Notes on Fractional and Irrational Calculus and Differential Equations for Engineers: Mathematics, Modeling and Numerical Methods

Bill Goodwine

March 16, 2021

Contents

Chapter 1

Introduction

1.1 Why Study Fractional Calculus?

Some people may study fractional calculus because it is inherently interesting to them. Other people need more or different reasons. A few reasons include:

- Fractional calculus expands the descriptive power of calculus beyond the familiar integer order derivatives and the basic concept of rate of change. This will yield more accurate descriptive equations when a system truly is fractional.
- Fractional calculus can expand scientific understanding when it models things that are traditionally very difficult to describe mathematically.
- Integer order derivatives are *local* needing information only in a neighborhood of a point. As will be seen, fractional derivatives are *non-local* and can describe systems with such a feature. For example, $F = ma$ right *now*. How something accelerates only depends on the forces right now, not the forces at any time in the past. For systems with "memory" effects, or analogous non-local spatial effects, the differential equation describing it most accurately may be fractional.

It is not surprising that "normal" integer-order calculus works well for most engineering applications over the past few centuries. Newton, who was one of the very early developers of calculus, clearly had a strong interest in mechanics [Newton, 1687]. However, there are a couple topics of more recent interest in engineering for which fractional calculus may be important. One area is bioengineering, where traditional mechanics and electromagnetism do not fully describe the system and in which non-local time effects may occur. Another is very large scale systems that may be easier to consider as having an infinite number of components, rather than keeping track of every single part. Examples of each type are given in this chapter. Finally, fractional calculus has been used as a new tool to try to match measured responses to "generally hard to model" system, *e.g.*, systems with friction, stiction, etc.

1.2 Introductory Concepts

Anyone reading this book should be familiar with the notion of the first, second and higher derivatives, *e.g.*, for $f(t) = t^3 + 5t^2 + 2$,

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = 3t^2 + 10t \tag{1.1}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}(t) = 6t + 10, \tag{1.2}$$

etc. Also we naturally think of integrals in an antiderivative sense, *e.g.*,

$$\int f(t)\,\mathrm{d}t = \frac{1}{4}t^4 + \frac{5}{3}t^3 + 5t^2 + c \tag{1.3}$$

and we adopt a notation of $f^{(1)}(t)$ as the first derivative, $f^{(2)}(t)$ as the second derivative and $f^{(n)}(t)$ as the nth derivative as well as $f^{(-1)}(t)$ as f integrated one time, $f^{(-2)}$ as f integrated two times, *etc.*

A mathematically curious reader may already be wondering if there are any derivatives "in between" the integer ones. For example, is there a one-half derivative:

$$\frac{\mathrm{d}^{\frac{1}{2}} f}{\mathrm{d}t^{\frac{1}{2}}}(t) = f^{\left(\frac{1}{2}\right)} = ?. \tag{1.4}$$

There is not an immediate obvious answer to this because of the fact that the integer order derivative (as is the integral) is defined as a limit

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \lim_{\Delta t \to 0} = \frac{f\,(t + \Delta t) - f\,(t)}{\Delta t} \tag{1.5}$$

and that is a discrete operation. There is not a natural half way to do it.

Basically we want to generalize the notion of the derivative. In a sense, if we define something to give the, say α derivative where $\alpha \in \mathbb{R}$, *i.e.*, α is a real number, then all we really need is that when α is an integer we get the usual definition of that integer order derivative. In between there may be lots of different options (there are!), but it makes sense to set some other basic requirements we want a fractional-order derivative to satisfy.

Example 1.1. *Consider* $f(t) = t^2$ *with the first and second derivatives* $f^{(1)}(t) = 2t$ *and* $f^{(2)}(t) = 2$*, respectively. We should expect that the* $1/2$ *derivative is, in some qualitative sense, "between"* $f(t)$ *and* $f^{(1)}(t)$*, and that the* $3/2$ *derivative is "between" the first and second derivatives, as is illustrated in Figures 1.1 and 1.2.*

We also expect that as we vary the order of the derivative, say from 0 *to* 1*, for low values of the order the result is near the zeroth derivative, and for values of the fractional order near one, it is near the first derivative. This is illustrated in Figure 1.3.*

This example motivates our first desirable attribute of a fractional derivative.

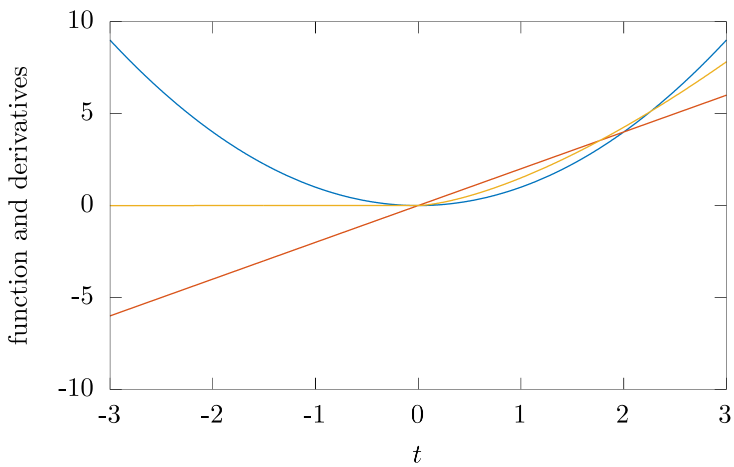

Figure 1.1: The zeroth and first derivatives of $f(t) = t^2$, (blue and red curves). A prospective half derivative (gold).

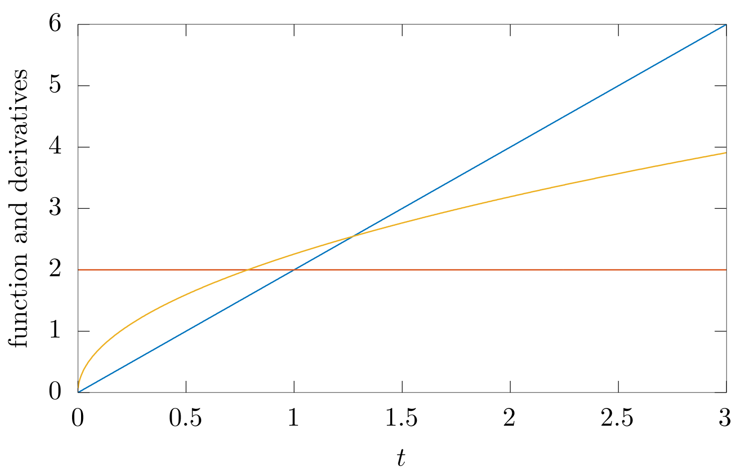

Figure 1.2: The first and second derivatives of $f(t) = t^2$ (blue and red curves) with a prospective 3/2 derivative (gold).

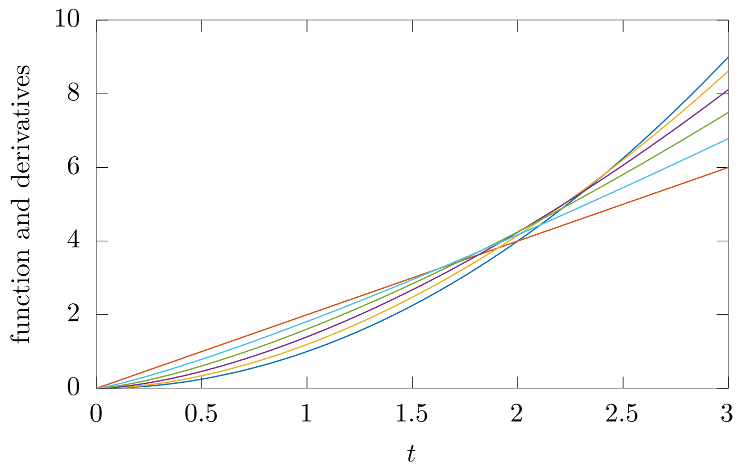

Figure 1.3: The function (blue) and the first derivative (red). The 0.2, 0.4, 0.6 and 0.8th (yellow, purple, green, light blue) order derivatives "move" from the zeroth derivative to the first derivative.

Attribute 1.1. *For a fractional derivative, $f^{(\alpha)}(t)$, if α is near an integer value, we expect the α derivative to be near that integer derivative of $f(t)$. As α varies between integer values, we expect that $f^{(\alpha)}(t)$ varies in a reasonable manner between those integer values.*

1.3 Fractional Derivatives of some Elementary Functions

1.3.1 Sine and Cosine Functions

As a first step into some real functions, we consider sine and cosine.

Example 1.2. *Consider $f(t) = \sin(t)$. The nice thing about sines and cosines are their relatively simple derivatives. In fact, from the pattern*

$$\frac{\mathrm{d}}{\mathrm{d}t}\sin(t) = \cos(t)$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sin(t) = -\sin(t)$$

as illustrated in Figure 1.4, it is clear that the derivative for this function just shifts it to the left by $\pi/2$. So clearly we would expect that the $1/2$ derivative just shifts it to the left by $\pi/4$, or the $1/2$ integral shifts it right by $\pi/3$, etc. Therefore, a reasonable guess for the half derivative would be as illustrated in Figure 1.5.

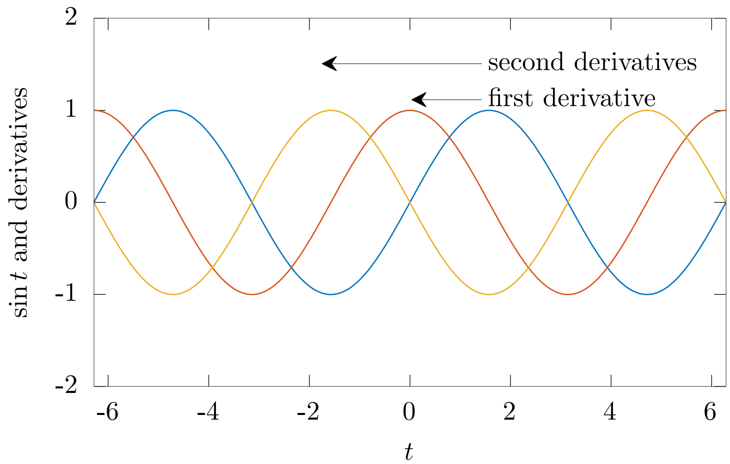

Figure 1.4: First (red) and second (gold) derivative of $\sin(t)$ (blue). The arrows indicate that derivative is just shifts the curve to the left by $\pi/2$ for each derivative.

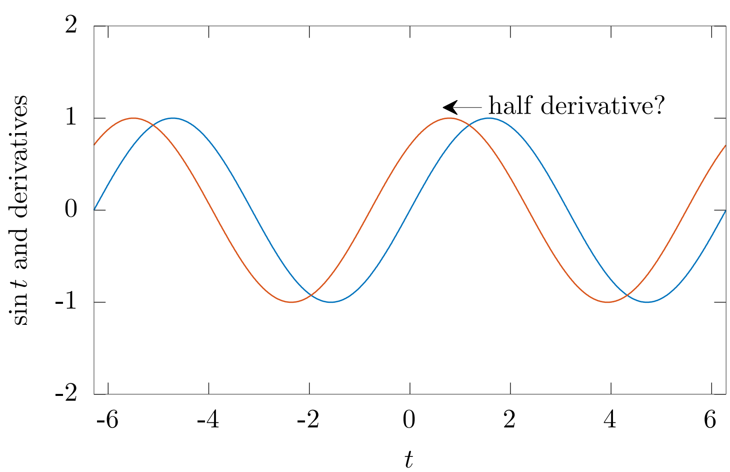

Figure 1.5: Reasonable assumption for the half derivative (red) of $\sin 9t)$ (blue).

