Notes on Fractional and Irrational Calculus and Differential Equations for Engineers: Mathematics, Modeling and Numerical Methods

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Contents

Chapter 1

Introduction

1.1 Why Study Fractional Calculus?

Some people may study fractional calculus because it is inherently interesting to them. Other people need more or different reasons. A few reasons include:

- Fractional calculus expands the descriptive power of calculus beyond the familiar integer order derivatives and the basic concept of rate of change. This will yield more accurate descriptive equations when a system truly is fractional.
- Fractional calculus can expand scientific understanding when it models things that are traditionally very difficult to describe mathematically.
- Integer order derivatives are *local* needing information only in a neighborhood of a point. As will be seen, fractional derivatives are *non-local* and can describe systems with such a feature. For example, $F = ma$ right *now*. How something accelerates only depends on the forces right now, not the forces at any time in the past. For systems with "memory" effects, or analogous non-local spatial effects, the differential equation describing it most accurately may be fractional.

It is not surprising that "normal" integer-order calculus works well for most engineering applications over the past few centuries. Newton, who was one of the very early developers of calculus, clearly had a strong interest in mechanics [Newton, 1687]. However, there are a couple topics of more recent interest in engineering for which fractional calculus may be important. One area is bioengineering, where traditional mechanics and electromagnitism do not fully describe the system and in which non-local time effects may occur. Another is very large scale systems that may be easier to consider as having an infinite number of components, rather than keeping track of every single part. Examples of each type are given in this chapter. Finally, fractional calculus has been used as a new tool to try to match measured responses to "generally hard to model" system, *e.g.*, systems with friction, stiction, etc.

1.2 Introductory Concepts

Anyone reading this book should be familiar with the notion of the first, second and higher derivatives, *e.g.*, for $f(t) = t^3 + 5t^2 + 2$,

$$\frac{df}{dt}(t) = 3t^2 + 10t \tag{1.1}$$

$$\frac{d^2f}{dt^2}(t) = 6t + 10, \tag{1.2}$$

etc. Also we naturally think of integrals in an antiderivative sense, *e.g.*,

$$\int f(t)dt = \frac{1}{4}t^4 + \frac{5}{3}t^3 + 5t^2 + c \tag{1.3}$$

and we adopt a notation of $f^{(1)}(t)$ as the first derivative, $f^{(2)}(t)$ as the second derivative and $f^{(n)}(t)$ as the nth derivative as well as $f^{(-1)}(t)$ as f integrated one time, $f^{(-2)}$ as f integrated two times, *etc.*

A mathematically curious reader may already be wondering if there are any derivatives "in between" the integer ones. For example, is there a one-half derivative:

$$\frac{d^{\frac{1}{2}}f}{dt^{\frac{1}{2}}}(t) = f^{\left(\frac{1}{2}\right)} = ?. \tag{1.4}$$

There is not an immediate obvious answer to this because of the fact that the integer order derivative (as is the integral) is defined as a limit

$$\frac{df}{dt}(t) = \lim_{\Delta t \to 0} = \frac{f\,(t + \Delta t) - f\,(t)}{\Delta t} \tag{1.5}$$

and that is a discrete operation. There is not a natural half way to do it.

Basically we want to generalize the notion of the derivative. In a sense, if we define something to give the, say α derivative where $\alpha \in \mathbb{R}$, *i.e.*, α is a real number, then all we really need is that when α is an integer we get the usual definition of that integer order derivative. In between there may be lots of different options (there are!), but it makes sense to set some other basic requirements we want a fractional-order derivative to satisfy.

Example 1.1. *Consider* $f(t) = t^2$ *with the first and second derivatives* $f^{(1)}(t) = 2t$ *and* $f^{(2)}(t) = 2$, *respectively. We should expect that the* $1/2$ *derivative is, in some qualitative sense, "between"* $f(t)$ *and* $f^{(1)}(t)$, *and that the* $3/2$ *derivative is "between" the first and second derivatives, as is illustrated in Figures 1.1 and 1.2.*

We also expect that as we vary the order of the derivative, say from 0 *to* 1, *for low values of the order the result is near the zeroth derivative, and for values of the fractional order near one, it is near the first derivative. This is illustrated in Figure 1.3.*

This example motivates our first desirable attribute of a fractional derivative.

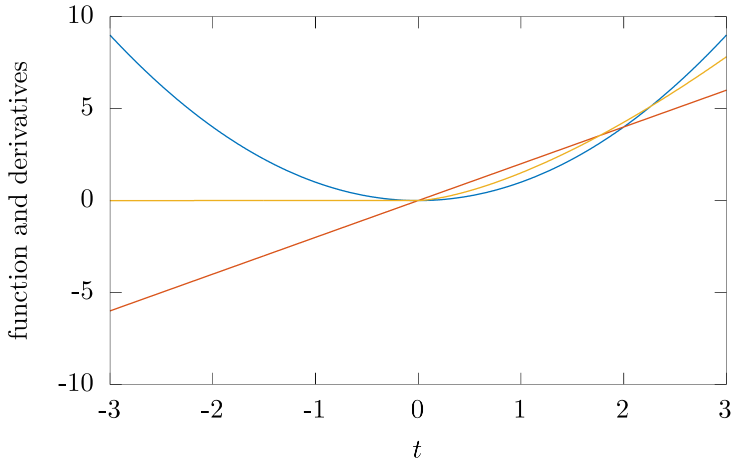

Figure 1.1: The zeroth and first derivatives of $f(t) = t^2$, (blue and red curves). A prospective half derivative (gold).

