THE LINEAR LUGIATO-LEFEVER EQUATION WITH FORCING AND NONZERO PERIODIC OR NONPERIODIC BOUNDARY CONDITIONS

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Abstract. We consider the linear Lugiato-Lefever equation formulated on a finite interval with *nonzero* boundary conditions. In particular, using the unified transform of Fokas, we obtain *explicit* solution formulae both for the general nonperiodic initial-boundary value problem and for the periodic Cauchy problem. These novel solution formulae involve integrals, as opposed to the infinite series associated with traditional solution techniques, and hence they have analytical as well as computational advantages. Importantly, as the linear Lugiato-Lefever can be related to the linear Schrödinger equation via a simple transformation, our results are directly applicable also to the linear Schrödinger equation posed on a finite interval with nonzero boundary conditions.

1. Introduction

The Lugiato-Lefever equation

$$u_t + i\beta u_{xx} + (1+i\alpha)\, u - i|u|^2 u = F \tag{1.1}$$

has recently gained significant attention within the broader applied mathematics community. Here, $u(x,t)$ is a complex-valued function, α and β are real parameters, and F is a positive constant. Equation (1.1) is an envelope model that originates from the Maxwell-Bloch equations and was introduced in [11] as an example for dissipative structure and pattern formation in nonlinear optics. Further information about the derivation and relevance of the Lugiato-Lefever equation as an optical model in various settings, including experimental results that illustrate its physical significance, can be found in [2,3,9,10,12–14]. Furthermore, several works have recently appeared in the literature on the rigorous mathematical study of equation (1.1), where a direction of particular emphasis concerns the stability of the periodic Cauchy problem — see, for example, the recent works [5–8, 32].

The Lugiato-Lefever equation (1.1) arises naturally in the periodic setting and also as a general initial-boundary value problem on a finite interval. In this work, we are concerned with the *forced linear counterpart* of equation (1.1), namely the equation

$$u_t + i\beta u_{xx} + (1+i\alpha)\, u = f(x,t), \tag{1.2}$$

where f is a given forcing function, for which we derive novel, explicit solution formulae in both the periodic and the nonperiodic case with *nonzero* boundary conditions — see expressions (1.7) and (1.5) respectively. Our main motivation behind this study is related to the task of showing well-posedness of the nonlinear equation (1.1) via contraction mapping techniques. In particular, we note that in recent years a new method has been introduced by one of the authors and collaborators for proving the local well-posedness of initial-boundary value problems for nonlinear evolution

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equations — see, for example, [21, 22, 27, 28]. This method is based on a contraction mapping argument that crucially relies on the solution formulae obtained for the forced linear counterparts of these problems via the *unified transform* of Fokas (also known as the Fokas method) [15, 19]. In this respect, the explicit solution formulae (1.5) and (1.7) derived in the present work for the forced linear equation (1.2) will serve as starting points for implementing the method of [21, 22, 27, 28] in order to establish the local well-posedness of the Lugiato-Lefever equation (1.1) on a finite interval with nonperiodic or periodic nonzero boundary conditions.

Specifically, in this work we explicitly solve the forced linear initial-boundary value problem

$$u_t + i\beta u_{xx} + (1+i\alpha)\, u = f(x,t), \quad 0 < x < \ell, \ t > 0, \tag{1.3a}$$

$$u(x,0) = u_0(x), \quad 0 < x < \ell, \tag{1.3b}$$

$$u(0,t) = g_0(t), \quad u(\ell,t) = h_0(t), \quad t > 0, \tag{1.3c}$$

where the initial data u_0 and the boundary data g_0, h_0 are assumed to be sufficiently smooth so that the various computations carried out in this work make sense. In particular, the precise characterization of the optimal regularity of the initial and boundary data is a task which becomes more relevant when aiming to prove the well-posedness of the nonlinear equation (1.1) and hence it is reserved for future work in that direction.

Observe that, in the special case $f(x,t) = F$, equation (1.3a) corresponds to the linearization of the Lugiato-Lefever equation (1.1). Furthermore, note that the initial-boundary value problem (1.3) for the forced linear Lugiato-Lefever equation can be easily related to a corresponding problem for the forced linear Schrödinger equation. Indeed, the transformation $u(x,t) = e^{-(1+i\alpha)t}\, v(x,t)$ turns problem (1.3) into

$$v_t + i\beta v_{xx} = e^{(1+i\alpha)t} f(x,t), \quad 0 < x < \ell, \ t > 0, \tag{1.4a}$$

$$v(x,0) = u_0(x), \quad 0 < x < \ell, \tag{1.4b}$$

$$v(0,t) = e^{(1+i\alpha)t} g_0(t), \quad v\,(\ell,t) = e^{(1+i\alpha)t} h_0(t), \quad t > 0, \tag{1.4c}$$

which is an initial-boundary value problem for the familiar forced linear Schrödinger equation with forcing and Dirichlet data on the finite interval. *Therefore, the results obtained in this work for the forced linear Lugiato-Lefever equation on the interval are directly applicable also to the solution of the forced linear Schrödinger equation on the interval.* In fact, although the cubic nonlinear Schrödinger equation on the interval has been considered in various works (e.g. [23, 24]), to the best of our knowledge the solution to the forced linear problem (1.4) via the unified transform has not been explicitly provided anywhere else in the literature. In particular, here we provide a complete derivation of the solution to this problem, with careful justification of the complex contour deformations that form the core of the unified transform.

It should be noted that the unified transform employed in this work is a universal method for solving initial-boundary value problems that involve evolution equations. The method was introduced in 1997 by Fokas [15] and has since been advanced in multiple settings by many researchers. Indicatively, we mention the works [16, 17, 25, 29, 31] which establish Fokas's method as the initial-boundary value problem analogue of the classical Fourier transform used for the solution of the initial value problem of *linear* evolution equations. Moreover, the unified transform also has a nonlinear component which is applicable to completely integrable equations such as the cubic nonlinear Schrödinger and Korteweg-de Vries equations on the half-line and the finite interval [1, 18, 23, 24] or the Davey-Stewartson and Kadomtsev-Petviashvili equations on the half-plane [20, 30]. A comprehensive presentation of the unified transform can be found in the monograph [19] as well as the review articles [4,26]. Finally, as noted above, the unified transform has recently inspired a new approach for proving well-posedness of general nonlinear evolution equations in the initial-boundary value problem setting, see [21,22,27,28].

The explicit solution formula for problem (1.4) as derived via the unified transform in Section 3 is the following:

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)\,dk \,+ \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(k,\tau)d\tau dk \\
&\quad - \frac{1}{2\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\left[e^{ik\ell}\,\widehat{u}_0(k) - e^{-ik\ell}\,\widehat{u}_0(-k)\right]dk \\
&\quad - \frac{\beta}{\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\,k\left[\widetilde{h}_0(\omega,t) - e^{-ik\ell}\,\widetilde{g}_0(\omega,t)\right]dk \\
&\quad - \frac{1}{2\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\int_{\tau=0}^{t} e^{i\omega\tau}\left[e^{ik\ell}\widehat{f}(k,\tau)d\tau - e^{-ik\ell}\widehat{f}(-k,\tau)\right]d\tau dk \\
&\quad + \frac{1}{2\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\left[\widehat{u}_0(k) - \widehat{u}_0(-k)\right]dk \\
&\quad + \frac{\beta}{\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\,k\left[e^{ik\ell}\,\widetilde{h}_0(\omega,t) - \widetilde{g}_0(\omega,t)\right]dk \\
&\quad + \frac{1}{2\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\int_{\tau=0}^{t} e^{i\omega\tau}\left[\widehat{f}(k,\tau) - \widehat{f}(-k,\tau)\right]d\tau dk,
\end{aligned} \tag{1.5}$$

where ω is given by (2.3), $\widehat{u}_0$ and $\widehat{f}$ denote the finite-interval Fourier transforms of the initial data and the forcing defined by (2.1), $\widetilde{g}_0$ and $\widetilde{h}_0$ are temporal transforms of the boundary data defined by (2.5), and the complex contours $\mathcal{C}^{\pm}$ are depicted in Figure 2.1.

