Analytic KL mode calculation for thin spherical shell

We wish to compute the signal-to-noise eigenmodes and eigenvalues explicitly for the case where our survey geometry is a thin spherical shell of thickness ϵ, at radial distance R, with constant number density $\bar{n}_0$ within the shell, and where our two-point correlation function is homogeneous and isotropic (i.e., in real space) with the functional form $\xi(r) = Ae^{-br^2}$. To recall, the signal and noise contributions to the covariance of the pixels $x_i = \int d^3x \; \psi_i(\boldsymbol{x}) n(\boldsymbol{x})$ are given by

$$S_{ij} = \int d^3x_1 \int d^3x_2 \; \psi_i^*(\boldsymbol{x}_1)\psi_j(\boldsymbol{x}_2)\bar{n}(\boldsymbol{x}_1)\bar{n}(\boldsymbol{x}_2)\xi(\boldsymbol{x}_1, \boldsymbol{x}_2), \tag{1}$$

$$N_{ij} = \int d^3x \; \psi_i^*(\boldsymbol{x})\psi_j(\boldsymbol{x})\bar{n}(\boldsymbol{x}). \tag{2}$$

We first show that, for a very thin shell, the eigenmodes are simply proportional to spherical harmonic functions, $\psi_i(\boldsymbol{x}) = C_i Y_\ell^m(\hat{x})$. To see this, first note that the noise matrix is already diagonal for this choice of ψ_i, owing to the orthogonality of spherical harmonics:

$$N_{ij} = \int_R^{R+\epsilon} r^2 dr \int d\Omega_{\hat{x}} \; C_i^* Y_\ell^{m*}(\hat{x}) \; C_j Y_{l'}^{m'}(\hat{x}) \; \bar{n}_0 = R^2 \epsilon \bar{n}_0 \; |C_i|^2 \delta_{ij}. \tag{3}$$

(Here and in the following, we treat i and j as double indices, with $i = \ell m$ and $j = \ell' m'$.)

The signal-to-noise eigenmodes are therefore just the functions that make the signal matrix diagonal. Since the correlation function is homogeneous and we are dealing with a thin shell, we have

$$\xi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \xi(|R\hat{x}_1 - R\hat{x}_2|) = \xi\left(R\sqrt{2(1 - \hat{x}_1 \cdot \hat{x}_2)}\right). \tag{4}$$

Importantly, this function depends only on the dot product of the two direction vectors, $\hat{x}_1 \cdot \hat{x}_2$. It can therefore be expanded as a sum of Legendre polynomials, $\xi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_\ell a_\ell P_\ell(\hat{x}_1 \cdot \hat{x}_2)$ for some real coefficients a_ℓ. Using the identity

$$P_\ell(\hat{x}_1 \cdot \hat{x}_2) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{x}_1) Y_\ell^{m*}(\hat{x}_2), \tag{5}$$

we see that the signal matrix

$$\begin{aligned} S_{ij} &= \int_R^{R+\epsilon} r_1^2 dr_1 \int d\Omega_{\hat{x}_1} \int_R^{R+\epsilon} r_2^2 dr_2 \int d\Omega_{\hat{x}_2} \; C_i^* Y_\ell^{m*}(\hat{x}_1) \; C_j Y_{\ell'}^{m'}(\hat{x}_2) \; \bar{n}_0^2 && (6) \\ &\quad \times \sum_{\ell'', m''} a_{\ell''} \frac{4\pi}{2\ell'' + 1} Y_{\ell''}^{m''}(\hat{x}_1) Y_{\ell''}^{m''*}(\hat{x}_2) && (7) \\ &= R^4 \epsilon^2 \bar{n}_0^2 \; a_\ell \frac{4\pi}{2\ell + 1} |C_i|^2 \delta_{ij}, && (8) \end{aligned}$$

is indeed diagonal, again owing to the orthogonality of spherical harmonics.