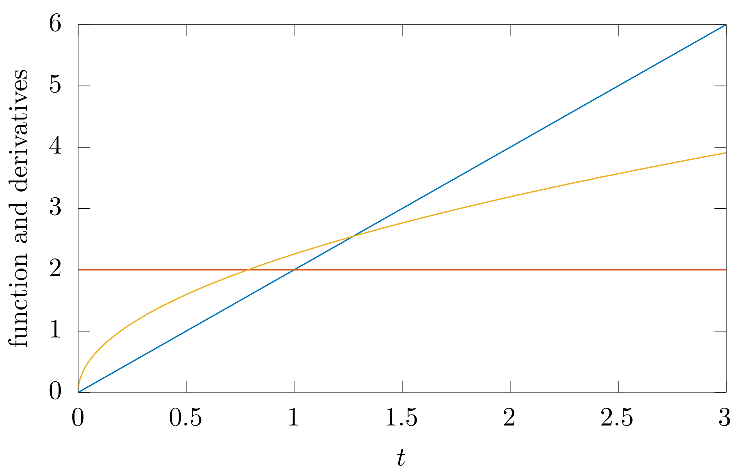

Figure 1.2: The first and second derivatives of $f(t) = t^2$ (blue and red curves) with a prospective 3/2 derivative (gold).

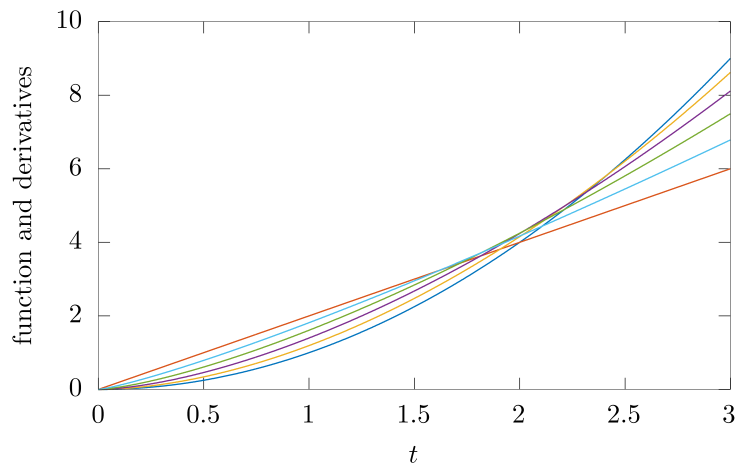

Figure 1.3: The function (blue) and the first derivative (red). The 0.2, 0.4, 0.6 and 0.8th (yellow, purple, green, light blue) order derivatives "move" from the zeroth derivative to the first derivative.

Attribute 1.1. *For a fractional derivative, $f^{(\alpha)}(t)$, if α is near an integer value, we expect the α derivative to be near that integer derivative of $f(t)$. As α varies between integer values, we expect that $f^{(\alpha)}(t)$ varies in a reasonable manner between those integer values.*

1.3 Fractional Derivatives of some Elementary Functions

1.3.1 Sine and Cosine Functions

As a first step into some real functions, we consider sine and cosine.

Example 1.2. *Consider $f(t) = \sin(t)$. The nice thing about sines and cosines are their relatively simple derivatives. In fact, from the pattern*

$$\frac{d}{dt}\sin(t) = \cos(t)$$
$$\frac{d^2}{dt^2}\sin(t) = -\sin(t)$$

as illustrated in Figure 1.4, it is clear that the derivative for this function just shifts it to the left by $\pi/2$. So clearly we would expect that the $1/2$ derivative just shifts it to the left by $\pi/4$, or the $1/2$ integral shifts it right by $\pi/3$, etc. Therefore, a reasonable guess for the half derivative would be as illustrated in Figure 1.5.

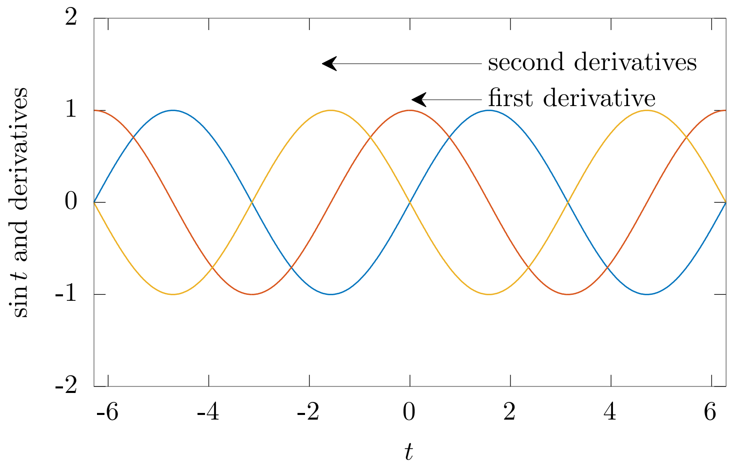

Figure 1.4: First (red) and second (gold) derivative of $\sin(t)$ (blue). The arrows indicate that derivative is just shifts the curve to the left by $\pi/2$ for each derivative.