1.3.2 Monomials and Polynomials

Being able to take the fractional derivative of sine and cosine functions is nice, but we would like to do more. The next easiest class of functions is polynomials. Consider

$$f(t) = t^k \tag{1.6}$$

which is easy to differentiate a few times and figure out the pattern:

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = kt^{k-1} \tag{1.7}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}(t) = k\,(k-1)\,t^{k-2} \tag{1.8}$$

$$\frac{\mathrm{d}^3 f}{\mathrm{d}t^3}(t) = k\,(k-1)\,(k-2)\,t^{k-3} \tag{1.9}$$

$$\vdots = \vdots \tag{1.10}$$

$$\frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t) = \frac{k!}{(k-n)!}t^{k-n}, \qquad n \le k. \tag{1.11}$$

Since we are looking to define a fractional derivative of t^k, we need to see what is allowed and not allowed for n to take on fractional values in Equation 1.11. The exponent of t can be a fraction (more about that later, but engineers are so used to it they may not remember where it came from). What is definitely a problem, though, is the factorial in the denominator: if k is a natural number and n is not an integer, then the factorial is not defined.

The factorial function is just a series of values, so it seems we can generalize the derivative of t^k if we can find a curve through the factorial values. In fact, of course, it has been done and it is the gamma function defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,\mathrm{d}x \tag{1.12}$$

which is plotted for positive real values of z in Figure 1.6. Clearly, for integer values

$$z! = \Gamma(z+1)\,. \tag{1.13}$$

So, returning to Equation 1.11, it seems that all we need to do to define a fractional derivative is to replace the factorial in the denominator with the gamma function shifted by one. Purely for aesthetics, we might as well use a gamma function in the numerator as well, which also then would allow for a fractional k. So, we have the following seemingly legitimate definition of a fractional derivative for a monomial.

Definition 1.1. *For the monomial, t^k, $k \in \mathbb{R}$, define a*[1] *proposed fractional derivative*

$$\boxed{\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}t^{k-\alpha}} \tag{1.14}$$

[1] This is *a* fractional derivative, rather than *the* fractional derivative because there are many definitions.

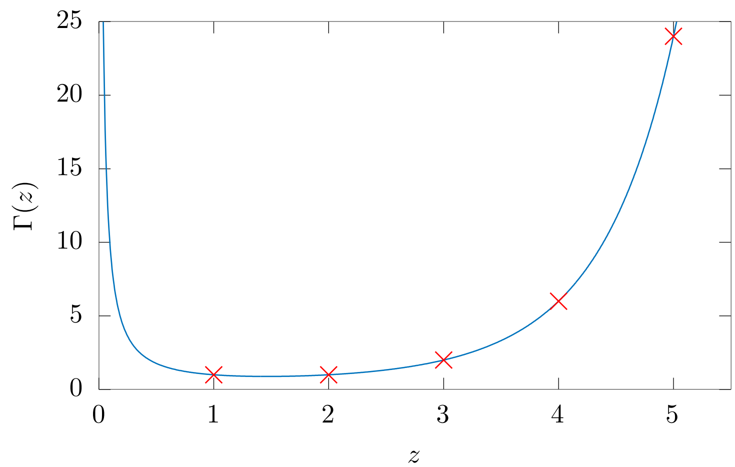

Figure 1.6: Plot of $\Gamma(z)$ (blue curve) and some factorials, $(z-1)!$ (red).

for real values of α, i.e., $\alpha \in \mathbb{R}$.

This definition was used to make the fractional derivative curves in Figures 1.1 through 1.3, so in a sense it has been validated. Also, because everything we have done is linear in t, we can use this definition for monomials and extend it term-by-term to polynomials.

For fun, we will do and example with a different k and also include negative values for α to see if integral-like ideas appear.

Example 1.3. *Consider $f(t) = t^{\frac{1}{2}} = \sqrt{t}$. Various fractional order derivatives and integrals computed using Equation 1.14 in Definition 1.1 are illustrated in Figure 1.7.*

1.3.3 Exponentials

Since

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} e^t = e^t$$

we would like it to also hold for when n is not an integer. Similarly,

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} e^{\alpha t} = \alpha^n e^{\alpha t}$$

we can require that the same, or something close to this, be true when n is not an integer.

At this point we can compute things we consider to be fractional derivatives and integrals of sines and cosines, monomials, by extension from monomials, polynomials if we do them term-by-term and exponentials. Before we generalize further, in order to develop a very important property of the fractional derivatives, we need to go a long way back and consider fractional exponents.

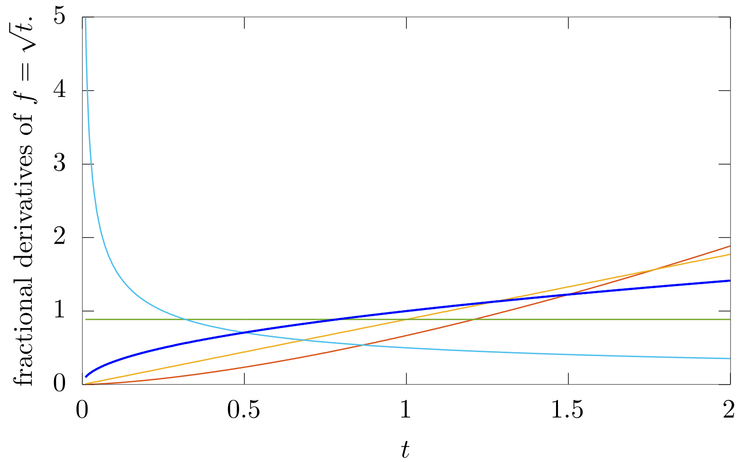

Figure 1.7: Various fractional derivatives using Definition 1.1 for $f(t) = \sqrt{t}$. The thick blue curve is the function, or zeroth derivative. The red curve is the $\alpha = -1$ derivative, which does correspond to the integral. The gold curve is the $\alpha = -1/2$ derivative, or the $1/2$ integral. The green and light blue curves are the $1/2$ and first derivatives, respectively.

1.4 The Law of Indices

Engineers deal with fractional and negative exponents so often that it is easy to lose track of why they actually make sense. The exponent is defined for natural numbers (integers greater than zero) as the number of times the base is multiplied by itself, *i.e.*,

$$t^n = \underbrace{t \times t \times t \cdots t \times t}_{n \text{ times}}. \tag{1.15}$$

An obvious property of this is that for two natural number exponents

$$(t^n) \times (t^m) = \left(\underbrace{t \times t \times t \cdots t \times t}_{n \text{ times}}\right) \times \left(\underbrace{t \times t \times t \cdots t \times t}_{m \text{ times}}\right) \tag{1.16}$$

$$= \underbrace{t \times t \times t \cdots t \times t}_{n+m \text{ times}} \tag{1.17}$$

$$= t^{n+m} \tag{1.18}$$

which also immediately leads to

$$(t^n)^p = t^{n \times p}. \tag{1.19}$$

This notion of adding indices can be used to *define* negative and fractional exponents by requiring that Equations 1.18 and 1.19 hold for all rational values as well (negative values, zero and fractional values).

For negatives values, consider n and m to be positive integers with $n > m$, and if we require that $(t^n) \times (t^{-m}) = t^{n-m}$, then the only way for the $-m$ to take away powers is for it to mean division, or in too much detail

$$t^n \times t^{-m} = \frac{\overbrace{t \times t \times t \cdots t \times t}^{n \text{ times}}}{\underbrace{t \times t \cdots t \times t}_{m \text{ times}}} = \underbrace{t \times t \cdots t \times t}_{n-m \text{ times}}. \tag{1.20}$$

For fractional values Equation we can use Equation 1.19 so that

$$t^{\frac{n}{m}} = y \qquad \Longrightarrow \qquad \left(t^{\frac{n}{m}}\right)^m = y^m \qquad \Longrightarrow \qquad t^n = y^m \tag{1.21}$$

which gives the meaning that y is the number that if you raise it to the mth power gives t to the nth power, e.g. in the simple case of $1/2$, y is the number that if you square you get t.

These exercises in indices are important, because they hold for integer order derivatives

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t)\right) = \frac{\mathrm{d}^{n+m} f}{\mathrm{d}t^{n+m}}(t). \tag{1.22}$$

By insisting that the same property hold when n and m are fractional, will help in generalizing the derivative to non-integer values for a larger class of functions that simple sines, cosines and polynomials. In fact, we may as well call it an attribute.

Attribute 1.2. *For real values of α and β*

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left(\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t)\right) = \frac{\mathrm{d}^{\alpha+\beta} f}{\mathrm{d}t^{\alpha+\beta}}(t). \tag{1.23}$$

Even in integer order calculus, integration and differentiation are not exactly inverses because an indefinite integral will have a constant of integration. In other words, if we take $f(t)$ and differentiate it and then integrate it, we get $f(t)+c$, but if we integrate and then differentiate, we get $f(t)$. The idea is clear enough, but it turns out that this complication does affect things.

1.5 Examples

1.5.1 Mechanical System and Frequency Domain Example

Multiplication by s in the frequency domain corresponds to differentiation by t in the time domain if we use the usual variables in the Laplace transform

$$\mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} f(t)e^{-st}\,\mathrm{d}t = F(s), \tag{1.24}$$

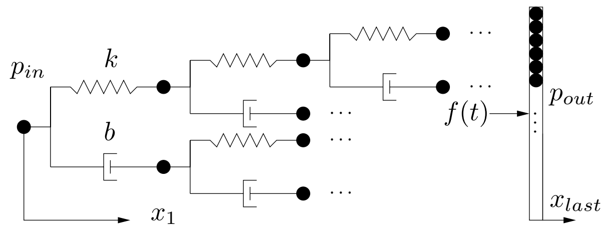

Figure 1.8: Infinite tree of springs and dampers.

i.e.,

$$\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right\} = sF(s) - f(0) \tag{1.25}$$

or assuming zero initial conditions

$$\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right\} = sF(s). \tag{1.26}$$

Higher derivatives are just increased exponents on the s, *e.g.*,

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t)\right\} = s^n F(s) \tag{1.27}$$

again assuming zero initial conditions.

Of course, the half derivative then would correspond to s raised to the one-half power:

$$\mathcal{L}\left\{\frac{\mathrm{d}^{\frac{1}{2}} f}{\mathrm{d}t^{\frac{1}{2}}}(t)\right\} = s^{\frac{1}{2}} F(s). \tag{1.28}$$

It turns out that irrational transfer functions can arise rather easily in two types of cases:

1. systems with an infinite number of components, and

2. systems with non-local interactions.

The following example illustrates the first case. Non-locality will be inherent in the more general definitions of fractional derivatives we develop subsequently, so examples will be deferred until later.

Example 1.4. *Consider the tree network of springs and dampers illustrated in Figure 1.8. The position of the left-most node is $x_1(t)$ and the right-most node by $x_{last}(t)$. Note all the nodes on the right are in the same position, so in effect they make one node. The equations of motion for this system can be determined by applying the relatively simple series and parallel rules for the springs and dampers. In order to change the network, a force must be exerted on one end, and an an equal and opposite force on the other, $f(t)$.*

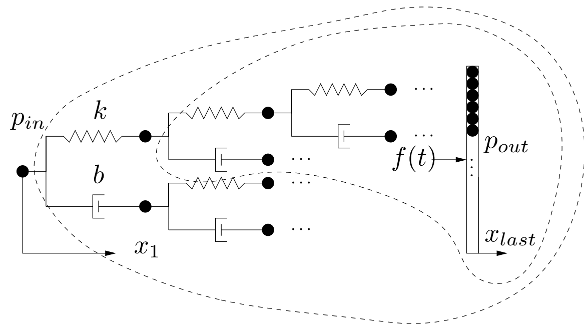

Figure 1.9: Self-similar network where the transfer function from the beginning to end of each outlined region must be equal.

It turns out that if we consider the network to be infinitely large, *i.e.*, an infinite number of bifurcating generations, it is easier to determine the transfer function describing we want, which is

$$G(s) = \frac{X_{last}(s) - X_1(s)}{F(s)} \tag{1.29}$$

and which describes the relationship between the applied force and deflection of the network. The reason it is easier, is that the network is *self-similar*. If there are an infinite number of generations, if we look at any specific node, then the transfer function from that node to the right end, is the same as any other node to the right end. In other words, from any node, there is an infinite tree growing to the right.