The general finite interval problem (1.3) is directly related to the periodic Cauchy problem

$$u_t + i\beta u_{xx} + (1+i\alpha)\,u = f(x,t), \quad 0<x<\ell,\ t>0, \tag{1.6a}$$
$$u(x,0) = u_0(x), \quad 0<x<\ell, \tag{1.6b}$$
$$u(0,t) = u(\ell,t), \quad u_x(0,t) = u_x(\ell,t), \quad t>0, \tag{1.6c}$$

where the initial data $u_0(x)$ is a periodic function such that $u_0(x+\ell) = u_0(x)$ for all $x\in\mathbb{R}$. Indeed, via a similar approach to the one used for problem (1.3), our analysis yields the following solution formula for the periodic problem (1.6):

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)dk + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(k,\tau)d\tau dk \\
&\quad - \frac{1}{2\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk - \frac{1}{2\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(k,\tau)d\tau dk \\
&\quad + \frac{1}{2\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk + \frac{1}{2\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(k,\tau)d\tau dk,
\end{aligned} \tag{1.7}$$

where, as before, ω is given by (2.3), $\widehat{u}_0$ and $\widehat{f}$ denote the finite-interval Fourier transforms of the initial data and the forcing defined by (2.1), and the complex contours $\mathcal{C}^{\pm}$ are depicted in Figure 2.1.

Structure. The solution formula (1.7) to the periodic problem (1.6) is derived in Section 2. Interesting reductions to the traditional "separation of variables/Fourier series" representation as well as to the solution of the direct linearization of the Lugiato-Lefever equation (1.1) (i.e. when the forcing f is constant and equal to $F > 0$) are also provided in that section (see expressions (2.14) and (2.10)). The analysis for the general nonperiodic problem (1.3) requires an additional crucial idea and is given in Section 3, leading to the solution formula (1.5). The reduction of this formula to the direct linearization of the Lugiato-Lefever equation (1.1) is also provided in that case (see expression (3.7)).

2. The Periodic Problem

Our analysis is done for the linear Lugiato-Lefever equation (1.6a) with general, non-constant forcing $f(x,t)$. The motivation for this is that the resulting solution formula can be used in the future for studying the nonlinear Lugiato-Lefever equation (1.1). Of course, setting $f(x,t) = F$ reduces equation (1.6a) to the linearization of (1.1).

2.1. *The global relation and an integral representation for the solution*

The Fourier transform pair for a function $\phi(x)$ on the interval $0 < x < \ell$ is defined by

$$\widehat{\phi}(k) = \int_{x=0}^{\ell} e^{-ikx}\phi(x)dx, \quad k \in \mathbb{C}, \quad \phi(x) = \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx}\,\widehat{\phi}(k)dk, \quad 0 < x < \ell. \tag{2.1}$$

Applying the above Fourier transform to equation (1.6a) yields

$$\begin{aligned}\partial_t \widehat{u}(k,t) + i\beta\left\{e^{-ik\ell}u_x(\ell,t) - u_x(0,t) + ik\left[e^{-ik\ell}u(\ell,t) - u(0,t) + ik\,\widehat{u}(k,t)\right]\right\}& \\ + (1+i\alpha)\,\widehat{u}(k,t) = \widehat{f}(k,t),&\end{aligned} \tag{2.2}$$

In view of the initial condition (1.6b), the periodic boundary conditions (1.6c), and the notation

$$\omega(k) = -\beta k^2 - i + \alpha, \quad u(0,t) = u(\ell,t) = h(t), \quad u_x(0,t) = u_x(\ell,t) = g(t), \tag{2.3}$$

we integrate (2.2) with respect to t to obtain what is known in the unified transform terminology as the *global relation*

$$e^{i\omega t}\,\widehat{u}(k,t) = \widehat{u}_0(k) + i\beta\left\{\left[\widetilde{g}(\omega,t) + ik\widetilde{h}(\omega,t)\right] - e^{-ik\ell}\left[\widetilde{g}(\omega,t) + ik\widetilde{h}(\omega,t)\right]\right\} + \int_{\tau=0}^{t} e^{i\omega\tau}\,\widehat{f}(k,\tau)d\tau, \tag{2.4}$$

where we have also introduced the notation

$$\widetilde{\phi}(\omega,t) = \int_{\tau=0}^{t} e^{i\omega\tau}\phi(\tau)d\tau. \tag{2.5}$$

We remark that the global relation (2.4) is valid for all $k \in \mathbb{C}$ in line with the domain of the interval Fourier transform (2.1).

Inverting the global relation (2.4) for $k \in \mathbb{R}$ by means of (2.1), we find the following *integral representation* for the solution u:

$$\begin{aligned}u(x,t) = \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)\,dk + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\,\widehat{f}(k,\tau)d\tau dk \\ + \frac{i\beta}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\left[\widetilde{g}(\omega,t) + ik\widetilde{h}(\omega,t)\right]dk - \frac{i\beta}{2\pi}\int_{k\in\mathbb{R}} e^{ik(x-\ell)-i\omega t}\left[\widetilde{g}(\omega,t) + ik\widetilde{h}(\omega,t)\right]dk.\end{aligned} \tag{2.6}$$

The integral representation (2.6) is not an explicit solution formula since it contains the unknown boundary values $h(t)$ and $g(t)$ through the transforms $\widetilde{h}(\omega,t)$ and $\widetilde{g}(\omega,t)$. However, it turns out that these unknown transforms can be eliminated from (2.6) by exploiting the analyticity and exponential decay of the relevant integrands in appropriate regions of the complex k-plane.

2.2. *Elimination of the unknowns and an explicit solution formula*

Noting that $\left|e^{ikx}\right| = e^{-\mathrm{Im}(k)x}$ and recalling that $x>0$, we see that e^{ikx} is bounded for $\mathrm{Im}(k) \geqslant 0$ and decays to zero as $|k| \to \infty$ whenever $\mathrm{Im}(k)>0$. Similarly, the exponential $e^{-i\omega(t-\tau)}$ with $0<t<\tau$ decays in the region $\mathcal{R} := \{k \in \mathbb{C} : \mathrm{Re}(i\omega)>0\} = \{2\beta \mathrm{Re}(k)\mathrm{Im}(k)+1>0\}$, which for all $\beta \in \mathbb{R}\setminus\{0\}$ corresponds to the region lying between the two branches of the hyperbola $2\beta\,\mathrm{Re}(k)\mathrm{Im}(k) = -1$, as shown in Figure 2.1.

