretation of the integral in (160) as the amount of work done by the force field $\vec{F}$ around the closed curve $\mathcal{C}$. As an object moves throughout the plane along the curve $\mathcal{C}$, the force $\vec{F}$ acts on it, does work, and therefore provides energy to the object. The line integral (160) measures how much energy the force adds to the object after going around the curve $\mathcal{C}$ once. For a conservative vector field the total energy provided to the object is exactly zero, suggesting that its energy is ***conserved***.

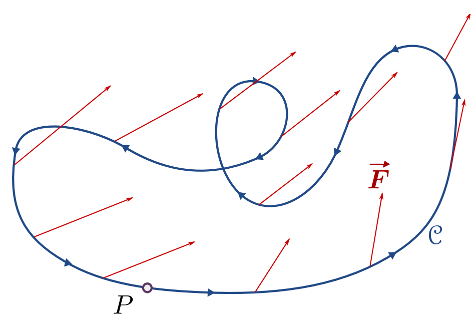


Figure 10. As an object moves along the closed curve $\mathcal{C}$ the force $\vec{F}$ acts on it. At times the force works in the direction of the motion, at other times it works against the motion. If the object starts at P, and goes around once, will it have gained energy when returning to P?

It follows from § 6.3 that any vector field $\vec{F}$ that is the gradient of a function is a conservative vector field. The following theorem says that these are actually the only conservative vector fields.

7.2. Theorem. *If $\vec{F}$ is a conservative vector field then there is a function f such that $\vec{F} = \vec{\nabla} f$.*

If $\vec{F} = \vec{\nabla} f$ the function

$$V = -f$$

is called a ***potential*** of the vector field $\vec{F}$. Thus a function V is a potential of the vector field F if

$$\vec{F} = -\vec{\nabla} V.$$

The potential V can be found by choosing one fixed point A, at which we declare $V(A) = 0$, and then computing the line integral

$$V(P) \stackrel{\text{def}}{=} -\int_A^P \vec{F} \cdot d\vec{x} \tag{161}$$

where the integral is a line integral over a path from the point A to the point P. The assumption that $\vec{F}$ is a conservative vector field implies that the integral in (161) does not depend on the path that is chosen.

8. Problems

1. Is the gravitational vector field

$$\vec{g}(x,y) = -g\vec{e}_2 = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

a conservative vector field?

2. Newton's gravitational vector field

$$\vec{F}(x,y) = -\frac{\vec{x}}{\|\vec{x}\|^3}$$

from §2.3, equation (146) is a conservative vector field. Show this by finding a potential of the form $f(x,y,z) = K\|\vec{x}\|^a$ for suitable constants a and K.

3. Reread the section in Chapter IV about Clairaut's theorem. You now have two ways to tell that a vector field $\vec{F} = P(x,y)\vec{e}_1 + Q(x,y)\vec{e}_2$ cannot be a gradient. Which are they?

4. **(a)** Compute the line integrals of the vector fields

$$\vec{F} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{G} = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

around the unit circle $\vec{x}(\theta) = \cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_2$.

(b) Which of the vector fields $\vec{F}$ or $\vec{G}$ ***cannot*** be a gradient, based on your answer to **(a)**?

(c) Can you conclude from your answer to **(a)** that any of the vector fields $\vec{F}$ or $\vec{G}$ must be a gradient?

9. Flux integrals

9.1. Definition of flux. In § 5.1 we defined the integral of a vector field along a curve $\mathcal{C}$ as the line integral of the tangential component of the vector field. If the curve $\mathcal{C}$ is not a space curve, but lies in the xy-plane, then one can also define the ***flux of the vector field across the curve***.

To define the flux we must first choose a unit normal vector $\vec{N}$ for the curve $\mathcal{C}$, i.e. at each point on $\mathcal{C}$ we must choose a vector $\vec{N}$ that has unit length and that is perpendicular to the curve:

$$\|\vec{N}\| = 1, \quad \text{and} \quad \vec{N}\cdot\vec{T} = 0.$$

Once a unit normal for the curve $\mathcal{C}$ has been chosen, the flux of a vector field $\vec{v}$ across the curve $\mathcal{C}$ in the direction of $\vec{N}$ is defined to be

$$\text{Flux} = \int_{\mathcal{C}} \vec{v}\cdot\vec{N}\, ds \tag{162}$$

The flux integral has a very natural interpretation if the vector field $\vec{v}$ is the velocity field of a two dimensional fluid flowing in the plane. If $\mathcal{C}$ is an arc in the plane, and if $\vec{N}$ is a unit normal to $\mathcal{C}$, then fluid will flow across this arc, and one can ask how much fluid flows across the arc in the direction of $\vec{N}$. The answer is given by the flux integral (162). For an explanation see Figure 11. There the arc is divided into many small sub arcs, which may be considered nearly straight. During a time interval of length Δt the fluid flowing through one such short arc sweeps out a parallelogram of which one side has length Δs, while the other is given by the vector $\vec{v}\Delta t$. The area of the small parallelogram is then $\vec{N}\cdot\vec{v}\Delta t\,\Delta s$. To get the rate at which fluid flows across the short arc we divide this by Δt to get $\vec{v}\cdot\vec{N}\Delta s$. Adding over all short arcs that comprise the curve $\mathcal{C}$ leads to the flux integral (162).

9.2. Flux across a closed curve. If $\mathcal{C}$ is a closed curve without self intersections, and $\mathcal{R}$ is the region it encloses then the flux of a vector field $\vec{v}$ across the curve $\mathcal{C}$ can again be interpreted as the rate at which fluid flows across the curve $\mathcal{C}$. Since the curve now encloses the bounded region $\mathcal{R}$, we can also say that the flux of $\vec{v}$ across the curve $\mathcal{C}$ is the net rate at which fluid leaves the region $\mathcal{R}$ (provided $\vec{N}$ is the outward pointing unit normal).

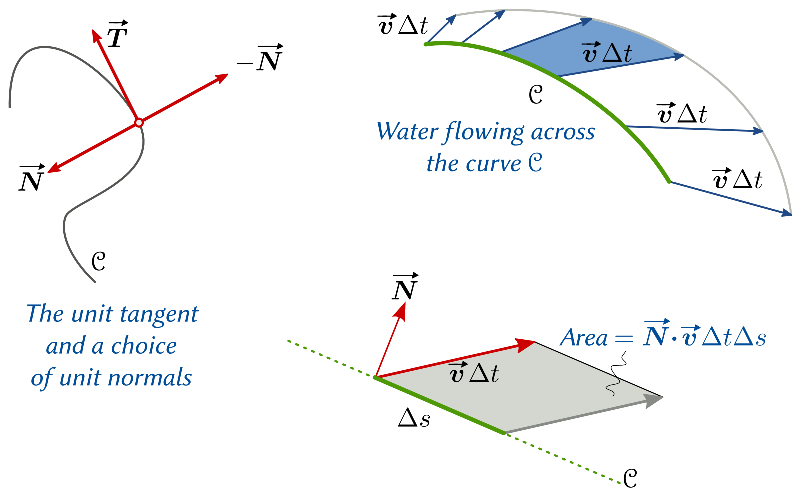


Figure 11. Left: At each point on a plane curve there are two choices of unit normal. If a unit tangent is given, then the most common choice of normal is to rotate the unit tangent counterclockwise by $90°$.

Top, right: if water is flowing over the plane with velocity field $\vec{v}$, then the rate at which water flows across the curve $\mathcal{C}$ in the direction of the normal $\vec{N}$ is given by the flux integral (162) of the velocity.

Bottom, right: the amount of water flowing across a short arc of length Δs on the curve $\mathcal{C}$ in time Δt is the area of a parallelogram one of whose sides is $\vec{v}\Delta t$. The area of this parallelogram is the length of the normal component of $\vec{v}\Delta t$ times Δs.

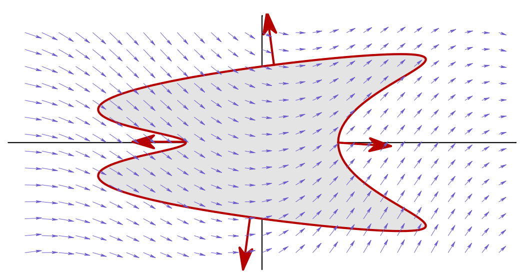


Figure 12. The flux of a vector field $\vec{v}$ across a closed curve measures the rate at which fluid is flowing out of the enclosed region, if $\vec{N}$ is the *outward* normal to the curve.

9.3. Example – water under the bridge. An endless river $\mathcal{R}$ occupies the strip

$$\mathcal{R} = \{(x, y) : -1 \le y \le 1\}$$

in the xy-plane. (The width of the river is 2.) The water in the river flows with velocity

$$\vec{v}(x, y) = V(1 - y^2)\vec{e}_1 = \begin{pmatrix} V(1-y^2) \\ 0 \end{pmatrix},$$

where V is a constant (it is the maximal velocity of the water, which is attained at $y = 0$, i.e. in the middle of the river; this flow is a two dimensional version of the Poiseuille flow from § 2.2.)

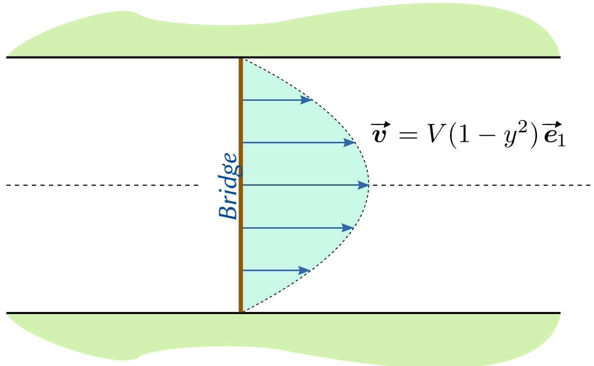


Figure 13. The shaded region represents the water that passed under the bridge during one time unit.

Question: *How much water flows from left to right through the line segment AB, where A is the point $(0,-1)$, and B is the point $(0,1)$?*

Solution: We parametrize the line segment by

$$\vec{x}(u) = \begin{pmatrix} 0 \\ u \end{pmatrix}, \quad -1 \le u \le 1.$$

Normally one refers to the parameter as "time," but since we are considering flowing water, time is already part of the problem. Therefore we have called the parameter on the curve u instead of t.

The line segment is vertical, so the unit normal is a horizontal vector of length 1, i.e. either $\vec{N} = \vec{e}_1$ or $\vec{N} = -\vec{e}_1$. We are asked to find how much water flows *from left to right*, so we need the normal that points to the right: $\vec{N} = +\vec{e}_1$.

We can now compute the integral. We begin with

$$ds = \|\vec{x}'(u)\| du = \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| du = du,$$

and

$$\vec{N} \cdot \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} V(1-y^2) \\ 0 \end{pmatrix} = V(1-y^2) = V\big(1-u^2\big),$$

which gives us

$$\int_{\mathcal{C}} \vec{v} \cdot \vec{N}\, ds = \int_{u=-1}^{1} V(1-u^2) du = V[u - \tfrac{1}{3}u^3]_{-1}^{1} = \tfrac{4}{3}V.$$

9.4. An expanding flow. A substance, perhaps a fluid, or a gas, is spreading from the origin and is moving with velocity field

$$\vec{v} = V \begin{pmatrix} x/R \\ y/R \end{pmatrix} = \frac{V}{R}\vec{x},$$

where V and R are constants: V has the units of a velocity, and R has the units of a length. The interpretation of these constants is that V is the speed at which fluid particles are moving when they are at a distance R from the origin.

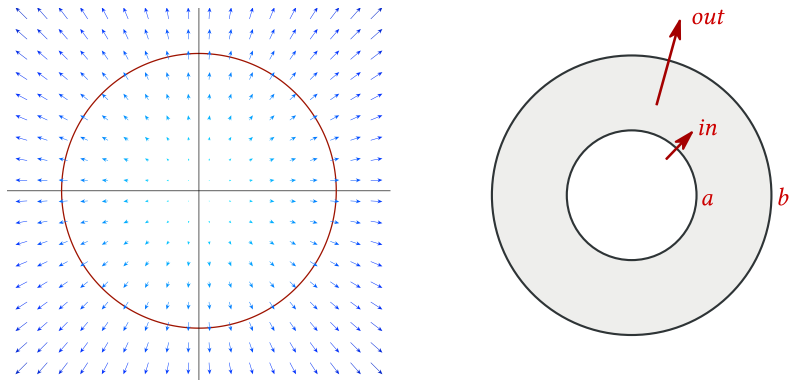


Figure 14. Left: The vector field $\vec{v}(x,y) = \frac{V}{R}\big(x\,\vec{e}_1 + y\,\vec{e}_2\big)$ and a circle with radius a. **Right:** This vector field cannot describe the flow of an "incompressible" fluid like water since more fluid flows out of the circle with radius b than through the circle with radius a: water would have to be created in the annular region between the two circles.

Question: How much fluid flows out of the circle with radius a?

Before we compute anything let us decide on the units that the answer should have. The question of "how much" fluid flows across the curve is ambiguous since we could answer in terms of mass (pounds or kilograms of fluid per second), or in terms of volume (gallons per second). These two are related by the density (pounds per gallon, kilos per liter, etc.) of the substance, and since we do not know anything about the density we will measure "how much" in terms of the volume of substance flowing across the curve per second. In fact, since we are dealing with a two dimensional model (the substance is flowing in the plane rather than three dimensional space, we will measure the area that flows across the curve instead of the volume.

Solution: We need to compute

$$\oint_{\mathcal{C}_a} \vec{v}\cdot\vec{N}\,ds$$

where $\mathcal{C}_a$ is the circle with radius a centered at the origin. The unit normal $\vec{N}$ is the outward pointing normal, because we are asked to find how much fluid flows *out* of the circle.

In this case $\vec{N}$ and $\vec{v}$ are parallel so that on the circle $\mathcal{C}_a$ we have

$$\vec{N}\cdot\vec{v} = \|\vec{v}\| = V\frac{a}{R}.$$

Therefore the flux integral is very simple, namely

$$\oint_{\mathcal{C}_a} \vec{v}\cdot\vec{N}\,ds = \oint_{\mathcal{C}_a} V\frac{a}{R}ds = V\frac{a}{R}\cdot \underbrace{\oint_{\mathcal{C}_a} ds}_{\text{Length of } \mathcal{C}_a} = 2\pi\frac{Va^2}{R}.$$

This answer is unrealistic if we assume that $\vec{v}$ really is the velocity field of a normal fluid (like water). To see what is wrong we compute how much fluid flows through circles of different radii a and b. If $a < b$ then the rate at which fluid flows through the smaller circle is less than the rate at which fluid flows out of the larger circle. The difference,

$$\frac{2\pi V}{R}\left(b^2 - a^2\right), \tag{163}$$

represents the amount of fluid that is (apparently) being created every second in the ring-shaped region between $\mathcal{C}_a$ and $\mathcal{C}_b$.

However, the computation could apply to a flowing gas. In this case we have computed the volume of gas that flows across each circle per time unit (or the area of gas, because we are using a two dimensional model here). A larger volume could flow across $\mathcal{C}_b$ than across $\mathcal{C}_a$, provided the gas is less dense at the circle $\mathcal{C}_b$ than it is at the smaller circle $\mathcal{C}_a$. This kind of reasoning is important for fluid and gas dynamics, and in fact appears in many other branches in physics.

10. Green's Theorem

We have seen that the line integral $\oint_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{T}}\,ds$ of a vector field along a closed curve vanishes if the vector field happens to be the gradient of some function (§ 6.3), but if the vector field $\vec{\boldsymbol{F}}$ is not the gradient of a function then its line integral around a closed curve need not vanish (see the example in § 5.8). We have also seen examples where a flux integral $\oint_{\mathcal{C}} \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{N}}\,ds$ is non-zero.

Green's theorem relates the line integral of any vector field on the boundary curve $\mathcal{C}$ of some domain $\mathcal{R}$ with a double integral involving partial derivatives of the vector field on the domain $\mathcal{R}$ itself. There are two versions of the theorem, depending on what kind of line integral one considers. The first version is for "work-type integrals," and is best written in differential form notation. The second version is about flux integrals.

10.1. Simply connected domains.

In both versions of Green's theorem one has a plane region $\mathcal{R}$ and its boundary curve(s). The boundary curves of a region can be somewhat complicated. The simplest situation is where the domain $\mathcal{R}$ is ***simply connected.*** This means that $\mathcal{R}$ is the region enclosed by *one* curve $\mathcal{C}$ (the curve $\mathcal{C}$ is not allowed to intersect itself.) Another way of describing what a simply connected region is, is to say that a region is simply connected if "it has no holes." See Figure 15. If a domain is not simply connected, then its boundary may consist of more than one curve (Figure 15 on the right).

Green's theorem. *Let $\mathcal{R}$ be a simply connected region in the plane, and let $\mathcal{C}$ be the boundary curve of the region $\mathcal{R}$, with the counter clockwise orientation. Let*

$$\vec{\boldsymbol{v}}(x,y) = P(x,y)\,\vec{\boldsymbol{e}}_1 + Q(x,y)\,\vec{\boldsymbol{e}}_2$$

be a vector field that is defined and has continuous derivatives ***everywhere*** *in $\mathcal{R}$. Then one has*

$$\oint_{\mathcal{C}} P(x,y)dx + Q(x,y)dy = \iint_{\mathcal{R}} \left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\}\, dA. \tag{164}$$

The second form of Green's Theorem is about flux integrals and is often called the "***divergence theorem.***"

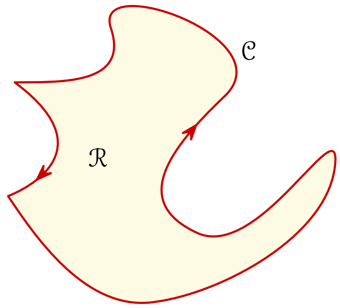


Figure 15. Left: A simply connected domain, i.e. a domain"without holes." **Right:** a non-simply connected domain, i.e. "a domain with a hole." For this non-simply connected domain the boundary consists of two closed curves rather than one.

Flux version of Green's theorem. *Let $\mathcal{R}$ be a bounded domain in the plane that is enclosed by a curve $\mathcal{C}$. If*

$$\vec{\boldsymbol{v}} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$

is a vector field that is everywhere defined and differentiable on $\mathcal{R}$, then

$$\oint_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{N}}\, ds = \iint_{\mathcal{R}} \Big\{ \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \Big\}\, dA \tag{165}$$

where $\vec{\boldsymbol{N}}$ is the outward unit normal for the domain $\mathcal{R}$.

The quantity

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \tag{166}$$

is called the ***divergence*** of the vector field $\vec{\boldsymbol{v}}$, and is written as "div $\vec{\boldsymbol{v}}$." It is one of several combinations of partial derivatives of vector fields that turn out to be useful. See § 16 for more of these.

10.2. Examples illustrating Green's Theorem.

An example where the line integral vanishes on any closed curve. Consider the vector field

$$\vec{\boldsymbol{F}}(x,y) = x\,\vec{\boldsymbol{e}}_1 + y\,\vec{\boldsymbol{e}}_2 = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right),$$

and let $\mathcal{C}$ be a closed curve in the plane, that encloses the region $\mathcal{R}$. Then the line integral of $\vec{\boldsymbol{F}}$ along $\mathcal{C}$ is given by

$$\begin{aligned} \oint_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} &= \iint_{\mathcal{R}} \Big\{ \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \Big\}\, dA \\ &= \iint_{\mathcal{R}} 0\, dA \\ &= 0. \end{aligned}$$

We find that the integral is always zero, no matter what the region $\mathcal{R}$ is. If we were lucky enough to note that this particular vector field is a gradient,

$$\vec{F} = \begin{pmatrix} x \\ y \end{pmatrix} = \vec{\nabla}\left(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2\right),$$

then we could also have used (159) to conclude that $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{x} = 0$ for any closed curve $\mathcal{C}$.