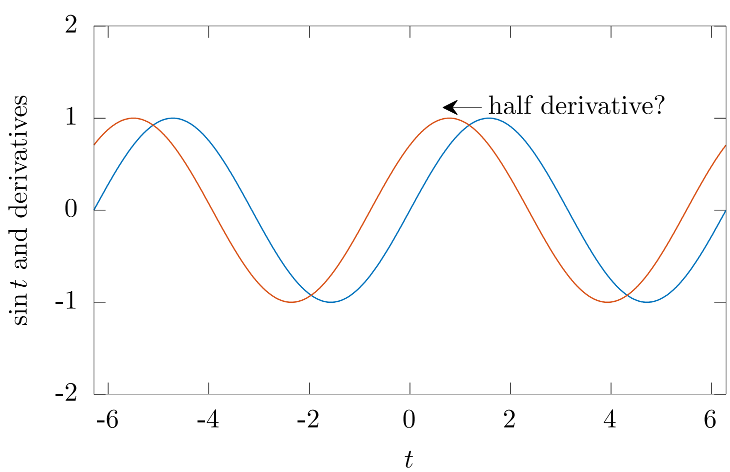

Figure 1.5: Reasonable assumption for the half derivative (red) of $\sin 9t)$ (blue).

1.3.2 Monomials and Polynomials

Being able to take the fractional derivative of sine and cosine functions is nice, but we would like to do more. The next easiest class of functions is polynomials. Consider

$$f(t) = t^k \tag{1.6}$$

which is easy to differentiate a few times and figure out the pattern:

$$\frac{df}{dt}(t) = kt^{k-1} \tag{1.7}$$

$$\frac{d^2 f}{dt^2}(t) = k(k-1)\,t^{k-2} \tag{1.8}$$

$$\frac{d^3 f}{dt^3}(t) = k(k-1)(k-2)\,t^{k-3} \tag{1.9}$$

$$\vdots = \vdots \tag{1.10}$$

$$\frac{d^n f}{dt^n}(t) = \frac{k!}{(k-n)!}t^{k-n}, \qquad n \leq k. \tag{1.11}$$

Since we are looking to define a fractional derivative of t^k, we need to see what is allowed and not allowed for n to take on fractional values in Equation 1.11. The exponent of t can be a fraction (more about that later, but engineers are so used to it they may not remember where it came from). What is definitely a problem, though, is the factorial in the denominator: if k is a natural number and n is not an integer, then the factorial is not defined.

The factorial function is just a series of values, so it seems we can generalize the derivative of t^k if we can find a curve through the factorial values. In fact, of course, it has been done and it is the gamma function defined by

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \tag{1.12}$$

which is plotted for positive real values of z in Figure 1.6. Clearly, for integer values

$$z! = \Gamma(z+1). \tag{1.13}$$

So, returning to Equation 1.11, it seems that all we need to do to define a fractional derivative is to replace the factorial in the denominator with the gamma function shifted by one. Purely for aesthetics, we might as well use a gamma function in the numerator as well, which also then would allow for a fractional k. So, we have the following seemingly legitimate definition of a fractional derivative for a monomial.

Definition 1.1. *For the monomial, t^k, $k \in \mathbb{R}$, define the fractional derivative*

$$\frac{d^\alpha}{dt^\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}t^{k-\alpha} \tag{1.14}$$

for real values of α, i.e., $\alpha \in \mathbb{R}$.

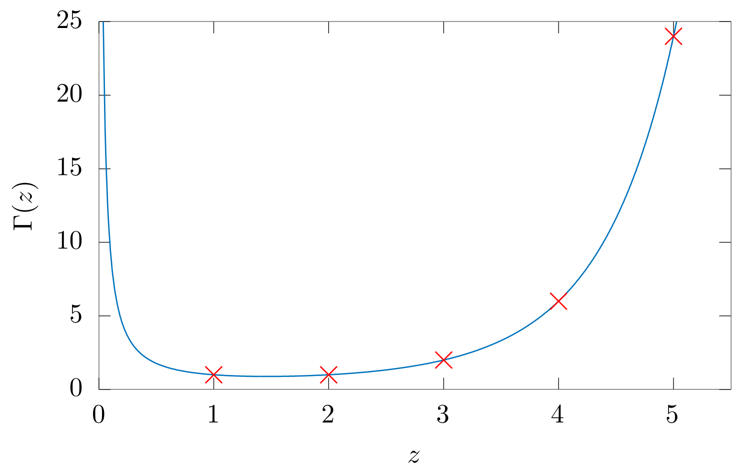

Figure 1.6: Plot of $\Gamma(z)$ (blue curve) and some factorials, $(z-1)!$ (red).

This definition was used to make the fractional derivative curves in Figures 1.1 through 1.3, so in a sense it has been validated. Also, because everything we have done is linear in t, we can use this definition for monomials and extend it term-by-term to polynomials.

For fun, we will do and example with a different k and also include negative values for α to see if integral-like ideas appear.

Example 1.3. *Consider $f(t) = t^{\frac{1}{2}} = \sqrt{t}$. Various fractional order derivatives and integrals are illustrated in Figure 1.7.*

At this point we can compute fractional derivatives and integrals of

- sines and cosines
- monomials
- by extension from monomials, polynomials if we do them term-by-term.

Before we generalize further, in order to develop a very important property of the fractional derivatives, we need to go a long way back and consider fractional exponents.