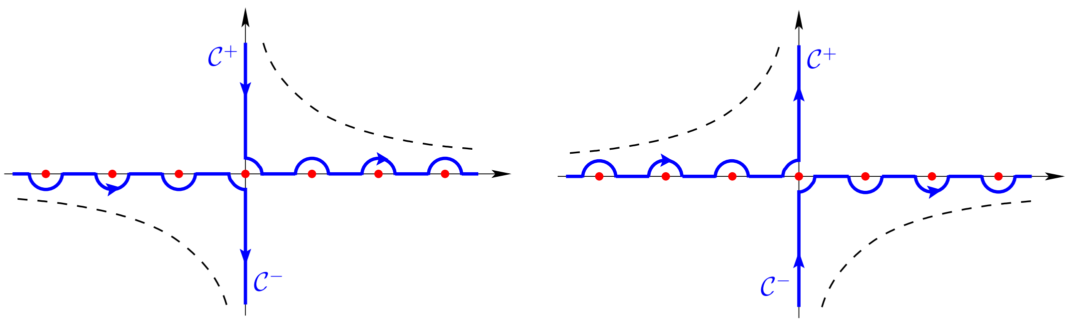

FIGURE 2.1. The hyperbolae $2\beta\,\mathrm{Re}(k)\mathrm{Im}(k) = -1$ (dashed) and the contours $\mathcal{C}^\pm$ (blue) for $\beta<0$ (left) and $\beta>0$ (right). The red dots along the real axis correspond to (i) the zeros $k_n = \frac{2n\pi}{\ell}$, $n \in \mathbb{Z}$, of the quantity $1-e^{-ik\ell}$ in the periodic problem, and (ii) the zeros $k_n = \frac{n\pi}{\ell}$, $n \in \mathbb{Z}$, of the quantity $e^{ik\ell}-e^{-ik\ell}$ in the finite interval (nonperiodic) problem.

Therefore, the exponential $e^{ikx-i\omega(t-\tau)}$ is bounded and decays to zero as $|k|\to\infty$ inside the region $\mathcal{R}\cap\{\mathrm{Im}(k)>0\}$. Thus, thanks to analyticity in k, Cauchy's integral theorem allows us to deform the path of integration of the third k-integral in (2.6) from $\mathbb{R}$ to the blue contour $\mathcal{C}^+$ in the upper half of the complex k-plane (see Figure 2.1). Similarly, since $\left|e^{ik(x-\ell)}\right| = e^{-\mathrm{Im}(k)(x-\ell)}$ and $x<\ell$, the exponential $e^{ik(x-\ell)-i\omega(t-\tau)}$ is bounded and decays to zero as $|k|\to\infty$ inside the region $\mathcal{R}\cap\{\mathrm{Im}(k)<0\}$. Therefore, we can deform the contour of integration of the fourth k-integral in (2.6) from $\mathbb{R}$ to the blue contour $\mathcal{C}^-$ in the lower half of the complex k-plane (see Figure 2.1). We note that both of those deformations can be rigorously justified along the lines of Proposition 2.1 proved later. Implementing them, we write the integral representation (2.6) in the form

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)\,dk + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(k,\tau)d\tau dk \\
&\quad + \frac{i\beta}{2\pi}\int_{k\in\mathcal{C}^+} e^{ikx-i\omega t}\left[\widetilde{g}(\omega,t)+ik\widetilde{h}(\omega,t)\right]dk - \frac{i\beta}{2\pi}\int_{k\in\mathcal{C}^-} e^{ik(x-\ell)-i\omega t}\left[\widetilde{g}(\omega,t)+ik\widetilde{h}(\omega,t)\right]dk.
\end{aligned}
\tag{2.7}
$$

Then, substituting for the unknown quantity $\widetilde{g}(\omega,t)+ik\widetilde{h}(\omega,t)$ via the global relation (2.4), which is valid for all $k\in\mathbb{C}$ and in particular for $k\in\mathcal{C}^\pm$, turns (2.7) into the *explicit solution formula* (1.7) for the periodic problem of the forced linear Lugiato-Lefever equation (1.6a), where we have made crucial use of the following result, which is proved at the end of this section.

Proposition 2.1. *For any $0 < x < \ell$ and any $t > 0$,*

$$\int_{k\in\mathcal{C}^+} \frac{e^{ikx}}{1-e^{-ik\ell}}\,\widehat{u}(k,t)dk = \int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)}}{1-e^{-ik\ell}}\,\widehat{u}(k,t)dk = 0. \tag{2.8}$$

Remark 2.1 (Need for deformation)**.** If we substitute for the unknown $\widetilde{g}(\omega,t)+ik\widetilde{h}(\omega,t)$ via (2.4) without first deforming the relevant contours to $\mathcal{C}^{\pm}$, i.e. at the level of (2.6), then we will obtain the tautology $u(x,t)=u(x,t)$. That is, the deformation to $\mathcal{C}^{\pm}$ is needed in order to eliminate the unknown.

Remark 2.2. The zeros of the quantity $1-e^{-ik\ell}$ occur at $k=k_n=\frac{2\pi n}{\ell}$, $n\in\mathbb{Z}$, and so they do not introduce any singularities in formula (1.7) since they are avoided by the contours $\mathcal{C}^{\pm}$ (see Figure 2.1). In fact, this is precisely the reason why we did not deform from $\mathbb{R}$ to the boundary of the domain $\mathcal{R}$, i.e. to the hyperbolae corresponding to the dashed red contours of Figure 2.1, as parts of these hyperbolae asymptotically tend to the real axis and hence eventually interfere with the singularities at $k=k_n$.

In the special case $f(x,t)=F$ that corresponds to the linearization of the Lugiato-Lefever equation (1.1), formula (1.7) becomes

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)dk - \frac{F}{2\pi}\int_{k\in\mathbb{R}} e^{ikx}\,\frac{\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{k\omega}\,dk \\
&\quad - \frac{1}{2\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk + \frac{F}{2\pi}\int_{k\in\mathcal{C}^+} e^{ikx}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk \\
&\quad + \frac{1}{2\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk - \frac{F}{2\pi}\int_{k\in\mathcal{C}^-} e^{ik(x-\ell)}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk.
\end{aligned} \tag{2.9}$$

This formula can be further simplified by observing that the singularities arising from the zeros of ω are all removable apart from $k=0$, which is a simple pole due to the presence of k in the denominator of the relevant integrands. Thus, denoting by $C_{\varepsilon,[0,\pi]}(0)$ and $C_{\varepsilon,[\pi,2\pi]}(0)$ the semicircles of radius $\varepsilon>0$ centered at the origin and oriented counterclockwise from 0 to π and from π to 2π respectively, we employ Cauchy's integral theorem to write

$$\int_{k\in\mathcal{C}^+} e^{ikx}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk = \left(\int_{k\in\mathbb{R}\setminus[-\varepsilon,\varepsilon]} - \int_{k\in C_{\varepsilon,[0,\pi]}(0)}\right) e^{ikx}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk$$

and

$$\int_{k\in\mathcal{C}^-} e^{ik(x-\ell)}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk = \left(\int_{k\in\mathbb{R}\setminus[-\varepsilon,\varepsilon]} + \int_{k\in C_{\varepsilon,[\pi,2\pi]}(0)}\right) e^{ik(x-\ell)}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk.$$

In turn, we have

$$\begin{aligned}
&\int_{k\in\mathcal{C}^+} e^{ikx}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk - \int_{k\in\mathcal{C}^-} e^{ik(x-\ell)}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk \\
&= \int_{k\in\mathbb{R}\setminus[-\varepsilon,\varepsilon]} e^{ikx}\left(1-e^{-ik\ell}\right)\frac{1-e^{-i\omega t}}{k\omega}\,dk - \int_{k\in C_{\varepsilon,[0,\pi]}(0)} e^{ikx}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk \\
&\quad - \int_{k\in C_{\varepsilon,[\pi,2\pi]}(0)} e^{ik(x-\ell)}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk
\end{aligned}$$

and so taking the limit $\varepsilon \to 0^+$ and then using Cauchy's residue theorem for the second and third integrals, as well as the fact that the singularity at $k=0$ is removable in the first integral, we find

$$\begin{aligned}
&\int_{k\in\mathcal{C}^+} e^{ikx}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk - \int_{k\in\mathcal{C}^-} e^{ik(x-\ell)}\,\frac{1-e^{-i\omega t}}{k\omega}\,dk \\
&= \int_{k\in\mathbb{R}} e^{ikx}\,\frac{\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{k\omega}\,dk + 2\pi\,\frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}.
\end{aligned}$$