As such, the transfer function from the first node to the end, is equal to the transfer function from one of the nodes in the second generation to the end, as is illustrated in Figure 1.9.

Let $G_\infty(s)$ represent the infinite transfer function from any node to the end, and let the transfer function corresponding to the individual components be

$$G_1(s) = \frac{1}{k} \qquad G_2(s) = \frac{1}{bs}. \tag{1.30}$$

If there are an infinite number of generations then

$$G_\infty(s) = \frac{1}{\frac{1}{G_1(s)+G_\infty(s)} + \frac{1}{G_2(s)+G_\infty(s)}},$$

and solving this for $G_\infty(s)$ gives

$$G_\infty(s) = \sqrt{G_1(s)G_2(s)} = \sqrt{\frac{1}{kbs}} = \frac{1}{\sqrt{kb}}\frac{1}{\sqrt{s}}.$$

where the $G_\infty(s)$ on the left hand side is the transfer function for the entire network, and the two $G_\infty(s)$ terms in the denominator are the transfer functions

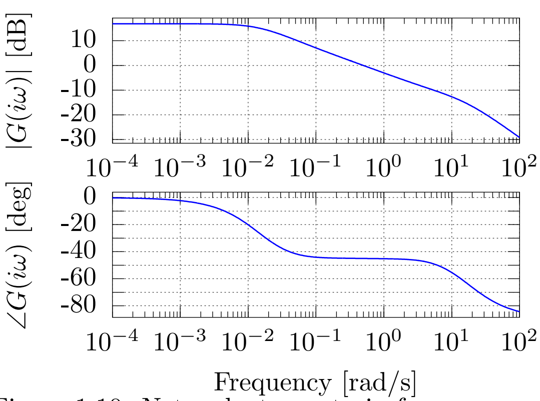

Figure 1.10: Network stress-strain frequency response.

from the two nodes in the second generation to the end. So this system should be characterized by half-order dynamics because $\sqrt{s} = s^{1/2}$.

Consistent with the idea above about fractional order systems "converging" to integer order ones, we plot the Bode plot for a network of this type where there are five generations with $k = 1$ and The Bode plot for this system is illustrated in Figure 1.10, which is characterized by two half-order dynamics features. First, the slope of the high frequency portion of the magnitude plot is -10dB/decade and the phase is $-45°$. Because first order terms are characterized by a slope of -20dB/decade and a phase of $-90°$, these features make sense as half order effects.

1.5.2 Ultrasound Example

Reference and summaries Holm's paper.

Chapter 2

Fractional Derivative Definitions

There are many definitions of fractional derivatives. In this chapter, we present a few of them along with their properties and compare and contrast them.

2.1 Summary of Important Functions

In engineering there is a relatively limited collection of functions that are so useful that their properties become second nature. Fractional calculus adds to that collection, and this section presents some of them along with some of their most important properties. First we consider some basic computations.

2.1.1 Preliminaries

A few computations are needed before we study the functions.

Gaussian Integral

The Gaussian integral is

$$\boxed{\int_{-\infty}^{\infty} e^{-z^2}\,\mathrm{d}z = \sqrt{\pi},} \tag{2.1}$$

and is the area under the curve in Figure 2.1. Clearly, also

$$\int_{0}^{\infty} e^{-z^2}\,\mathrm{d}z = \frac{\sqrt{\pi}}{2}.$$

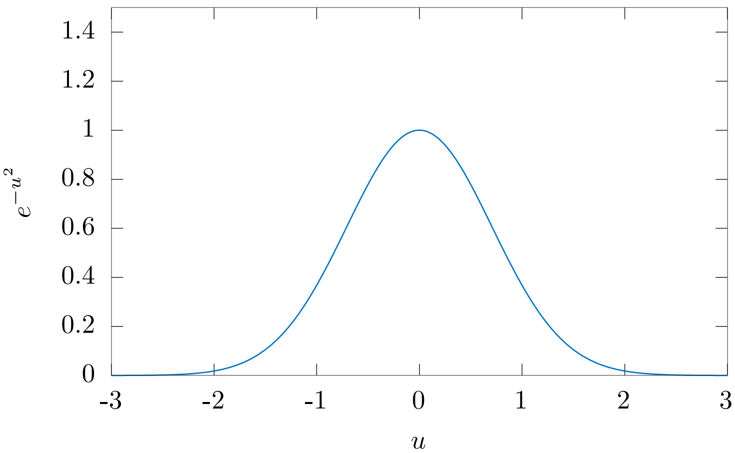

Figure 2.1: Gaussian integral.

To see this, consider the square of the integral and switch to polar coordinates

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-z^2}\,\mathrm{d}z\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-z_1^2}\,\mathrm{d}z_1\right)\left(\int_{-\infty}^{\infty} e^{-z_2^2}\,\mathrm{d}z_2\right)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-z_1^2-z_2^2}\,\mathrm{d}z_1\,\mathrm{d}z_2\\ &= \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta\\ &= 2\pi\int_0^{\infty} re^{-r^2}\,\mathrm{d}r\\ &= \pi\int_{-\infty}^{0} e^{u}\,\mathrm{d}u \qquad (u=-r^2)\\ &= \pi,\end{aligned}$$

which shows

$$\int_{-\infty}^{\infty} e^{-z^2}\,\mathrm{d}z = \sqrt{\pi}.$$

Notation

We will need notation for the “floor” and “ceiling” operations. For $\alpha \in \mathbb{R}$, $\lceil \alpha \rceil$ represents the smallest integer greater than α. For example, $\lceil 3.05 \rceil = 4$. Correspondingly, $\lfloor \alpha \rfloor$ is the largest integer less than α, so $\lfloor 4.99 \rfloor = 4$.

Because there will be multiple definitions of a fractional derivative, we need a way to distinguish them. In general, D^{α} will stand for the derivative operator of order α, and $D^{-\alpha}$ will represent integrating by a fractional number of times.

As will be seen, because most definitions will involve integrals, the limits of integration will also impact the operation, so we need to have those in the definition, so ${}_tD_0^\alpha$ will be the derivative operator if α is positive and the 0 and t will be the limits of integration (described subsequently).

2.1.2 The Gamma Function

The gamma function will appear just about everywhere where we deal with fractional derivatives. We have already seen an example. The integral representation of the gamma function is

$$\boxed{\Gamma(t) = \int_0^\infty e^{-z} z^{t-1}\,\mathrm{d}z.} \tag{2.2}$$

In the case where t is an integer, they way to compute the integral by hand would be to do so repeated by parts to work the exponent of z in the integrand down to zero:

$$\begin{aligned}
\Gamma(t) &= \int_0^\infty e^{-z} z^{t-1}\,\mathrm{d}z \\
&= \left[z^{t-1}\left(-e^{-z}\right)\right]_0^\infty + (t-1)\int_0^\infty e^{-z} z^{t-2}\,\mathrm{d}z \\
&= 0 - 0 + (t-1)\,\Gamma(t-1).
\end{aligned}$$

Comparing the last line to the right hand side of the line above it gives a recursion relation analogous to $n\,(n-1)! = n!$,

$$\boxed{\Gamma(t) = (t-1)\,\Gamma(t-1).} \tag{2.3}$$

Also, continuing to integrate by parts and knowing that the boundary terms will always continue to be zero, we have

$$\begin{aligned}
\Gamma(t) &= (t-1)\,\Gamma\,(t-1) \\
&= [(t-1)\,(t-2)]\,\Gamma\,(t-2) \\
&\vdots \\
&= [(t-1)\,(t-2)\cdots 1]\,\Gamma(1) \\
&= (t-1)!,
\end{aligned}$$

which proves that $\boxed{\Gamma(t) = (t-1)!, t \in \mathbb{Z}}$, where $\mathbb{Z}$ is the set of natural numbers.

While the gamma function provides a nice generalization of the factorial for positive t, it is singular at zero and negative integer values as is illustrated in Figure 2.2.[1] This is a feature we will have to expect to see in fractional

[1] In fact, this integral definition we used in Equation 1.12 is only valid for positive arguments. Other definitions, such as a series one, has to be used for zero and negative values for the definition to be complete and rigorous.

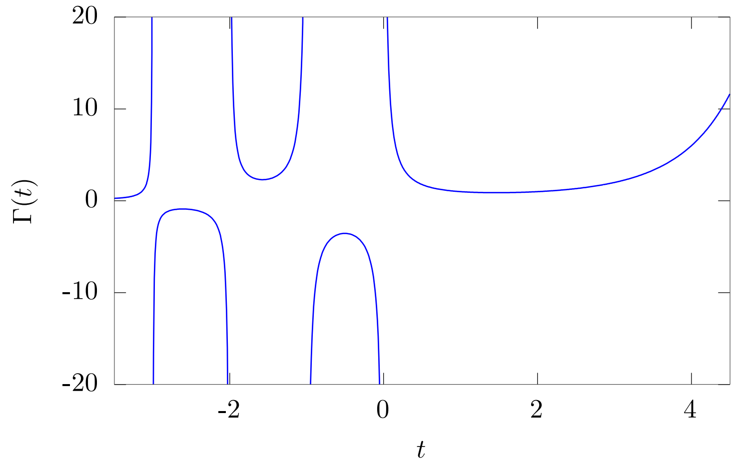

Figure 2.2: Gamma function for positive and negative real t values.

derivatives that use the gamma function. Singularities are usually considered "bad things" but they actually make some sense in this context as the following example illustrates.

Example 2.1. *Consider $f(t) = t$ and the fractional derivatives computed using Equation 1.14 that are illustrated in Figure 2.3. Note that the singularity of the gamma function at $t = 0$ can be seen as a way for the fractional derivatives between the zeroth and first derivatives to move between the two.*

The value of the gamma function at some special values should be cataloged.

- $\boxed{\Gamma(1) = 1.}$ This can be directly computed
$$\Gamma(1) = \int_0^\infty e^{-z} z^{1-1}\,\mathrm{d}z = -e^{-z}\Big|_0^\infty = 1.$$

- $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.}$ This can also be directly computed using the Gaussian integral from Equation 2.1
$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty e^{-z} z^{\frac{1}{2}-1}\,\mathrm{d}z = \int_0^\infty e^{-z} z^{-\frac{1}{2}}\,\mathrm{d}z \\ &= 2\int_0^\infty e^{-u^2}\,\mathrm{d}u \qquad (u^2 = z) \\ &= \sqrt{\pi}.\end{aligned}$$

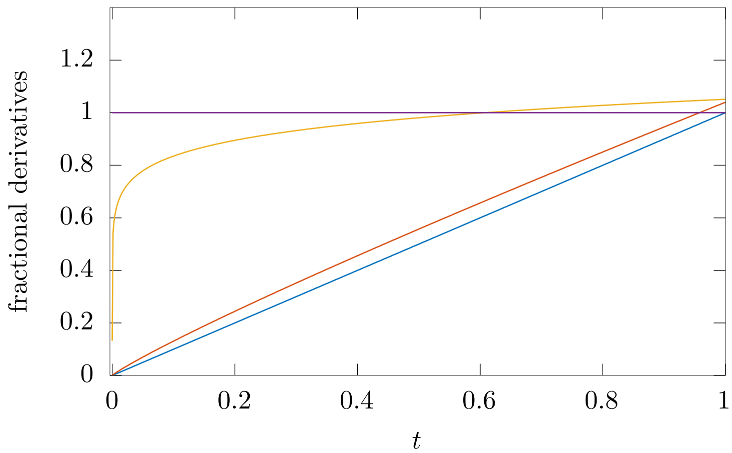

Figure 2.3: Plot of $f(t) = t$ and its 0.1, 0.9 and first derivative.