For convenience we choose the normalization constant C_i so as to make the noise matrix precisely equal to the identity matrix, i.e., $C_i = (R^2 \epsilon \bar{n}_0)^{-1/2}$. Then $N_{ij} = \delta_{ij}$ and

$$S_{ij} = 4\pi R^2 \epsilon \bar{n}_0 \, \frac{a_\ell}{2\ell+1} \, \delta_{ij}. \tag{9}$$

Note that the eigenvalues $\lambda_i = S_{ii}$ are independent of m. This gives us the freedom to choose our eigenmodes to be real-valued functions, by taking appropriate combinations of Y_ℓ^m's. Specifically we define real spherical harmonics by

$$Y_{\ell m} = \begin{cases} Y_\ell^0 & \text{if } m = 0, \\ \frac{1}{\sqrt{2}}(Y_\ell^m + (-1)^m Y_\ell^{-m}) \propto P_\ell^m(\cos\theta)\cos m\phi & \text{if } m > 0, \\ \frac{1}{i\sqrt{2}}(Y_\ell^{-m} - (-1)^m Y_\ell^m) \propto P_\ell^m(\cos\theta)\sin m\phi & \text{if } m < 0, \end{cases} \tag{10}$$

and choose our eigenmodes to be $\psi_i = (R^2 \epsilon \bar{n}_0)^{-1/2} Y_{\ell m}$.

Thus far we have seen that, so long as the two-point correlation function depends only on the directional dot product $\hat{x}_1 \cdot \hat{x}_2$, the signal-to-noise eigenmodes can be taken to be real spherical harmonics. To compute the eigenvalues we need to consider the specific form of the correlation function, which we take to be $\xi(r) = Ae^{-br^2}$, or

$$\xi(\boldsymbol{x}_1, \boldsymbol{x}_2) = Ae^{-b|\boldsymbol{x}_1 - \boldsymbol{x}_2|^2} = Ae^{-2bR^2(1-\hat{x}_1\cdot\hat{x}_2)} = Ae^{-y}e^{y(\hat{x}_1\cdot\hat{x}_2)}, \tag{11}$$

where we define $y \equiv 2bR^2$ for convenience. After Taylor-expanding the exponential and using the identity [1]

$$x^n = \sum_{\ell=n,n-2,\ldots} \frac{(2\ell+1)n!}{2^{(n-\ell)/2}\left(\frac{1}{2}(n-\ell)\right)!(\ell+n+1)!!} P_\ell(x), \tag{12}$$

we have

$$\begin{aligned} e^{y(\hat{x}_1\cdot\hat{x}_2)} &= \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{\ell=n,n-2,\ldots} \frac{(2\ell+1)n!}{2^{(n-\ell)/2}\left(\frac{1}{2}(n-\ell)\right)!(\ell+n+1)!!} P_\ell(\hat{x}_1\cdot\hat{x}_2) &(13)\\ &= \sum_{\ell=0}^{\infty} \left(\sum_{n=\ell,\ell+2,\ldots} \frac{y^n}{n!} \frac{(2\ell+1)n!}{2^{(n-\ell)/2}\left(\frac{1}{2}(n-\ell)\right)!(\ell+n+1)!!} \right) P_\ell(\hat{x}_1\cdot\hat{x}_2) &(14)\\ &= \sum_{\ell=0}^{\infty} \left(\sum_{k=0}^{\infty} y^{\ell+2k} \frac{(2\ell+1)}{2^k k!(2\ell+2k+1)!!} \right) P_\ell(\hat{x}_1\cdot\hat{x}_2). &(15) \end{aligned}$$

The coefficients of the Legendre expansion $\xi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_\ell a_\ell P_\ell(\hat{x}_1 \cdot \hat{x}_2)$ are therefore

$$a_\ell = Ae^{-y} \sum_{k=0}^{\infty} y^{\ell+2k} \frac{(2\ell+1)}{2^k k!(2\ell+2k+1)!!}, \tag{16}$$

which gives us for the signal-to-noise eigenvalues

$$\lambda_i = S_{ii} = 4\pi R^2 \epsilon \bar{n}_0 \; Ae^{-y} \sum_{k=0}^{\infty} \frac{y^{\ell+2k}}{2^k k!(2\ell+2k+1)!!}. \tag{17}$$

Note that the factor $4\pi R^2 \epsilon \bar{n}_0$ is just the expected number of galaxies in the shell. The function $f_\ell(y) = e^{-y} \sum_k \frac{y^{\ell+2k}}{2^k k!(2\ell+2k+1)!!}$ is plotted in the upper panel of Figure 1 for $0 \leq \ell \leq 5$. For $\ell = 0$ the series evaluates to $f_0(y) = y^{-1}e^{-y}\sinh y$, which decays asymptotically as $1/(2y)$. The lower panel of Figure 1 shows that, in fact, all the f_ℓ's have the same asymptotic behavior.