Via the above calculations, expression (2.9) yields the following solution formula for the linearization of the Lugiato-Lefever equation (1.1) with periodic boundary conditions:

$$\begin{aligned}
u(x,t) = \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)dk - \frac{1}{2\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk \\
+ \frac{1}{2\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk + \frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}\,F.
\end{aligned} \tag{2.10}$$

2.3. *Reduction to the traditional separation of variables/Fourier series representation*

We emphasize that the solution formula (2.10) could not have been obtained without deforming from $\mathbb{R}$ to the contours $\mathcal{C}^\pm$. This is because if one were to employ the global relation (2.4) without making these deformations (i.e. directly at the level of (2.6) instead of (2.7)) then one would obtain the tautology $u(x,t)=u(x,t)$. Indeed, without the deformations, the three terms involving the initial datum in (1.7) would cancel one another, and so would the forcing terms, while the last two terms that involve $\widehat{u}(k,t)$ would combine to yield $u(x,t)$ via the inverse Fourier transform.

At the same time, if desired, it is possible to collapse the contours $\mathcal{C}^\pm$ involved in formula (2.10) to the real axis. However, in doing so, one must take into account the residue contributions from the poles at $k_n=\frac{2\pi n}{\ell}$, $n\in\mathbb{Z}$, arising from the term $1-e^{-ik\ell}$. In particular, using analyticity (Cauchy's theorem) and exponential decay, we can write the solution formula (2.10) as

$$\begin{aligned}
u(x,t) = \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)dk - \frac{1}{2\pi}\int_{k\in\mathcal{L}_\varepsilon^+} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk \\
+ \frac{1}{2\pi}\int_{k\in\mathcal{L}_\varepsilon^-} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk + \frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}\,F,
\end{aligned}$$

with the contours $\mathcal{L}_\varepsilon^\pm$ given by

$$\mathcal{L}_\varepsilon^+ = \mathbb{R}_\varepsilon \cup \bigcup_{n\in\mathbb{Z}} -C_{\varepsilon,[0,\pi]}(k_n), \quad \mathcal{L}_\varepsilon^- = \mathbb{R}_\varepsilon \cup \bigcup_{n\in\mathbb{Z}} C_{\varepsilon,[\pi,2\pi]}(k_n), \tag{2.11}$$

where for $0<\varepsilon\leqslant \frac{|k_{n+1}-k_n|}{3}=\frac{2\pi}{3\ell}$ we define

$$\mathbb{R}_\varepsilon = \bigcup_{n\in\mathbb{Z}} \left[k_n+\varepsilon, k_{n+1}-\varepsilon\right], \quad C_{\varepsilon,[a,b]}(k_n) = \Big\{ |k-k_n|=\varepsilon,\ a\leqslant \arg(k)\leqslant b\Big\} \tag{2.12}$$

with positive orientation. In fact, breaking down the contours $\mathcal{L}_\varepsilon^\pm$ into their individual components and noting that the integrals along $\mathbb{R}_\varepsilon$ combine to a single integral that does not involve a singular integrand, we take the limit $\varepsilon\to 0^+$ to obtain

$$\begin{aligned}
u(x,t) = \frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{k\in\bigcup_{n\in\mathbb{Z}} C_{\varepsilon,[0,\pi]}(k_n)} \frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk \\
+ \frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{k\in\bigcup_{n\in\mathbb{Z}} C_{\varepsilon,[\pi,2\pi]}(k_n)} \frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk + \frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}\,F.
\end{aligned} \tag{2.13}$$

Finally, using Cauchy's residue theorem we compute

$$\lim_{\varepsilon\to 0^+}\int_{k\in C_{\varepsilon,[0,\pi]}(k_n)}\frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk = i\pi\lim_{k\to k_n}\left[\frac{e^{ikx-i\omega t}}{1-e^{-ik\ell}}\,(k-k_n)\,\widehat{u}_0(k)\right] = \frac{\pi}{\ell}\,e^{ik_nx-i\omega(k_n)t}\,\widehat{u}_0(k_n)$$

and, similarly,

$$\lim_{\varepsilon\to 0^+}\int_{k\in C_{\varepsilon,[\pi,2\pi]}(k_n)}\frac{e^{ik(x-\ell)-i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}_0(k)dk = \frac{\pi}{\ell}\,e^{ik_nx-i\omega(k_n)t}\,\widehat{u}_0(k_n).$$

Therefore, (2.13) becomes

$$u(x,t)=\frac{1}{\ell}\sum_{n\in\mathbb{Z}}e^{ik_nx-i\omega(k_n)t}\,\widehat{u}_0(k_n)+\frac{1-e^{-(1+i\alpha)t}}{1+i\alpha}\,F,\quad k_n=\frac{2\pi n}{\ell},\tag{2.14}$$

which is the formula one can expect to obtain via separation of variables and the traditional Fourier series method.

2.4. *Proof of Proposition 2.1*

We only provide the proof for the integral along $\mathcal{C}^+$, as the argument for the integral along $\mathcal{C}^-$ is entirely analogous. Integrating twice by parts and rearranging, we have

$$\int_{k\in\mathcal{C}^+}\frac{e^{ikx}}{1-e^{-ik\ell}}\,\widehat{u}(k,t)dk=\int_{k\in\mathcal{C}^+}e^{ikx}\,\frac{1}{1-e^{-ik\ell}}\int_{y=0}^{\ell}e^{-iky}\,u(y,t)\,dy\,dk\tag{2.15}$$

$$=u(0,t)\int_{k\in\mathcal{C}^+}e^{ikx}\,\frac{1}{ik}\,dk-u_y(0,t)\int_{k\in\mathcal{C}^+}e^{ikx}\,\frac{1}{k^2}\,dk-\int_{y=0}^{\ell}u_{yy}(y,t)\int_{k\in\mathcal{C}^+}e^{ikx}\,\frac{1}{k^2}\,\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk\,dy.$$

Note that we have used Fubini's theorem in order to interchange the order of integration in the double integral since

$$\left|\int_{y=0}^{\ell}u_{yy}(y,t)\int_{k\in\mathcal{C}^+}e^{ikx}\,\frac{1}{k^2}\,\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk\,dy\right|\leqslant\int_{y=0}^{\ell}|u_{yy}(y,t)|\int_{k\in\mathcal{C}^+}\frac{1}{|k|^2}\,\frac{e^{-\mathrm{Im}(k)(x+\ell-y)}}{|e^{ik\ell}-1|}dk\,dy<\infty$$

via similar steps with those in the estimation of I_R and J_R below.