The expanding gas example again. Let

$$\vec{v}(x,y) = \frac{V}{R}\vec{x} = \frac{V}{R}x\vec{e}_1 + \frac{V}{R}y\vec{e}_2$$

be the velocity field of the expanding gas from § 9.4, and let $\mathcal{C}$ be any closed curve that is the boundary curve of some domain $\mathcal{R}$. We again compute the flux of the velocity field across the curve $\mathcal{C}$ in the direction of its outward normal, but this time we use Green's Theorem.

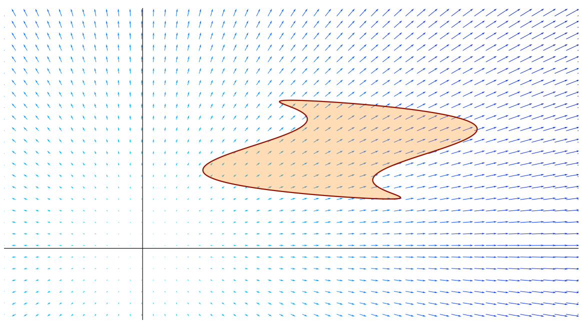


Figure 16. A "gas" is flowing in the plane with velocity field $\vec{v}$. At what rate is gas flowing out of the shaded region?

The answer turns out to be proportional to the area of the region.

According to Green's Theorem we have

$$\oint_{\mathcal{C}} \vec{v}\cdot\vec{N}\, ds = \iint_{\mathcal{R}} \operatorname{div} \vec{v}\, dA$$

where $\operatorname{div}\vec{v}$ is the divergence of $\vec{v}$, defined in (166). Thus

$$\operatorname{div}\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{\partial\{Vx/R\}}{\partial x} + \frac{\partial\{Vy/R\}}{\partial y} = \frac{V}{R} + \frac{V}{R} = 2\frac{V}{R},$$

and finally,

$$\oint_{\mathcal{C}} \vec{v}\cdot\vec{N}\, ds = \iint_{\mathcal{R}} 2\frac{V}{R}\, dA = 2\frac{V}{R}\cdot \text{area of } \mathcal{R}. \tag{167}$$

This is consistent with our previous computation in § 9.4. There we found in (163) that the amount of fluid produced in an annulus of inner and outer radii a and b is $2\pi\frac{V}{R}(b^2 - a^2)$. Since the area of the annulus is $\pi b^2 - \pi a^2$ this is the same result that we just found in (167).

11. Conservative vector fields and Clairaut's theorem

Let $\vec{F}(x,y) = P(x,y)\vec{e}_1 + Q(x,y)\vec{e}_2$ be a vector field on some region $\mathcal{R}$ in the plane. The fundamental theorem for line integrals and Clairaut's Theorem (III.13.3) provide connections between conservative vector fields, gradient vector fields, and the partial derivatives of P and Q. To summarize what we have seen so far, recall that…

- *if $\vec{F} = \vec{\nabla} f$ for some function $f(x,y)$ then $\vec{F}$ is conservative,*

- *if $\vec{F}$ is conservative then $\vec{F} = \vec{\nabla} f$ for some function $f(x, y)$*
- *if $\vec{F} = \vec{\nabla} f$, then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.*

Looking at this list we see that the missing statement would be that "$P_y = Q_x$ implies that $\vec{F}$ is a gradient vector field." This turns out only to be true if we impose an extra assumption on the domain $\mathcal{R}$, namely, $\mathcal{R}$ must be simply connected (see § 10.1.) We formulate this more precisely in a theorem.

11.1. Theorem. *If the domain $\mathcal{R}$ is simply connected and if $\vec{F} = P\vec{e}_1 + Q\vec{e}_2$ is a vector field on $\mathcal{R}$ for which*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \tag{168}$$

then $\vec{F}$ is conservative, and hence $\vec{F} = \vec{\nabla} f$ for some function f.

The proof is an instructive application of Green's theorem, so we include it here:

Proof. We will show that (168) implies that $\vec{F}$ is conservative, i.e. that the line integral of $\vec{F}$ around any closed curve in $\mathcal{R}$ vanishes.

Let $\mathcal{C}$ be a closed curve in $\mathcal{R}$, and assume to begin with that the curve does not intersect itself. Then it must enclose a domain $\mathcal{D}$, and since $\mathcal{R}$ is simply connected, the domain $\mathcal{D}$ enclosed by the curve $\mathcal{C}$ lies entirely within $\mathcal{R}$. We can therefore apply Green's theorem to the curve $\mathcal{C}$ and conclude that

$$\oint_{\mathcal{C}} \vec{F}\cdot d\vec{x} = \oint_{\mathcal{C}} P dx + Q dy = \iint_{\mathcal{D}} \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dA = 0.$$

This is what we have to show. For a complete proof we would still have to remove the assumption we made that the curve $\mathcal{C}$ does not intersect itself. We will not do this in detail, but merely point out that if $\mathcal{C}$ has one self intersection, then one can break the

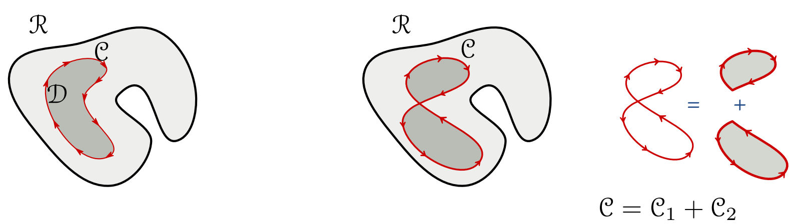


Figure 17. Left: In the proof of Theorem 11.1 the case that $\mathcal{C}$ has no self intersections. **Right:** the case where $\mathcal{C}$ has at least one self intersection.

curve into pieces, each of which forms a closed curve without self intersections, to which we can apply the previous arguments.

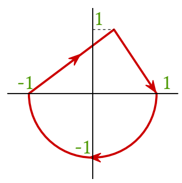
∎

12. Problems

1. Use Green's theorem to compute the line integrals

$$I = \oint_{\mathcal{C}} y\,dx - x\,dy$$
$$J = \oint_{-\mathcal{C}} y\,dx - x\,dy$$
$$K = \oint_{\mathcal{C}} (x - \sin y)\,dy$$

where $\mathcal{C}$ is this curve:



In this drawing the circle has radius 1, and the height of the triangle is also 1. The orientation of the curve is in the direction of the arrows.

2. Let $\mathcal{R}$ be the unit square, i.e. $\mathcal{R} = \{(x, y) : 0 \le x, y \le 1\}$. Let $\mathcal{C}$ be the boundary of the square $\mathcal{R}$ traversed in counterclockwise sense.

(a) Compute $\int_{\mathcal{C}} 2y\,dx + 3x\,dy$ by finding parametrizations of the edges and applying the definition of the line integral.

(b) Compute $\int_{\mathcal{C}} 2y\,dx + 3x\,dy$ by applying Green's theorem and computing a suitable double integral over $\mathcal{R}$. •

3. Compute $\oint_{\mathcal{C}} \vec{\nabla}(x^2y^2)\cdot\vec{T}\,ds$ where $\mathcal{C}$ is the counter clockwise traversed boundary of the region $\mathcal{R}$ defined by $x^2 + y^2 < 16$. •

4. A gas is flowing in the plane with velocity field

$$\vec{v}(x, y) = \begin{pmatrix} 1 \\ -y \end{pmatrix}.$$

(a) Draw the vector field.

(b) How much gas flows out of the rectangle $\mathcal{R}$ defined by $0 < x < L$, $-H < y < H$?

5. In each of the following problems $\mathcal{C}$ is the counter clockwise traversed boundary of the region $\mathcal{D}$ and you are asked to compute the indicated line integral in two ways: directly, and by using Green's Theorem.

(a) $\oint_{\mathcal{C}} xy\,dx + xy\,dy$, $\quad\mathcal{R} : 0 \le x, y \le 1$. •

(b) $\oint_{\mathcal{C}} e^{2x+3y}\,dx + e^{xy}\,dy$, $\quad\mathcal{R} : -2 \le x \le 2, -1 \le y \le 1$. •

(c) $\oint_{\mathcal{C}} \vec{F}\cdot\vec{T}ds$, $\quad\vec{F}(x, y) = \left(\begin{smallmatrix} y\cos x \\ y\sin x \end{smallmatrix}\right)$, $\quad\mathcal{R} : 0 \le x \le \pi/2, 1 \le y \le 2$. •

(d) $\oint_{\mathcal{C}} xy^2\,dx + x^2y\,dy$, $\quad\mathcal{R} : 0 \le x \le 1, 0 \le y \le x$. •

(e) $\oint_{\mathcal{C}} x^2y\,dx + xy^2\,dy$, $\quad\mathcal{R} : 0 \le x \le 1, 0 \le y \le x$. •

(f) $\oint_{\mathcal{C}} x\sqrt{y}\,dx + \sqrt{x+y}\,dy$, $\quad\mathcal{R} : 1 \le x \le 2, 2x \le y \le 4$. •

(g) $\oint_{\mathcal{C}} (x/y)\,dx + (2 + 3x)\,dy$, $\quad\mathcal{R} : 1 \le x \le 2, 1 \le y \le x^2$. •

(h) $\oint_{\mathcal{C}} \sin y\,dx + \sin x\,dy$, $\quad\mathcal{R} : 0 \le x \le \pi/2, x \le y \le \pi/2$. •

(i) $\oint_{\mathcal{C}} x\ln y\,dx$, $\quad\mathcal{R} : 1 \le x \le 2, e^x \le y \le e^{x^2}$. •

(j) $\oint_{\mathcal{C}} \sqrt{1+x^2}\,dy$, $\quad\mathcal{R} : -1 \le x \le 1, x^2 \le y \le 1$. •

(k) $\displaystyle\oint_{\mathcal{C}} x^2y\,dx - xy^2\,dy,$ $\qquad \mathcal{R} : x^2 + y^2 \le 1.$

(l) $\displaystyle\oint_{\mathcal{C}} \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{N}}\,ds, \quad \vec{\boldsymbol{v}}(x,y) = \left(\begin{smallmatrix} xy^2 \\ x^2y \end{smallmatrix}\right),$ $\qquad \mathcal{R} : x^2 + y^2 \le 1,$ $\quad \vec{\boldsymbol{N}}$ the outward normal.

(m) $\displaystyle\oint_{\mathcal{C}} y^3\,dx + 2x^3\,dy,$ $\qquad \mathcal{R} : x^2 + y^2 \le 4.$

13. Surfaces and Surface integrals

In addition to integrals over two and three dimensional domains, and line integrals over curves in the plane or in space, one can also integrate over surfaces. In this section we will give a quick introduction to surfaces and surface integrals. For an in-depth study of the subject, students should consider taking a more advanced course on vector calculus, such as Math 321.

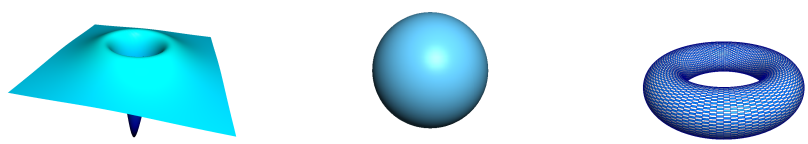


Figure 18. Two dimensional surfaces.

13.1. Surfaces and surface patches. We can think of a curve as the result of taking a line and bending it into some curved shape. In the same way a surface can be thought of as the result of taking a portion of a flat plane and bending and twisting it into some other shape. Just as some curves appear as the boundaries (or edges) of plane domains, some surfaces appear as boundaries of domains in three dimensional space. For example, the sphere centered at the origin and with radius R

$$x^2 + y^2 + z^2 = R^2 \tag{169}$$

is the boundary of the three dimensional ball it encloses.

Surfaces can be described using "defining equations," i.e. by specifying an equation whose zero set is the intended surface. For example, the sphere of radius R has (169) as defining equation. For purposes of integration it is more convenient to represent surfaces in terms of ***surface patches.*** These are the surface analog of parametrized curves.

Definition. A surface patch is a differentiable vector function of two variables

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(u,v), \qquad a \le u \le b, \quad c \le v \le d.$$

13.2. Example – the graph of a function is a surface patch. If $z = f(x,y)$ is a function defined for $a \le x \le b$, $c \le y \le d$, then its graph can be thought of as a surface patch, where

$$\vec{\boldsymbol{x}}(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}. \tag{170}$$

In words: we take the x and y coordinates as parameters, setting $x = u$ and $y = v$. The z component of any point on the patch is then $z = f(x,y) = f(u,v)$.

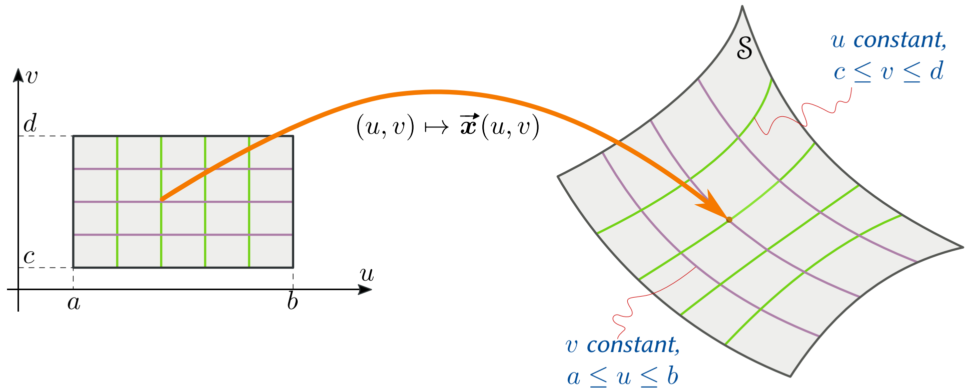


Figure 19. A surface patch. A vector function $\vec{x}$ of two variables u and v maps a piece of the uv-plane into three dimensional space. The rectangular grid in the uv domain gets mapped onto a network of curves on the surface patch $\mathcal{S}$. If the rectangular grid in the uv-domain is sufficiently fine, then the corresponding curves on the surface divide the surface patch into small pieces that are approximately parallelograms.

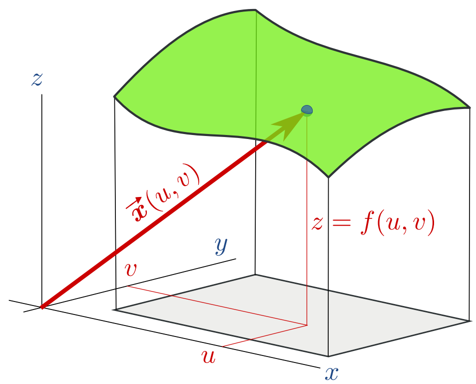


Figure 20. A graph as a surface patch: the graph of a function $z = f(x, y)$ can be represented as a surface patch. The vector function $\vec{x}$ that parametrizes the graph is $\vec{x}(u, v) = u\vec{e}_1 + v\vec{e}_2 + f(u, v)\vec{e}_3$.

13.3. Example – the sphere as a surface patch. The sphere is a two dimensional surface, and one way to parametrize it is to use spherical coordinates. Thus

$$\vec{x}(\theta, \varphi) = \begin{pmatrix} R\cos\varphi\sin\theta \\ R\sin\varphi\sin\theta \\ R\cos\theta \end{pmatrix} \tag{171}$$

with

$$0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$$

is a surface patch that parametrizes the sphere: it is a parametrization of the sphere. See §VI-6.2 where spherical coordinates were defined, and see Figure 21 for a picture.

All points with $\theta = 0$ are mapped to the "north pole"; all points with $\theta = \pi$ correspond to the "south pole"; the points with $\theta = \frac{1}{2}\pi$ form the "equator."

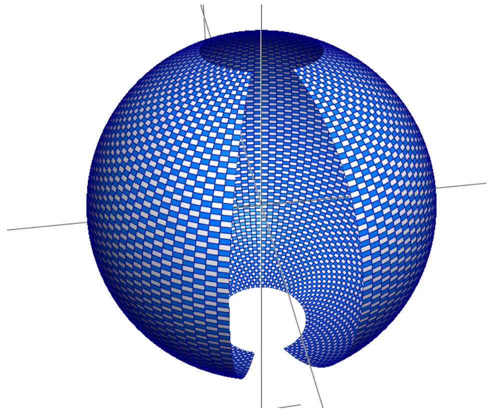


Figure 21. Sphere: a piece of the sphere parametrized by the surface patch in (171). Shown is the piece with $0.1\pi \le \theta \le 0.9\pi$ and $0.1\pi \le \varphi \le 1.9\pi$.

13.4. Area of a surface patch. For any given surface we can ask "what is its surface area?" The intuitive interpretation of this could be

(172) *"how much paint do we need to cover one side of the surface?"*

or

(173) *"how much paper to we need to make the surface?"*

Neither interpretation stands up to closer scrutiny: there are surfaces, like the Möbius strip in Figure 22, that only have one side, so that questions (172) and (173) will give

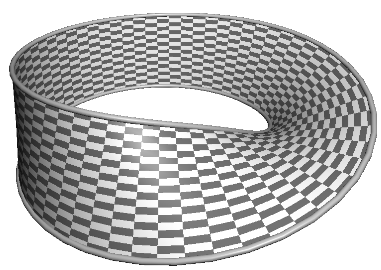


Figure 22. A Möbius strip. What is the surface area of this strip, and how many square inches of paper do we need to make one?

different answers. On the other hand, while it is possible to take a flat piece of paper and bend it in the shape of a cylinder, a cone, or a Möbius strip, it is not possible to bend a flat piece of paper into a sphere without ripping or stretching it (and thus changing its area.)

In spite of these (and other) issues we will argue from intuition and derive a formula for the area of a surface patch. The story is very similar to the derivation of the arc length of a parametrized curve in § II.13.