1.3.3 Exponentials

Since

$$\frac{d^n}{dt^n} e^t = e^t$$

we would like it to also hold for when n is not an integer. Similarly,

$$\frac{d^n}{dt^n} e^{\alpha t} = \alpha^n e^{\alpha t}$$

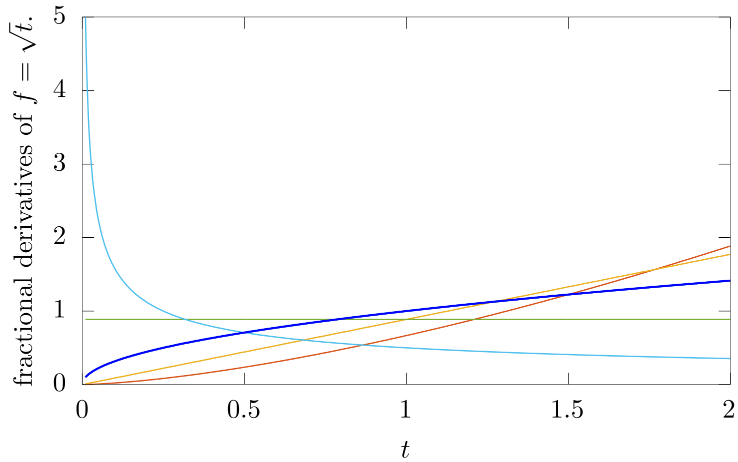

Figure 1.7: Various fractional derivatives using Definition 1.1 for $f(t) = \sqrt{t}$. The thick blue curve is the function, or zeroth derivative. The red curve is the $\alpha = -1$ derivative, which does correspond to the integral. The gold curve is the $\alpha = -1/2$ derivative, or the 1/2 integral. The green and light blue curves are the 1/2 and first derivatives, respectively.

we can require that the same be true when n is not an integer.

1.4 The Law of Indices

Engineers deal with fractional and negative exponents so often that it is easy to lose track of why they actually make sense. The exponent is defined for natural numbers (integers greater than zero) as the number of times the base is multiplied by itself, *i.e.*,

$$t^n = \underbrace{t \times t \times t \cdots t \times t}_{n \text{ times}}. \tag{1.15}$$

An obvious property of this is that for two natural number exponents

$$(t^n) \times (t^m) = \left(\underbrace{t \times t \times t \cdots t \times t}_{n \text{ times}}\right) \times \left(\underbrace{t \times t \times t \cdots t \times t}_{m \text{ times}}\right) \tag{1.16}$$

$$= \underbrace{t \times t \times t \cdots t \times t}_{n+m \text{ times}} \tag{1.17}$$

$$= t^{n+m} \tag{1.18}$$

which also immediately leads to

$$(t^n)^p = t^{n \times p}. \tag{1.19}$$

This notion of adding indices can be used to *define* negative and fractional exponents by requiring that Equations 1.18 and 1.19 hold for all rational values as well (negative values, zero and fractional values).

For negatives values, consider n and m to be positive integers with $n > m$, and if we require that $(t^n) \times (t^{-m}) = t^{n-m}$, then the only way for the $-m$ to take away powers is for it to mean division, or in too much detail

$$t^n \times t^{-m} = \frac{\overbrace{t \times t \times t \cdots t \times t}^{n \text{ times}}}{\underbrace{t \times t \cdots t \times t}_{m \text{ times}}} = \underbrace{t \times t \cdots t \times t}_{n-m \text{ times}}. \tag{1.20}$$

For fractional values Equation we can use Equation 1.19 so that

$$t^{\frac{n}{m}} = y \qquad \Longrightarrow \qquad \left(t^{\frac{n}{m}}\right)^m = y^m \qquad \Longrightarrow \qquad t^n = y^m \tag{1.21}$$

which gives the meaning that y is the number that if you raise it to the mth power gives t to the nth power, e.g. in the simple case of $1/2$, y is the number that if you square you get t.

These exercises in indices are important, because they hold for integer order derivatives

$$\frac{d^n}{dt^n}\left(\frac{d^m}{dt^m} f(t)\right) = \frac{d^{n+m} f}{dt^{n+m}}(t). \tag{1.22}$$

By insisting that the same property hold when n and m are fractional, will help in generalizing the derivative to non-integer values for a larger class of functions that simple sines, cosines and polynomials. In fact, we may as well call it an attribute.

Attribute 1.2. *For real values of α and β*

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{d^\beta}{dt^\beta} f(t)\right) = \frac{d^{\alpha+\beta} f}{dt^{\alpha+\beta}}(t). \tag{1.23}$$

Even in integer order calculus, integration and differentiation are not exactly inverses because an indefinite integral will have a constant of integration. In other words, if we take $f(t)$ and differentiate it and then integrate it, we get $f(t)+c$, but if we integrate and then differentiate, we get $f(t)$. The idea is clear enough, but it turns out that this complication does affect things.

1.5 The Frequency Domain

Multiplication by s in the frequency domain corresponds to differentiation by t in the time domain if we use the usual variables in the Laplace transform

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt = F(s), \tag{1.24}$$

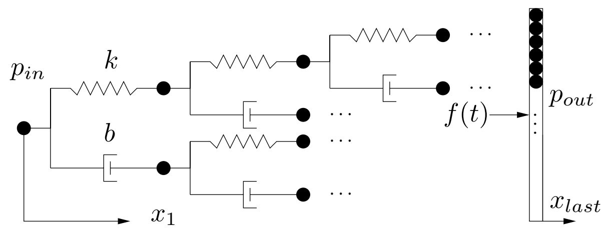

Figure 1.8: Infinite tree of springs and dampers.

i.e.,

$$\mathcal{L}\left\{\frac{df}{dt}(t)\right\} = sF(s) - f(0) \tag{1.25}$$

or assuming zero initial conditions

$$\mathcal{L}\left\{\frac{df}{dt}(t)\right\} = sF(s). \tag{1.26}$$

Higher derivatives are just increased exponents on the s, *e.g.*,

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}(t)\right\} = s^n F(s) \tag{1.27}$$

again assuming zero initial conditions.

Of course, the half derivative then would correspond to s raised to the one-half power:

$$\mathcal{L}\left\{\frac{d^{\frac{1}{2}} f}{dt^{\frac{1}{2}}}(t)\right\} = s^{\frac{1}{2}} F(s). \tag{1.28}$$

It turns out that irrational transfer functions can arise rather easily in two types of cases:

1. systems with an infinite number of components, and
2. systems with non-local interactions.