- $\boxed{\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.}$ This can be computed using Equation 2.3

$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2} - 1\right)\Gamma\left(\frac{3}{2} - 1\right) = \frac{1}{2}\sqrt{\pi}.$$

- $\boxed{\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}.}$ Similar to the previous one

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2} - 1\right)\Gamma\left(\frac{5}{2} - 1\right) = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3}{4}\sqrt{\pi}.$$

- $\boxed{\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.}$ Similarly, for some negative values, this follows from the recursion relation in Equation 2.3

$$\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2} - 1\right)\Gamma\left(\frac{1}{2} - 1\right) \qquad \Longleftrightarrow \qquad \Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}.$$

The gamma function also appears in fractional-order inverse Laplace transforms. Similarly to what we did previously, we will start with integer-order computations and then generalize. Consider the usual Laplace transform of

$f(t) = t^n$ where $n \in \mathbb{Z}$

$$\begin{aligned}
\mathcal{L}\left\{t^n\right\} &= \int_{0^-}^{\infty} t^n e^{-st}\,\mathrm{d}t \\
&= \left(-t^n \frac{1}{s} e^{-st}\right)\Big|_0^{\infty} + \frac{n}{s}\int_{0^-}^{\infty} t^{n-1} e^{-st}\,\mathrm{d}t \\
&= \frac{n}{s}\int_{0^-}^{\infty} t^{n-1} e^{-st}\,\mathrm{d}t \\
&\vdots \\
&= \frac{n!}{s^{n+1}}\int_{0^-}^{\infty} e^{-st}\,\mathrm{d}t \\
&= \frac{n!}{s^{n+1}}.
\end{aligned}$$

A perfectly reasonable, albeit not mathematically rigorous, inference at this point would be

$$\boxed{\mathcal{L}\left\{t^{\alpha}\right\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.}$$

In fact, we can compute it directly

$$\begin{aligned}
\mathcal{L}\left\{t^{\alpha}\right\} &= \int_{0^-}^{\infty} t^{\alpha} e^{-st}\,\mathrm{d}t \\
&= \int_{0^-}^{\infty} \left(\frac{u}{s}\right)^{\alpha} \frac{e^{-u}}{s}\,\mathrm{d}u \qquad (u = st) \\
&= \frac{1}{s^{\alpha+1}}\int_{0^-}^{\infty} e^{-u} u^{\alpha}\,\mathrm{d}u \\
&= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1.
\end{aligned}$$

2.1.3 Binomial Coefficient

In the integer case, the binomial coefficient for $n, k \in \mathbb{Z}$ is

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!\,(n-k)!}. \tag{2.4}$$

Common interpretations include entries in Pascal's triangle and "n choose k" because it is the number of different ways to choose k elements from a set with n elements. It will be convenient, but perhaps not so rigorous, to consider the binomial coefficient to be zero "outside" of Pascal's triangle. This makes sense for the probability: *e.g.*, there are zero ways to choose 5 elements from a set of 4.

We will use it because it naturally arises in finite difference expansions, and we will naturally want to allow the entries to be non-integers. So we will have

k	n=4	n=3.95	n=3.5
0	1	1	1
1	4	3.95	3.5
2	6	5.82625	4.375
3	4	3.78706	2.1875
4	1	0.899427	0.273438
5	0	-0.00899427	-0.0273438
6	0	0.001574	0.00683594
7	0	-0.000460957	-0.00244141
8	0	0.00017574	0.00106812
9	0	-0.00007908	-0.000534058
10	0	0.00003994	0.000293732

Table 2.1: Binomial coefficients with $n = 4$, $n = 3.95$ and $n = 3.5$ for various integer values of k

the generalization

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\,\Gamma(\alpha-\beta+1)}, \tag{2.5}$$

where $\alpha, \beta \in \mathbb{R}$, *i.e.*, they can be fractional.

The following example illustrates the manner in which this will play a role in a subsequent fractional derivative definition.

Example 2.2. *Consider the binomial coefficient with* $n = 4$ *for various values of* k*, as shown in the first row of Table 2.1, where we have adopted the convention that the value is zero where the factorials are not defined. These values are also plotted in Figure 2.4.*

Note that for a small change in n *from* 4 *to* $3;95$ *the values of the binomial coefficient are close to the values for* 4*. However, while the values for increasing* β *are tending towards zero in absolute value, they are not equal to zero and the rate of convergence is not all that fast.*

2.1.4 The Error Function and Complementary Error Functions

These functions will be important as solutions to equations like

$$\frac{\mathrm{d}^{\frac{1}{2}}x}{\mathrm{d}t^{\frac{1}{2}}}(t) + x(t) = 1.$$

The error function is defined by

$$\boxed{\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-z^2}\,\mathrm{d}z.} \tag{2.6}$$

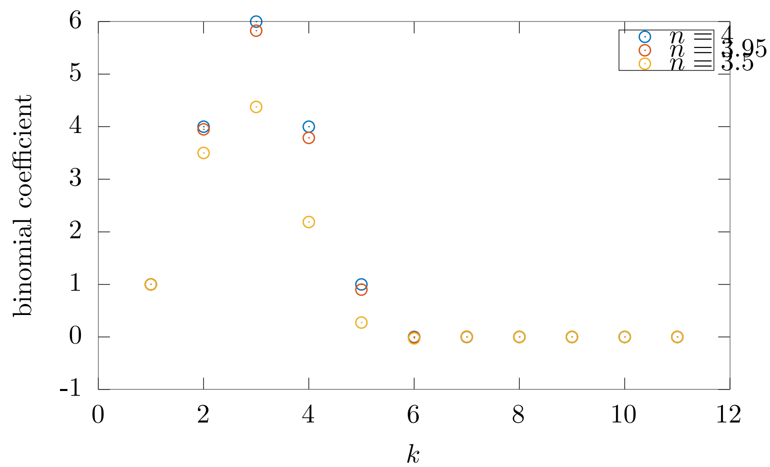

Figure 2.4: Binomial coefficient values for $n = 4, 3.95 and 3.5$ for various k values.

Note that it is like the Gaussian integral, but only over a subset of the range of the definite integral. The *complementary error function* is the integral over the remaining part of the domain

$$\boxed{\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-z^2}\, \mathrm{d}z.} \tag{2.7}$$

Plots of both the error function and the complementary error function appear in Figure 2.5. It is clear from the definitions and the plots that

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t). \tag{2.8}$$

Evaluating erf at specific values of t:

- $\boxed{\operatorname{erf}(0) = \frac{2}{\sqrt{\pi}} \int_0^{0} e^{-u^2}\, \mathrm{d}u = 0.}$
- $\boxed{\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2}\, \mathrm{d}u = 1.}$
- $\boxed{\operatorname{erf}(-\infty) = -1.}$

2.1.5 Mittag-Leffler Functions

Mittag-Leffler Functions are generalizations of the exponential function, and play a role in solutions to constant-coefficient, homogeneous linear fractional-order ordinary differential equations analogous to the exponential for integer

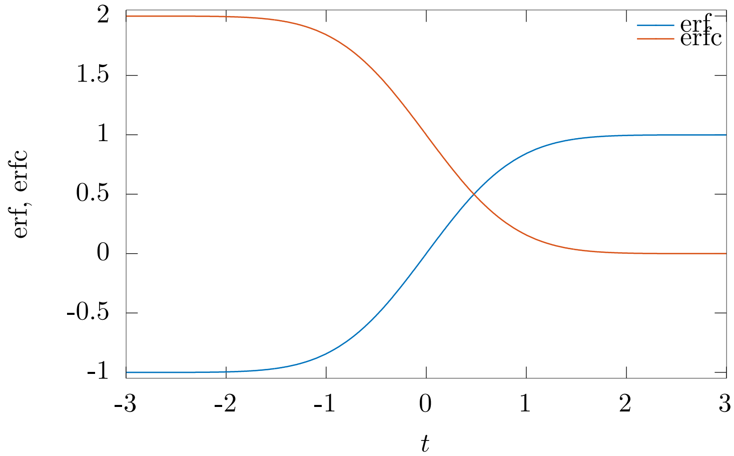

Figure 2.5: The error function and complementary error function.

order differential equations. As will be shown subsequently, just as $x(t) = ce^{-at}$ is the solution to

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) + ax(t) = 0$$

the function $ct^{\alpha-1}E_{\alpha,\alpha}\left(-at^{\alpha}\right)$, where $E_{\alpha,\alpha}$ is the Mittag-Leffler function to be defined shortly, is the solution to

$$\frac{\mathrm{d}^{\alpha}x}{\mathrm{d}t^{\alpha}}(t) + ax(t) = 0.$$

Recall the Taylor series of the exponential function about $t = 0$ is

$$e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}.$$

These days we can not help but replace factorials with gamma functions. However, just doing that in the previous equation does not generalize anything because

$$\sum_{k=0}^{\infty} \frac{t^k}{\Gamma\left(k+1\right)} = e^{t},$$

and nothing is really changed.

The *one parameter* and *two parameter* Mittag-Leffler functions put a coefficient in front of the k and 1 in the gamma function

$$\boxed{E_{\alpha}(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma\left(\alpha k+1\right)}, \quad \alpha > 0} \tag{2.9}$$

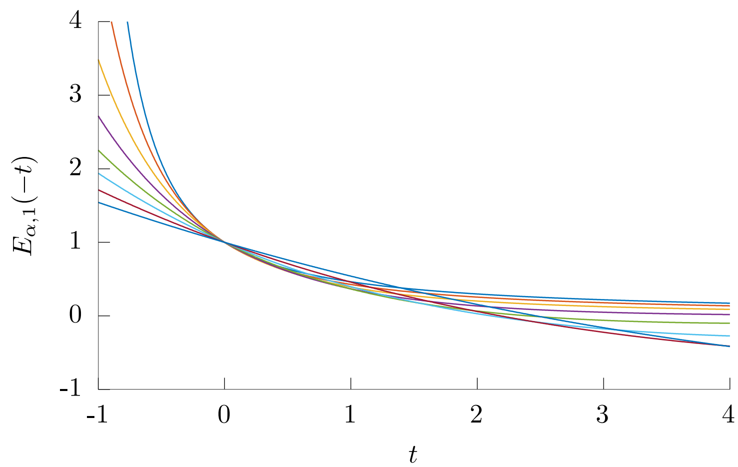

Figure 2.6: Mittag-Leffler functions, $E_{\alpha,1}(-t)$ for $\alpha = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75$ and 2. Looking at the left part of the plot near $t = -1$, $\alpha = 0.25$ is the top curve, and they are in order down to $\alpha = 2$ for the bottom curve.

and

$$\boxed{E_{\alpha,\beta}(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0.} \tag{2.10}$$

In order to gain some insight into these functions, let us see what the effect of varying the two parameters does. Figure 2.6 plots $E_{\alpha,t}(-t)$ for various values of α. Observe that for negative values, smaller α values are "stronger" whereas for positive values of t the opposite is basically the case. All of the curves go through the value of 1 at $t = 0$. The curves are more "curved" than the exponential for α values less than one, and less curved for α values greater than one.

Figure 2.7 illustrates $E_{1,\beta}(-t)$ for various β values. The trend to observe is that around time $t = 0$, the lowest curve corresponds to the smallest β values, and each subsequent curve increased from that one correspond to increasing β values.