For the first k-integral on the right-hand side of (2.15), by Cauchy's theorem we have

$$\int_{k\in\mathcal{C}^+}e^{ikx}\,\frac{1}{ik}\,dk=-\lim_{R\to\infty}\int_{k\in C_{R,\theta_0}(0)}e^{ikx}\,\frac{1}{ik}\,dk,$$

where $C_{R,\theta_0}(0)$ denotes the circular arc

$$C_{R,\theta_0}(0)=\begin{cases}\left\{Re^{i\theta}:\,\theta_0\leqslant\theta\leqslant\frac{\pi}{2}\right\}, & \beta<0,\\ \left\{Re^{i\theta}:\,\frac{\pi}{2}\leqslant\theta\leqslant\pi-\theta_0\right\}, & \beta>0,\end{cases}\tag{2.16}$$

with $\theta_0=0$ if $|R-k_n|\geqslant\frac{2\pi}{3\ell}$ for all $n\in\mathbb{Z}$, and $\theta_0\in\left(0,\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)\right]$ if there exists $n\in\mathbb{Z}$ such that $|R-k_n|<\frac{2\pi}{3\ell}$. Since $\sin\theta=\sin(\pi-\theta)$, the cases $\beta<0$ and $\beta>0$ can be handled in the same way to yield

$$\left|\int_{k\in C_{R,\theta_0}(0)}e^{ikx}\,\frac{1}{ik}\,dk\right|\leqslant\int_{\theta=0}^{\frac{\pi}{2}}e^{-xR\sin\theta}d\theta.$$

Thus, using the well-known inequality

$$\sin\theta\geqslant\frac{2}{\pi}\,\theta,\quad\theta\in\left[0,\frac{\pi}{2}\right],\tag{2.17}$$

we find

$$\left|\int_{k\in C_{R,\theta_0}(0)} e^{ikx}\frac{1}{ik}\,dk\right| \leqslant \int_{\theta=0}^{\frac{\pi}{2}} e^{-xR\cdot\frac{2}{\pi}\theta}d\theta = \frac{\pi}{2xR}\left(1-e^{-xR}\right) \longrightarrow 0, \quad R\to\infty,$$

and hence we conclude that

$$\int_{k\in\mathcal{C}^+} e^{ikx}\frac{1}{ik}\,dk = 0. \tag{2.18}$$

Similarly, it follows that the second integral on the right-hand side of (2.15) also equals zero:

$$\int_{k\in\mathcal{C}^+} e^{ikx}\frac{1}{k^2}\,dk = 0. \tag{2.19}$$

Concerning the k-integral inside the double integral of (2.15), Cauchy's theorem implies

$$\int_{k\in\mathcal{C}^+} e^{ikx}\,\frac{1}{k^2}\,\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk = -\lim_{R\to\infty}\int_{k\in C_{R,\theta_0}(0)} e^{ikx}\,\frac{1}{k^2}\,\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk \tag{2.20}$$

with $C_{R,\theta_0}(0)$ defined as above. We will estimate this integral by decomposing it into two pieces which we handle separately:

$$\int_{k\in C_{R,\theta_0}(0)} e^{ikx}\,\frac{1}{k^2}\,\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk = I_R + J_R, \tag{2.21}$$

where

$$I_R = \int_{k\in C_{R,\theta_0}(0),\,\mathrm{Im}(k)\geqslant\frac{2\pi}{3\ell}} e^{ikx}\,\frac{1}{k^2}\,\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk, \tag{2.22}$$

$$J_R = \int_{k\in C_{R,\theta_0}(0),\,\mathrm{Im}(k)\leqslant\frac{2\pi}{3\ell}} e^{ikx}\,\frac{1}{k^2}\,\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk. \tag{2.23}$$

Estimation of I_R. This term is the easiest of the two. First, observe that, since $y\leqslant\ell$ and $\mathrm{Im}(k)\geqslant 0$, we have

$$\left|\frac{e^{-iky}}{1-e^{-ik\ell}}\right| = \frac{e^{-\mathrm{Im}(k)(\ell-y)}}{|e^{ik\ell}-1|} \leqslant \frac{1}{|e^{ik\ell}-1|} \tag{2.24}$$

and so

$$|I_R| \leqslant \int_{k\in C_{R,\theta_0}(0),\,\mathrm{Im}(k)\geqslant\frac{2\pi}{3\ell}} \left|e^{ikx}\,\frac{1}{k^2}\right|\frac{1}{|e^{ik\ell}-1|}\,|dk|\,. \tag{2.25}$$

Therefore, noting that for $k\in C_{R,\theta_0}(0)$ with $\mathrm{Im}(k)\geqslant\frac{2\pi}{3\ell}$ we have $\arg(\theta)\in\left[\sin^{-1}\left(\frac{2\pi}{3\ell R}\right),\frac{\pi}{2}\right]$ when $\beta<0$ and $\arg(\theta)\in\left[\frac{\pi}{2},\pi-\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)\right]$ when $\beta>0$, we employ the triangle inequality twice to get

$$|I_R| \leqslant \left\{\begin{array}{ll} \dfrac{1}{R}\displaystyle\int_{\theta=\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)}^{\frac{\pi}{2}} \frac{e^{-xR\sin\theta}}{\left|e^{iRe^{i\theta}\ell}-1\right|}\,d\theta, & \beta<0 \\ \dfrac{1}{R}\displaystyle\int_{\theta=\frac{\pi}{2}}^{\pi-\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)} \frac{e^{-xR\sin\theta}}{\left|e^{iRe^{i\theta}\ell}-1\right|}\,d\theta, & \beta>0 \end{array}\right\} \leqslant \frac{1}{R}\int_{\theta=\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)}^{\frac{\pi}{2}} \frac{e^{-xR\sin\theta}}{1-e^{-\ell R\sin\theta}}\,d\theta,$$

where we have made the change of variable $\theta\mapsto\pi-\theta$ in the case $\beta>0$. Thus, noting in addition that $R\sin\theta\geqslant\frac{2\pi}{3\ell}$ and so $1-e^{-\ell R\sin\theta}\geqslant 1-e^{-\frac{2\pi}{3}}$, we find

$$|I_R| \leqslant \frac{1}{R}\left(1-e^{-\frac{2\pi}{3}}\right)^{-1}\int_{\theta=\theta_0}^{\frac{\pi}{2}} e^{-xR\sin\theta}\,d\theta$$

and using inequality (2.17) we conclude that

$$\lim_{R\to\infty} I_R = 0, \quad x, y \in (0, \ell)\,. \tag{2.26}$$

Estimation of J_R. This term is trickier only because it is harder to obtain a positive lower bound for $\left|e^{ik\ell}-1\right|$. First, like for I_R, using the triangle inequality and the bound (2.24) we find

$$|J_R| \leqslant \int_{k\in C_{R,\theta_0}(0),\,\mathrm{Im}(k)\leqslant \frac{2\pi}{3\ell}} \left|e^{ikx}\,\frac{1}{k^2}\right| \frac{1}{|e^{ik\ell}-1|}\,|dk|\,. \tag{2.27}$$

Next, we use the following result.