The following example illustrates the first case. Non-locality will be inherent in the more general definitions of fractional derivatives we develop subsequently, so examples will be deferred until later.

Example 1.4. *Consider the tree network of springs and dampers illustrated in Figure 1.8. The position of the left-most node is $x_1(t)$ and the right-most node by $x_{last}(t)$. Note all the nodes on the right are in the same position, so in effect they make one node. The equations of motion for this system can be determined by applying the relatively simple series and parallel rules for the springs and dampers. In order to change the network, a force must be exerted on one end, and an an equal and opposite force on the other, $f(t)$.*

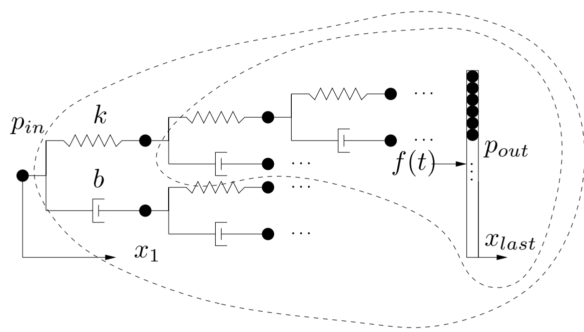

Figure 1.9: Self-similar network where the transfer function from the beginning to end of each outlined region must be equal.

It turns out that if we consider the network to be infinitely large, *i.e.*, an infinite number of bifurcating generations, it is easier to determine the transfer function describing we want, which is

$$G(s) = \frac{X_{last}(s) - X_1(s)}{F(s)} \tag{1.29}$$

and which describes the relationship between the applied force and deflection of the network. The reason it is easier, is that the network is *self-similar*. If there are an infinite number of generations, if we look at any specific node, then the transfer function from that node to the right end, is the same as any other node to the right end. In other words, from any node, there is an infinite tree growing to the right.

As such, the transfer function from the first node to the end, is equal to the transfer function from one of the nodes in the second generation to the end, as is illustrated in Figure 1.9.

Let $G_\infty(s)$ represent the infinite transfer function from any node to the end, and let the transfer function corresponding to the individual components be

$$G_1(s) = \frac{1}{k} \qquad G_2(s) = \frac{1}{bs}. \tag{1.30}$$

If there are an infinite number of generations then

$$G_\infty(s) = \frac{1}{\frac{1}{G_1(s)+G_\infty(s)} + \frac{1}{G_2(s)+G_\infty(s)}},$$

and solving this for $G_\infty(s)$ gives

$$G_\infty(s) = \sqrt{G_1(s)G_2(s)} = \sqrt{\frac{1}{kbs}} = \frac{1}{\sqrt{kb}}\frac{1}{\sqrt{s}}.$$

where the $G_\infty(s)$ on the left hand side is the transfer function for the entire network, and the two $G_\infty(s)$ terms in the denominator are the transfer functions

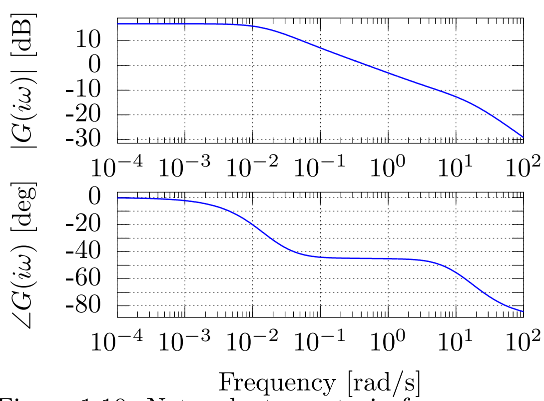

Figure 1.10: Network stress-strain frequency response.

from the two nodes in the second generation to the end. So this system should be characterized by half-order dynamics because $\sqrt{s} = s^{1/2}$.

Consistent with the idea above about fractional order systems "converging" to integer order ones, we plot the Bode plot for a network of this type where there are five generations with $k = 1$ and The Bode plot for this system is illustrated in Figure 1.10, which is characterized by two half-order dynamics features. First, the slope of the high frequency portion of the magnitude plot is -10dB/decade and the phase is $-45°$. Because first order terms are characterized by a slope of -20dB/decade and a phase of $-90°$, these features make sense as half order effects.

Chapter 2

Fractional Derivative Definitions

There are many definitions of fractional derivatives. In this chapter, we present a few of them along with their properties and compare and contrast them.

2.1 Preliminaries

From last chapter, it was obvious that one element to generalize from integer-order derivatives to allow for fractional or real-ordered derivatives, was the gamma function in cases where the only barrier to allowing a derivative to take real values was a factorial. This section covers a couple other similar tools that we will need shortly.

Cauchy's Formula for Repeated Integration

It turns out we will more easily find a general formula for a fractional number of integrations, as opposed to differentiation. That is no problem, though, because, for example, if we want the 1/3 derivative, we can integrate a function 2/3 times and then compute the integer-order first derivative of the result, the law of indices (through the Fundamental Theorem of Calculus) gives that the result what we want.

Theorem 2.1. *Let $f(t)$ be continuous. Then the nth repeated integral of $f(t)$ is given by*

$$\begin{aligned} f^{(-n)}(t) &= \int_a^t \int_a^{\sigma_1} \int_a^{\sigma_2} \int_a^{\sigma_3} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n) d\sigma_n d\sigma_{n-1} \cdots d\sigma_1 \\ &= \frac{1}{(n-1)!} \int_a^t (x-z)^{n-1} f(z) dz. \end{aligned} \tag{2.1}$$

Proof. add the proof. □

This theorem should make some intuitive sense. If you had to evaluate the single integral, the way to to it would be to integrate by parts n times to eliminate the $(x-z)$ term, which would give the multiple integral form of it.