If $\vec{x}(u, v)$ is a surface patch with domain $a \le u \le b$, $c \le v \le d$, then we divide its domain into many small rectangular pieces of size Δu by Δv by partitioning both the u and v intervals. See the left half of Figure 23. This leads to a partitioning of the surface patch into small regions, each of which is approximately a parallelogram (on the right

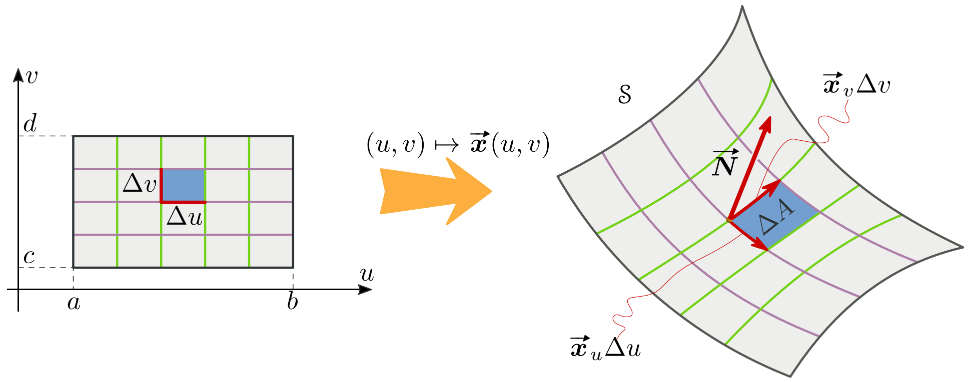


Figure 23. Computing the area and normal to a surface patch. The small rectangle in the uv-domain gets mapped to a small region on the surface patch. This small region is almost a parallelogram whose sides are given by the vectors $\vec{\boldsymbol{x}}_u \Delta u$ and $\vec{\boldsymbol{x}}_v \Delta v$.

in Figure 23). We compute the area of the surface patch by adding the areas of all these smaller pieces. Since any such piece is approximately a parallelogram, we can find its area by computing the cross product of the vectors defined by its edges. To find these

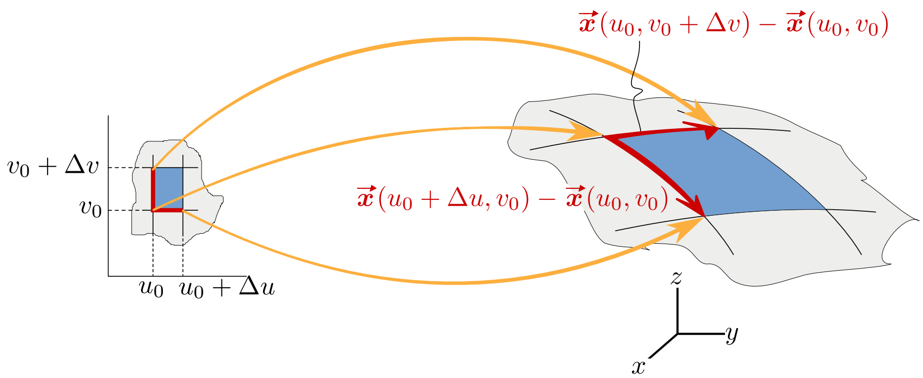


Figure 24. The small blue rectangle in the uv-plane from Figure 23, and its image on the surface patch.

edges consider Figure 24. In a small partition piece on the surface patch, the parameter u is allowed to vary between some value u_0 and $u_0 + \Delta u$, while the other parameter v is allowed to vary between some v_0 and $v_0 + \Delta v$. One edge of the surface patch (on the right in Figure 24) represents the change in $\vec{\boldsymbol{x}}(u, v)$ as u is increased by Δu, while keeping v constant; i.e. it is

$$\vec{\boldsymbol{x}}(u_0 + \Delta u, v_0) - \vec{\boldsymbol{x}}(u_0, v_0) \approx \frac{\partial \vec{\boldsymbol{x}}}{\partial u}(u_0, v_0) \cdot \Delta u.$$

The other edge represents the change in $\vec{\boldsymbol{x}}(u, v)$ when v is increased by Δv and is thus given by

$$\vec{\boldsymbol{x}}(u_0, v_0 + \Delta v) - \vec{\boldsymbol{x}}(u_0, v_0) \approx \frac{\partial \vec{\boldsymbol{x}}}{\partial v}(u_0, v_0) \cdot \Delta v.$$

The area of the small parallelogram on the surface patch is therefore the length of the cross-product of these two vectors:

$$\Delta A \approx \left\| \frac{\partial \vec{\boldsymbol{x}}}{\partial u}(u_0, v_0) \cdot \Delta u \times \frac{\partial \vec{\boldsymbol{x}}}{\partial v}(u_0, v_0) \cdot \Delta v \right\|$$
$$= \|\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v\| \, \Delta u \, \Delta v.$$

Adding this over all pieces that make up the surface patch gives us the total area of the patch:

$$\text{Area of } \mathcal{S} = \int_c^d \int_a^b \|\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v\| \, du \, dv. \tag{174}$$

The quantity that appears in this integral appears in many other surface integrals and is called "the area element" of the surface patch $\vec{\boldsymbol{x}}$. The usual notation for this quantity is

$$dA = \|\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v\| \, du \, dv, \tag{175}$$

and it is thought of as the "area of an infinitesimally small piece of the surface."

13.5. Surface integrals. If $f(x, y, z)$ is some function that is defined on the surface (e.g. a density of some kind), then one defines its integral over the surface to be

$$\iint_{\mathcal{S}} f(x, y, z) \, dA = \int_c^d \int_a^b f(\vec{\boldsymbol{x}}(u, v)) \, \|\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v\| \, du \, dv. \tag{176}$$

Here $f(\vec{\boldsymbol{x}}(u, v))$ is the result of substituting the surface parametrization $\vec{\boldsymbol{x}}(u, v)$ in the function.

13.6. Unit normal to a surface patch. From Figures 23 and 24 it appears that both vectors $\vec{\boldsymbol{x}}_u$ and $\vec{\boldsymbol{x}}_v$ are tangent to the surface, and that their cross product $\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v$ is perpendicular to the surface. We adopt this as the definition of the tangent plane and normal direction to the surface:

Definition. Let $\vec{\boldsymbol{x}}$ be a surface patch, and let X_0 be a point with position vector $\vec{\boldsymbol{x}}(u_0, v_0)$ on the surface patch. If

$$\vec{\boldsymbol{m}} \stackrel{\text{def}}{=} \vec{\boldsymbol{x}}_u(u_0, v_0) \times \vec{\boldsymbol{x}}_v(u_0, v_0) \neq \vec{\boldsymbol{0}},$$

then the vector $\vec{\boldsymbol{m}}$ defines the normal direction to the surface. The tangent plane to the surface through X_0 is the plane with normal vector $\vec{\boldsymbol{m}}$ that goes through X_0.

In general the vector $\vec{\boldsymbol{m}}$ does not have unit length, and one often needs a normal vector with length one for the surface. Thus one defines

$$\vec{\boldsymbol{N}} = \frac{\vec{\boldsymbol{m}}}{\|\vec{\boldsymbol{m}}\|} = \frac{\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v}{\|\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v\|} \tag{177}$$

to be the ***unit normal*** for the surface patch $\vec{\boldsymbol{x}}$. Note that $-\vec{\boldsymbol{N}}$ also is a unit vector that is normal to the surface.

13.7. Flux across a surface patch. In § 9 we defined the flux across a curve of a vector field $\vec{v}$ (which we think of as the velocity field of some flowing liquid or gas). The set-up in § 9 was purely two dimensional. Now that we have introduced surface integrals we can formulate the same concept for the more realistic situation of a fluid flowing through three dimensional space with velocity field $\vec{v}$. We define the ***flux of a vector field $\vec{v}$ across a surface patch*** to be

$$\mathsf{Flux} = \iint_{S} \vec{v} \cdot \vec{N}\, dA \tag{178}$$

We have expressions for both $\vec{N}$ and dA (namely, (175) and (177)). When put together, they simplify to

$$\vec{N}\, dA = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|} \cdot \|\vec{x}_u \times \vec{x}_v\|\, du\, dv = \vec{x}_u \times \vec{x}_v\, du\, dv$$

Therefore the flux integral can be computed as

$$\iint_{S} \vec{v} \cdot \vec{N}\, dA = \int_c^d \int_a^b \vec{v} \cdot (\vec{x}_u \times \vec{x}_v)\, du\, dv. \tag{179}$$

14. Examples

14.1. Area and unit normal of a sphere. The sphere with radius R can be represented by the surface patch

$$\vec{x}(\theta, \varphi) = \begin{pmatrix} R\cos\varphi\sin\theta \\ R\sin\varphi\sin\theta \\ R\cos\theta \end{pmatrix}, \tag{180}$$

for which we have

$$\vec{x}_\theta = R\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{pmatrix}, \qquad \vec{x}_\varphi = R\begin{pmatrix} -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta \\ 0 \end{pmatrix}$$

and hence

$$\begin{aligned}
\vec{x}_\theta \times \vec{x}_\varphi &= R^2 \begin{pmatrix} \cos\varphi\sin^2\theta \\ \sin\varphi\sin^2\theta \\ \cos^2\varphi\sin\theta\cos\theta + \sin^2\varphi\sin\theta\cos\theta \end{pmatrix} \\
&= R^2 \begin{pmatrix} \cos\varphi\sin^2\theta \\ \sin\varphi\sin^2\theta \\ \sin\theta\cos\theta \end{pmatrix} \\
&= R^2\sin\theta \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}.
\end{aligned}$$

The length of $\vec{x}_\theta \times \vec{x}_\varphi$ is

$$\|\vec{x}_\theta \times \vec{x}_\varphi\| = R^2\sin\theta \left\| \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix} \right\| = R^2\sin\theta.$$

and the area element on the sphere is

$$dA = R^2\sin\theta\, d\theta\, d\varphi.$$

Integrating over the sphere gives us the area of the sphere:

$$\text{Area of sphere} = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} R^2 \sin\theta \, d\theta \, d\varphi = 4\pi R^2, \tag{181}$$

which is the familiar answer.

We also find from our formula for $\vec{x}_\theta \times \vec{x}_\varphi$ that the unit normal at the point with position vector $\vec{x}(\theta, \varphi)$ is

$$\vec{N} = \frac{\vec{x}_\theta \times \vec{x}_\varphi}{\|\vec{x}_\theta \times \vec{x}_\varphi\|} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$$

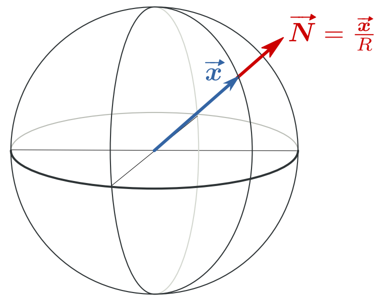


Figure 25. The unit normal at a point on a sphere centered at the origin has the same direction as the position vector of the point.

Looking back at the definition (180) of our surface patch we see that

$$\vec{x} = R\vec{N}, \text{ or, } \vec{N} = \frac{\vec{x}}{R}.$$

In words: the unit normal is just the position vector $\vec{x}$ rescaled to length one. Perhaps with hindsight, this should be clear from a drawing of the sphere (e.g. Figure 25). In many geometrically simple situations it is often easier to guess the unit normal from a drawing than by going through a computation like the one we did in this example. And sometimes it is even possible to compute the area element without working out $\vec{x}_u$, $\vec{x}_v$, and their cross product. For instance, it is possible to derive our formula for the area element $dA = R^2 \sin\theta d\theta d\varphi$ from a drawing like Figure VI.16.

14.2. The flux of a vector field across the sphere. We consider the velocity field of the expanding gas from § 9.4 again, except we now consider a gas occupying three dimensional space:

$$\vec{v} = \frac{V_0}{R_0}\vec{x}.$$

Here V_0 and R_0 are constants: V_0 is the velocity of the gas when it has reached distance R_0 from the origin.

We compute the flux

$$\textit{Flux} = \iint_{\mathcal{S}_R} \vec{v} \cdot \vec{N} \, dA$$

of this velocity field across the sphere $\mathcal{S}_R$ with radius R in two ways.

First, we use the formula for $\vec{\boldsymbol{N}}\,dA$

$$\vec{\boldsymbol{N}}\,dA = \vec{\boldsymbol{x}}_\theta \times \vec{\boldsymbol{x}}_\varphi \, d\varphi\, d\theta = R^2 \sin\theta \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix} d\varphi\, d\theta$$

and compute

$$\begin{aligned}
\textit{Flux} &= \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \frac{V_0}{R_0}\vec{\boldsymbol{x}}\cdot R^2\sin\theta \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix} d\varphi\, d\theta \\
&= \frac{V_0}{R_0}R^2 \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \begin{pmatrix} R\cos\varphi \sin\theta \\ R\sin\varphi\sin\theta \\ R\cos\theta \end{pmatrix}\cdot\begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix} \sin\theta\, d\varphi\, d\theta \\
&= \frac{V_0}{R_0}R^3 \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \sin\theta\, d\varphi\, d\theta \\
&= 4\pi\frac{V_0}{R_0}R^3.
\end{aligned}$$

The second approach is more geometrical and avoids computing any integrals. We begin by noting that the unit normal on the sphere at the point with position vector $\vec{\boldsymbol{x}}$ is $\vec{\boldsymbol{N}} = \vec{\boldsymbol{x}}/R$, and hence that

$$\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{N}} = \frac{V_0}{R_0}\vec{\boldsymbol{x}}\cdot\frac{\vec{\boldsymbol{x}}}{R} = \frac{V_0}{R_0}\frac{\vec{\boldsymbol{x}}\cdot\vec{\boldsymbol{x}}}{R} = \frac{V_0}{R_0}\frac{R^2}{R} = V_0\frac{R}{R_0}.$$

The quantity we want to integrate is therefore constant. We find that the flux is

$$\textit{Flux} = \iint_{\mathcal{S}} V_0\frac{R}{R_0}dA = V_0\frac{R}{R_0}\cdot \textit{Area of } \mathcal{S} = V_0\frac{R}{R_0}\cdot 4\pi R^2,$$

which is the same as we got using the first approach.

15. The divergence theorem and Stokes' theorem

15.1. The divergence theorem in three dimensions. *If $\mathcal{S}$ is a surface that encloses a three dimensional region $\mathcal{R}$, if $\vec{\boldsymbol{v}}$ is a vector field that is defined and differentiable on all of $\mathcal{R}$, and if $\vec{\boldsymbol{N}}$ is the outward unit normal on $\mathcal{S}$, then*

$$\iint_{\mathcal{S}} \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{N}}\,dA = \iiint_{\mathcal{R}} \operatorname{div}\vec{\boldsymbol{v}}\,dV \tag{182}$$

where $\operatorname{div}\vec{\boldsymbol{v}}$ *is the divergence of the vector field* $\vec{\boldsymbol{v}}$.

By definition the divergence of the vector field

$$\vec{\boldsymbol{v}} = \begin{pmatrix} v_1(x,y,z) \\ v_2(x,y,z) \\ v_3(x,y,z) \end{pmatrix} = v_1(x,y,z)\vec{\boldsymbol{e}}_1 + v_2(x,y,z)\vec{\boldsymbol{e}}_2 + v_3(x,y,z)\vec{\boldsymbol{e}}_3$$

is

$$\operatorname{div}\vec{\boldsymbol{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

15.2. Stokes' Theorem. *If $\mathcal{S}$ is a surface patch, if the curve $\mathcal{C}$ is the boundary of $\mathcal{S}$, and if $\vec{\boldsymbol{F}}$ is a differentiable vector field defined everywhere on the surface, then*

$$\oint_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} = \iint_{\mathcal{S}} (\mathbf{curl}\, \vec{\boldsymbol{F}}) \cdot \vec{\boldsymbol{N}}\, dA \tag{183}$$

where the "curl" of a vector field is defined by

$$\mathbf{curl}\, \vec{\boldsymbol{F}} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

15.3. Example involving the divergence theorem. We return to the computation in § 14.2 of the flux across the sphere $\mathcal{S}$ of radius R of the expanding gas vector field $\vec{\boldsymbol{v}} = \frac{V_0}{R_0}\vec{\boldsymbol{x}}$. According to the divergence theorem we have

$$\iint_{\mathcal{S}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{N}}\, dA = \iiint_{\mathcal{B}} \operatorname{div} \vec{\boldsymbol{v}}\, dV$$

where $\mathcal{B}$ is the region enclosed by the sphere (the ball of radius R).

The divergence of $\vec{\boldsymbol{v}}$ is easy to compute:

$$\begin{aligned} \operatorname{div} \vec{\boldsymbol{v}} &= \frac{\partial}{\partial x}\left\{\frac{V_0 x}{R_0}\right\} + \frac{\partial}{\partial y}\left\{\frac{V_0 y}{R_0}\right\} + \frac{\partial}{\partial z}\left\{\frac{V_0 z}{R_0}\right\} \\ &= 3\frac{V_0}{R_0} \end{aligned}$$

Since the divergence is constant its integral over $\mathcal{B}$ is easy:

$$\begin{aligned} \iiint_{\mathcal{B}} \operatorname{div} \vec{\boldsymbol{v}}\, dV &= 3\frac{V_0}{R_0} \cdot \text{Volume of } \mathcal{B} \\ &= 3\frac{V_0}{R_0}\frac{4}{3}\pi R^3 \\ &= 4\pi\frac{V_0}{R_0}R^3 \end{aligned}$$

where we have used that the volume of the ball $\mathcal{B}$ is $\frac{4}{3}\pi R^3$.

16. $\vec{\nabla}$ – differentiating vector fields

The components of a vector field are functions, and therefore we can differentiate them. As we have seen in the divergence theorem and Stokes' theorem, various combinations of the partial derivatives of vector fields turn out to be very useful. The easiest way to describe these is to introduce the so-called "nabla operator" (or "del operator") defined by

$$\vec{\boldsymbol{\nabla}} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \frac{\partial}{\partial x}\vec{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\vec{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\vec{\boldsymbol{k}}. \tag{184}$$

At first sight something is missing here: there are partial derivatives, but the function whose derivative is supposed to be taken is missing. This is intentional, and the way $\vec{\boldsymbol{\nabla}}$ is to be interpreted is as follows:

in any formula containing $\vec{\nabla}$,
the partial derivatives are to be taken of
*all functions appearing to the **right** of the $\vec{\nabla}$.*

For example, if $f(x, y, z)$ is a function of (x, y, z), then

$$\vec{\nabla} f = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} f(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x}(x, y, z) \\ \frac{\partial f}{\partial y}(x, y, z) \\ \frac{\partial f}{\partial z}(x, y, z) \end{pmatrix}.$$

So $\vec{\nabla} f$ is the gradient of the function f, just as we had defined it before. Sometimes a different notation is used, namely

$$\vec{\nabla} f = \mathbf{grad}\, f.$$

Next, supposing we have a vector field

$$\vec{\boldsymbol{v}} = \begin{pmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{pmatrix}$$

what would be the result of "multiplying" $\vec{\nabla}$ with $\vec{\boldsymbol{v}}$? Since we think of $\vec{\nabla}$ as a vector, the multiplication can be either a dot product, or a cross product. If we "take the dot product" of $\vec{\nabla}$ and $\vec{\boldsymbol{v}}$, we get

$$\vec{\nabla} \cdot \vec{\boldsymbol{v}} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Other commonly used notation for the divergence is

$$\operatorname{div} \vec{\boldsymbol{v}} = \vec{\nabla} \cdot \vec{\boldsymbol{v}}.$$

This combination of derivatives of the components of $\vec{\boldsymbol{v}}$ is called the ***divergence of the vector field*** $\vec{\boldsymbol{v}}$.