Lemma 2.1. *For any $k \in \left\{0 \leqslant \mathrm{Im}(k) \leqslant \frac{2\pi}{3\ell}\right\} \setminus \bigcup_{n\in\mathbb{Z}} D_{\frac{2\pi}{3\ell}}(k_n)$, we have $\left|e^{ik\ell}-1\right| \geqslant 1 - e^{-\frac{2\pi}{3}}$.*

Before proving this lemma, we use it to complete the estimation of J_R. From (2.27), we have

$$|J_R| \leqslant \frac{1}{R}\left(1-e^{-\frac{2\pi}{3}}\right)^{-1} \int_{\theta=\theta_0}^{\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)} e^{-xR\sin\theta} d\theta. \tag{2.28}$$

Hence, employing once again inequality (2.17), we obtain

$$|J_R| \leqslant \left(1-e^{-\frac{2\pi}{3}}\right)^{-1} \frac{\pi}{2xR^2}\left(e^{-\frac{2xR}{\pi}\theta_0} - e^{-\frac{2xR}{\pi}\sin^{-1}\left(\frac{2\pi}{3\ell R}\right)}\right) \xrightarrow{R\to\infty} 0, \quad x, y \in (0, \ell)\,. \tag{2.29}$$

Altogether, combining the above with (2.20) and (2.21) we conclude that

$$\int_{k\in\mathcal{C}^+} e^{ikx}\,\frac{1}{k^2}\,\frac{e^{-iky}}{1-e^{-ik\ell}}\,dk = 0, \quad x, y \in (0, \ell)\,, \tag{2.30}$$

and so, in view of (2.15), (2.18) and (2.19),

$$\int_{k\in\mathcal{C}^+} e^{ikx-i\omega t}\,\frac{e^{i\omega t}}{1-e^{-ik\ell}}\,\widehat{u}(k,t)\,dk = 0, \quad x, y \in (0, \ell)\,,$$

as desired. It remains to establish Lemma 2.1.

Proof of Lemma 2.1. We minimize $\left|e^{ik\ell}-1\right|$ as a function of two variables simply by using calculus techniques. For any $k \in \left\{0 \leqslant \mathrm{Im}(k) \leqslant \frac{2\pi}{3\ell}\right\} \setminus \bigcup_{n\in\mathbb{Z}} D_{\frac{2\pi}{3\ell}}(k_n)$, there exists $n \in \mathbb{Z}$ such that $k = k_n + \rho e^{i\phi}$ with $\frac{2\pi}{3\ell} \leqslant \rho \leqslant \frac{\sqrt{13}\pi}{3\ell}$ and $0 \leqslant \phi \leqslant \pi$ (see Figure 2.2 for the bounds on ρ). Then,

$$\left|e^{ik\ell}-1\right|^2 = e^{-2\rho\ell\sin\phi} - 2e^{-\rho\ell\sin\phi}\cos\left(\rho\ell\cos\phi\right) + 1 =: f(\rho, \phi).$$

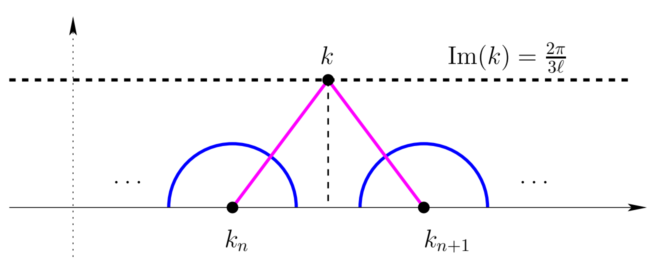

FIGURE 2.2. The upper bound for the radius ρ used in the proof of Lemma 2.1.

First, we look for critical points of $f(\rho, \phi)$ inside $(\rho, \phi) \in \left[\frac{2\pi}{3\ell}, \frac{\sqrt{13}\pi}{3\ell}\right] \times [0, \pi]$ by solving the system $f_\rho(\rho, \phi) = 0$ and $f_\phi(\rho, \phi) = 0$. We note that if $\cos\phi = 0$ then $\sin\phi = 1$ and

$f_\rho = 2e^{-\rho\ell}\left(1-e^{-\rho\ell}\right) \neq 0$, while if $\sin\phi = 0$ then $\cos\phi = \pm 1$ and $f_\phi = \mp 2\rho\ell\left[1-\cos(\rho\ell)\right] \neq 0$. Hence, no critical points arise inside our domain when $\cos\phi\sin\phi = 0$. Therefore, assuming $\cos\phi\sin\phi \neq 0$, we multiply the equation $f_\rho(\rho,\phi) = 0$ by $\cos\phi$ and the equation $f_\phi(\rho,\phi) = 0$ by $\frac{\sin\phi}{\rho\ell}$ and then subtract the resulting equations to obtain

$$\sin\left(\rho\ell\cos\phi\right) = 0 \ \Leftrightarrow\ \rho\ell\cos\phi = \kappa\pi,\ \kappa\in\mathbb{Z}.$$

Since $\frac{2\pi}{3\ell} \leqslant \rho \leqslant \frac{\sqrt{13}\pi}{3\ell}$, it follows that $\kappa = 0, \pm 1$. However, none of these values corresponds to critical points inside our domain since either $f_\rho \neq 0$ (when $\kappa = 0$) or $f_\phi \neq 0$ (when $\kappa = \pm 1$).

Since f is continuous and there are no critical points inside $\left[\frac{2\pi}{3\ell}, \frac{\sqrt{13}\pi}{3\ell}\right] \times [0,\pi]$, the minimum will be attained at the boundary of the domain. If $\rho = \frac{2\pi}{3\ell}$, then

$$f\left(\tfrac{2\pi}{3\ell},\phi\right) = e^{-\frac{4\pi}{3}\sin\phi} - 2e^{-\frac{2\pi}{3}\sin\phi}\cos\left(\tfrac{2\pi}{3}\cos\phi\right) + 1 =: g(\phi), \quad \phi\in[0,\pi]\,.$$

We compute

$$g'(\phi) = -\tfrac{4\pi}{3}\, e^{-\frac{2\pi}{3}\sin\phi}\left[\cos\phi\, e^{-\frac{2\pi}{3}\sin\phi} - \cos\left(\phi + \tfrac{2\pi}{3}\cos\phi\right)\right]$$

and observe that $g'(\frac{\pi}{2}) = 0$. To show that $\phi = \frac{\pi}{2}$ is the unique zero of g', we consider the intervals $\left[0,\frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2},\pi\right]$ separately.

If $\phi\in\left[0,\frac{\pi}{2}\right]$, then the function $h(\phi) := \phi + \frac{2\pi}{3}\cos\phi$ has derivative $h'(\phi) = 1 - \frac{2\pi}{3}\sin\phi$. Thus, the only critical point occurs at $\phi = \sin^{-1}\left(\frac{3}{2\pi}\right)$ with corresponding value $\sin^{-1}\left(\frac{3}{2\pi}\right) + \frac{2\pi}{3}\sqrt{1-\frac{9}{4\pi^2}} \simeq 2.34$. Moreover, $h(0) = \frac{2\pi}{3}$ and $h(\frac{\pi}{2}) = \frac{\pi}{2}$. Thus, $\frac{\pi}{2} \leqslant h(\phi) < \frac{3\pi}{2}$ and so $\cos\left(\phi + \frac{2\pi}{3}\cos\phi\right) \leqslant 0$ for $\phi\in[0,\frac{\pi}{2}]$. In turn,

$$\cos\phi\, e^{-\frac{2\pi}{3}\sin\phi} - \cos\left(\phi + \tfrac{2\pi}{3}\cos\phi\right) \geqslant e^{-\frac{2\pi}{3}}\left[\cos\phi - \cos\left(\phi + \tfrac{2\pi}{3}\cos\phi\right)\right] \geqslant 0.$$

Moreover, for this non-negative lower bound to vanish we must have

$$\cos\phi = 3\kappa \quad \text{or} \quad \cos\phi + \tfrac{3}{\pi}\,\phi + 3\kappa = 0, \quad \kappa\in\mathbb{Z}.$$

The first of these equations has unique solution $\phi = \frac{\pi}{2}$. The second equation has no solution since $\cos\phi + \frac{3}{\pi}\,\phi$ has global maximum equal to $\sqrt{1-\frac{9}{\pi^2}} + \frac{3}{\pi}\sin^{-1}\left(\frac{3}{\pi}\right) \simeq 1.51$ (via its unique critical point on $[0,\frac{\pi}{2}]$ at $\phi = \sin^{-1}\left(\frac{3}{\pi}\right)$) and global minimum equal to 1 (via the end point $\phi = 0$). Thus, we conclude that for $\phi\in\left[0,\frac{\pi}{2}\right]$ the only zero of g' is at $\phi = \frac{\pi}{2}$.