There are certain combinations of α and β where $E_{\alpha,\beta}(t)$ is equal to a known function. Specifically

- $\boxed{E_{1,1}(t) = E_1(t) = e^t.}$
- $\boxed{E_{\frac{1}{2},1}(t) = E_{\frac{1}{2}}(t) = x^{t^2}\,\mathrm{erfc}(-t)}$
- $\boxed{E_{1,2}(t) = \dfrac{e^t - 1}{t}}$

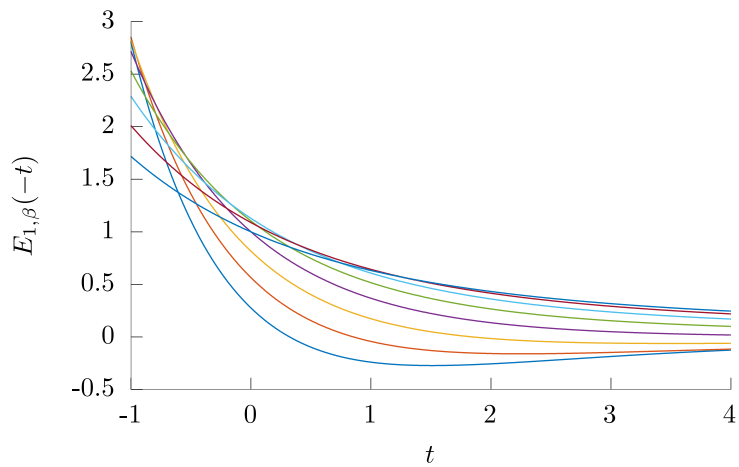

Figure 2.7: Mittag-Leffler functions, $E_{1,\beta}(-t)$ for $\beta = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75$ and 2. Near $t = 0$ the lowest curve is for $\beta = 0.25$ and increased values in β correspond to the curves above that in order.

- $\boxed{E_{2,1}(t^2)\cosh(t)}$
- $\boxed{E_{2,2}(t^2) = \dfrac{\sinh(t)}{t}.}$

2.2 Fractional Integration and Fractional Derivatives

From last chapter, it was obvious that one element to generalize from integer-order derivatives to allow for fractional or real-ordered derivatives, was the gamma function in cases where the only barrier to allowing a derivative to take real values was a factorial. This section covers a couple other similar tools that we will need shortly.

2.2.1 Fractional Integration

It turns out we will more easily find a general formula for a fractional number of integrations, as opposed to differentiation. That is no problem, though, because, for example, if we want the 1/3 derivative, we can integrate a function 2/3 times and then compute the integer-order first derivative of the result, the law of indices (through the Fundamental Theorem of Calculus) gives that the result what we want.

The following theorem contains what is commonly called *Cauchy's formula for repeated integration.*

Theorem 2.1. *Let $f(t)$ be continuous. Then the nth repeated integral of $f(t)$ is given by*

$$\begin{aligned} f^{(-n)}(t) &= \int_a^t \int_a^{\sigma_1} \int_a^{\sigma_2} \int_a^{\sigma_3} \cdots \int_a^{\sigma_{n-1}} f(\sigma_n) d\sigma_n d\sigma_{n-1} \cdots d\sigma_1 \\ &= \frac{1}{(n-1)!} \int_a^t (x-z)^{n-1} f(z)\, \mathrm{d}z. \end{aligned} \tag{2.11}$$

Proof. This is fairly apparent. In the case where n is an integer, integrate the right hand side by parts $n-1$ times to obtain the left hand side. □

This theorem should make some intuitive sense. If you had to evaluate the single integral, the way to to it would be to integrate by parts n times to eliminate the $(x-z)$ term, which would give the multiple integral form of it.

If we ask how can we integrate a function a fractional number of times, though, it is similar to what was done in the first chapter. If we have

$$f^{(-n)}(t) = \frac{1}{(n-1)!} \int_a^t (t-z)^{n-1} f(z)\, \mathrm{d}z$$

the only term containing the order of integration, n where n can not be a fraction is, again, the factorial. So we can just replace it with a gamma function

$$\boxed{{}_a\mathrm{D}_t^{(-\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-z)^{\alpha-1} f(z)\, \mathrm{d}z} \tag{2.12}$$

where the new notation for the operator D will be used going forward.

Because it is so common to have initial conditions specified at time $t = 0$, we will adopt the notation

$$\boxed{\mathrm{D}^{(-\alpha)}(t) = {}_0\mathrm{D}_t^{(-\alpha)}(t)}$$

i.e., we will not bother with adding to the notation when the limits of integration are from 0 to t.

Let us compute some fractional-order integrals of some common functions.

Example 2.3. *Consider $f(t) = t$. We know that $\mathrm{D}^{(-1)}\, t = 1/2t^2$. The half*

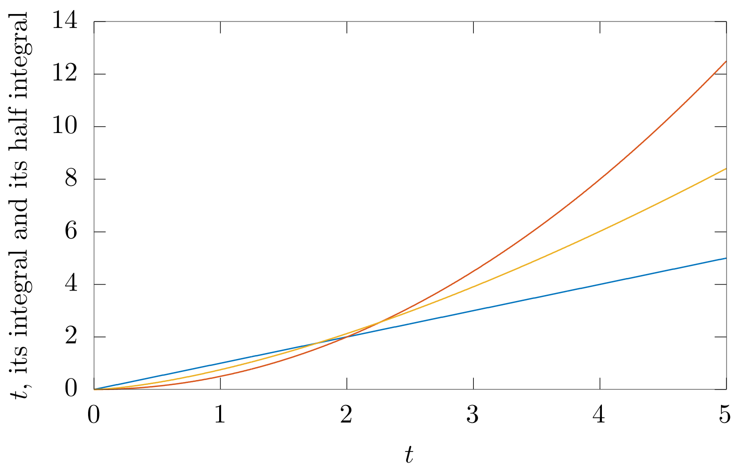

Figure 2.8: The function $f(t) = t$ (blue), ${}_0\mathrm{D}_t^{(-1)}\, t$ (red) and ${}_0\mathrm{D}_t^{(-1/2)}\, t$ (yellow).

integral should be something "in between" t and t^2. In detail

$$\begin{aligned}
\mathrm{D}^{-\frac{1}{2}}\, t &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-z)^{(1/2-1)}\, z \,\mathrm{d}z \\
&= -\frac{1}{\sqrt{\pi}} \int_t^0 \frac{t-u}{\sqrt{u}} \,\mathrm{d}u \qquad (u = t - z) \\
&= \frac{1}{\sqrt{\pi}} \int_0^t \frac{t}{\sqrt{u}} - \sqrt{u} \,\mathrm{d}u \\
&= \frac{1}{\sqrt{\pi}} \left[2tu^{\frac{1}{2}} - \frac{2}{3} u^{\frac{3}{2}} \right]_0^t \\
&= \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}}.
\end{aligned}$$

Figure 2.8 illustrates t, ${}_0\mathrm{D}_t^{(-1/2)}\, t$ and ${}_0\mathrm{D}_t^{(-1)}\, t$.

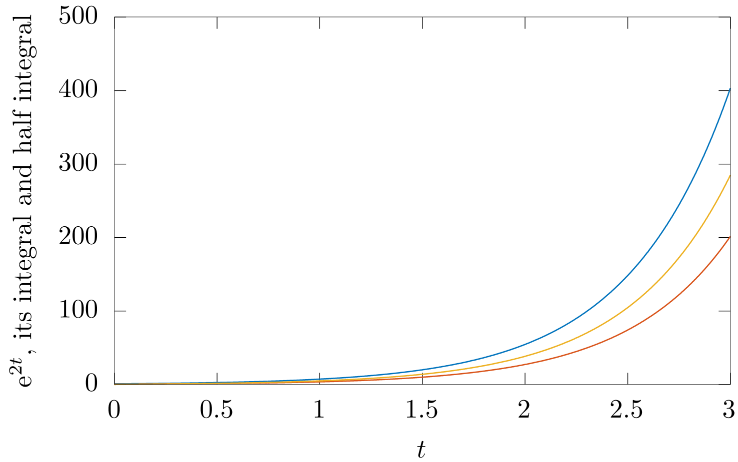

Figure 2.9: The function $f(t) = \mathrm{e}^{2t}$ and ${}_0\mathrm{D}_t^{(-1)}\,\mathrm{e}^{2t}$ (red) and ${}_0\mathrm{D}_t^{(-1/2)}\,\mathrm{e}^{2t}$ (yellow).

Example 2.4. *Consider $f(t) = \mathrm{e}^{2t}$. Then*

$$\begin{aligned}
\mathrm{D}^{(-1/2)}\,\mathrm{e}^{2t} &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-z)^{(-1/2)}\,\mathrm{e}^{2z}\,\mathrm{d}z \\
&= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \mathrm{e}^{2\left(t-u^2\right)}\,\mathrm{d}u \qquad \left(u = \sqrt{t-z}\right) \\
&= \mathrm{e}^{2t}\,\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \mathrm{e}^{\left(\sqrt{2}u\right)^2}\,\mathrm{d}u \\
&= \mathrm{e}^{2t}\,\frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\sqrt{2t}} \mathrm{e}^{-v^2}\,\mathrm{d}v \qquad \left(v = \sqrt{2}u\right) \\
&= \frac{1}{\sqrt{2}}\,\mathrm{e}^{2t}\,\mathrm{erf}\,\sqrt{2t}
\end{aligned}$$

Figures 2.9 and 2.10 illustrate e^{2t}, ${}_0\mathrm{D}_t^{(-1/2)}\,\mathrm{e}^{2t}$ and ${}_0\mathrm{D}_t^{(-1)}\,\mathrm{e}^{2t}$. Note that because of the square root, the error function part of the solution is not defined for negative values of t.

Example 2.5. *Consider $f(t) = \cos 3t$. We have*

$$\mathrm{D}^{-1/2}\cos 3t = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-z)^{(-1/2)} \cos(3z)\,\mathrm{d}z.$$

There is a closed-form solution to this integral in terms of the Fresnel integrals, but an easier check in this case is probably a numerical approach. Numerical

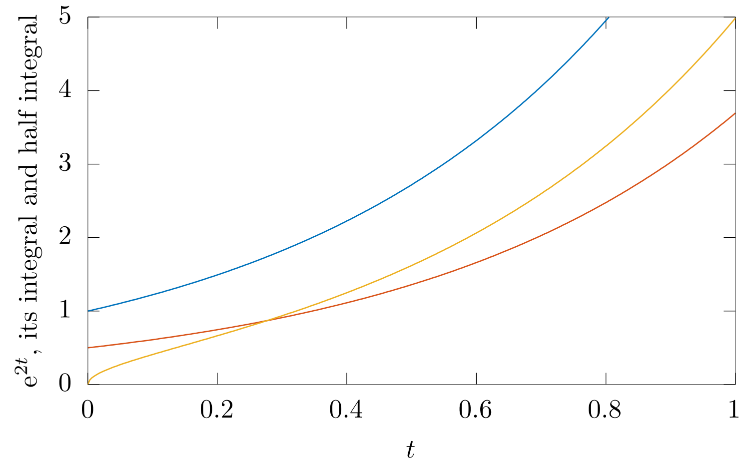

Figure 2.10: The function $f(t) = \mathrm{e}^{2t}$ and ${}_0\mathrm{D}_t^{(-1)}\,\mathrm{e}^{2t}$ (red) and ${}_0\mathrm{D}_t^{(-1/2)}\,\mathrm{e}^{2t}$ (yellow).

methods are considered in greater detail subsequently. For now, we can simply evaluate the integral using the `integrate()` *function in octave or Matlab. The plot of this integral is illustrated in Figure 2.11 and displays the characteristics one would expect of the half integral of the cosine function, including both the magnitude and the phase shift.*

Note that the initial part of the solution is not exactly a shifted cosine function, however. In fact, the fractional derivative is equal to zero at zero whereas a simple phase shift would result in a non-zero value. Specifically, if the fractional integral only shifted the function in phase and scaled the magnitude in a manner expected by the coefficient of t in the argument of the cosine function, we would have expected the value at zero to be equal to $2/3 * cos(\pi/4) = \sqrt{2}/3 \approx 0.4714$.