If we ask how can we integrate a function a fractional number of times, though, it is similar to what was done in the first chapter. If we have

$$f^{(-n)}(t) = \frac{1}{(n-1)!}\int_a^t (t-z)^{n-1} f(z)dz$$

the only term containing the order of integration, n where n can not be a fraction is, again, the factorial. So we can just replace it with a gamma function

$$\boxed{f^{(-\alpha)}(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-z)^{\alpha-1} f(z)dz.} \tag{2.2}$$

2.2 Summary of Important Functions

In engineering there is a relatively limited collection of functions that are so useful that their properties become second nature. Fractional calculus adds to that collection, and this section presents some of them along with some of their most important properties. First we consider some basic computations.

2.2.1 A Collection of Computations

Gaussian Integral

The Gaussian integral is

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}, \tag{2.3}$$

and is the area under the curve in Figure 2.1. Clearly, also

$$\int_0^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{2}.$$

2.2.2 The Gamma Function

The gamma function will appear just about everywhere where we deal with fractional derivatives. We have already seen an example. The integral representation of the gamma function is

$$\boxed{\Gamma(t) = \int_0^{\infty} e^{-z} z^{t-1} dz.} \tag{2.4}$$

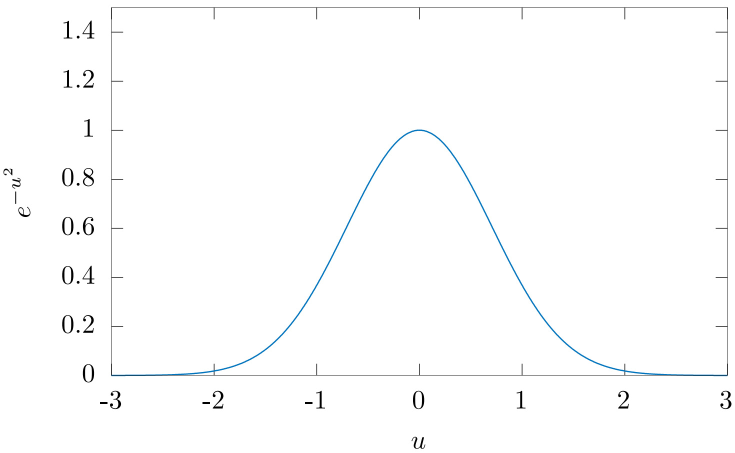

Figure 2.1: Gaussian integral.

In the case where t is an integer, they way to compute the integral by hand would be to do so repeated by parts to work the exponent of z in the integrand down to zero:

$$\begin{aligned}\Gamma(t) &= \int_0^\infty e^{-z} z^{t-1} dz \\ &= \left[z^{t-1}\left(-e^{-z}\right)\right]_0^\infty + (t-1)\int_0^\infty e^{-z} z^{t-2} dz \\ &= 0 - 0 + (t-1)\,\Gamma(t-1).\end{aligned}$$

Comparing the last line to the right hand side of the line above it gives a recursion relation analogous to $n\,(n-1)! = n!$,

$$\boxed{(t-1)\,\Gamma(t-1) = \Gamma(t).} \tag{2.5}$$

Also, continuing to integrate by parts and knowing that the boundary terms will always continue to be zero, we have

$$\begin{aligned}\Gamma(t) &= (t-1)\,\Gamma\,(t-1) \\ &= \left[(t-1)\,(t-2)\right]\Gamma\,(t-2) \\ &\vdots \\ &= \left[(t-1)\,(t-2)\cdots 1\right]\Gamma(1) \\ &= (t-1)!,\end{aligned}$$

which proves that $\boxed{\Gamma(t) = (t-1)!, t \in \mathbb{Z}}$, where $\mathbb{Z}$ is the set of natural numbers.

While the gamma function provides a nice generalization of the factorial for positive t, it is singular at zero and negative integer values as is illustrated in

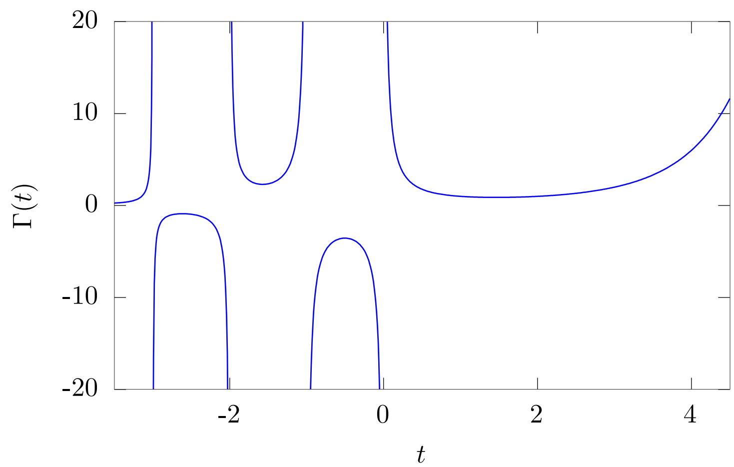

Figure 2.2: Gamma function for positive and negative real t values.

Figure 2.2. This is a feature we will have to expect to see in fractional derivatives that use the gamma function. Singularities are usually considered "bad things" but they actually make some sense in this context as the following example illustrates.