Similarly, if $\phi\in\left[\frac{\pi}{2},\pi\right]$ then $\cos\left(\phi + \frac{2\pi}{3}\cos\phi\right) \geqslant 0$ and so

$$\cos\phi\, e^{-\frac{2\pi}{3}\sin\phi} - \cos\left(\phi + \tfrac{2\pi}{3}\cos\phi\right) \leqslant e^{-\frac{2\pi}{3}}\left[\cos\phi\ - \cos\left(\phi + \tfrac{2\pi}{3}\cos\phi\right)\right] \leqslant 0.$$

As before, we can show that the non-positive upper bound vanishes only at $\phi = \frac{\pi}{2}$ and is otherwise negative. Thus, for $\phi\in\left[\frac{\pi}{2},\pi\right]$ the only zero of g' is at $\phi = \frac{\pi}{2}$.

Overall, since the only critical point of g on $[0,\pi]$ is at $\phi = \frac{\pi}{2}$, comparing the values $g(\frac{\pi}{2}) = \left(1-e^{-\frac{2\pi}{3}}\right)^2$ and $g(0) = g(\pi) = 3$ we conclude that $g(\phi) = f(\frac{2\pi}{3\ell},\phi)$ is minimized at $\phi = \frac{\pi}{2}$ with corresponding value

$$f\left(\tfrac{2\pi}{3\ell},\tfrac{\pi}{2}\right) = \left(1-e^{-\frac{2\pi}{3}}\right)^2.$$

Similarly, if $\rho = \frac{\sqrt{13}\pi}{3\ell}$ then we find

$$f\left(\tfrac{\sqrt{13}\pi}{3\ell},\phi\right) \geqslant f\left(\tfrac{\sqrt{13}\pi}{3\ell},\tfrac{\pi}{2}\right) = \left(1-e^{-\frac{\sqrt{13}\pi}{3}}\right)^2.$$

Finally, if $\phi = 0$ or $\phi = \pi$, then recalling that $\rho\ell \in \left[\frac{2\pi}{3}, \frac{\sqrt{13}\pi}{3}\right]$ we have

$$f(\rho, 0) = f(\rho, \pi) = 2\left[1 - \cos\left(\rho\ell\right)\right] \geqslant 2\left[1 - \cos\left(\tfrac{2\pi}{3}\right)\right] = 3.$$

Therefore, the global minimum of f is equal to $\left(1 - e^{-\frac{2\pi}{3}}\right)^2$ as desired, completing the proof of Lemma 2.1.

3. The Finite Interval Problem

We now consider initial-boundary value problem (1.3) for the linear Lugiato-Lefever equation with general forcing. While the first part of our derivation is essentially the same as the one for the periodic problem, the elimination of the unknown terms from the integral representation now requires an additional idea, namely the use of a symmetry transformation for the quantity ω.

3.1. *The global relation and an integral representation for the solution*

As in the periodic case, taking the Fourier transform (2.1) of equation (1.3a), we obtain equation (2.2). Then, using the boundary conditions (1.3c) and the notation $u_x(0, t) = g_1(t)$, $u_x(\ell, t) = h_1(t)$ for the (unknown) Neumann values, we have

$$\partial_t \widehat{u}(k, t) + i\omega(k)\widehat{u}(k, t) = i\beta\Big\{g_1(t) - e^{-ik\ell}h_1(t) + ik\big[g_0(t) - e^{-ik\ell}h_0(t)\big]\Big\} + \widehat{f}(k, t), \tag{3.1}$$

where ω is defined by (2.3). Hence, integrating with respect to t and using the initial condition (1.6b) as well as the notation (2.5), we obtain the *global relation*

$$e^{i\omega t}\, \widehat{u}(k, t) = \widehat{u}_0(k) + i\beta\left\{\left[\widetilde{g}_1(\omega, t) + ik\widetilde{g}_0(\omega, t)\right] - e^{-ik\ell}\left[\widetilde{h}_1(\omega, t) + ik\widetilde{h}_0(\omega, t)\right]\right\} + \int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(k, \tau)d\tau, \tag{3.2}$$

where $k \in \mathbb{C}$ and the time transform $\widetilde{\phi}(\omega, t)$ of a function $\phi(t)$ is defined by (2.5).

Using the global relation (3.2) for $k \in \mathbb{R}$ together with the inverse Fourier transform (2.1), we obtain the *integral representation*

$$\begin{aligned} u(x, t) = {} & \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx - i\omega t}\,\widehat{u}_0(k)\,dk + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(k, \tau)d\tau dk \\ & + \frac{i\beta}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\left[\widetilde{g}_1(\omega, t) + ik\widetilde{g}_0(\omega, t)\right]dk - \frac{i\beta}{2\pi}\int_{k\in\mathbb{R}} e^{ik(x-\ell)-i\omega t}\left[\widetilde{h}_1(\omega, t) + ik\widetilde{h}_0(\omega, t)\right]dk. \end{aligned} \tag{3.3}$$

The integral representation (3.3) is not an explicit solution formula as it involves the unknown Neumann boundary values $u_x(0, t)$, $u_x(\ell, t)$ through the transforms $\widetilde{g}_1(\omega, t)$, $\widetilde{h}_1(\omega, t)$. Fortunately, as in the periodic case, these unknowns can be eliminated. However, *unlike the periodic case*, where it sufficed to simply re-employ the global relation after deforming the paths of integration from $\mathbb{R}$ to the complex contours $\mathcal{C}^{\pm}$ (see Figure 2.1 and the deformed representation (2.7)), here we must additionally exploit a certain *symmetry* of ω.

3.2. *Elimination of the unknowns and an explicit solution formula*

First, similarly to the periodic case (see Subsection 2.2), thanks to analyticity in k and use of Cauchy's theorem from complex analysis, we write the integral representation (3.3) in the form

$$u(x, t) = \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)\,dk + \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(k, \tau)d\tau dk \tag{3.4}$$

$$+\frac{i\beta}{2\pi}\int_{k\in\mathcal{C}^+} e^{ikx-i\omega t}\left[\widetilde{g}_1(\omega,t)+ik\widetilde{g}_0(\omega,t)\right]dk-\frac{i\beta}{2\pi}\int_{k\in\mathcal{C}^-} e^{ik(x-\ell)-i\omega t}\left[\widetilde{h}_1(\omega,t)+ik\widetilde{h}_0(\omega,t)\right]dk,$$

where the contours $\mathcal{C}^{\pm}$ are shown in Figure 2.1.