The octave code that generated this figure is:

```
t = linspace(0,10,1000);
soln = zeros(1,length(t));
f = @(z,t) ((t - z).^(-1/2)).*cos(3*z);
for i=1:length(t)
  soln(i) = 1/gamma(1/2)*integral(@(z) f(z,t(i)),0,t(i));
end
plot(t,cos(3*t),'linewidth',2);
hold on;
plot(t,1/3*sin(3*t),'linewidth',2);
plot(t,soln,'linewidth',2);
```

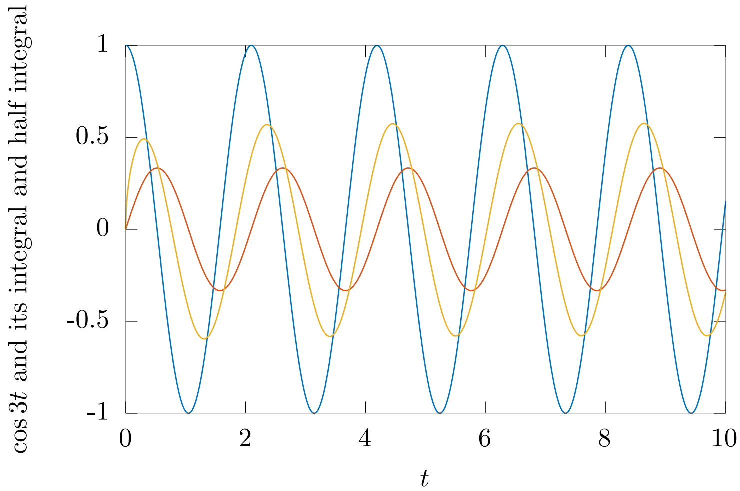

Figure 2.11: The function $f(t) = \cos 3t$ and ${}_0\mathrm{D}_t^{(-1)} \cos 3t$ (red) and ${}_0\mathrm{D}_t^{(-1/2)} \cos 3t$ (yellow).

2.2.2 Fractional Derivative Definitions

We will consider two basic approaches to generalizing the derivative to fractional orders. The first will use the fractional integration idea above along with the fundamental theorem of calculus. The second is an extension of the usual limit definition of the derivative that has a nice extension to the finite difference method in numerical methods.

At this point we have the ability to integrate a function by a fractional amount, *e.g.*, $\mathrm{D}^{(-1/2)} f(t)$ is the one-half integral of f. The basic idea is that we can use the fact that integrals and derivatives are inverse operations and, for example, integrate a function by, say, $2/3$ and then differentiate once to get the $1/3$ derivative. The difference between the first two definitions we will consider is simply whether we differentiate first and then fractionally integrate, or vice versa. That seemingly small difference actually has large consequences.

Riemann-Liouville Fractional Derivative

For the Riemann-Liouville derivative definition, we integrate first and then differentiate, specifically

$$\begin{aligned} {}^{RL}_{\ 0}\mathrm{D}_t^{\alpha} f(t) &= {}^{RL}\mathrm{D}^{\alpha} f(t) \\ &= \frac{\mathrm{d}^{\lceil \alpha \rceil}}{\mathrm{d}t^{\lceil \alpha \rceil}} \mathrm{D}^{-(\lceil \alpha \rceil - \alpha)} f(t) \\ &= \frac{\mathrm{d}^{\lceil \alpha \rceil}}{\mathrm{d}t^{\lceil \alpha \rceil}} \left(\frac{1}{\Gamma\left(\lceil \alpha \rceil - \alpha\right)} \int_0^t (t-z)^{\lceil \alpha \rceil - \alpha - 1} f(z)\,\mathrm{d}z \right). \end{aligned} \tag{2.13}$$

Remark 2.1. *Note that the nonlocal nature of a fractional derivative is apparent* *from this definition. Information from the function evaluated over the entire range of the integral will effect the fractional derivative.*

Example 2.6. *Consider $f(t) = t$ and assume we want to compute ${}^{RL}\mathrm{D}^{1/2}\, t$. So for this problem, $\alpha = 1/2$ and $\lceil\alpha\rceil = 1$. In Example 2.3 we computed $\mathrm{D}^{(-1/2)}\, t = 4/\left(3\sqrt{\pi}\right) t^{3/2}$. So we have from Equation 2.13*

$$\begin{aligned}
{}^{RL}\mathrm{D}^{1/2}\, t &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma\left(1/2\right)}\,\mathrm{D}^{-(1-1/2)}\, t\right) \\
&= \frac{4}{3\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}t} t^{\frac{3}{2}} \\
&= \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}.
\end{aligned}$$

Note that this is the same as the $1/2$ derivative that we could compute from Definition 1.1.

We will do a slightly different example, because its fractional derivative will be different from what we will get with the next definition.

Example 2.7. *Add one to the function from the previous example:*

$$\begin{aligned}
{}^{RL}\mathrm{D}^{1/2}\left(t+1\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma\left(1/2\right)}\,\mathrm{D}^{-(1-1/2)}\left(t+1\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} + \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}\right) \\
&= \frac{2}{\sqrt{\pi}}\sqrt{t} + \frac{1}{\sqrt{\pi}\sqrt{t}}.
\end{aligned}$$

See Example 2.8 for the substitution to integrate the second term in the integral.

In integer-order calculus, adding a constant to a function has no effect on its derivative. However, for fractional derivatives, this is not the case. The second term came directly from the added constant in $f(t)$.

Caputo Fractional Derivative

The Caputo definition simply switches the order of fractional-order integration and integer-order differentiation

$$\begin{aligned}
{}^{C}_{0}\mathrm{D}^{\alpha}_{t}\, f(t) &= {}^{C}\mathrm{D}^{\alpha}\, f(t) \\
&= \mathrm{D}^{-(\lceil\alpha\rceil-\alpha)}\left(\frac{\mathrm{d}^{\lceil\alpha\rceil} f}{\mathrm{d}t^{\lceil\alpha\rceil}}(t)\right) \\
&= \frac{1}{\Gamma\left(\lceil\alpha\rceil-\alpha\right)}\int_0^t \left(t-z\right)^{\lceil\alpha\rceil-\alpha-1}\frac{\mathrm{d}^{\lceil\alpha\rceil} f}{\mathrm{d}z^{\lceil\alpha\rceil}}(z)\,\mathrm{d}z.
\end{aligned} \tag{2.14}$$

Example 2.8. *Let us repeat Example 2.6 using the Caputo definition:*

$$\begin{aligned}
{}^{C}\mathrm{D}^{1/2}\, t &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-z)^{-\frac{1}{2}} \frac{\mathrm{d}z}{\mathrm{d}z}\, \mathrm{d}z \\
&= \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-z}}\, \mathrm{d}z \\
&= \frac{1}{\sqrt{\pi}} \int_t^0 -\frac{1}{\sqrt{u}}\, \mathrm{d}u \qquad (u = t - z) \\
&= \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{u}}\, \mathrm{d}u \\
&= \frac{1}{\sqrt{\pi}} \left[2\sqrt{u}\right]_0^t \\
&= \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}.
\end{aligned}$$

This is the same as the Riemann-Liouville definition.

Example 2.9. *Let us repeat Example 2.7 using the Caputo definition:*

$$\begin{aligned}
{}^{C}\mathrm{D}^{1/2}\, t &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-z)^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}z} (z+1)\, \mathrm{d}z \\
&= \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-z}}\, \mathrm{d}z \\
&= \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}.
\end{aligned}$$

This is not *the same as the Riemann-Liouville definition! It is due to the fact that the constant term in the function was eliminated by differentiating first using the Caputo definition, but was retained when it was integrated first using the Riemann-Liouville definition. Figure 2.12 illustrates the difference between these two fractional derivatives.*

Grünwald-Letnikov Fractional Derivative

The third definition of a fractional derivative we will consider is appealing for two reasons. First, it is a limit definition, which corresponds in a sense to our usual consideration of an integer-order derivative. Second, because it is a limit as $\Delta t \to 0$, we can directly adopt it in numerical methods by eliminating the limit and simply taking a small Δt. To understand the basis for the definition, we will do what we have done a lot so far, which is look for a pattern.

Consider the usual definition of the first derivative of the function $f(t)$ at time t

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \lim_{\Delta t \to 0} \frac{f(t) - f(t - \Delta t)}{\Delta t}$$

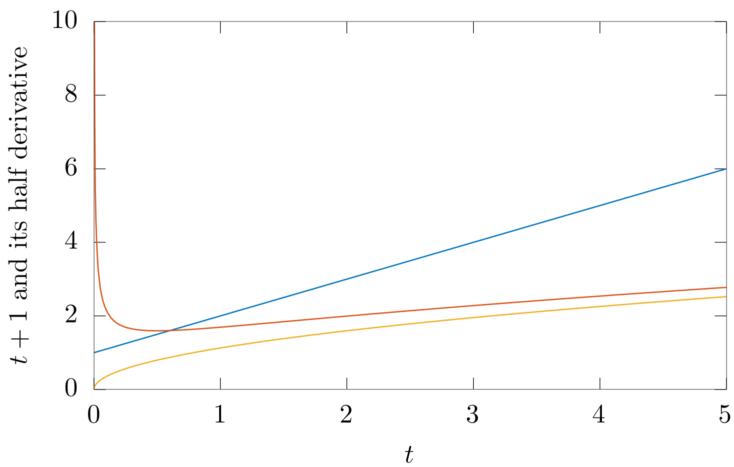

Figure 2.12: The function $f(t) = t+1$ (blue), the 1/2 derivative computed using the Riemann-Liouville definition (red) and the 1/2 derivative computed using the Caputo definition (yellow).

and the second derivative

$$\begin{aligned}\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}(t) &= \lim_{\Delta t \to 0} \left(\frac{\frac{f(t)-f(t-\Delta t)}{\Delta t} - \frac{f(t-\Delta t)-f(t-2\Delta t)}{\Delta t}}{\Delta t} \right) \\ &= \lim_{\Delta t \to 0} \left(\frac{f(t) - 2f(t-\Delta t) + f(t-2\Delta t)}{(\Delta t)^2} \right).\end{aligned}$$

Continuing to compute higher-order derivatives makes it easy to see the pattern that gives

$$\frac{\mathrm{d}^n f}{\mathrm{d}t^n}(t) = \lim_{\Delta t \to 0} \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t-k\Delta t)}{(\Delta t)^n}$$

where $t = n\Delta t$. We have the usual business of factorials inside the binomial coefficient, which we have already generalized in Equation 2.5, which gives us the following definition.

Definition 2.1. *The Grünwald-Letnikov fractional derivative is given by*

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}(t) = \lim_{\Delta t \to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-k\Delta t)}{(\Delta t)^\alpha}. \tag{2.15}$$

Note that the sum goes back over all values of $f(t)$ from t to $t = -\infty$, i.e., all of the history of $f(t)$ contributes to the definition.

If all values of $f(t)$ are zero for $t < 0$, then we can write

$$\boxed{\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}(t) = \lim_{\Delta t \to 0} \frac{\sum_{k=0}^{\left\lfloor \frac{t}{\Delta t} \right\rfloor} (-1)^k \binom{\alpha}{k} f(t - k\Delta t)}{(\Delta t)^\alpha}} \tag{2.16}$$

so the sum only goes back over values of $f(t)$ to zero. In the limit the sum will still contain an infinite number of terms, but those infinite number of evaluations of $f(t)$ will only be between the current t and 0.

Note that the nonlocal nature of the fractional-derivative is also present in this definition because the sum incorporates values of the function back to $t = 0$, regardless of the order. In cases where the order is an integer order, then many of the binomial coefficients will be zero, making the definition local. In the limit, fractional derivatives will contain an infinite number of terms in the summation, much like the integrals do for the Riemann-Liouville and Caputo definitions.

We will use this definition frequently because taking a small Δt should give us decent numerical approximations to fractional derivatives, and, as we will see shorty, allow us to compute numerical solutions to fractional-order differential equations.

It may seem like this definition is the best because it will alleviate the need to compute a lot of integrals that may not have solutions in terms of simple functions. However, there is a hidden complication. Note that k, the index of the summation, will become increasingly large as t gets large, and will generally be large if Δt is small, which we want for good numerical accuracy. However, referring back to the generalization of the binomial coefficient given by Equation 2.5, the denominator will contain two gamma functions with possibly large arguments. Recall that the gamma function can be thought of as the generalization of the factorial, so this will grow very large very quickly. Numerical issues arise surprisingly quickly.