Example 2.1. *Consider $f(t) = t$ and the fractional derivatives computed using Equation 1.14 that are illustrated in Figure 2.3. Note that the singularity of the gamma function at $t = 0$ can be seen as a way for the fractional derivatives between the zeroth and first derivatives to move between the two.*

The value of the gamma function at some special values should be cataloged.

- $\boxed{\Gamma(1) = 1.}$ This can be directly computed
$$\Gamma(1) = \int_0^\infty e^{-z} z^{1-1} dz = -e^{-z}\Big|_0^\infty = 1.$$
- $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.}$ This can also be directly computed using the Gaussian integral from Equation 2.3
$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz = \int_0^\infty e^{-z} z^{-\frac{1}{2}} dz = 2\int_0^\infty e^{-y^2} dy = \sqrt{\pi}.$$
- $\boxed{\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.}$ This can be computed using Equation 2.5
$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2} - 1\right)\Gamma\left(\frac{3}{2} - 1\right) = \frac{1}{2}\sqrt{\pi}.$$

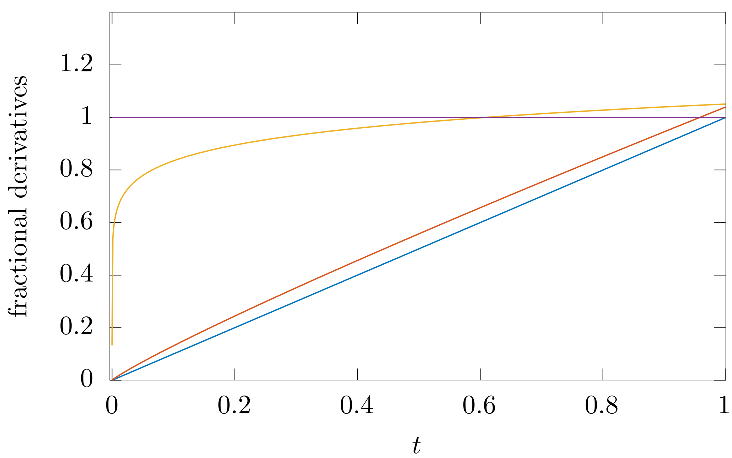

Figure 2.3: Plot of $f(t) = t$ and its 0.1, 0.9 and first derivative.

- $\boxed{\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}.}$ Similar to the previous one

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2} - 1\right)\Gamma\left(\frac{5}{2} - 1\right) = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3}{4}\sqrt{\pi}.$$

- $\boxed{\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.}$ Similarly, for some negative values, this follows from the recursion relation in Equation 2.5

$$\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2} - 1\right)\Gamma\left(\frac{1}{2} - 1\right) \qquad \Longleftrightarrow \qquad \Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}.$$

The gamma function also appears in fractional-order inverse Laplace transforms. Similarly to what we did previously, we will start with integer-order computations and then generalize. Consider the usual Laplace transform of

$f(t) = t^n$ where $n \in \mathbb{Z}$

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \int_{0^-}^{\infty} t^n e^{-st} dt \\
&= \left(-t^n \frac{1}{s} e^{-st}\right)\Big|_0^{\infty} + \frac{n}{s}\int_{0^-}^{\infty} t^{n-1} e^{-st} dt \\
&= \frac{n}{s}\int_{0^-}^{\infty} t^{n-1} e^{-st} dt \\
&\vdots \\
&= \frac{n!}{s^{n+1}}\int_{0^-}^{\infty} e^{-st} dt \\
&= \frac{n!}{s^{n+1}}.
\end{aligned}$$

A perfectly reasonable, albeit not mathematically rigorous, inference at this point would be

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

In fact, we can compute it directly

$$\begin{aligned}
\mathcal{L}\{t^\alpha\} &= \int_{0^-}^{\infty} t^\alpha e^{-st} dt \\
&= \int_{0^-}^{\infty} \left(\frac{u}{s}\right)^\alpha \frac{e^{-u}}{s} du \qquad (u = st) \\
&= \frac{1}{s^{\alpha+1}}\int_{0^-}^{\infty} e^{-u} u^\alpha du \\
&= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1.
\end{aligned}$$

2.2.3 The Error Function and Complementary Error Functions

These functions will be important as solutions to equations like

$$\frac{d^{\frac{1}{2}} x}{dt^{\frac{1}{2}}}(t) + x(t) = 1.$$

The error function is defined by

$$\boxed{\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-z^2} dz.} \tag{2.6}$$

Note that it is like the Gaussian integral, but only over a subset of the range of the definite integral. The *complementary error function* is the integral over the

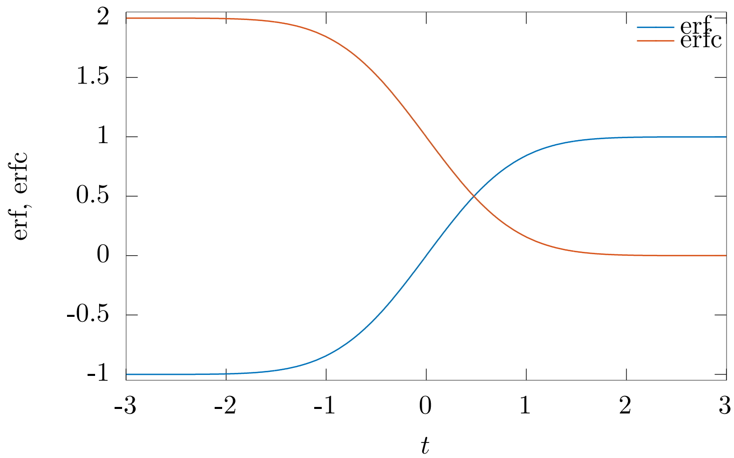

Figure 2.4: The error function and complementary error function.