If we take the cross product of $\vec{\nabla}$ and $\vec{\boldsymbol{v}}$ we find the so-called ***curl of the vector field*** $\vec{\boldsymbol{v}}$,

$$\vec{\nabla} \times \vec{\boldsymbol{v}} = \begin{vmatrix} \vec{\boldsymbol{\imath}} & \frac{\partial}{\partial x} & P \\ \vec{\boldsymbol{\jmath}} & \frac{\partial}{\partial y} & Q \\ \vec{\boldsymbol{k}} & \frac{\partial}{\partial z} & R \end{vmatrix} = \begin{pmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{pmatrix}.$$

The curl of a vector field $\vec{\boldsymbol{v}}$ is sometimes called the "rotation of $\vec{\boldsymbol{v}}$," and the following alternative notations also get used:

$$\vec{\nabla} \times \vec{\boldsymbol{v}} = \mathbf{curl}\, \vec{\boldsymbol{v}} = \mathbf{rot}\, \vec{\boldsymbol{v}}.$$

16.1. Example – compute the divergence of $\vec{\boldsymbol{v}}(x, y, z) = \vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{w}} = \rho \vec{\boldsymbol{x}}$. The vector fields are

$$\vec{\boldsymbol{v}}(x, y, z) = \vec{\boldsymbol{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{ and } \vec{\boldsymbol{w}}(x, y, z) = \rho \vec{\boldsymbol{x}} = \begin{pmatrix} \rho x \\ \rho y \\ \rho z \end{pmatrix},$$

in which ρ is the radius from spherical coordinates, i.e.

$$\rho = \sqrt{x^2 + y^2 + z^2}.$$

The divergence of $\vec{\boldsymbol{v}}$ is easy:

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{v}} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3, \text{ or } \operatorname{div}\vec{\boldsymbol{v}} = 3.$$

The divergence of $\vec{\boldsymbol{w}}$ is a little harder. To begin with, we have

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{w}} = \frac{\partial \rho x}{\partial x} + \frac{\partial \rho y}{\partial y} + \frac{\partial \rho z}{\partial z}.$$

It helps to find the partial derivatives of ρ separately. They are

$$\frac{\partial\rho}{\partial x} = \frac{x}{\rho}, \qquad \frac{\partial\rho}{\partial y} = \frac{y}{\rho}, \qquad \frac{\partial\rho}{\partial z} = \frac{z}{\rho}.$$

These formulas look nicer in vector form, namely

$$\vec{\boldsymbol{\nabla}}\rho = \begin{pmatrix} x/\rho \\ y/\rho \\ z/\rho \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{\vec{\boldsymbol{x}}}{\rho}. \tag{185}$$

(Problem 17.8 will ask you to check this.) Armed with these partial derivatives we find

$$\frac{\partial \rho x}{\partial x} = \frac{x}{\rho}x + \rho\frac{\partial x}{\partial x} = \frac{x^2}{\rho} + \rho.$$

We get similar terms for $\frac{\partial \rho y}{\partial y}$ and $\frac{\partial \rho z}{\partial z}$. Adding these together leads to

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{w}} = \frac{x^2}{\rho} + \frac{y^2}{\rho} + \frac{z^2}{\rho} + 3\rho = \frac{x^2+y^2+z^2}{\rho} + 3\rho = \frac{\rho^2}{\rho} + 3\rho = 4\rho.$$

16.2. Example – compute the curl of the Poiseuille flow from § 2.2. The flow is given in Equation (145). For simplicity we will assume $R = 1$ and $v_c = 1$. If we assume that the central axis is the x axis, then the distance r to the central axis is $r = \sqrt{y^2+z^2}$, and the velocity field in the cylinder is given by

$$\vec{\boldsymbol{v}}(x,y,z) = \begin{pmatrix} 1-y^2-z^2 \\ 0 \\ 0 \end{pmatrix}.$$

Its curl is then

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{v}} = \begin{vmatrix} \vec{\boldsymbol{\imath}} & \frac{\partial}{\partial x} & 1-y^2-z^2 \\ \vec{\boldsymbol{\jmath}} & \frac{\partial}{\partial y} & 0 \\ \vec{\boldsymbol{k}} & \frac{\partial}{\partial z} & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ -2z \\ +2y \end{pmatrix}$$

16.3. The curl of a gradient always vanishes. If $f(x,y,z)$ is any function of three variables, then its gradient is a vector field. What is the curl of this vector field? The computation is straightforward,

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}f = \vec{\boldsymbol{\nabla}}\times\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{vmatrix} \vec{\boldsymbol{\imath}} & \frac{\partial}{\partial x} & f_x \\ \vec{\boldsymbol{\jmath}} & \frac{\partial}{\partial y} & f_y \\ \vec{\boldsymbol{k}} & \frac{\partial}{\partial z} & f_z \end{vmatrix} = \begin{pmatrix} (f_z)_y - (f_y)_z \\ (f_x)_z - (f_z)_x \\ (f_y)_x - (f_x)_y \end{pmatrix}. \tag{186}$$

We know that for any function of several variables "mixed partials are equal" (when they are continuous), meaning $(f_x)_y = (f_y)_x$, etc. Another look at the curl we just computed tells us that

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\nabla}}f = \vec{\boldsymbol{0}}, \text{ or, } \mathbf{curl}\,\mathbf{grad}\,f = \vec{\boldsymbol{0}}, \tag{187}$$

for any function f (whose second derivatives are continuous).

$$\textit{Function} \xrightarrow{\textbf{grad}} \textit{Vector field} \xrightarrow{\textbf{curl}} \textit{Vector field} \xrightarrow{\text{div}} \textit{Function}$$
$$f \xrightarrow{\textbf{grad}} \vec{\nabla}(f), \quad \vec{v} \xrightarrow{\textbf{curl}} \vec{\nabla}\times\vec{v}, \quad \vec{w} \xrightarrow{\text{div}} \vec{\nabla}\cdot\vec{w}$$

Figure 26. The three basic operations of vector calculus. If we apply two consecutive operations in this diagram, we get zero. See Equations (187) and (188).

16.4. The divergence of a curl always vanishes.

A computation just like the one above shows that if we have a vector field $\vec{v}$ and we compute the divergence of its curl, we always get zero:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0, \text{ or, } \operatorname{div}\mathbf{curl}\,\vec{v} = 0. \tag{188}$$

Both Equations (187) and (188) are easy to remember in their "$\vec{\nabla}$" form, if we pretend that $\vec{\nabla}$ is a real vector.

To get (187) remember that the cross product of any vector with itself always vanishes: $\vec{a}\times\vec{a} = \vec{0}$ for any $\vec{a}$. The expression $\vec{\nabla}\times\vec{\nabla}f$ contains the cross product of $\vec{\nabla}$ with itself, and so it should vanish. The argument doesn't hold because $\vec{\nabla}$ is not really a vector, but our computation (186) shows that the conclusion is true anyway.

To get (188), we use that $\vec{a}\times\vec{b}$ is always perpendicular to $\vec{b}$, no matter what $\vec{a}$ and $\vec{b}$ are, so that $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$ always holds. Equation (188) is exactly that, with "$\vec{a} = \vec{\nabla}$" and "$\vec{b} = \vec{v}$."

16.5. Other combinations of gradient, curl and divergence.

The divergence of the gradient does not normally vanish. If we expand the definitions we find

$$\vec{\nabla}\cdot\vec{\nabla}f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

This combination of second derivatives of a function, which occurs very often is called the ***Laplacian*** of the function f. The following notation is used:

$$\triangle(f) = \vec{\nabla}\cdot\vec{\nabla}f = f_{xx} + f_{yy} + f_{zz}.$$

The other combination of derivatives that one can consider is "the curl of the curl." If $\vec{v}$ is a vector field then its curl $\vec{\nabla}\times\vec{v}$ is again a vector field, and thus one can compute the curl of the curl: $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})$. This combination usually does not vanish.

For a given vector field one can also consider its divergence, $\vec{\nabla}\cdot\vec{v}$, which is a function, and of which one can compute the gradient, $\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$. This quantity usually also does not vanish.

There is a relation between the curl of the curl and the gradient of the divergence, which is useful in mathematical physics, and which we state here for reference only: for any vector field $\vec{v}$ one has

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \triangle(\vec{v}) - \vec{\nabla}(\vec{\nabla}\cdot\vec{v}).$$

17. Problems

1. If the central axis of the cylinder in Figure 2 is the x-axis, and if the vector field is as given in (145), then write $\vec{v}$ in terms of x, y, z instead of r.

2. It is always said that Newton discovered the "inverse square law" for gravitation. According to this law the strength of the gravitational force is inversely proportional to the ***square*** of the distance to the center of the Earth. But the exponent in our equation (146) is *three* instead of two!

Could this be a different law? A typo? To find out, compute the length $\|\vec{F}\|$ of the gravitational force in (146).

3. Show that the magnetic field in (148) can be written as

$$\vec{B}(x,y,z) = C\frac{\vec{k}\times\vec{x}}{\|\vec{k}\times\vec{x}\|^n}$$

for some integer n and some constant C. Find the right n and C.

4. Let $\vec{a}$ and $\vec{m}$ be two constant vectors, with components

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{ and } \vec{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}.$$

Let $\vec{v}(x,y,z)$ be the vector field

$$\vec{v} = (\vec{m}\cdot\vec{x})\vec{a}.$$

(a) Write $\vec{v}$ in terms of its components:

$$\vec{v} = \begin{pmatrix} \cdots ? \cdots \\ \cdots ? \cdots \\ \cdots ? \cdots \end{pmatrix}.$$

(b) Compute $\vec{\nabla}\cdot\vec{v}$.

(c) Compute $\vec{\nabla}\times\vec{v}$.

(d) If $\vec{v}$ is the gradient of some function f, what can you say about the vectors $\vec{a}$ and $\vec{m}$?

(e) If $\vec{v}$ is the curl of some vector field $\vec{w}$, what can you say about the vectors $\vec{a}$ and $\vec{m}$?

5. Let $\vec{a}$ and $\vec{m}$ be as in the previous problem. Consider the vector field

$$\begin{aligned}\vec{v}(x,y,z) &= e^{\vec{m}\cdot\vec{x}}\vec{a} \\ &= e^{m_1x+m_2y+m_3z}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\end{aligned}$$

(a) Show by computing the derivatives that $\vec{\nabla}(e^{\vec{m}\cdot\vec{x}}) = e^{\vec{m}\cdot\vec{x}}\vec{m}$.

(b) Compute $\vec{\nabla}\cdot\vec{v}$. (Find the shortest way to write the answer.)

(c) Compute $\vec{\nabla}\times\vec{v}$. Again, simplify your answer.

(d) Which condition must the vectors $\vec{a}$ and $\vec{m}$ satisfy if $\vec{v}$ is to be "divergence free," i.e. if $\operatorname{div}\vec{v} = 0$?

(e) Suppose that $\vec{v} = \vec{\nabla}\phi$ for some function. What do you know about $\vec{a}$ and $\vec{m}$?

6. If $\vec{v} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ is a vector field and f is a function, then what is $\vec{v}\cdot\vec{\nabla}f$?

7. ***Product rules.*** Let f be a function of three variables, and let $\vec{v}$ be a three dimensional vector field.

(a) $\vec{\nabla}\cdot(f\vec{v}) = (\vec{\nabla}f)\cdot\vec{v} + f\vec{\nabla}\cdot\vec{v}$

(b) Guess a product rule for $\vec{\nabla}\times(f\vec{v})$ and prove it.

8. In this problem, as in all the problems in this section, $\rho = \sqrt{x^2+y^2+z^2} = \|\vec{x}\|$ is the radius in spherical coordinates.

Check the following formulas

$$\vec{\nabla}\rho = \frac{\vec{x}}{\rho}, \text{ and } \vec{\nabla}\cdot\vec{x} = 3.$$

9. Use the product rule from Problem 17.7 and the formulas from problem 17.8 to compute the following quantities

(a) $\vec{\nabla}\cdot(\rho^2\vec{x})$

(b) $\vec{x}\cdot\vec{\nabla}\rho$

(c) $\operatorname{div}\dfrac{\vec{x}}{\|\vec{x}\|^3}$. What does this say about the Earth's gravitational field?

10.

(a) Show that $\vec{x} = \frac{1}{2}\vec{\nabla}(\rho^2)$.

(b) Compute $\vec{\nabla}\times\vec{x}$ without doing any derivatives.

(c) Compute $\vec{\nabla}\times(\rho\vec{x})$ using the product rule from problem 17.7.

11. Compute $\vec{\nabla}\times\vec{v}$ for the vector field $\vec{v}(x,y,z) = \vec{k}\times\vec{x}$.

12. Consider the vector field

$$\vec{\boldsymbol{v}}(x,y,z) = \rho^n \vec{\boldsymbol{x}},$$

where n is a constant. (Both Newton's law of gravitation and Coulomb's law have this vector field with $n = -3$.)

(a) Write $\vec{\boldsymbol{v}}(x,y,z)$ in the form $\begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}$, using only Cartesian coordinates x, y, z.

(b) Compute $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$. (Use one of the product rules from Problem 17.7; you can also avoid computing the derivatives of ρ by looking them up in the text.)

(c) For which value(s) of n does one have $\operatorname{div} \vec{\boldsymbol{v}} = 0$?

13. A function of three variables is called ***radially symmetric*** if it only depends on the radius $\rho = \sqrt{x^2+y^2+z^2}$, i.e. if it can be written as $F(\rho)$ for some function F of one variable. E.g. $f(x,y,z) = \rho^{-2}$, or $g(x,y,z) = e^{-\rho}$ are radially symmetric functions.

Find the gradient of a radially symmetric function $F(\rho)$.

(You may want to use $\rho_x = x/\rho$, etc. from (185) to speed up the computation.)

(a) Let $\vec{\boldsymbol{v}} = \rho^n \vec{\boldsymbol{x}}$, as in problem 17.12. Does there exist a function $f(x,y,z)$ such that $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\nabla}} f$? (Hint: try a radially symmetric function, and use problem 17.13.)

Math 234 – Answers and Hints

(I12.3e) (a) 3 (b) $\begin{pmatrix}2\\-4\\4\end{pmatrix}$ (c) 36 (d) $\begin{pmatrix}3\\-3\\3\end{pmatrix}$ (e) $\begin{pmatrix}1\\-5\\5\end{pmatrix}$

(I12.4) Every vector is a position vector. To see of which point it is the position vector translate it so its initial point is the origin.

Here $\overrightarrow{AB} = \begin{pmatrix}-3\\3\end{pmatrix}$, so $\overrightarrow{AB}$ is the position vector of the point $(-3, 3)$.

(I12.5) One always labels the vertices of a parallelogram counterclockwise (see §??).

$ABCD$ is a parallelogram if $\overrightarrow{AB}+\overrightarrow{AD} = \overrightarrow{AC}$. $\overrightarrow{AB} = \begin{pmatrix}1\\1\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}2\\3\end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix}3\\1\end{pmatrix}$. So $\overrightarrow{AB}+\overrightarrow{AD} \neq \overrightarrow{AC}$, and $ABCD$ is not a parallelogram.

(I12.6a) As in the previous problem, we want $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. If D is the point (d_1, d_2, d_3) then $\overrightarrow{AB} = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix}d_1\\d_2-2\\d_3-1\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}4\\-1\\3\end{pmatrix}$, so that $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ will hold if $d_1 = 4$, $d_2 = 0$ and $d_3 = 3$.

(I12.6b) Now we want $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$, so $d_1 = 4$, $d_2 = 2$, $d_3 = 5$.

(I12.9) Compute the dot product: $\vec{a}\cdot\vec{b} = 2s + 3(1-s) = 3 - s$. When the dot-product vanishes the vectors are perpendicular; this happens when $s = 3$. The angle between the vectors is acute is the dot-product is positive. This happens when $3 - s > 0$, i.e. when $s < 3$.

(I12.11a) The problem is open-ended because it doesn't specify what "draw" means.

If you are allowed to use a calculator and a protractor, then you could use the dot product to compute the angle θ between the two vectors; then, using your protractor, draw two line segments that make this angle, and mark off lengths 3 and 5 to get the vectors. From the dot-product and the two lengths you find $3 \times 5 \times \cos\theta = -12$, so $\cos\theta = -\frac{12}{15} = -0.8$, which implies $\theta = \arccos(-0.8) \approx 2.498\dots$ radians, or $\theta \approx 143.13\dots$ degrees.

This turns out to be only half the answer: we have forgotten that the equation $\cos\theta = -0.8$ has many more solutions than just $\arccos(-0.8)$. One other solution is $-\arccos(-0.8)$. This gives us two vectors $\vec{b}$ with $\|\vec{b}\| = 5$ and $\|\vec{b}\| = 5$ and $\vec{a}\cdot\vec{b} = -12$.

A different approach goes like this: you could assume $\vec{a} = 3\vec{e}_1$, which has length 3, and $\vec{b} = \begin{pmatrix}b_1\\b_2\end{pmatrix}$. The condition that $\vec{b}$ have length 5 then says $b_1^2 + b_2^2 = 5^2 = 25$, while the dot-product is $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 = 3b_1$. Since the dot-product must be -12 we find $b_1 = -\frac{12}{3} = -4$. Using the length of $\vec{b}$ leads to $b_2 = \sqrt{25-(-4)^2} = \pm 3$. Thus we find two solutions: $\vec{b} = \begin{pmatrix}-4\\\pm 3\end{pmatrix} = -4\vec{e}_1 \pm 3\vec{e}_2$.

You make the drawing.

(I12.11b) No. The inner product of two vectors is $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$, and therefore it can never be larger than $\|\vec{a}\|\,\|\vec{b}\|$.

(I12.13a) True:

$$\begin{aligned}(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) &= (\vec{a}+\vec{b})\cdot\vec{a} - (\vec{a}+\vec{b})\cdot\vec{b}\\ &= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{b}\\ &= \|\vec{a}\|^2 + \|\vec{b}\|^2.\end{aligned}$$

(I12.13b) True: This is Pythagoras' theorem. Here is an algebraic derivation:

$$\begin{aligned}\|\vec{a}+\vec{b}\|^2 &= (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})\\ &= (\vec{a}+\vec{b})\cdot\vec{a} + (\vec{a}+\vec{b})\cdot\vec{b}\\ &= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}\\ &= \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2\\ &= \|\vec{a}\|^2 + \|\vec{b}\|^2.\end{aligned}$$

(I12.13c) Not so. The same computation as for the previous problem shows

$$\begin{aligned}\|\vec{a}-\vec{b}\|^2 &= (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})\\ &= (\vec{a}-\vec{b})\cdot\vec{a} - (\vec{a}-\vec{b})\cdot\vec{b}\\ &= \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}\\ &= \|\vec{a}\|^2 - 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2\\ &= \|\vec{a}\|^2 + \|\vec{b}\|^2.\end{aligned}$$

Therefore

$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 - \|\vec{b}\|^2$$

only is true if $\vec{b} = \vec{0}$.

(I12.15a) $(\vec{a}+\vec{b})\times(\vec{a}+\vec{b}) = \vec{0}$

(I12.15b) $(\vec{a}+\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c}) = \vec{0}$

(I12.15c) $(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = 2\vec{a}\times\vec{b}$.

(I12.16a) $\vec{a}\cdot\vec{c} = \vec{a}\cdot(\vec{a}\times\vec{b}) = 0$, but for the two given vectors in the problem $\vec{a}\cdot\vec{c} = -1 \neq 0$, so there cannot be a vector $\vec{b}$ with $\vec{a}\times\vec{b} = \vec{c}$ as $\vec{c}$ is not perpendicular to $\vec{a}$.

(I12.16b) In this case $\vec{a} \perp \vec{c}$, so the argument from the first part of this problem doesn't rule out that there might be a solution. So let's try $\vec{b} = \begin{pmatrix} b_1\\ b_2\\ b_3\end{pmatrix}$. Then

$$\vec{a}\times\vec{b} = \begin{pmatrix} b_2\\ -b_1-2b_3\\ 2b_2\end{pmatrix} \stackrel{?}{=} \vec{c} = \begin{pmatrix}1\\3\\2\end{pmatrix}.$$

Solving this for b_1, b_2, and b_3 leads to $b_2 = 1$, and $-b_1 - 2b_3 = 3$ as only remaining equation. Since we have found b_2 there are still two unknowns left. We can choose an arbitrary b_3 and set $b_1 = -3 - 2b_3$, e.g. $b_3 = 0$ works, provided we choose $b_1 = -3$.

(II17.7d) $\kappa(x) = \dfrac{e^x}{\left(1+e^{2x}\right)^{3/2}}$.

To find the point with largest curvature: $\kappa'(t) = \dfrac{e^t}{\left(1+e^{2t}\right)^{5/2}}\left(1-2e^{2t}\right)$, so the maximal curvature (smallest radius of curvature) occurs when $x = -\frac{1}{2}\ln 2$.

(III5.1) $-d(x,y)$.

(III5.2) $a < 0, b > 0, c > 0$.

(III5.3a) $x = -2$ for the x-axis, $y = 6$ for the y-axis, $z = 6$ for the z-axis.

(III5.3b) $z = 3 - \frac{3}{4}x - \frac{3}{2}y$.

(III5.3c) $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ is a nice symmetric way of writing the equation.

(III5.4) The distance is $\dfrac{|c|}{\sqrt{1+a^2+b^2}}$.

(III5.5a) This one is already the sum of squares. We don't have to do anything, and can immediately conclude that $f(x,y) > 0$ for all (x,y) in the plane except the origin, where $x = y = 0$ and $f(x,y) = 0$.

(III5.5b) The square containing x is already complete (no xy terms) and we can immediately factor $Q(x,y) = (x-y)(x+y)$.