Unless we specify to the contrary, in this course we will assume everything has a value of zero prior to $t = 0$ and hence use the definition in Equation 2.16.

Example 2.10. *Use the Grünwald-Letnikov definition to compute a numerical approximation for the $1/2$ derivative of*

$$f(t) = \begin{cases} 0, & t < 0 \\ \cos 3t, & t \geq 0 \end{cases} \tag{2.17}$$

for $t \in (0, 10]$.

The period of oscillation of $f(t)$ is approximately 2, so if we take $\Delta t = 1/100$, each period should contain approximately 200 data points, which, as a first guess, should give a reasonable approximation. So we have

$$\frac{\mathrm{d}^{\frac{1}{2}} f}{\mathrm{d}t^{\frac{1}{2}}}(t) \approx \sum_{k=0}^{\lfloor 100t \rfloor} \frac{(-1)^k \binom{\frac{1}{2}}{k} \cos 3\left(t - \frac{k}{100}\right)}{\frac{1}{100}}$$

The octave code that produced this result is

```
t = linspace(0,10,1001);
dt = t(2)-t(1);
alpha = 1.1;
deriv = 0;
f = cos(3*t);
coefs = 0;
coefs(1) = bincoeff(alpha,0);
deriv(1) = 0;
for n = 2:length(t)
   coefs(n) = (-1)^(n-1)*bincoeff(alpha,(n-1));
   sum = dot(fliplr(f(1:n)),coefs)/dt^alpha;
   deriv(n) = sum;
end
plot(t,f,'linewidth',2);
hold on;
plot(t,-3*sin(3*t),'linewidth',2);
plot(t,deriv,'linewidth',2);
xlabel('$t$');
ylabel('half derivative of $\cos 3 t$');
```

2.3 Operational Calculus

This section deals with operational calculus, specifically oriented towards fractional cases, which also includes the more common notions of Laplace transforms.

2.3.1 History and Basic Ideas

2.3.2 Rigorous Operational Calculus

2.3.3 Laplace Transforms

In many ways, Laplace transforms will be one of the more powerful tools. First we will consider the Laplace transform of some of the functions described in the previous sections, and it will be apparent that they are related to some fractional powers of s. Then we will consider fractional differentiation and integration in the Laplace transform context.

First, though, recall the definition of the Laplace transform

$$\boxed{F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} f(t)\,\mathrm{e}^{-st}\,dt} \tag{2.18}$$

and the inverse

$$f(t) = \frac{1}{2\pi\imath} \lim_{b\to\infty} \int_{a-\imath b}^{a+\imath b} F(s)e^{st} ds. \tag{2.19}$$

2.3.4 Laplace Transform Pairs

A list of the Laplace transform of some functions appears in Table 2.2. We will work out some of the entries and leave the others as exercises.

First, consider $f(t) = t^n$ where n is a positive integer. Using the definition and integrating by parts gives

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \int_0^\infty t^n \, \mathrm{e}^{-st} \, dt \\
&= \left[t^n \, \mathrm{e}^{-st}\right]_{0^-}^\infty + \frac{n}{s} \int_{0^-}^\infty t^{n-1} \, \mathrm{e}^{-st} \, dt \\
&= \frac{n}{s} \int_{0^-}^\infty t^{n-1} \, \mathrm{e}^{-st} \, dt \\
&= \frac{n}{s} \left(\left[t^{n-1} \, \mathrm{e}^{-st}\right]_{0^-}^\infty + \frac{n-1}{s} \int_{0^-}^\infty t^{n-1} \, \mathrm{e}^{-st} \, dt \right) \\
&= \vdots \\
&= \frac{n!}{s^{n+1}}.
\end{aligned}$$

From this, we have a pretty clear generalization for $f(t) = t^\alpha$ where α is not necessarily an integer, specifically

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

This leads easily to a few special cases for specific values of α:

- $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}}$

- $\mathcal{L}\left\{\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$

	$f(t)$	$F(s)$
1	1	$\frac{1}{s}$
2	$t^n, \quad n \in \mathbb{Z}$	$\frac{n!}{s^{n+1}}$
3	$t^\alpha, \quad \alpha \in \mathbb{R}$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$
4	$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
5	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
6	$t^{\beta-1}\,\mathrm{E}_{\alpha,\beta}\left(\pm a t^\alpha\right)$	$\frac{s^{\alpha-\beta}}{s^\alpha \mp a}$
7	$\frac{1}{\sqrt{t}}\,\mathrm{E}_{\frac{1}{2},\frac{1}{2}}\left(\pm a\sqrt{t}\right)$	$\frac{1}{\sqrt{s} \mp a}$
8	$\mathrm{E}_\alpha\left(\pm a t^\alpha\right)$	$\frac{s^\alpha}{s(s^\alpha \mp a)}$

Table 2.2: Laplace Transform Pairs.

Let us consider the Laplace transform of Mittag-Leffler functions:

$$\begin{aligned}
\mathcal{L}\left\{t^{\beta-1}\,\mathrm{E}_{\alpha,\beta}\left(a t^\alpha\right)\right\} &= \int_{0^-}^{\infty} t^{\beta-1}\,\mathrm{E}_{\alpha,\beta}\left(a t^\alpha\right) \mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_{o^-}^{\infty} t^{\beta-1} \sum_{k=0}^{\infty} \frac{\left(a t^\alpha\right)^k}{\Gamma\left(\alpha k+\beta\right)} \mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_{o^-}^{\infty} \sum_{k=0}^{\infty} \frac{a^k t^{\alpha k+\beta-1}}{\Gamma\left(\alpha k+\beta\right)} \mathrm{e}^{-st}\,\mathrm{d}t \\
&= \sum_{k=0}^{\infty} \frac{a^k}{\Gamma\left(\alpha k+\beta\right)} \int_{o^-}^{\infty} t^{\alpha k+\beta-1}\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \sum_{k=0}^{\infty} \frac{a^k}{\Gamma\left(\alpha k+\beta\right)} \frac{\Gamma\left(\alpha k+\beta\right)}{s^{\alpha k+\beta}} \\
&= \frac{1}{s^\beta} \sum_{k=0}^{\infty} \left(\frac{a}{s^\alpha}\right)^k \\
&= \frac{1}{s^\beta} \frac{1}{1-\frac{a}{s^\alpha}} \\
&= \frac{s^{\alpha-\beta}}{s^\alpha - a}.
\end{aligned}$$

Remark 2.2.

1. *Note that convergence properties are needed in order to switch the order of integration and the sum. While the conditions are met to make this valid, we have not shown them here.*

2. *Recall the basic geometric series property for the last step*

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

for $|r| < 1$.

A few special cases that may arise frequently are also in the table.

2.3.5 Integration and Differentiation in the Frequency Domain

Now we will consider calculus in the frequency domain. First, recall the basic property of Laplace transforms that multiplication in the time domain is equal to convolution in the frequency domain, specifically

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t f_1(t-\tau)f_2(\tau)\,\mathrm{d}\tau\right\} &= \int_{0^-}^{\infty}\int_0^t f_1(t-\tau)f_2(\tau)\,\mathrm{d}\tau\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_{0^-}^{\infty}\int_0^{\infty} f_1(t-\tau)\mathbb{1}(t-\tau)f_2(\tau)\,\mathrm{d}\tau\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_0^{\infty}\int_{0^-}^{\infty} f_1(t-\tau)\mathbb{1}(t-\tau)f_2(\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \int_0^{\infty}\left(\int_{0^-}^{\infty} f_1(t-\tau)\mathbb{1}(t-\tau)\,\mathrm{e}^{-st}\,\mathrm{d}t\right)f_2(\tau)\,\mathrm{d}\tau \\
&= \int_0^{\infty}\left(\int_{-\tau}^{\infty} f_1(\lambda)\mathbb{1}(\lambda)\,\mathrm{e}^{-s(\lambda+\tau)}\,\mathrm{d}\lambda\right)f_2(\tau)\,\mathrm{d}\tau \\
&= \int_0^{\infty}\left(\int_0^{\infty} f_1(\lambda)\,\mathrm{e}^{-s\lambda}\,\mathrm{d}\lambda\right)\mathrm{e}^{-s\tau}\,f_2(\tau)\,\mathrm{d}\tau \\
&= \left(\int_0^{\infty} f_1(\lambda)\,\mathrm{e}^{-s\lambda}\,\mathrm{d}\lambda\right)\left(\int_0^{\infty}\mathrm{e}^{-s\tau}\,f_2(\tau)\,\mathrm{d}\tau\right) \\
&= F_1(s)F_2(s).
\end{aligned}$$

This is a fundamental fact important outside the fractional context. We will use it to derive the formulas we want; namely, that division by s^α corresponds to the fractional-order integral operation and correspondingly for multiplication and differentiation.

Consider the Laplace transform of the fractional-order integral given in Equation 2.12

$$\begin{aligned}
\mathcal{L}\left\{{}_0\mathrm{D}_t^{-\alpha}f(t)\right\} &= \int_0^{\infty}\frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1}f(z)\,\mathrm{d}z\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)}\mathcal{L}\left\{t^{\alpha-1}\right\}\mathcal{L}\left\{f(t)\right\} \\
&= \frac{1}{\Gamma(\alpha)}\left(\frac{\Gamma(\alpha-1+1)}{s^{\alpha-1+1}}\right)(F(s)) \\
&= \frac{1}{s^\alpha}F(s)
\end{aligned}$$

which is exactly what we would expect. Fractional integration of order α in the frequency domain is division by s^α. It is important enough to put it in a box and give it a number:

$$\boxed{\mathcal{L}\left\{\mathrm{D}^{(-\alpha)}\, f(t)\right\} = \frac{1}{s^{\alpha}} F(s), \qquad \alpha > 0.} \tag{2.20}$$

For fractional differentiation in the frequency domain, we will get a different answer for the Riemann-Liouville and Caputo definitions. For the Riemann-Liouville fractional derivative we have

$$\begin{aligned}
\mathcal{L}\left\{ {}^{RL}\mathrm{D}^{\alpha}\, f(t)\right\} &= \mathcal{L}\left\{\frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}}\left(\mathrm{D}^{-(\lceil\alpha\rceil-\alpha)}\, f(t)\right)\right\} \\
&= s^{\lceil\alpha\rceil}\left(\frac{F(s)}{s^{\lceil\alpha\rceil-\alpha}}\right) - s^{\lceil\alpha\rceil-1}\left[\mathrm{D}^{-(\lceil\alpha\rceil-\alpha)}\, f(t)\right]_{t=0} \\
&\quad - s^{\lceil\alpha\rceil-2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{D}^{-(\lceil\alpha\rceil-1)}\, f(t)\right)\right]_{t=0} \\
&\quad - \cdots - \left[\frac{\mathrm{d}^{\lceil\alpha\rceil-1}}{\mathrm{d}t^{\lceil\alpha\rceil-1}}\left(\mathrm{D}^{-(\lceil\alpha\rceil-1)}\, f(t)\right)\right]_{t=0}.
\end{aligned}$$

Note that the first term simplifies to $s^{\alpha}F(s)$ just like we want. However, the initial conditions involve fractional integrals of $f(t)$, which may (or may not!) be convenient for a given problem. In the usual situation in controls where we assume that all initial conditions, including fractional initial conditions, are zero, we have

$$\mathcal{L}\left\{ {}^{RL}\mathrm{D}^{\alpha}\, f(t)\right\} = s^{\alpha}F(s).$$

For the Caputo derivative, we have

$$\begin{aligned}
\mathcal{L}\left\{ {}^{C}\mathrm{D}^{\alpha}\, f(t)\right\} &= \mathcal{L}\left\{\mathrm{D}^{-(\lceil\alpha\rceil-\alpha)}\left(\frac{\mathrm{d}^{\lceil\alpha\rceil}f}{\mathrm{d}t^{\lceil\alpha\rceil}}(t)\right)\right\} \\
&= \mathcal{L}\left\{\frac{1}{\Gamma\left(\lceil\alpha\rceil-\alpha\right)}\int_0^t (t-z)^{\lceil\alpha\rceil-\alpha-1}\,\frac{\mathrm{d}^{\lceil\alpha\rceil}f}{\mathrm{d}z^{\lceil\alpha\rceil}}(z)\,\mathrm{d}z\right\} \\
&= \frac{1}{\Gamma\left(\lceil\alpha\rceil-\alpha\right)}\mathcal{L}\left\{(t-z)^{\lceil\alpha\rceil-\alpha-1}\right\}\mathcal{L}\left\{\frac{\mathrm{d}^{\lceil\alpha\rceil}f}{\mathrm{d}t^{\lceil\alpha\rceil}}(t)\right\} \\
&= \frac{1}{s^{\lceil\alpha\rceil-\alpha}}\left(s^{\lceil\alpha\rceil}F(s) - s^{\lceil\alpha\rceil-1}f(0) - s^{\lceil\alpha\rceil-2}\frac{\mathrm{d}f}{\mathrm{d}t}(0) - \right. \\
&\quad \left. \cdots - \frac{\mathrm{d}^{\lceil\alpha\rceil-1}f}{\mathrm{d}t^{\lceil\alpha\rceil-1}}(0)\right)
\end{aligned}$$

Note that the Laplace transform of the Caputo derivative involves evaluating *integer-order* initial conditions, in contrast to the fractional-order initial conditions from the Riemann-Liouville derivative.