To proceed, we must use an idea which was *not* needed in the periodic case, namely we must exploit the symmetries of ω. In particular, solving the equation $\omega(\nu)=\omega(k)$ we find that the only nontrivial symmetry of ω is $k\mapsto -k$. In turn, since the unknown transforms $\widetilde{g}_1(\omega,t)$, $\widetilde{h}_1(\omega,t)$ depend on k only through ω, they are invariant under the transformation $k\mapsto -k$. Hence, under this transformation the global relation (3.2) yields the additional identity

$$\begin{aligned}e^{i\omega t}\,\widehat{u}(-k,t)&=\widehat{u}_0(-k)+i\beta\left\{\left[\widetilde{g}_1(\omega,t)-ik\widetilde{g}_0(\omega,t)\right]-e^{ik\ell}\left[\widetilde{h}_1(\omega,t)-ik\widetilde{h}_0(\omega,t)\right]\right\}\\&\quad+\int_{\tau=0}^{t} e^{i\omega\tau}\widehat{f}(-k,\tau)d\tau,\quad k\in\mathbb{C}.\end{aligned}\tag{3.5}$$

Equations (3.2) and (3.5) form a 2×2 system for the unknowns $\widetilde{g}_1(\omega,t)$ and $\widetilde{h}_1(\omega,t)$. Solving this system for these two quantities and then substituting the resulting expressions into (3.4), we obtain the *explicit solution formula* (1.5), where we have made use of the following analogue of Proposition 2.1.

Proposition 3.1. *For any $0<x<\ell$ and any $t>0$,*

$$\int_{k\in\mathcal{C}^+}\frac{e^{ikx}}{e^{ik\ell}-e^{-ik\ell}}\left[e^{ik\ell}\,\widehat{u}(k,t)-e^{-ik\ell}\,\widehat{u}(-k,t)\right]dk=\int_{k\in\mathcal{C}^-}\frac{e^{ik(x-\ell)}}{e^{ik\ell}-e^{-ik\ell}}\left[\widehat{u}(k,t)-\widehat{u}(-k,t)\right]dk=0.$$

The proof of Proposition 3.1 is omitted since it is entirely analogous to the one of Proposition 2.1 for the periodic case, which is provided in detail in Section 2.

Remark 3.1. The zeros of $e^{ik\ell}-e^{-ik\ell}$ do not introduce any singularities in formula (1.5) since they occur at $k=k_n=\frac{n\pi}{\ell}$, $n\in\mathbb{Z}$, and hence are avoided by the contours $\mathcal{C}^{\pm}$ (see Figure 2.1).

In the special case $f(x,t)=F$ that corresponds to the linearization of the Lugiato-Lefever equation (1.1), formula (1.5) reduces to

$$\begin{aligned}u(x,t)&=\frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)\,dk-\frac{F}{2\pi}\int_{k\in\mathbb{R}} e^{ikx}\,\frac{\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{k\omega}\,dk\\&\quad-\frac{1}{2\pi}\int_{k\in\mathcal{C}^+}\frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\left[e^{ik\ell}\,\widehat{u}_0(k)-e^{-ik\ell}\,\widehat{u}_0(-k)\right]dk\\&\quad-\frac{\beta}{\pi}\int_{k\in\mathcal{C}^+}\frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\,k\left[\widetilde{h}_0(\omega,t)-e^{-ik\ell}\,\widetilde{g}_0(\omega,t)\right]dk\\&\quad+\frac{F}{2\pi}\int_{k\in\mathcal{C}^+}\frac{e^{ikx}\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{\left(1+e^{-ik\ell}\right)k\omega}\,dk\\&\quad+\frac{1}{2\pi}\int_{k\in\mathcal{C}^-}\frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\left[\widehat{u}_0(k)-\widehat{u}_0(-k)\right]dk\\&\quad+\frac{\beta}{\pi}\int_{k\in\mathcal{C}^-}\frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\,k\left[e^{ik\ell}\,\widetilde{h}_0(\omega,t)-\widetilde{g}_0(\omega,t)\right]dk\\&\quad+\frac{F}{2\pi}\int_{k\in\mathcal{C}^-}\frac{e^{ik(x-\ell)}\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{\left(1+e^{-ik\ell}\right)k\omega}\,dk.\end{aligned}\tag{3.6}$$

This formula can be further simplified by evaluating the integrals multiplied by F via Cauchy's residue theorem. In particular, we note that $k=0$ as well as the zeros of ω in the integrals multiplied by F in (3.6) correspond to removable singularities due to the presence of $1-e^{-ik\ell}$ and $1-e^{-i\omega t}$ respectively. That is, recalling that in the nonperiodic case $k_n = \frac{n\pi}{\ell}$, $n \in \mathbb{Z}$, we see that the only singularities in these integrals arise at $k = k_{2n+1}$, $n \in \mathbb{Z}$, due to the term $1+e^{-ik\ell}$. Hence, introducing the notation $\widetilde{\mathbb{R}}_\varepsilon = \bigcup_{n\in\mathbb{Z}} \left[k_{2n+1}+\varepsilon, k_{2n+3}-\varepsilon\right]$, and using Cauchy's residue theorem, we find

$$\begin{aligned}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)}\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{\left(1+e^{-ik\ell}\right)k\omega}\,dk &= \lim_{\varepsilon\to 0}\int_{k\in\widetilde{\mathbb{R}}_\varepsilon} \frac{e^{ik(x-\ell)}\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{\left(1+e^{-ik\ell}\right)k\omega}\,dk \\ &\quad - \frac{2\pi}{\ell}\sum_{n\in\mathbb{Z}} \frac{e^{ik_{2n+1}x}\left(1-e^{-i\omega(k_{2n+1})t}\right)}{k_{2n+1}\,\omega(k_{2n+1})}\end{aligned}$$

and

$$\begin{aligned}\int_{k\in\mathcal{C}^+} \frac{e^{ikx}\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{\left(1+e^{-ik\ell}\right)k\omega}\,dk &= \lim_{\varepsilon\to 0}\int_{k\in\widetilde{\mathbb{R}}_\varepsilon} \frac{e^{ikx}\left(1-e^{-ik\ell}\right)\left(1-e^{-i\omega t}\right)}{\left(1+e^{-ik\ell}\right)k\omega}\,dk \\ &\quad - \frac{2\pi}{\ell}\sum_{n\in\mathbb{Z}} \frac{e^{ik_{2n+1}x}\left(1-e^{-i\omega(k_{2n+1})t}\right)}{k_{2n+1}\,\omega(k_{2n+1})}.\end{aligned}$$

Therefore, adding these two expressions and noting that the resulting integral along $\widetilde{\mathbb{R}}_\varepsilon$ no longer contains singularities and hence can be replaced to one along $\mathbb{R}$, we obtain a simplified, final form of formula (3.6) as

$$\begin{aligned}u(x,t) &= \frac{1}{2\pi}\int_{k\in\mathbb{R}} e^{ikx-i\omega t}\,\widehat{u}_0(k)dk - \frac{1}{2\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\left[e^{ik\ell}\,\widehat{u}_0(k) - e^{-ik\ell}\,\widehat{u}_0(-k)\right]dk \\ &\quad + \frac{1}{2\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\left[\widehat{u}_0(k)-\widehat{u}_0(-k)\right]dk - \frac{2F}{\ell}\sum_{n\in\mathbb{Z}} \frac{e^{ik_{2n+1}x}\left(1-e^{-i\omega(k_{2n+1})t}\right)}{k_{2n+1}\,\omega(k_{2n+1})} \\ &\quad - \frac{\beta}{\pi}\int_{k\in\mathcal{C}^+} \frac{e^{ikx-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\,k\left[\widetilde{h}_0(\omega,t) - e^{-ik\ell}\,\widetilde{g}_0(\omega,t)\right]dk \\ &\quad + \frac{\beta}{\pi}\int_{k\in\mathcal{C}^-} \frac{e^{ik(x-\ell)-i\omega t}}{e^{ik\ell}-e^{-ik\ell}}\,k\left[e^{ik\ell}\,\widetilde{h}_0(\omega,t) - \widetilde{g}_0(\omega,t)\right]dk. \end{aligned} \tag{3.7}$$

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