(III5.5c) We complete the square:

$$g(x,y) = (x-2y)^2 - y^2.$$

We get the difference of two squares, so we can factor the quadratic form:

$$g(x,y) = (x-2y-y)(x-2y+y) = (x-3y)(x-y).$$

(III5.5d) This one is positive definite:

$$Q = 9\big(s^2 - 4st + 9t^2\big) = 9\big[(s-2t)^2 - 4t^2 + 9t^2\big] = 9\big[(s-2t)^2 + 5t^2\big] = 9(s-2t)^2 + 45t^2.$$

(III5.5e) Positive definite:

$$M = \frac{1}{2}\big\{\alpha^2 - 2\alpha\beta + 2\beta^2\big\} = \frac{1}{2}\big\{(\alpha-\beta)^2 + \beta^2\big\}.$$

(III5.5f) This quadratic form has no x^2 term. When that happens you cna immediately factor the form, because all terms contain y:

$$Q(x,y) = xy + y^2 = (x+y)y.$$

This form is indefinite.

(III5.5g) Now this form does have an x^2 term, so we *can* complete the square if we want to ... but if we look carefully then we see that there's not y^2 term. Because of this we can factor out x, and we get

$$Q = x^2 + 2xy = x(x+2y).$$

The form is indefinite.

What if we don't notice that y^2 is missing and just blindly complete the square? Nothing goes wrong and we get the same answer:

$$Q = x^2 + 2xy = x^2 + 2xy + y^2 - y^2 = (x+y)^2 - y^2 = (x+y-y)(x+y+y) = x(x+2y).$$

We did work too hard though :-(

(III5.6) Complete the square:

$$Q = (x+ky)^2 - k^2y^2 + y^2 = (x+ky)^2 + (1-k^2)y^2.$$

If $1 - k^2 > 0$ then we have the sum of two squares. If $1 - k^2 < 0$, then we can rewrite Q as the difference of two squares

$$Q = (x+ky)^2 - (k^2-1)y^2 = (x+ky)^2 - \big(\sqrt{k^2-1}\,y\big)^2$$

which is indefinite. That is all we need to know: we are not actually asked to factor the form when it is indefinite. But in case you're wondering, the somewhat ugly formula is thus:

$$Q = \Big(x + (k + \sqrt{k^2-1})y\Big)\Big(x + (k - \sqrt{k^2-1})y\Big).$$

The conclusion is that $Q(x,y)$ is positive definite if $-1<k<1$ and indefinite when $k>1$ or $k<-1$. In the remaining cases $k=\pm 1$ we have

$$Q=(x+ky)^2-k^2y^2+y^2=(x+ky)^2+(1-k^2)y^2=(x\pm y)^2,$$

i.e. the form is a square (it is semidefinite).

(III5.7a) The graph is the saddle surface, the function is defined at all (x,y). The level set is given by $xy=c$. If $c\neq 0$ then this set consists of both branches of the hyperbola $y=\frac{c}{x}$. If $c=0$ then $xy=0$ is equivalent with $x=0$ or $y=0$, so the level set is the union of the x-and y-axes.

(III5.7b) $z-x^2=0$. Domain $\mathbb{R}^2$. Graph is a ***parabolic cylinder*** and consists of horizontal lines perpendicular to the xz-plane, going through the parabola $y=x^2$ in that plane.

Level sets: parallel straight lines $x=\pm\sqrt{z}$ if $z>0$, the x axis if $z=0$, the empty set if $z<0$.

(III5.7c) $z^2-x=0$. Implicit function. At least two functions are defined, namely $z=\pm\sqrt{x}$. Domain: all points (x,y) with $x\geq 0$. Graph is ***half a parabolic cylinder*** and consists of horizontal lines perpendicular to the xz-plane, going through the parabola $z=\sqrt{x}$ (or $z=-\sqrt{x}$, depending on which function you choose) in that plane.

Level sets (assuming we choose the function $z=+\sqrt{x}$): the line $x=z^2$ if $z\geq 0$, empty set otherwise.

(III5.7d) $z-x^2-y^2=0$. Domain is the whole plane. Graph is a paraboloid of revolution, obtained by rotating the parabola $z=x^2$ in the xz-plane around the z axis.

Level sets: circle with radius $\sqrt{z}$ for $z>0$, the origin for $z=0$ (note: this level set is a point rather than a curve), empty for $z<0$.

(III5.7e) $z^2-x^2-y^2=0$. Implicit function. Domain all of $\mathbb{R}^2$. Possible functions are $z=\pm\sqrt{x^2+y^2}$. Graph is the cone obtained by rotating the half line $z=x, x\geq 0$ in the xz-plane around the z axis (or the half line $z=-x, x\geq 0$, if you chose $z=-\sqrt{x^2+y^2}$.)

Level sets (assuming we choose $z=+\sqrt{x^2+y^2}$): circle with radius z when $z>0$, origin when $z=0$, empty when $z<0$.

(III5.7f) $xyz=1$. Domain the whole plain with the x and y-axes removed, i.e. all points (x,y) with $xy\neq 0$. Function is $f(x,y)=\frac{1}{xy}$. For each y the graph is the hyperbola $z=1/(yx)$ which is just the standard hyperbola $z=1/x$ stretched vertically by a factor $1/y$. As $y\to 0$ this factor goes to ∞.

(III5.7g) $xy/z^2=1$. Implicit function. Domain first and third quadrants (all points with $xy>0$). Functions $z=\pm\sqrt{xy}$. Cross sections with planes $y=$constant are half parabolas.

Note: Harder to see, but the surface with equation $xy=z^2$ is in fact the cone obtained by rotating the x-axis around the line $x=y$ in the xy-plane.

(III5.8a) $x>0$. This one is in the text.

(III5.8b) $x<0$.

(III5.8c) $x>0$. This is the same region as in part (a): remember that the polar angle is only determined up to a multiple of 2π.

(III5.8d) In the upper half plane, $y>0$.

(III5.8e) In the whole plane, except the origin, and the negative x-axis. This formula for the polar angle θ clearly is valid in a larger region than the other formulas, but it does not look half as nice.

(III5.9) The level set for $c=-24$ is the empty set, since it consists of all points on the lake surface where the lake is -24 meters deep–i.e. where the water reaches 24meters ***above*** the lake.

Similarly, the level set for $c=+400$ is also empty since the lake is not that deep anywhere.

The level set $d^{-1}(0)$ consists of those points where the lake is 0meters deep. This is exactly the shore line.

The level set $d^{-1}(24)$ consists of all points on the lake surface where the lake is exactly 24meters deep. Form the map it looks like this happens on two separate curves near the center of the lake.

(III5.10) See § 4.

(III5.11a) The two rectangular strips $-3 \leq x \leq 3, 2 \leq y < \infty$ and $-3 \leq x \leq 3, -\infty < y \leq -2$.

(III5.11b) By definition $\arcsin(x)$ is only defined if $-1 \leq x \leq 1$. For $\arcsin(x^2 + y^2 - 2)$ to be defined, we must therefore have $-1 \leq x^2 + y^2 - 2 \leq 1$, i.e. $1 \leq x^2 + y^2 \leq 3$.

The domain of this function is the ring-shaped region between the circles with radii 1 and $\sqrt{3}$, both centered at the origin. Circles are included in the domain.

(III5.11c) The way this function is written both $\sqrt{x}$ and $\sqrt{y}$ must be defined, so the domain consists off all (x, y) with $x \geq 0$ and $y \geq 0$.

(III5.11d) $\sqrt{xy}$ must exist, which happens for all (x, y) in the first and third quadrants (axes included.)

(III5.11f) The region in the plane given by $x^2 + 4y^2 \leq 16$, which is the region enclosed by an ellipse with major axis of length 4, along the x axis, and minor axis of length 2 along the y-axis. The ellipse is included.

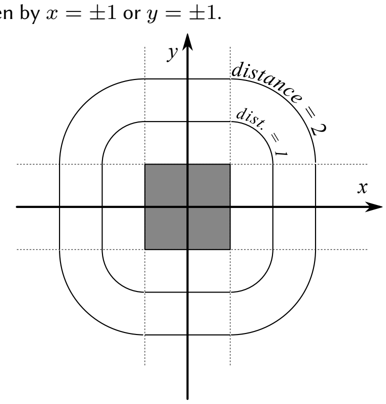
(III5.12) The level sets of the function whose graph is a cone are equally spaced circles (the level set at level c is a circle with radius c). Hence the one on the right corresponds to the cone, and the one on the left corresponds to the paraboloid.

(III5.13a) $(0, \frac{1}{2})$ is in the square Q, so it is the point closest to $(0, \frac{1}{2})$.
The point $(0, 1)$ on the top edge of the square is closest to $(0, 2)$.
The corner point $(1, 1)$ is closest to $(3, 4)$.

(III5.13b) $f(0, \frac{1}{2}) = 0$; $f(0, 2) = 1$ and $f(3, 4)) = \sqrt{2^2 + 3^2} = \sqrt{13}$.

(III5.13c) The zero set of f is the square Q.

(III5.13d) The level set at level -1 is empty. The others are “rounded rectangles,” see this drawing, in which the square is grey, the dashed lines are given by $x = \pm 1$ or $y = \pm 1$.



(III5.13e) The lines $x = \pm 1$ and $y = \pm 1$ divide the plane into nine regions. On each region the function is given by a different formula. Here they are:

$f(x,y)$	**if** …
0	(x, y) in Q
$x - 1$	$x \geq 1, \lvert y\rvert \leq 1$
$y - 1$	$\lvert x\rvert \leq 1, y \geq 1$
$-x - 1$	$x \leq -1, \lvert y\rvert \leq 1$
$-y - 1$	$\lvert x\rvert \leq 1, y \leq -1$
$\sqrt{(x-1)^2 + (y-1)^2}$	$x \geq 1$ and $y \geq 1$
$\sqrt{(x-1)^2 + (y+1)^2}$	$x \geq 1$ and $y \leq -1$
$\sqrt{(x+1)^2 + (y-1)^2}$	$x \leq -1$ and $y \geq 1$
$\sqrt{(x+1)^2 + (y+1)^2}$	$x \leq -1$ & $y \leq -1$

(III5.14a) At time t we have a line through the origin with slope $\sin t$. As time progresses this lines turns up and down, and up and down, etc.

(III5.14b) Same as previous problem, but twice as fast.

(III5.14c) At all times one sees the graph of $y = \sin x$ stretched vertically by a factor t.

(III5.14d) Same as previous problem, but twice as fast.

(III5.14e) The graph of $y = \sin 2x$ stretched vertically by a factor t.

(III5.14f) Parabola with its minimum on the x-axis at $x = t$. So we see the parabola $y = x^2$ translating from the left to the right with constant speed 1.

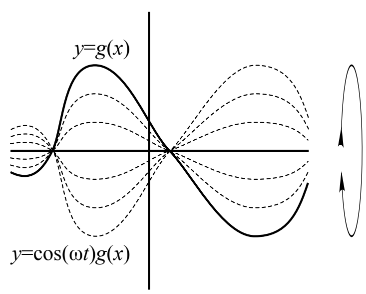
(III5.14g) Parabola with its minimum on the x-axis at $x = \sin t$. So we see the parabola $y = x^2$ translating back and forth horizontally every 2π time units.

(III5.14j) At time t we see Agnesi's witch, i.e. the graph $y = a/(1 + x^2)$ with amplitude $a = 1/(1 + t^2)$. Thus we see a bump whcich starts out small at $t = -\infty$, grows to its maximal size at time $t = 0$, and then decays again, until it vanishes at $t = +\infty$.

(III5.16) The graph of $y = g(x - a)$ is obtained from the graph of $y = g(x)$ by translating the graph of $y = g(x)$ by a units to the right.

Hence the graph of $g(x - ct)$ is the graph of $g(x)$ translated by ct units to the right. As time changes the graph of $g(x - ct)$ therefore moves with velocity c to the right.

(III5.17) If you know the graph of a function $y = g(x)$, then you get the graph of $y = cg(x)$ by stretching the graph of g vertically by a factor c (here c is a constant.) If you allow this constant to depend on time, e.g. as in this problem by setting $c = \cos(\omega t)$, then the "movie" you get is of a version of the graph of g which is growing and shrinking vertically.



(IV3.2b) $-2xy\sin(x^2y), -x^2\sin(x^2y) + 3y^2$

(IV3.2c) $(y^2 - x^2y)/(x^2 + y)^2, x^3/(x^2 + y)^2$

(IV3.2g) $2xe^{x^2+y^2}, 2ye^{x^2+y^2}$

(IV3.2h) $y\ln(xy) + y, x\ln(xy) + x$

(IV3.2i) $-x/\sqrt{1 - x^2 - y^2}, -y/\sqrt{1 - x^2 - y^2}$

(IV3.2l) $\tan y, x/\cos^2 y$

(IV3.2m) $-1/(x^2y), -1/(xy^2)$

(IV3.4a) $\dfrac{\partial\theta}{\partial x} = -\dfrac{y}{x^2 + y^2}, \dfrac{\partial\theta}{\partial x} = \dfrac{x}{x^2 + y^2}.$

(IV3.5) The distance to the origin is exactly the radius in polar coordinates, so $f(x,y) = \sqrt{x^2+y^2}$, and

$$f_x = \frac{x}{\sqrt{x^2+y^2}}, \qquad f_y = \frac{y}{\sqrt{x^2+y^2}}.$$

This is the same as in problem 3.3. The only quantity that we did not compute before is

$$(f_x)^2 + (f_y)^2 = \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1.$$

(IV3.6a) $\frac{\partial z}{\partial x} = f'(x)g(y)$, $\frac{\partial z}{\partial y} = f(x)g'(y)$.

(IV3.6b) $\frac{\partial z}{\partial x} = yf'(xy)$, $\frac{\partial z}{\partial y} = xf'(xy)$.

(IV3.6c) $\frac{\partial z}{\partial x} = \frac{1}{y} f'(\frac{x}{y})$, $\frac{\partial z}{\partial y} = -\frac{x}{y^2} f'(\frac{x}{y})$.

(IV7.1a) The linear approximation formula is equation (60), in which $x_0 = a = 3, y_0 = b = 1$, and $\Delta x = x-a = x-3$, $\Delta y = y-b = y-1$. So for this problem the linear approximation of $f(x,y) = xy^2$ at $(3,1)$ is

$$f(x,y) \approx 3 + (x-3) + 6(y-1) = x + 6y - 6.$$

This approximation is only expected to be good when (x,y) is close to $(3,1)$. The approximation contains an error which is small compared to $|x-3|$ and $|y-1|$.
FAQ: What is the relation between the linear approximation and the tangent plane?
Answer: They are very closely related: the tangent plane is the graph of the linear approximation. The linear approximation is the equation for the tangent plane. To compute either you have to do the same thing.

(IV7.1b) $x/y^2 \approx 3 + (x-3) - 6(y-1) = x - 6y + 6$ when x is close to 3 and y is close to 1.

(IV7.1c) $\sin x + \cos y \approx -1 + (-1)(x-\pi) + (0)(y-\pi) = \pi - 1 - x$ when x is close to π and y is close to π.

(IV7.1d) $\frac{xy}{x+y} \approx \frac{3}{4} + \frac{1}{16}(x-3) + \frac{9}{16}(y-1)$ when x is close to 3 and y is close to 1.

(IV7.2) $z = 1$

(IV7.3) $z = 6(x-3) + 3(y-1) + 10$

(IV7.4) $z = (x-2) + 4(y-1/2)$

(IV7.5a) Solve for z: $z = \pm\sqrt{2x^2+3y^2-4}$. In this problem we are looking at the point $(1,1,-1)$ so we have the graph of $z = f(x,y) = -\sqrt{2x^2+3y^2-4}$. The partials are

$$\frac{\partial f}{\partial x} = \frac{-2x}{\sqrt{2x^2+3y^2-4}}, \qquad \frac{\partial f}{\partial y} = \frac{-3y}{\sqrt{2x^2+3y^2-4}}$$

so that, at $(1,1,-1)$ you get $f_x = -2$, $f_y = -3$. There for the equation for the tangent plane is $z = -2(x-1) - 3(y-1) - 1$

(IV7.6a) The tangent plane has equation $z = z_0 + A(x-x_0) + B(y-y_0)$. By putting the variables x, y, z on one side, and all the constants on the other, you can write this as

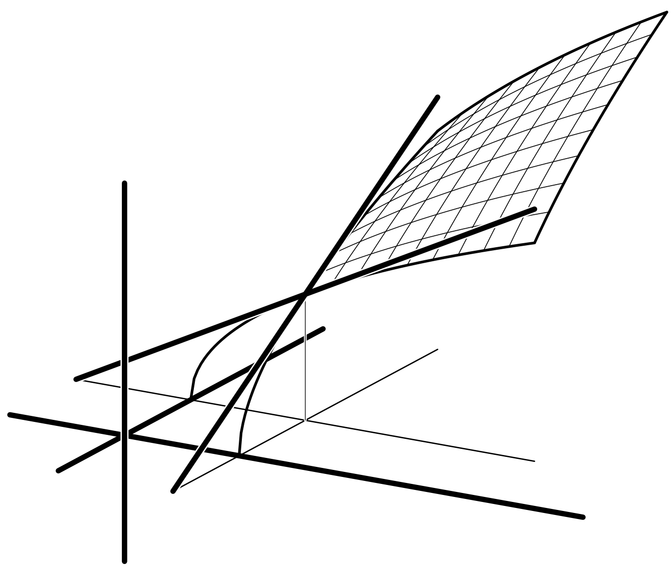
$$Ax + By - z = Ax_0 + By_0 - z_0.$$

This is the equation for a plane whose normal is $\vec{\boldsymbol{n}} = \begin{pmatrix} A \\ B \\ -1 \end{pmatrix}$. Any other multiple of this vector is also a valid normal to the plane, in particular, $\begin{pmatrix} -A \\ -B \\ +1 \end{pmatrix}$ is OK.

(IV7.6b) We want a normal to the graph of $z = f(x,y) = \frac{1}{2}x^2 + 2y^2$ at the point P. By the previous problem a normal is given by $\vec{\boldsymbol{n}} = \begin{pmatrix} f_x(2,1) \\ f_y(2,1) \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$.

A line through P in the direction of $\vec{\boldsymbol{n}}$ is given by $\vec{\boldsymbol{r}}(t) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$

(IV7.7) Below you see the graph of a function and two (solid) lines which are tangent to the graph. On one line you have $x = a$ (hence constant), and its slope is $f_x(a,b)$; on the other you have $y = b$, and it has slope $f_y(a,b)$.



The tangent plane to the graph (not drawn here, but see Figure 4 in the notes) is the plane containing the two lines in the drawing.

(IV7.8) The function is $f(x,y) = x\ln(xy)$. We have $f(2,\frac{1}{2}) = 2\ln(2\cdot\frac{1}{2}) = \ln 1 = 0$. The gradient of the function is $\vec{\nabla} f = \begin{pmatrix} \ln(xy)+1 \\ x/y \end{pmatrix}$. At the point $(2,\frac{1}{2})$ this is $\vec{\nabla} f = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, so the linear approximation is

$$f(x,y) \approx f(2,\frac{1}{2}) + 1\cdot(x-2) + 4\cdot(y-\frac{1}{2}),$$

i.e.

$$f(x,y) \approx 1(x-2) + 4(y-\frac{1}{2}).$$

(This is also the answer to problem 7.4.)

Here we don't want to describe the tangent plan, but we want to find the value of $f(x,y)$ for $(x,y) = (1.98, 0.4)$. Substituting these values of x and y in the linear approximation we get $f(1.98, 0.4) \approx (1.98 - 2) + 4(0.4-0.5) = -0.42$.