[1] http://mathworld.wolfram.com/LegendrePolynomial.html

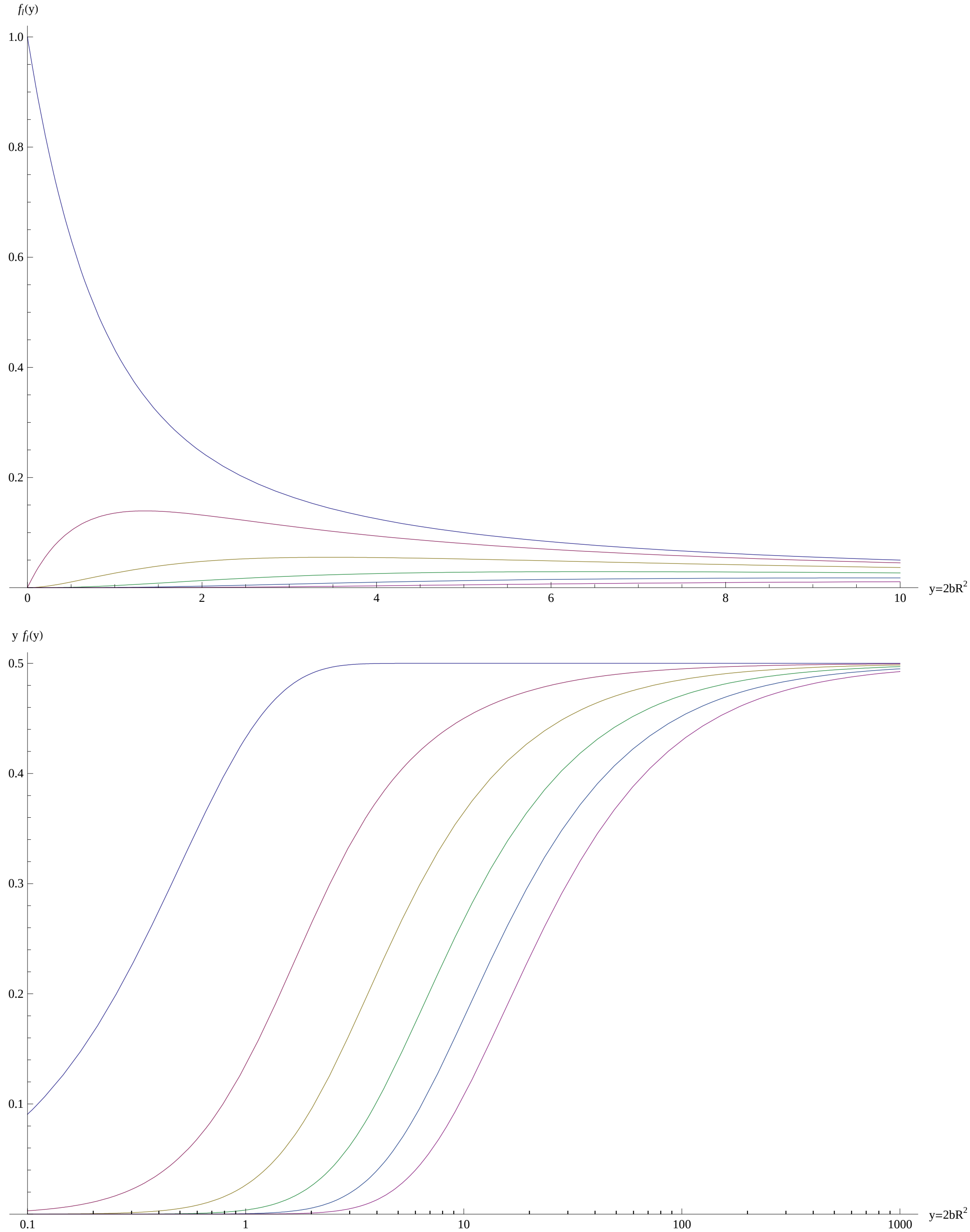

Figure 1: (Upper panel) The function $f_\ell(y) = e^{-y} \sum_{k=0}^{\infty} \frac{y^{\ell+2k}}{2^k k!(2\ell+2k+1)!!}$ plotted for $\ell = 0$ (top-most curve) to $\ell = 5$ (bottom-most curve). (Lower panel) The function $y f_\ell(y)$, showing that asymptotically $f_\ell(y) \sim 1/(2y)$.