remaining part of the domain

$$\boxed{\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-z^2} dz.} \tag{2.7}$$

Plots of both the error function and the complementary error function appear in Figure 2.4. It is clear from the definitions and the plots that

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t). \tag{2.8}$$

Evaluating erf at specific values of t:

- $\boxed{\operatorname{erf}(0) = \frac{2}{\pi} \int_0^0 e^{-u^2} du = 0.}$
- $\boxed{\operatorname{erf}(\infty) = \frac{2}{\pi} \int_0^\infty e^{-u^2} du = 1.}$
- $\boxed{\operatorname{erf}(-\infty) = -1.}$

2.2.4 Mittag-Leffler Functions

Mittag-Leffler Functions are generalizations of the exponential function, and play a role in solutions to constant-coefficient, homogeneous linear fractional-order ordinary differential equations analogous to the exponential for integer order differential equations. Recall the Taylor series of the exponential function about $t = 0$ is

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \sum_{k=0}^\infty \frac{t^k}{k!}.$$

These days we can not help but replace factorials with gamma functions. However, just doing that in the previous equation does not generalize anything because

$$\sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = e^t,$$

and nothing is really changed. The *one parameter* and *two parameter* Mittag-Leffler functions put a coefficient in front of the k and 1 in the gamma function

$$\boxed{E_\alpha(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0} \tag{2.9}$$

and

$$\boxed{E_{\alpha,\beta}(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \beta > 0.} \tag{2.10}$$

There are certain combinations of α and β where $E_{\alpha,\beta}(t)$ is equal to a known function. Specifically

- $\boxed{E_{1,1}(t) = E_1(t) = e^t.}$
- $\boxed{E_{\frac{1}{2},1}(t) = E_{\frac{1}{2}}(t) = x^{t^2} \operatorname{erfc}(-t)}$
- $\boxed{E_{1,2}(t) = \frac{e^t - 1}{t}}$
- $\boxed{E_{2,1}(t^2) \cosh(t)}$
- $\boxed{E_{2,2}(t^2) = \frac{\sinh(t)}{t}.}$

In order to gain some insight into these functions, let us see what the effect of varying the two parameters does. Figure 2.5 plots $E_{\alpha,t}(-t)$ for various values of α. Observe that for negative values, smaller α values are "stronger" whereas for positive values of t the opposite is basically the case. All of the curves go through the value of 1 at $t = 0$. The curves are more "curved" than the exponential for α values less than one, and less curved for α values greater than one.

Figure 2.6 illustrates $E_{1,\beta}(-t)$ for various β values. The trend to observe is that around time $t = 0$, the lowest curve corresponds to the smallest β values, and each subsequent curve increased from that one correspond to increasing β values.

- Operator theory?

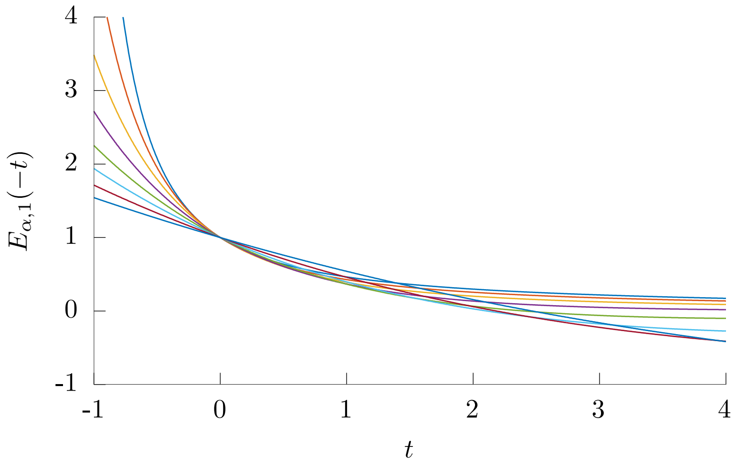

Figure 2.5: Mittag-Leffler functions, $E_{\alpha,1}(-t)$ for $\alpha = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75$ and 2. Looking at the left part of the plot near $t = -1$, $\alpha = 0.25$ is the top curve, and they are in order down to $\alpha = 2$ for the bottom curve.

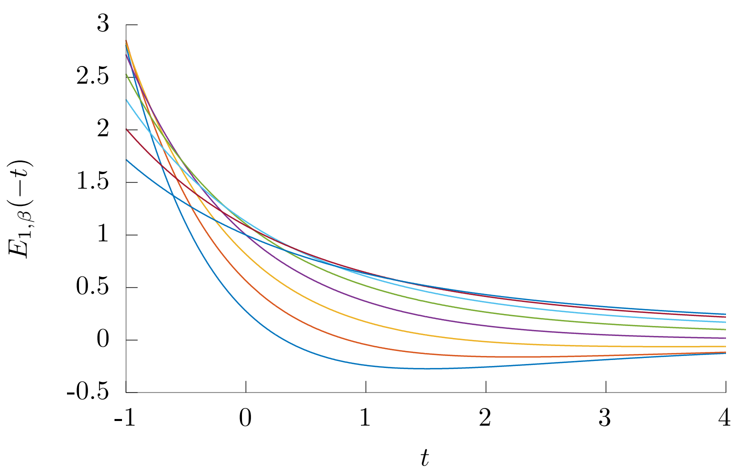

Figure 2.6: Mittag-Leffler functions, $E_{1,\beta}(-t)$ for $\beta = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75$ and 2. Near $t = 0$ the lowest curve is for $\beta = 0.25$ and increased values in β correspond to the curves above that in order.

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