This is only an approximation, and you wonder how good it is. We have $\Delta x = 1.98 - 2 = -0.02$, and $\Delta y = 0.4 - \frac{1}{2} = -0.1\ldots$ are these numbers "small"? To find the error in the approximation you could use a Lagrange-type remainder term, but that's not part of math 234. Instead we grab a calculator and compute $f(1.98, 0.4) = 1.98\cdot\ln(1.98\cdot 0.4) = -0.46172\cdots$. So our linear approximation formula is off by $0.04\cdots$.

(IV7.9a) The x-and y-axes.

(IV7.9b) The heights are the z-coordinates, so $z = xy$ and $z_* = -2 + x + 2y$. The difference is

$$z - z_* = xy - (-2 + x + 2y) = xy - x - 2y + 2.$$

(IV7.10a) The tangent plane has equation $z = ab + b(x-a) + a(y-b) = bx + ay - ab$.

(IV7.10b) The point (x, y, z) lies on the intersection if $z = xy$ and $z = bx + ay - ab$. Therefore x and y must satisfy $xy - bx - ay + ab = 0$. This equation factors as follows:

$$xy - bx - ay + ab = (x - a)(y - b) = 0,$$

so that the intersection contains the line $x = a$, $z = ay$, and also the line $y = b$, $z = bx$.

(IV10.2) $\frac{\partial(f+g)}{\partial x} = f_x + g_x$, and $\frac{\partial(f+g)}{\partial y} = f_y + g_y$, so

$$\begin{pmatrix} \frac{\partial(f+g)}{\partial x} \\ \\ \frac{\partial(f+g)}{\partial y} \end{pmatrix} = \begin{pmatrix} f_x + g_x \\ f_y + g_y \end{pmatrix} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} + \begin{pmatrix} g_x \\ g_y \end{pmatrix}$$

Hence $\vec{\nabla}(f + g) = \vec{\nabla} f + \vec{\nabla} g$.

The product and quotient rules follow in the same way.

(IV10.3b) The gradient is $\vec{\nabla} f = \binom{2x}{8y}$. This vector is parallel to $\binom{1}{1}$ if there is a number s such that $\vec{\nabla} f = s\binom{1}{1}$, i.e. $\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \binom{s}{s}$. This happens if $f_x(x, y) = f_y(x, y)$. From our computation of the partial derivatives of f we find that $\vec{\nabla} f$ is parallel to $\binom{1}{1}$ when $2x = 8y$. This happens at every point on the line $y = \frac{1}{4}x$.

We are asked which points on the level set $f = 4$ satisfy this condition, so we must find where the line $y = \frac{1}{4}x$ intersects the level set $x^2 + 4y^2 = 4$. Solving the two equations gives two points $(\frac{4}{5}\sqrt{5}, \frac{1}{5}\sqrt{5})$ and $(-\frac{4}{5}\sqrt{5}, -\frac{1}{5}\sqrt{5})$.

(IV10.3c) $\vec{\nabla} g = \begin{pmatrix} 4y^2 \\ 8xy \end{pmatrix}$. This is parallel to $\binom{1}{1}$ when $y = 2x$. This line intersects the level set $g = 4$ in the point $(\frac{1}{2}\sqrt[3]{2}, \sqrt[3]{2})$.

Note: when you solve the equations $\vec{\nabla} g = \binom{s}{s}$, you find $y = 2x$, but also the line $y = 0$ (x-axis). On this line the gradient actually vanishes, i.e. $\vec{\nabla} g = \vec{0}$ and has no direction, so you can't really say it is parallel to $\binom{1}{1}$.

(IV10.4a) It's a paraboloid of revolution.

(IV10.4b) $\vec{\nabla} f = \begin{pmatrix} 2x \\ 2y \\ -2 \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ if $-2 = 2s$, i.e. $s = -1$. This then implies $2x = -1, 2y = -1$, so that $x = y = -\frac{1}{2}$. Since the point has to lie on the zero set of f, we find $z = \frac{1}{2}(x^2 + y^2) = \frac{1}{4}$.

(IV10.5a) At $(2, 1)$ the gradient is $\vec{\nabla} T = \begin{pmatrix} -2x \\ -9y^2 \end{pmatrix} = \begin{pmatrix} -4 \\ -9 \end{pmatrix}$. To cool off as fast as possible the bug should go in the opposite direction, i.e. in the direction of $\binom{4}{9}$, or any positive multiple of this vector.

(IV10.5b) At $(1, 3)$ the gradient is $\vec{\nabla} T = \begin{pmatrix} -2 \\ -81 \end{pmatrix}$. To keep its temperature constant the bug should walk in any direction perpendicular to the gradient. The vector $\binom{81}{-2}$ is perpendicular to the gradient, so the bug should go in the direction of $\binom{81}{-2}$ or the opposite direction, $\binom{-81}{2}$.

Any non-zero multiple of $\binom{-81}{2}$ is also a valid answer, since we can only give the *direction* and not the speed.

Remember: the vector $\binom{-b}{a}$ is perpendicular to $\binom{a}{b}$.

(IV10.6) The zero set doesn't have to be a curve. For example the zero set of the function $f(x, y) =$distance from (x, y) to the square Q (Problems 5.13 and 3.7) is the whole square Q.

(IV10.7) $\|\vec{\nabla} f\|$ is larger at the top right, because there the function f changes faster.

(IV10.8a) The gradient at the origin is the zero vector. This was explained in the text.

(IV10.8b) The function increases in the direction of the gradient. Since it vanishes on the curve in Figure 8, the function will be positive in the region above the curve, and it will be negative both below the curve and inside the little loop.

(IV10.12b) The result of a rather long calculation is that $\|\vec{\nabla} f\| = 1$ everywhere outside the square, and $\|\vec{\nabla} f\| = 0$ inside the square (because f is constant in the square.)

(IV10.14) $ax + by + cz = R^2$.

(IV12.1) $4xt\cos(x^2 + y^2) + 6yt^2\cos(x^2 + y^2)$

(IV12.2) $2xy\cos t + 2x^2t$

(IV12.3) $2xyt\cos(st) + 2x^2s$, $2xys\cos(st) + 2x^2t$

(IV12.4) $2xy^2t - 4yx^2s$, $2xy^2s + 4yx^2t$

(IV12.6a) $\frac{\partial T_B}{\partial Y} = -\sin\alpha\frac{\partial T_A}{\partial x} + \cos\alpha\frac{\partial T_A}{\partial y}$.

(IV12.6b) Take the formulas for $\frac{\partial T_B}{\partial X}$ and $\frac{\partial T_B}{\partial Y}$ and work out the right hand side in this problem.

(IV12.9a) $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\ln r = \frac{1}{r^2}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.

(IV12.9b) $\|\vec{\boldsymbol{E}}\| = 1/r = \frac{1}{\sqrt{x^2+y^2}}$.

(IV12.13a) Height $= -(x^2 - y^2)/(x^2 + y^2)$

(IV12.13b) Height $= \sin 2\theta$.

(IV12.13c) Height $= \cos 2\varphi$.

(IV15.1) $f_x = 3x^2y^2$, $f_y = 2x^3y + 5y^4$, $f_{xx} = 6xy^2$, $f_{yy} = 2x^3 + 20y^3$, $f_{xy} = 6x^2y$

(IV15.2) $f_x = 12x^2 + y^2$, $f_y = 2xy$, $f_{xx} = 24x$, $f_{yy} = 2x$, $f_{xy} = 2y$

(IV15.3) $f_x = \sin y$, $f_y = x\cos y$, $f_{xx} = 0$, $f_{yy} = -x\sin y$, $f_{xy} = \cos y$

(IV15.9) A function of two variables has

$$f_{xx}, \qquad f_{xy} = f_{yx}, \qquad f_{yy},$$

so it has ***three*** different partial derivatives of second order.

A function of three variables has these partial derivatives:

$$\begin{matrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{matrix}$$

The ones "below the diagonal" are the same as corresponding derivatives above the diagonal, so there are only six different partial derivatives of second order, namely these:

$$\begin{matrix} f_{xx} & f_{xy} & f_{xz} \\ & f_{yy} & f_{yz} \\ & & f_{zz} \end{matrix}$$

A function of two variables has

$$\begin{gathered} f_{xxx}, \\ f_{xxy} = f_{xyx} = f_{yxx}, \\ f_{xyy} = f_{yxy} = f_{yyx}, \\ \text{and } f_{yyy} \end{gathered}$$

so ***four*** different partial derivatives of third order.

(IV15.15a) We have $g(u, v) = f(u + v, u - v)$, so

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial(u+v)}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial(u-v)}{\partial u} = f_x(u+v, u-v) + f_y(u+v, u-v).$$

Similarly,

$$\frac{\partial g}{\partial v} = f_x(u+v, u-v) - f_y(u+v, u-v).$$

Differentiate again to get $\dfrac{\partial^2 g}{\partial u^2} = f_{xx}(u+v, u-v) + 2f_{xy}(u+v, u-v) + f_{yy}(u+v, u-v)$.

(IV15.15b) $\dfrac{\partial^2 g}{\partial v^2} = f_{xx}(u+v,u-v) - 2f_{xy}(u+v,u-v) + f_{yy}(u+v,u-v)$

Note that this is almost the same as $\dfrac{\partial^2 g}{\partial u^2}$: the only change is in the minus sign before f_{xy}.

(IV15.15c) $\dfrac{\partial^2 g}{\partial u \partial v} = f_{xx}(u+v,u-v) - f_{yy}(u+v,u-v)$

(IV15.15d) $\dfrac{\partial^2 g}{\partial u^2} - \dfrac{\partial^2 g}{\partial v^2} = -4f_{xy}$

(IV15.15e) $\dfrac{\partial^2 g}{\partial u^2} + \dfrac{\partial^2 g}{\partial v^2} = 2\big(f_{xx} + f_{yy}\big).$

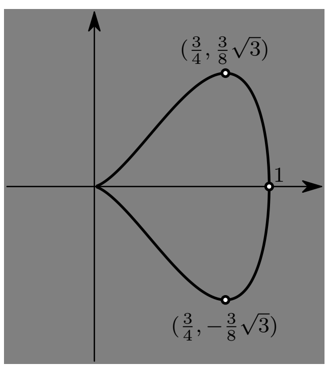
(V3.1a) If $y \neq 0$ then you can increase $x^2 - x^3 - y^2$ by setting $y = 0$. To put it differently, no matter what you choose for y, you always have

$$f(x,y) = x^2 - x^3 - y^2 \leq x^2 - x^3 = f(x,0).$$

(V3.1b) The maximum has to appear on the x axis, so the question is *which* $x \geq 0$ *maximizes* $f(x,0) = x^2 - x^3$*?* This is a Math 221 question. The answer is at $x = 2/3$.

(V3.1c) No, $\lim_{x\to-\infty} f(x,y) = +\infty$, so f has no largest value.

(V3.3)



The quantity $4(x^3 - x^4) = 4x^3(1-x)$ is negative when $x < 0$ or $x > 1$, so the region is confined to the vertical strip $0 \leq x \leq 1$. Within this strip R is comprised of those points which satisfy $-\sqrt{4(x^3 - x^4)} \leq y \leq +\sqrt{4(x^3 - x^4)}$. The largest x value is attained at the point with $x = 1$, where $y = 0$, so, at the point $(1,0)$. The smallest x value is attained at the point $(0,0)$. The largest y value is attained at the point where $y^2 = 4x^3 - 4x^4$ is maximal. This happens when $x = \frac{3}{4}$, and the largest y value is therefore $\sqrt{4[(3/4)^3 - (3/4)^4]} = \frac{3}{8}\sqrt{3}$. The smallest y value also occurs at $x = \frac{3}{4}$ and is given by $y = -\frac{3}{8}\sqrt{3}$.

(V6.1a) $f_x = 2x - 2$, $f_y = 8y + 8$, $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 8$.

There is exactly one critical point, at $(x,y) = (1,-1)$.

The 2nd order Taylor expansion at this point is

$$f(1 + \Delta x, -1 + \Delta y) = f(1,-1) + (\Delta x)^2 + 4(\Delta y)^2 + \cdots$$

The quadratic part is positive definite, therefore f has a ***local minimum*** at $(1,-1)$.

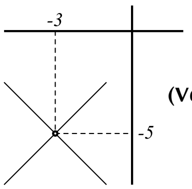
(V6.1b) $f_x = 2x + 6$, $f_y = -2y - 10$, $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = -2$.

There is exactly one critical point, at $(x,y) = (-3,-5)$.

The 2nd order Taylor expansion at this point is

$$\begin{aligned} f(-3 + \Delta x, -5 + \Delta y) &= f(-3,-5) + (\Delta x)^2 - (\Delta y)^2 + \cdots \\ &= f(-3,-5) + \big(\Delta x - \Delta y\big)\big(\Delta x + \Delta y\big) + \cdots \end{aligned}$$

The quadratic part factors, therefore f has a ***saddle point*** at $(-3,-5)$. The level set near the critical point consists of two crossing curves whose tangents are given by the equations $\Delta x = \Delta y$ and $\Delta x = -\Delta y$. Since



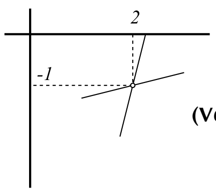
Critical point and level set near the critical point.

$\Delta x = x - a = x + 3$ and $\Delta y = y - b = y + 5$, the two tangent lines have equations $x + 3 = y + 5$ and $x + 3 = -(y + 5)$.

(V6.1c) $f_x = 2x + 4y$, $f_y = 4x + 2y$, $f_{xx} = 2$, $f_{xy} = 4$, $f_{yy} = 2$. There is one critical point: $(x, y) = (2, -1)$.

The 2nd order Taylor expansion at this point is

$$\begin{aligned} f(2 + \Delta x, -1 + \Delta y) &= f(2, -1) + (\Delta x)^2 + 4\Delta x\Delta x + (\Delta y)^2 + \cdots \\ &= f(2, -1) + \big(\Delta x + 2\Delta y\big)^2 - 3(\Delta y)^2 + \cdots \\ &= f(2, -1) + \big(\Delta x + (2 + \sqrt{3})\Delta y\big)\big(\Delta x + (2 - \sqrt{3})\Delta y\big) + \cdots \end{aligned}$$



Critical point and level set near the critical point.

The quadratic part factors, therefore f has a ***saddle point*** at $(2, -1)$. The level set near the critical point consists of two crossing curves whose tangents are given by the equations $\Delta x = -(2 + \sqrt{3})\Delta y$ and $\Delta x = -(2 - \sqrt{3})\Delta y$. Since $\Delta x = x - a = x - 2$ and $\Delta y = y - b = y + 1$, the two tangent lines have equations $x - 2 = -(2 + \sqrt{3})(y + 1)$ and $x - 2 = -(2 - \sqrt{3})(y + 1)$.

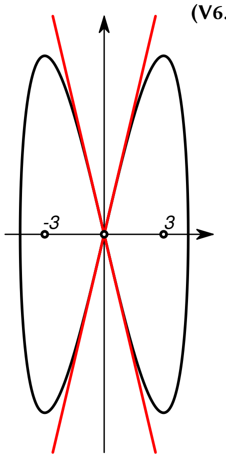
(V6.1d) $f_x = 2x - y - 5$, $f_y = -x + 4y + 6$, $f_{xx} = 2$, $f_{xy} = -1$, $f_{yy} = 4$.

There is again one critical point: $x = 2$, $y = -1$.

The 2nd order Taylor expansion at this point is

$$\begin{aligned} f(2 + \Delta x, -1 + \Delta y) &= f(2, -1) + (\Delta x)^2 - \Delta x\Delta x + 2(\Delta y)^2 + \cdots \\ &= f(2, -1) + \big(\Delta x - \tfrac{1}{2}\Delta y\big)^2 + \tfrac{7}{4}(\Delta y)^2 + \cdots \end{aligned}$$

The second order part of the Taylor expansion is positive, so $(2, -1)$ is a ***local minimum***.



Critical points and zero set.

(V6.1e) $f_x = -36x + 4x^3$, $f_y = 2y$, $f_{xx} = -36 + 12x^2$, $f_{xy} = 0$, $f_{yy} = 2$.

The equation $f_x = 0$ has three solutions, $x = 0$ and $x = \pm 3$. The equation $f_y = 0$ has only one solution $y = 0$. Therefore there are three critical points, the origin and the points $(\pm 3, 0)$.

The taylor expansions at these points are

$$\begin{aligned} f(\Delta x, \Delta y) &= f(0, 0) - 18(\Delta x)^2 + (\Delta y)^2 + \cdots \\ &= f(0, 0) + \big(\Delta y - \sqrt{18}x\big)\big(\Delta y + \sqrt{18}x\big) + \cdots \\ f(3 + \Delta x, \Delta y) &= f(3, 0) + 36(\Delta x)^2 + (\Delta y)^2 + \cdots \\ f(-3 + \Delta x, \Delta y) &= f(-3, 0) + 36(\Delta x)^2 + (\Delta y)^2 + \cdots \end{aligned}$$

The second order terms in the Taylor expansions at $(3, 0)$ and at $(-3, 0)$ are both positive for all Δx and Δy, so both points $(\pm 3, 0)$ are local minima. The second order part of the expansion at the origin factors and hence the origin is a saddle point. The tangents to the zeroset at the origin are the lines $\Delta y = \pm\sqrt{18}\Delta x = \pm 3\sqrt{2}\Delta x$. Since here $\Delta x =$ "$x - a$" $= x$, and $\Delta y = y$, the tangents are the lines through the origin given by $y = \pm 3\sqrt{2}x$.

You can try to draw the zeroset of this function and analyze it in the same way as the "fishy example" in 4.4. The zeroset of f consists of the graphs of $y = \pm\sqrt{18x^2 - x^4} = \pm|x|\sqrt{18 - x^2}$. It looks like a squashed "∞" or a butterfly (you decide.)

(V6.1f) There are nine critical points. Four global minima at $(\pm 3, \pm\sqrt{3})$, four saddle points at $(0, \pm\sqrt{3})$ and $(\pm 3, 0)$ respectively, and finally, a local but not global maximum at the origin.

(V6.1g) critical point at $(1, -1/6)$ $f_x = 4 - 4x$, $f_y = -1 - 6y$, $f_{xx} = -4$, $f_{xy} = 0$, $f_{yy} = -6$.

Second order Taylor expansion at the critical point:

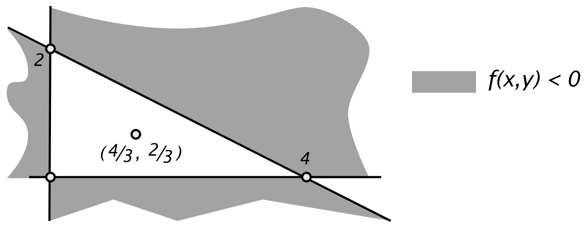
$$f(-1 + \Delta x, -\tfrac{1}{6} + \Delta y) = f(1, -\tfrac{1}{6}) - 2(\Delta x)^2 - 3(\Delta y)^2 + \cdots$$

The second order terms are always negative so $(1, -\frac{1}{6})$ is a local maximum.

(V6.1h) The derivatives are:

$$f_x = 4y - 2xy - 2y^2, \quad f_y = 4x - x^2 - 4xy, \quad f_{xx} = -2y, \quad f_{xy} = 4 - 2x - 4y, \quad f_{yy} = -4x.$$

This function is given in factored form, so without solving the equations $f_x = 0$, $f_y = 0$ you can say the following about this problem. The zero set consists of the three lines: the y-axis ($x = 0$), the x-axis ($y = 0$) and the line with equation $4 - x - 2y = 0$. It follows that the intersection points $(0, 0)$, $(4, 0)$, and $(0, 2)$ of these lines are saddle points. Since $f > 0$ in the triangle formed by the three lines this triangle must contain at least one local maximum.



