Every square can be tiled with T-tetrominos and no more than 5 monominos

Jack Grahl

July 12, 2018

Abstract

If n is a multiple of 4, then a square of side n can be tiled with T-tetrominos, using a well-known construction. If n is even but not a multiple of four, then there exists an equally well-known construction for tiling a square of side n with T-tetrominos and exactly 4 monominos. On the other hand, it was shown by Walkup in [3] that it is not possible to tile the square using only T-tetrominos. Now consider the remaining cases, where n is odd. It was shown by Zhan in [4] that it is not possible to tile such a square using only one monomino. Hochberg showed in [2] that no more than 9 monominos are ever needed. We give a construction for all odd n which uses exactly 5 monominos, thereby resolving this question.

1 Introduction

The sequence [1] gives the maximal number of T-tetrominos which can be used to tile the $n \times n$ square with t-tetrominos and monominos. Theorem 2.1 shows that this sequence is trivially given by $\frac{n^2}{4}$, $\frac{(n^2-1)}{4}-1$, $\frac{n^2}{4}-1$, $\frac{(n^2-1)}{4}-1$, depending on the value of n modulo 4.

2 Tiling every square

Theorem 2.1. *Every square can be tiled with T-tetrominos and at most 5 monominos.*

This theorem follows immediately from propositions 2.2, 2.3 and 2.4.

Proposition 2.2. *Every square of side $n = 4m$ can be tiled with T-tetrominos.*

Proposition 2.3. *Every square of side $n = 4m + 2$ can be tiled with T-tetrominos and 4 monominos, and 4 monominos are always needed.*

For $n = 2$ this is the same as pointing out that a single T-tetromino will not fit in the $2x2$ square.

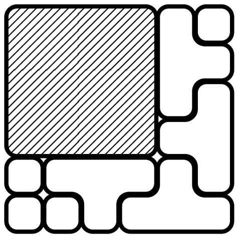

Figure 1: Extending the 4×4 tiling to 6×6, adding 4 monominos and 4 T-tetrominos.

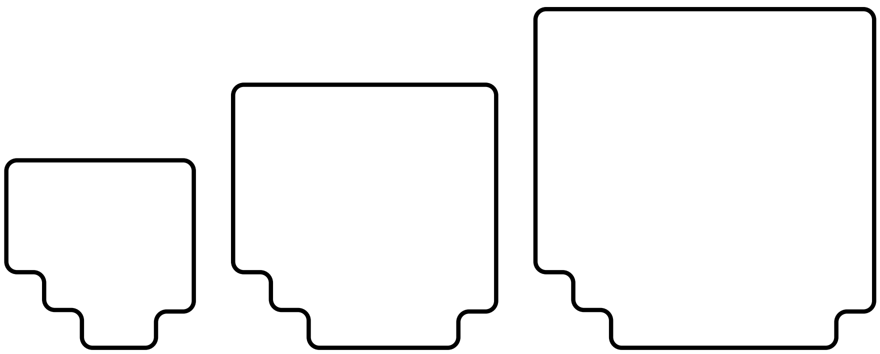

Figure 2: A_5, the 5×5 square with four lattice squares removed, A_7 and A_9.

For $n = 4m+2$, where m is a positive integer, we can extend the tiling of the $4m$-square without monominos to a tiling of the $4m+2$-square, adding only 4 monominos. The tiling of the the L-shaped strip which extends the 4×4 square to a 6×6 square is given in figure 1. We can increase the length of the arms of the strip, by replacing the two T-tetrominos with a longer sequence taken from the 'frieze', or tiling of a strip of width 2.

Proposition 2.4. *Every square of side $n = 2m + 1$ can be tiled with T-tetrominos and 5 monominos, and 5 monominos are always needed (except for $n = 1$).*

Zhan's ([4]) Theorem 2 states that it is not possible to tile any rectangle with T-tetrominos and only one monomino. It must therefore be the case that at least 5 are needed. We show that exactly 5 are sufficient.

Definition 2.5. *Call A_n the set of lattice squares given by the square of side n, with the lattice squares at $(0, 0), (0, 1), (1, 0)$ and $(0, n - 1)$ removed. This shape has area $n^2 - 4 = 4(m^2 + m - 1) + 1$.*

Lemma 2.6. *For all $m \in \mathbb{N}$, A_{2m+1} can be tiled with $m^2 + m - 1$ T-tetrominos and one monomino.*

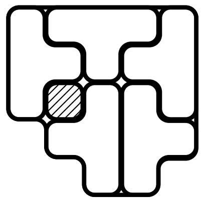

Figure 3: Tiling of A_5 with a single monomino.

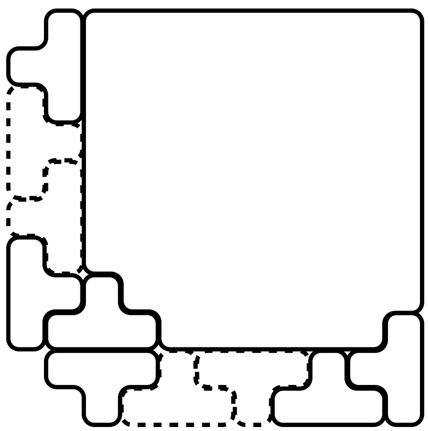

Figure 4: A tiling of A_{4k+1} can be extended to a tiling of a reflected copy of A_{4k+3}.

Proof. The proof is by induction on n. In figure 3 we show how A_5 can be tiled by 5 tetrominos and a single monomino. (It is trivial to tile A_3 with a single tetromino and a single monomino, but it is slightly clearer to start the induction with $n = 5$.) If A_n can be tiled with one monomino, then so can A_{n+1}. There are two constructions for the cases $n = 4k + 1$ and $n = 4k + 3$.

References

[1] Jack Grahl. Sequence A256535 of the Online Encyclopedia of Integer Sequences. `http://oeis.org/A256535`, 2015.

[2] Robert Hochberg. The gap number of the T-tetromino. 2014.

[3] D. W. Walkup. Covering a rectangle with T-tetrominos. *The American Mathematical Monthly*, 72(9), November 1965.

[4] Shuxin Zhan. Tiling a deficient rectangle with T-tetrominos. 2012.

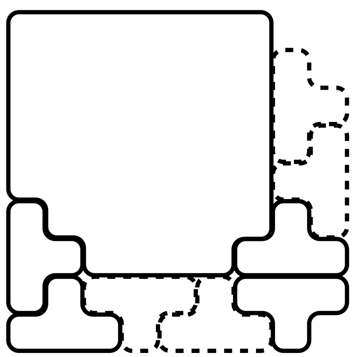

Figure 5: A tiling of A_{4k+3} can be extended to a tiling of a reflected copy of $A_{4(k+1)+1}$.