To find all critical points solve these equations:

$$f_x = 4y - 2xy - 2y^2 = 0 \iff \boxed{y = 0 \text{ or } 4 - 2x - 2y = 0}$$

and

$$f_y = 4x - x^2 - 4xy = 0 \iff \boxed{x = 0 \text{ or } 4 - x - 4y = 0}$$

Since both equations $f_x = 0$ and $f_y = 0$ lead to two possibilities, we have to consider $2 \times 2 = 4$ cases:
- $y = 0$ **&** $x = 0$**:** This tells us the origin is a critical point
- $y = 0$ **&** $4 - x - 4y = 0$**:** Solving these equations leads to $x = 4, y = 0$, so $(4, 0)$ is a critical point.
- $4 - 2x - 2y = 0$ **&** $x = 0$**:** Solve and you find that $(0, 2)$ is a critical point.
- $4 - 2x - 2y = 0$ **&** $4 - x - 4y = 0$**:** Solve these equations and you get $(x, y) = (\frac{4}{3}, \frac{2}{3})$.

The first three critical points are the saddle points we predicted. The fourth critical point must be a local maximum, since there has to be one in the triangle, and of all the critical points we have found the others are all saddle points.

(V6.1i) Two saddle points: $(0, 0)$ and $(1, 1)$.

(V6.1j) Two saddle points: $(2, 2)$ and $(-2, -2)$

(V6.1l) The origin. Neither a local max, min, nor saddle. The graph of this function is called the "Monkey Saddle" as it accommodates two legs and a tail too. Draw it in your graphing program to see this.

(V6.1m) Zero set is the parabola with equation $x = y^2$, and the line $x = 1$. They intersect at $(1, \pm 1)$, so the function has two saddle points $(1, 1)$ and $(1, -1)$. The region between the line $x = 1$ and the parabola must contain local minimum. It is located at $(\frac{1}{2}, 0)$.

(V6.1n) Two saddle points : $(2, 2)$ and $(-2, -2)$. Yes, this problem appeared twice.

(V6.1o) All points on the y-axis are critical points. They are all global minima, but the second derivative test doesn't tell you so.

(V6.1p) All points on the y-axis are again critical points. Those with $y > 0$ are local minima, those with $y < 0$ are local maxima, and the origin is neither. The second derivative test applies to none of these points.

(V6.1q) All points on the unit circle are global minima, because the function vanishes there, and is positive everywhere else. The origin is a local maximum. The 2nd derivative test applies to the origin, but not to any of the other critical points.

(V6.1r) All points on the y-axis are again critical points. Those with $y > 0$ are local minima, those with $y < 0$ are local maxima, and the origin is neither. The second derivative test applies to none of these points.

(V6.5a) $(3, 4/3)$

(V6.5c) $x = (a + c + e)/3, y = (b + d + f)/3$.

(V6.6) You have to show that $f_x(a, b) = f_y(a, b) = 0$. By the product rule $f_x(a, b) = g_x(a, b)h(a, b) + g(a, b)h_x(a, b)$. Since both $g(a, b) = 0$ and $h(a, b) = 0$, it follows that $f_x(a, b) = 0$. The same reasoning applies to $f_y(a, b)$.

(V8.1a) One variable calculus! There is only one variable, a, and we must solve $E'(a) = 0$.

(V8.1b) $a = (x_1 + \cdots + x_N)/N$, i.e. the average provides "the best fit."

(V8.2a) Three: a, b, and c.

(V8.2b) The equations for (a, b, c) are:

$$\begin{array}{rcrcrcl}
(\sum x_k^4)\ a &+& (\sum x_k^3)\ b &+& (\sum x_k^2)\ c &=& \sum x_k^2 y_k \\
(\sum x_k^3)\ a &+& (\sum x_k^2)\ b &+& (\sum x_k)\ c &=& \sum x_k y_k \\
(\sum x_k^2)\ a &+& (\sum x_k)\ b &+& N\ c &=& \sum y_k
\end{array}$$

(V8.3) The equations are

$$\begin{array}{rcrcrcl}
(\sum x_k^2)\ a &+& (\sum x_k y_k)\ b &+& (\sum x_k)\ c &=& \sum x_k z_k \\
(\sum x_k y_k)\ a &+& (\sum y_k^2)\ b &+& (\sum y_k)\ c &=& \sum y_k z_k \\
(\sum x_k)\ a &+& (\sum y_k)\ b &+& N\ c &=& \sum z_k
\end{array}$$

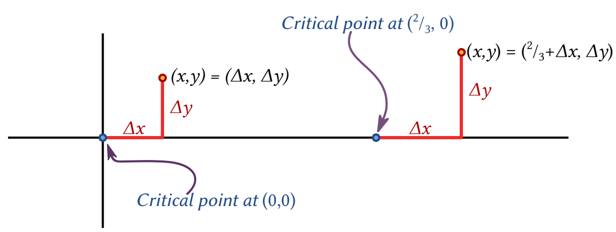
(V10.1) The two Δx and Δy's are different. The first set of $(\Delta x, \Delta y)$ are

$$\Delta x = x - 0, \qquad \Delta y = y - 0,$$

$(0, 0)$ being the coordinates of the first critical point we studied. The second set of $(\Delta x, \Delta y)$ is

$$\Delta x = x - \tfrac{2}{3}, \qquad \Delta y = y - 0,$$

where $(\frac{2}{3}, 0)$ is the other critical point. In a drawing:



(V10.2a) $f(\Delta x, \Delta y) = \left(1 - \Delta x + \Delta x \Delta y\right)^2 = 1 - 2\Delta x + \Delta x^2 + 2\Delta x \Delta y + \cdots$

(V10.2b) $f(1 + \Delta x, 1 + \Delta y) = \left(1 - (1 + \Delta x) + (1 + \Delta x)(1 + \Delta y)\right)^2 = 1 + 2\Delta y + 2\Delta x \Delta y + 2(\Delta y)^2 + \cdots$

(V10.2c) $f(\Delta x, \Delta y) = e^{\Delta x - (\Delta y)^2} = 1 + \Delta x + \frac{1}{2}(\Delta x)^2 - (\Delta y)^2 + \cdots$

(V10.2d) $f(1 + \Delta x, 1 + \Delta y) = e^{(1+\Delta x) - (1+\Delta y)^2} = 1 + \Delta x - 2\Delta y + \frac{1}{2}(\Delta x)^2 - 2\Delta x \Delta y + (\Delta y)^2 + \cdots$

(V10.4) Complete the square and you get

$$Q(x, y) = \left(x - ay\right)^2 + \left(1 - a^2\right)y^2.$$

When $1 - a^2 > 0$, i.e. when $-1 < a < 1$ the form is positive definite. When $a = \pm 1$ the form is a perfect square, namely,

$$x^2 \pm 2xy + y^2 = \left(x \pm y\right)^2.$$

When $1 - a^2 < 0$, i.e. when $a > 1$ or $a < -1$, the form is indefinite:

$$x^2 + 2axy + y^2 = \left(x - ay - \sqrt{a^2 - 1}y\right)\left(x - ay + \sqrt{a^2 - 1}y\right) = (x - k_+ y)(x - k_- y),$$

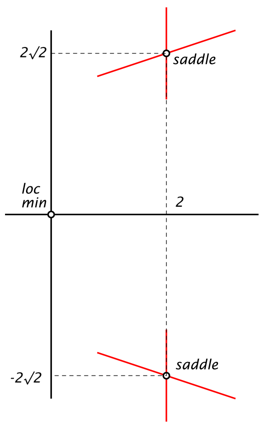
where $k_\pm = -a \pm \sqrt{a^2 - 1}$.

(V10.5) See the solutions to Problem 6.1 for the solutions to this problem.

(V10.7a) $f_x = 2x - \frac{1}{2}y^2$, $f_y = 2y - xy$. The equation $f_y = y(2-x) = 0$ leads to two possibilities: $x = 2$ or $y = 0$. If $y = 0$ then $f_x = 0$ implies $x = 0$, which gives us one critical point, the origin $(0,0)$. If on the other hand $x = 2$, then $f_x = 0$ implies $y^2 = 8 \iff y = \pm 2\sqrt{2}$. We therefore get two more critical points $(2, \pm 2\sqrt{2})$.

The second derivatives are $f_{xx} = 2$, $f_{xy} = -y$, $f_{yy} = 2 - x$. Therefore we have the following Taylor expansions at the three critical points:

$$\begin{aligned}
f(\Delta x, \Delta y) &= f(0,0) + (\Delta x)^2 + (\Delta y)^2 + \cdots && \Longrightarrow \text{loc.min.}\\
f(2+\Delta x, 2\sqrt{2}+\Delta y) &= f(2,2\sqrt{2}) + (\Delta x)^2 - 2\sqrt{2}\Delta x\Delta y + 0(\Delta y)^2 + \cdots \\
&= f(2,2\sqrt{2}) + \big(\Delta x - 2\sqrt{2}\Delta y\big)\Delta x + \cdots && \Longrightarrow \text{saddle}\\
f(2+\Delta x, -2\sqrt{2}+\Delta y) &= f(2,-2\sqrt{2}) + (\Delta x)^2 + 2\sqrt{2}\Delta x\Delta y + 0(\Delta y)^2 + \cdots \\
&= f(2,-2\sqrt{2}) + \big(\Delta x + 2\sqrt{2}\Delta y\big)\Delta x + \cdots && \Longrightarrow \text{saddle}
\end{aligned}$$



The origin is therefore a local minimum, and the points $(2, \pm 2\sqrt{2})$ are saddle points. At $(0, 2\sqrt{2})$ the level set consists of two crossing curves, whose tangents are given by $\Delta x = 0$ (a vertical line) and $\Delta x = 2\sqrt{2}\Delta y$ (a line with slope $1/2\sqrt{2} = \frac{1}{4}\sqrt{2}$).

(V10.7c) $f_x = 1 - y^2$, $f_y = 2 - 2xy$. Critical points: $f_x = 0$ holds when $y = \pm 1$. If $y = +1$, then $f_y = 0$ implies $x = 1$, and if $y = -1$ then $f_y = 0$ implies $x = -1$. There are therefore two critical points, $(1,1)$ and $(-1,-1)$.

(V13.1) $f(x,y) = xy$, $g(x,y) = x^2 + \frac{1}{4}y^2$. $\vec{\boldsymbol\nabla} f = \binom{y}{x}$, $\vec{\boldsymbol\nabla} g = \begin{pmatrix} 2x \\ y/2 \end{pmatrix}$.

First we check for possible max/minima which satisfy $\vec{\boldsymbol\nabla} g = \vec{\mathbf{0}}$. But the only point (x,y) satisfying $\vec{\boldsymbol\nabla} g(x,y) = \binom{0}{0}$ is the origin $(x,y) = (0,0)$, and this point does not lie on the constraint set.

Therefore, **if** there is a minimum it is attained at a solution of Lagrange's equations

$$\begin{aligned}
f_x = \lambda g_x &\iff y = 2\lambda x\\
f_y = \lambda g_y &\iff x = \lambda y/2\\
g(x,y) = 1 &\iff x^2 + \tfrac{1}{4}y^2 = 1
\end{aligned}$$

Multiply the first equation with y and the second with $4x$, then you get

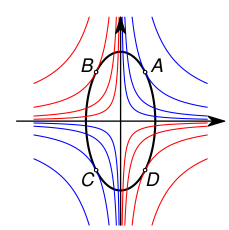
$$y^2 = 2\lambda xy \text{ and } 4x^2 = 2\lambda xy$$

Hence $y^2 = 4x^2$. Put that in the constraint, and you find

$$1 = x^2 + \tfrac{1}{4}y^2 = 2x^2.$$

Thus $x = \pm\sqrt{1/2} = \pm\frac{1}{2}\sqrt{2}$ and $y = \pm\sqrt{2}$. In all we have found ***four*** possible solutions. Lagrange's method does not tell us which, if any, of these are minima.

By looking at the constraint set (it's an ellipse with horizontal axis of length 1 and vertical axis of length 2) and taking into account that $f(x,y) = xy$ is positive in the first and third quadrants, and negative in the second and fourth, you find out that the two points $(\frac{1}{2}\sqrt{2}, \sqrt{2})$ and $(-\frac{1}{2}\sqrt{2}, -\sqrt{2})$ (A and C in the figure) are maximum points, while $(-\frac{1}{2}\sqrt{2}, \sqrt{2})$ and $(\frac{1}{2}\sqrt{2}, -\sqrt{2})$ (B and D in the figure) are minimum points.



Level sets of the function $f(x,y) = xy$ and the constraint set $x^2 + \frac{1}{4}y^2 = 1$

(V13.2a) Let the sides of the box be x, y, z. We want to minimize the quantity $A = 2xy+2yz+2xz$, with the constraint $V = xyz = \frac{1}{2}$. The constraint implies that $x \neq 0, y \neq 0$ and $z \neq 0$ moreover, given x and y the only z which satisfies the constraint is $z = 1/(2xy)$. Thus we must minimize the following function of two variables

$$A(x,y) = xy + \frac{1}{2x} + \frac{1}{2y}$$

over all $x > 0, y > 0$.

A minimum must be an interior minimum (can't be on the x or y-axis since these are excluded), and thus must be a critical point.

$$\frac{\partial A}{\partial x} = y - \frac{1}{2x^2}, \qquad \frac{\partial A}{\partial y} = x - \frac{1}{2y^2}.$$

Solving $A_x = A_y = 0$ for (x, y) leads to $x = y = \sqrt[3]{2}$, so the solution is a cube $1/\sqrt[3]{2}$ on a side

(V13.2b) We wish to minimize $A(x, y, z) = 2yz+2xz+2xy$ with constraint $V(x, y, z) = xyz = \frac{1}{2}$, using Lagrange's method.

First we check for exceptional points on the constraint set, i.e. points (x, y, z) that satisfy both $V(x, y, z) = \frac{1}{2}$ and $\vec{\nabla} V(x, y, z) = \vec{\mathbf{0}}$. Since

$$\vec{\nabla} V = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

the gradient $\vec{\nabla} V$ vanishes if at least two of the three coordinates x, y, z are zero. But such a point can never satisfy the constraint $xyz = \frac{1}{2}$. Therefore, if there is a box with least area, its sides x, y, z must satisfy Lagrange's equations.

Lagrange's equations are

$$\begin{aligned} A_x = \lambda V_x &\iff 2y + 2z = \lambda yz \\ A_y = \lambda V_y &\iff 2x + 2z = \lambda xz \\ A_z = \lambda V_z &\iff 2x + 2y = \lambda xy \end{aligned}$$

To get rid of λ multiply the first equation with x and the second with y to get

$$y(2x + 2z) = \lambda xyz = x(2y + 2z) \implies 2xy + 2yz = 2xy + 2xz \implies 2yz = 2xz.$$

Therefore we find that either $z = 0$ or $x = y$. But $z = 0$ is not possible, because (x, y, z) must satisfy the constraint $xyz = 0$. Therefore we get $x = y$.

If you multiply the second Lagrange equation with y and the third with z then the same reasoning as above tells you that $y = z$.

So, ***if there is a minimum*** then it happens when $x = y = z$, i.e. when the box is a cube. The only cube that satisfies the constraint has sides $x = y = z = 2^{-1/3}$.

As always, Lagrange's method does not rule out the possibility that the cube we have found actually maximizes the surface area, rather than minimizing it. That this is actually not the case is something you would have to prove by other means. We will not do that in this course.

(V13.3) Answer: the shortest distance is $\sqrt{100/3}$.

Solution: If (x, y, z) is any point than its distance to the origin is $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. We want to minimize $d(x, y, z)$ over all points (x, y, z) which satisfy the constraint $g(x, y, z) = x+y+z = 10$. Instead of minimizing $d(x, y, z)$ we will minimize $f(x, y, z) = d(x, y, z)^2 = x^2 + y^2 + z^2$. You can do this problem directly with the function $d(x, y, z)$ and you will get the same answer – the computations are just a little longer because f has easier derivatives than d.

We use Lagrange's method. First we check for exceptional points, i.e. points on the constraint set which satisfy $\vec{\nabla} g = \vec{\mathbf{0}}$. Since $\vec{\nabla} g = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ the gradient of g can never be the zero vector, so there are no exceptional points. If there is a minimum of f on the constraint set, it must be a solution of Lagrange's equations.

The Lagrange equations are

$$\begin{aligned} f_x = \lambda g_x &\iff 2x = \lambda \\ f_y = \lambda g_y &\iff 2y = \lambda \\ f_z = \lambda g_z &\iff 2z = \lambda \end{aligned}$$

Therefore ***if there is a nearest point to the origin on the plane*** then it must satisfy $x = y = z = \lambda/2$ as well as the constraint. The only point satisfying these conditions is $(\frac{10}{3}, \frac{10}{3}, \frac{10}{3})$.

Lagrange's method does not tell us that this is the nearest point. As far as Lagrange is concerned it could also be the furthest point from the origin. (But because we know what a plane looks like we "know" that there has to be a nearest point to the origin.)

(V13.5a) Minimize $f(x, y, z) = (x-2)^2 + (y-1)^2 + (z-4)^2$ subject to the constraint $g(x, y, z) = 2x - y + 3z = 1$.

First, since $\vec{\nabla} g - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \neq \vec{0}$, there are no exceptional points, so the nearest point (if it exists) is a solution of Lagrange's equations. These are

$$2(x-2) = 2\lambda, \quad 2(y-1) = -\lambda, \quad 2(z-4) = 3\lambda.$$

Eliminate λ to get

$$x = -2y + 4, \qquad z = -3y + 7.$$

Combined with the constraint you then find

$$y = 2, \quad x = 0, \quad z = 1.$$

The Lagrange multiplier is $\lambda = x - 2 = -2$.

The distance from the point we found to the given point $(2, 1, 4)$ is

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (z-4)^2} = \sqrt{14}$$

(V13.5b) $|ax_0 + by_0 + cz_0 - d| / \sqrt{a^2 + b^2 + c^2}$

(V13.8) a cube

(V13.10) $65/3 \times 65/3 \times 130/3$

(V13.11) It has a square base, and is one and one half times as tall as wide. If the volume is V the dimensions are $\sqrt[3]{2V/3} \times \sqrt[3]{2V/3} \times \sqrt[3]{9V/4}$.

(V13.12) $(0, 0, 1)$, $(0, 0, -1)$

(V13.13) $\sqrt[3]{4V} \times \sqrt[3]{4V} \times \sqrt[3]{V/16}$

(V13.14) Farthest: $(-\sqrt{2}, \sqrt{2}, 2 + 2\sqrt{2})$; closest: $(2, 0, 0)$, $(0, -2, 0)$

(VI3.1a) 2

(VI3.1b) 8

(VI3.1c) $2/3$

(VI3.1d) $\displaystyle\int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy = \int_0^\pi \frac{\sin y}{y} \cdot y\, dy = \int_0^\pi \sin y\, dy = 2.$

(VI3.1e) Except for a change in notation ($y \to \theta$ and $x \to r$) this is the same integral as in the previous problem. The answer is again 2.

(VI3.1f) Which function is being integrated? It's the function $f(x, y) = 1$.
$\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = \int_0^1 [y]_{y=0}^{y=\sqrt{1-x^2}} dx = \int_0^1 \sqrt{1-x^2}\, dx$. The last integral is the area of a quarter circle with radius 1, so the answer is $\pi/4$.

(VI3.2) Once you compute the inner integral

$$\int_0^1 \sin(\pi x) dx = \left[-\frac{1}{\pi}\cos \pi x\right]_{x=0}^1 = -\frac{1}{\pi}\cos \pi - \frac{1}{\pi}/4(-\cos 0) = 2,$$

you get

$$\int_x^1 \left\{\int_0^1 \sin(\pi x) dx\right\} dy = \int_x^1 2dy = [2y]_{y=x}^1 = 2(1-x).$$

The result depends on x. The x in the answer and the two x-es in the inner integral refer to different quantities. This is at best confusing, and should really never be done.

(VI3.3a) Not true! To give a counterexample for the statement in the problem, almost any two functions f and g will do. For instance, if you choose $f(x) = x$, $g(y) = 1$, then you get

$$\int_0^1 \int_0^2 f(x)g(y)\;dx\;dy = \int_0^1 \int_0^2 xdxdy = 2.$$

but

$$\int_0^1 f(x)\;dx\;\times\;\int_0^2 g(y)\;dy = \int_0^1 x\;dx\;\times\;\int_0^2 dy = \frac{1}{2}\times 2 = 1.$$

(VI3.3b) True!

$$\int_0^1 \int_0^2 f(x)g(y)dydx = \int_0^1 \left\{\int_0^2 f(x)g(y)dy\right\} dx.$$

Since $f(x)$ does not depend on y, we have

$$\int_0^2 f(x)g(y)dy = f(x)\int_0^2 g(y)\,dy.$$

Therefore

$$\int_0^1 \left\{\int_0^2 f(x)g(y)dy\right\} dx = \int_0^1 f(x)\left\{\int_0^2 g(y)dy\right\} dx.$$

The integral $\int_0^2 g(y)dy$ is a constant, and does therefore not depend on x, so we can factor it out of the x-integral:

$$\int_0^1 f(x)\left\{\int_0^2 g(y)dy\right\} dx = \int_0^1 f(x)\,dx\cdot\int_0^2 g(y)\,dy,$$

which is what we had to show.

(VI3.3c) This is false, and there is no simple way of fixing it. To see that this fails evaluate both sides with $f(x) = 1$ and $g(y)$. On the left you get the area of the disc D, which is π, and on the right you get $2 \cdot 2 = 4$.

(VI3.4) The volume under the graph is $\frac{1}{3}ba^3 + \frac{1}{3}ab^3 = \frac{1}{3}ab(a^2 + b^2)$. The volume of the surrounding block is $a \times b \times (a^2 + b^2)$, so the region beneath the graph occupies one third of the surrounding block, no matter which a or b you choose.

(VI3.5a) 16

(VI3.5b) 4

(VI3.5c) $15/8$

(VI3.5d) $1/2$

(VI3.5e) $5/6$

(VI3.5f) $12 - 65/(2e)$.

(VI3.5g) $1/2$

(VI3.5h) $(2/9)2^{3/2} - (2/9)$

(VI3.5i) $(1 - \cos(1))/4$

(VI3.5j) $(2\sqrt{2} - 1)/6$

(VI3.5k) $\pi - 2$

(VI3.6a) 8π

(VI3.6b) 2

(VI3.6c) $5/3$

(VI3.6d) $81/2$

(VI3.6e) $2a^3/3$

(VI3.6f) 8π

(VI3.6g) $\pi/32$

(VI3.8a) A

(VI3.8b) $B/2$

(VI3.9a) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{2xy}{x^2+y^2}\, dy\, dx.$

(VI3.9b) In P.C. the function simplifies to $F(r,\theta) = 2\sin\theta\cos\theta$, so the volume is

$$V = \int_0^1 \int_0^{\pi/2} 2\sin\theta\cos\theta r\, d\theta\, dr = \int_0^1 \left[\sin^2\theta\right]_0^{\pi/2} r\, dr = \tfrac{1}{2}.$$

(VI7.1a) A cone around the positive z axis, with opening angle $\pi/6$.

(VI7.1b) The negative half of the z axis.

(VI7.1c) The xy plane.

(VI7.1d) The half of the yz plane which contains the positive y axis, and which ends at the z-axis.

(VI7.2a) $0 \le \theta \le \pi/2, 0 \le \rho \le a, 0 \le \phi \le \pi/2.$

(VI7.2b) $0 \le \theta \le \pi/2, 0 \le r \le a, 0 \le z \le \sqrt{a^2 - r^2}$, or:
$0 \le \theta \le \pi/2, 0 \le z \le a, 0 \le r \le \sqrt{a^2 - z^2}.$

(VI7.3) Figures 15 and 16.

(VI7.4a) Large circle has radius 1, the smaller has radius $\sqrt{1-z^2}$.

(VI7.4b) $x = \sqrt{1-y^2-z^2}$ for the point in front, and $x = -\sqrt{1-y^2-z^2}$ for the point in the back (furthest away from you, the viewer).

(VI7.5a) The potential energy is "mass$\times$height$\times g$". The mass of the small piece of honey is $\Delta m = \mu \times \Delta V$, where ΔV is the volume occupied by the small piece of honey. This is not an exact formula, but only an approximation, since not all particles in the small piece of honey have exactly the same height. However, as one considers smaller and smaller pieces the approximation gets better.

(VI7.5b) The total potential energy is

$$\text{P.E.} = \iiint_D \mu g z\, dV.$$

Interpretation: this is the total energy that would be released if you put all the honey at height zero (e.g. by pouring it out of the jar onto the floor.)

(VI7.5c) The iterated integral is

$$\text{P.E.} = \int_{x=0}^{A} \int_{y=0}^{B} \int_{z=0}^{f(x,z)} \mu g z\, dz\, dy\, dx = \frac{1}{2}\mu g \int_{x=0}^{A} \int_{y=0}^{B} f(x,y)^2\, dy\, dx.$$

(VI7.6a) The kinetic energy in a small region of the airmass is $\frac{1}{2}\Delta m \times v^2$, where Δm is the mass of the air in the small region. This mass is $\mu \times \Delta V$, with ΔV the volume of the small region, so the kinetic energy of the small region is $\frac{1}{2}\mu \times v^2 \times \Delta V$. Partitioning the whole airmass, and adding the kinetic energies of all the small pieces leads to this integral:

$$\text{K.E.} = \iiint_D \tfrac{1}{2}\mu v(r)^2\, dV = \tfrac{1}{2}\mu \iiint_D v(r)^2\, dV.$$

(VI7.6b) In cylindrical coordinates the domain is defined by $0 \le r \le R$ and $0 < z \le H$, so the integral is

$$\text{K.E.} = \frac{1}{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{H}\int_{r=0}^{R} \frac{r}{1+r^2}\, dr\, dz\, d\theta = \frac{\pi}{2} H \ln\bigl(1+R^2\bigr).$$

(VI7.7a) $623/60$

(VI7.7b) $-3e^2/4 + 2e - 3/4$

(VI7.7c) $1/20$

(VI7.7d) $\pi/48$

(VI7.7e) $11/84$

(VI7.7f) $151/60$

(VI7.8) 32

(VI7.9) $64/3$

(VI7.10) $\bar{x} = \bar{y} = 0$, $\bar{z} = 16/15$

(VI7.11) $\bar{x} = \bar{y} = 0$, $\bar{z} = 1/3$

(VI7.12a) $I = V_+$, $J = -V_-$ (note the minus sign), $K = V_+ - V_-$, $L = V_+ + V_-$.

(VI7.13) $\pi/12$

(VI7.14) $5\pi/4$

(VI7.15) 0

(VI7.16) $5\pi/4$

(VI7.17) $4/5$

(VI7.18) $256\pi/15$

(VI7.19) $4\pi^2$

(VI7.20) $\pi k h^2 a^2/12$

(VI7.21) $\pi k h a^3/6$

(VI7.22) $\pi^2/4$

(VI7.23) $4\pi/5$

(VI7.24) 15π

(VII4.1a) The answer is 1. You could compute that, but you don't have to. The distance is 1 everywhere, so its average should also be 1.

(VII4.1b) $\frac{\int_0^{\pi/2} \theta\, d\theta}{\pi/2} = \pi/4$

(VII4.3) The average x coordinate is zero, and the average y coordinate is $2/\pi$.

(VII4.4) $\vec{x}(t) = \binom{t}{t^2}$ is a parametrization, so the integral becomes

$$\int_{\mathcal{C}} x\, ds = \int_{t=0}^{1} \underbrace{t}_{x=t}\ \underbrace{\sqrt{1+4t^2}}_{\|\vec{x}\,'(t)\|}\, dt = \left[\frac{2}{3}\frac{1}{8}\bigl(1+4t^2\bigr)^{3/2}\right]_0^1 = \frac{5\sqrt{5}-1}{12}.$$

(VII4.5a) a, H, L are lengths; T_0 is a temperature.

(VII4.5b) $a = 1$ is the radius of the cylinder on which the helix lies, and $H = \pi/2$ is the height of one turn of the helix.

(VII4.5c) The average is

$$\text{average temp.} = \frac{\int_{\mathcal{C}} T\ ds}{\int_{\mathcal{C}}\ ds}.$$

With the given parametrization $ds = \|\vec{\boldsymbol{x}}'(t)\|\ dt = \sqrt{a^2 + H^2/4\pi^2}\ dt$ – an ugly expression, but it's constant, which is good for integrating. You get

$$\int_{\mathcal{C}} ds = \int_0^{2\pi} \sqrt{a^2 + H^2/4\pi^2}\ dt = 2\pi\sqrt{a^2 + H^2/4\pi^2} = \sqrt{4\pi^2 a^2 + H^2}.$$

and

$$\begin{aligned}\int_{\mathcal{C}} T\ ds &= \int_0^{2\pi} T_0 e^{-Ht/2\pi L}\sqrt{a^2 + H^2/4\pi^2}\ dt \\ &= T_0\sqrt{a^2 + H^2/4\pi^2}\left[-\frac{2\pi L}{H}e^{-Ht/2\pi L}\right]_{t=0}^{2\pi} \\ &= T_0\sqrt{a^2 + H^2/4\pi^2}\,\frac{2\pi L}{H}\left[1 - e^{-H/L}\right].\end{aligned}$$

Therefore the average temperature is

$$\text{average temp.} = \frac{L}{H}\left(1 - e^{-H/L}\right) T_0.$$

(VII8.1) Yes. It is the gradient of $f(x, y) = gy$.

(VII8.3) By Clairaut's theorem, if $\vec{\boldsymbol{F}}$ is a gradient, then $P_y = Q_x$.

By the fundamental theorem for line integrals, if $\vec{\boldsymbol{F}}$ is a gradient, then $\oint_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} = 0$ (or, equivalently, $\oint_{\mathcal{C}} Pdx + Qdy = 0$) for every closed curve $\mathcal{C}$.

(VII8.4a) $\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} = \int_{\mathcal{C}} xdx = 0.$

$\int_{\mathcal{C}} \vec{\boldsymbol{G}}\cdot d\vec{\boldsymbol{x}} = \int_{\mathcal{C}} xdy = \pi.$

(VII8.4b) Since $\int_{\mathcal{C}} \vec{\boldsymbol{G}}\cdot d\vec{\boldsymbol{x}} \neq 0$ the vector field $\vec{\boldsymbol{G}}$ cannot be a gradient.

(VII8.4c) The integral $\int_{\mathcal{C}} \vec{\boldsymbol{f}}\cdot d\vec{\boldsymbol{x}}$ vanishes, but to check that $\vec{\boldsymbol{F}}$ is conservative one has to check $\int_{\mathcal{C}} \vec{\boldsymbol{f}}\cdot d\vec{\boldsymbol{x}} = 0$ for ***all*** closed curves $\mathcal{C}$, and not just the unit circle. So our integral computation does not imply that $\vec{\boldsymbol{F}}$ is a gradient.

You can use different arguments to show directly that $\vec{\boldsymbol{F}}$ is a gradient, for instance, by noting that $\vec{\boldsymbol{\nabla}}(\frac{1}{2}x^2) = \left(\begin{smallmatrix}x\\0\end{smallmatrix}\right)$, or if you're not that lucky, by using the methods of § IV.14.

(VII12.2b) The answer in both cases is the same (because they are two different ways of computing the same integral). The second approach, using Green's theorem leads to

$$\int_{\mathcal{C}} 2y\,dx + 3x\,dy = \iint_{\mathcal{R}} \Big(\frac{\partial 3x}{\partial x} - \frac{\partial 2y}{\partial y}\Big)dA = \iint_{\mathcal{R}} (3-2)\,dA,$$

so the answer is the area of the square, i.e. 1

(VII12.3) Using Green's theorem we get zero. But here we do not need Green's theorem: the Fundamental Theorem for line integrals (see § sec:integral-over-closed-curve-of-gradient-vanishes) tells us that this integral must be zero.

(VII12.5a) 0

(VII12.5b) $1/(2e) - 1/(2e^7) + e/2 - e^7/2$

(VII12.5c) $1/2$

(VII12.5d) 0

(VII12.5e) $-1/6$

(VII12.5f) $(2\sqrt{3} - 10\sqrt{5} + 8\sqrt{6})/3 - 2\sqrt{2}/5 + 1/5$

(VII12.5g) $11/2 - \ln(2)$

(VII12.5h) $2 - \pi/2$

(VII12.5i) $-17/12$

(VII12.5j) 0

(VII12.5k) $-\pi/2$

(VII12.5l) $-\pi/2$

(VII12.5m) 12π

(VII17.1) The distance to the central axis is $r^2 = y^2 + z^2$, so

$$\vec{\boldsymbol{v}}(x, y, z) = v_c \left(1 - \frac{y^2 + z^2}{R^2}\right) \vec{\boldsymbol{\imath}}$$

(VII17.2) The inverse square law holds:

$$\|\vec{\boldsymbol{F}}\| = \left\| -C \frac{\vec{\boldsymbol{x}}}{\|\vec{\boldsymbol{x}}\|^3} \right\| = \frac{C}{\|\vec{\boldsymbol{x}}\|^3} \|\vec{\boldsymbol{x}}\| = \frac{C}{\|\vec{\boldsymbol{x}}\|^2}.$$

(VII17.3) $n = 2$ and $C = \mu_0 I / 2\pi$.

(VII17.5a) $\left(e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}\right)_{x_1} = m_1 e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}$, and the same for the x_2 and x_3 derivatives. Therefore

$$\vec{\boldsymbol{\nabla}}\left(e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}\right) = \begin{pmatrix} m_1 e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}} \\ m_2 e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}} \\ m_3 e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}} \end{pmatrix} = e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}} \left(\begin{smallmatrix} m_1 \\ m_2 \\ m_3 \end{smallmatrix}\right).$$

(VII17.5b) After simplifying you get $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{v}} = \vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{a}} e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}$.

(VII17.5c) $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{v}} = \vec{\boldsymbol{m}}\times\vec{\boldsymbol{a}} e^{\vec{\boldsymbol{m}}\cdot\vec{\boldsymbol{x}}}$.

(VII17.5d) $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$ must be perpendicular.

(VII17.5e) If $\vec{\boldsymbol{v}}$ is the gradient of some function, then its curl must vanish. Therefore $\vec{\boldsymbol{a}}\times\vec{\boldsymbol{m}} = \vec{\boldsymbol{0}}$ in view of part 3 of this problem. The conclusion is that $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{m}}$ must be parallel.

(VII17.6) $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{\nabla}} f = P f_x + Q f_y + R f_z$.

(VII17.7a) By definition,

$$\begin{aligned}
\vec{\boldsymbol{\nabla}}\cdot(f\vec{\boldsymbol{v}}) = \vec{\boldsymbol{\nabla}}\cdot \begin{pmatrix} fP \\ fQ \\ fR \end{pmatrix} &= \frac{\partial fP}{\partial x} + \frac{\partial fQ}{\partial y} + \frac{\partial fR}{\partial z} \\
&= f_x P + f P_x + f_y Q + f Q_y + f_z R + f R_z \\
&= f_x P + f_y Q + f_z R + f\big(P_x + Q_y + R_z\big) \\
&= \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix} + f \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{v}} \\
&= \vec{\boldsymbol{\nabla}} f\cdot\vec{\boldsymbol{v}} + f\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{v}},
\end{aligned}$$

as claimed.

(VII17.7b) $\vec{\nabla}\times(f\vec{v}) = (\vec{\nabla}f)\times\vec{v} + f\vec{\nabla}\times\vec{v}$ is the rule. The derivation goes along the same lines as in the previous product rule.

(VII17.8) This is example 16.1.

(VII17.9a) $5\rho^2$.

(VII17.9b) $\vec{x}\cdot\frac{\vec{x}}{\rho} = \|\vec{x}\|^2/\rho = \rho^2/\rho = \rho$.

(VII17.9c) Note that $\|\vec{x}\| = \rho$, so you have to compute $\vec{\nabla}\cdot(\vec{x}/\rho^3)$. The answer is zero.
It says that the divergence of the gravitational field of the Earth is zero.

(VII17.10b) Since $\vec{x}$ is the gradient of some function its curl must vanish.

(VII17.10c) $\vec{\nabla}\times(\rho\vec{x}) = (\vec{\nabla}\rho)\times\vec{x} + \rho\vec{\nabla}\times\vec{x} = \vec{0}$

(VII17.11) $\vec{v}(x,y,z) = \left(\begin{smallmatrix}-y\\x\\0\end{smallmatrix}\right)$ so $\vec{\nabla}\times\vec{v} = \left(\begin{smallmatrix}0\\0\\2\end{smallmatrix}\right) = 2\vec{k}$.

(VII17.12a) $\vec{v}(x,y,z) = \begin{pmatrix}x(x^2+y^2+z^2)^{n/2}\\y(x^2+y^2+z^2)^{n/2}\\z(x^2+y^2+z^2)^{n/2}\end{pmatrix}$.

(VII17.12b) Using the product rule, you get

$$\vec{\nabla}(\rho^n\vec{x}) = (\vec{\nabla}\rho^n)\cdot\vec{x} + \rho^n\vec{\nabla}\cdot\vec{x} = -n\rho^{n-1}(\vec{\nabla}\rho)\cdot\vec{x} + \rho^n\vec{\nabla}\cdot\vec{x}.$$

Now recall (or compute again):

$$\vec{\nabla}\rho = \frac{\vec{x}}{\rho},\ \text{and}\ \vec{\nabla}\cdot\vec{x} = 3.$$

This leads to

$$\vec{\nabla}(\rho^n\vec{x}) = n\rho^{n-1}\frac{\vec{x}}{\rho}\cdot\vec{x} + 3\rho^n = n\rho^{n-2}\|\vec{x}\|^2 + 3\rho^n = (n+3)\rho^n)$$

(VII17.12c) $n = -3$.

(VII17.13) There are a long and a short answer. The long(er) computation goes likes this:

$$\vec{\nabla}F(\rho) = \begin{pmatrix}F(\rho)_x\\F(\rho)_y\\F(\rho)_z\end{pmatrix} = \begin{pmatrix}F'(\rho)\rho_x\\F'(\rho)\rho_y\\F'(\rho)\rho_z\end{pmatrix} = F'(\rho)\begin{pmatrix}\rho_x\\\rho_y\\\rho_z\end{pmatrix}.$$

Now recall (185), and you find

$$\vec{\nabla}F(\rho) = F'(\rho)\begin{pmatrix}x/\rho\\y/\rho\\z/\rho\end{pmatrix} = \frac{1}{\rho}F'(\rho)\vec{x}.$$

The short computation is essentially the same, but you never write the components of the vectors:

$$\vec{\nabla}F(\rho) = F'(\rho)\vec{\nabla}\rho = \frac{1}{\rho}F'(\rho)\vec{x}.$$

(VII17.13a) If $f(x,y,z) = F(\rho)$, then by the previous problem we have $\vec{\nabla}f = \rho^{-1}F'(\rho)\vec{x}$. We want this to be equal to $\rho^{-n}\vec{x}$, so $F(\rho)$ must satisfy

$$\rho^{-1}F'(rho) = \rho^n \implies F'(\rho) = \rho^{1+n} \implies F(\rho) = \frac{\rho^{2+n}}{2+n} + C$$

for some constant C. We are only asked to find on function f, so we find that the given vector field is indeed the gradient of a radially symmetric function:

$$\vec{v} = \rho^n\vec{x} = \vec{\nabla}\left(\frac{\rho^{2+n}}{2+n}\right).$$

The exceptional case is when $n = -2$, in which case you get $F(\rho) = \ln\rho$.