Chapter 3

Fractional-Order Differential Equations

Example 3.1. *Determine the solution to*

$$\frac{\mathrm{d}^{\frac{1}{2}}x}{\mathrm{d}t^{\frac{1}{2}}}(t) + ax(t) = f(t)$$

where we use the Riemann-Liouville fractional derivative, or

$$^{RL}\mathrm{D}^{\frac{1}{2}}\,x(t) + ax(t) = f(t). \tag{3.1}$$

Computing the Laplace transform of each side of the equation gives

$$s^{\frac{1}{2}}X(s) - \left[\mathrm{D}^{-\frac{1}{2}}\,x(t)\right]_{t=0} + aX(s) = F(s).$$

For the time being, let us assume that the initial condition term is not zero, and call it c. Solving for $X(s)$ gives

$$X(s) = \frac{c}{\sqrt{s}+a} + \frac{F(s)}{\sqrt{s}+a}.$$

If $f(t) = 0$ and $c = 1$, then

$$X(s) = \frac{1}{\sqrt{s}+a}$$

and the Laplace transform table gives that

$$x(t) = \frac{1}{\sqrt{t}}\,\mathrm{E}_{\frac{1}{2},\frac{1}{2}}\left(a\sqrt{t}\right).$$

The solution to this equation is illustrated in Figure 3.1. Note for increasing a the solution decays more rapidly. A graph of $x(t) = \mathrm{e}^{-2t}$ is also shown for comparison.

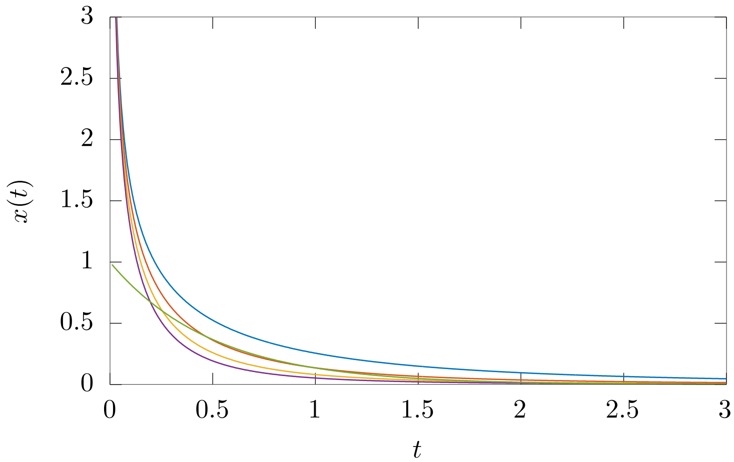

Figure 3.1: Solutions to Equation 3.1 for various $a = 1/2$ (blue), $a = 1$ (red), $a = 3.2$ gold and $a = 2$ (purple) and $f(t) = 0$ and $x(t) = e^{-2t}$ (green) for comparison.

Now, assume that $f(t) = 1$, so that $F(s) = 1/s$, in which case

$$X(s) = \frac{1}{\sqrt{s} + a} + \frac{1}{s\left(\sqrt{s} + a\right)}. \tag{3.2}$$

From Table 2.2, for the second term, we need that $\alpha = 1/2$ and in order to get the other s term in the denominator, we need $\beta - \alpha = 1$, so $\beta = 3/2$, which gives

$$x(t) = \frac{1}{\sqrt{t}} \mathrm{E}_{\frac{1}{2},\frac{1}{2}}\left(a\sqrt{t}\right) + \sqrt{t}\,\mathrm{E}_{\frac{1}{2},\frac{3}{2}}\left(a\sqrt{t}\right).$$

Figure 3.2 illustrates the solutions when $c = 0$, i.e., it is the "step response" portion of the solution. Figure 3.3 illustrates the full solution including the term with $c = 1$.

Example 3.2. *As a second example, consider a mass attached to a wall and subjected to a force as illustrated in Figure 3.4. We will consider the attachment to the wall as having some sort of mechanical impendence, which could be a spring, or a damper or perhaps a fractional-order type network such as the 1/2-order tree network of springs and dampers considered earlier in Section 1.5.1. The top part of the figure illustrates the half-order connection, and the bottom part represents the more general situation where the order of the relationship between the force resisting the motion and the displacement is of order γ.*

Newton's law on the mass gives

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + q\frac{\mathrm{d}^\gamma x}{\mathrm{d}t^\gamma}(t) = f(t).$$

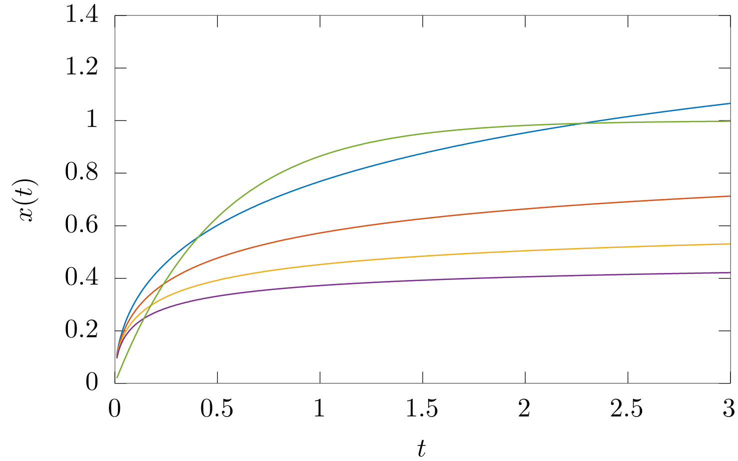

Figure 3.2: Solutions to Equation 3.1 for various $a = 1/2$ (blue), $a = 1$ (red), $a = 3.2$ gold and $a = 2$ (purple), $f(t) = 1$ and $c = 0$ and $x(t) = 1 - e^{-2t}$ (green) for comparison.

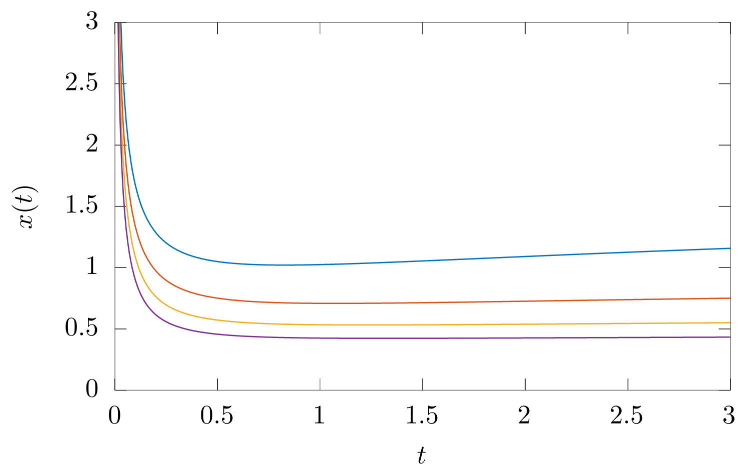

Figure 3.3: Solutions to Equation 3.1 for various $a = 1/2$ (blue), $a = 1$ (red), $a = 3.2$ gold and $a = 2$ (purple), $f(t) = 1$ and $c = 1$.

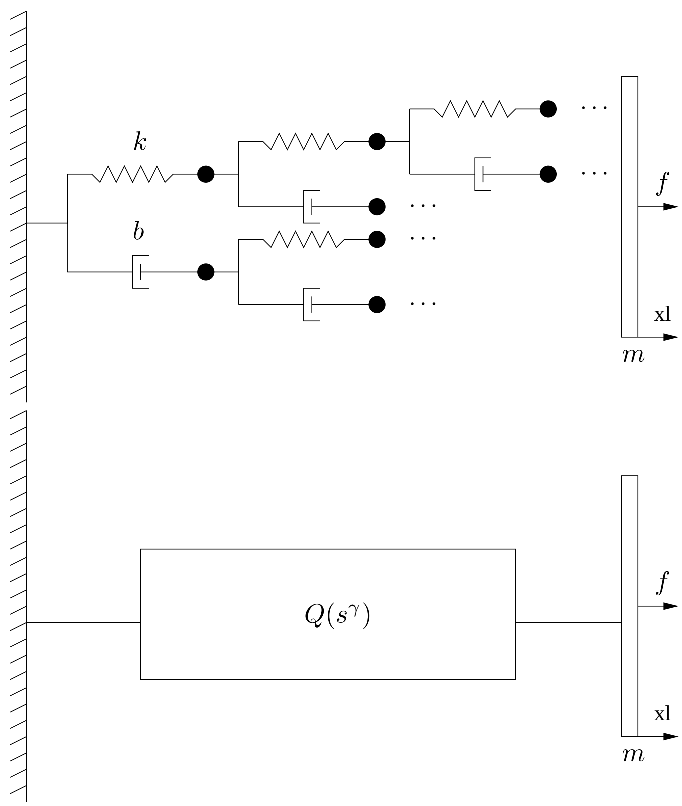

Figure 3.4: Mass attached to a wall.

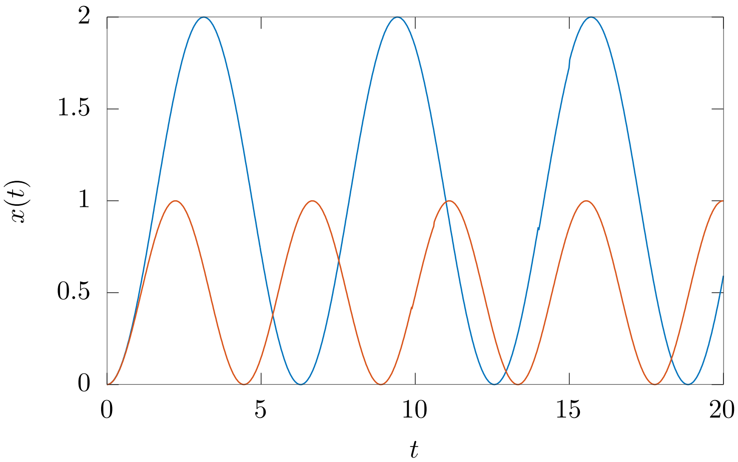

Figure 3.5: Solution to Equation 3.3 in the case where $\gamma = 0$, corresponding to a spring element connecting the mass to the wall. The blue curve is for $q/m = 1$ and the red curve is for $q/m = 2$.

In the case where the connection is a spring, we would use $q = k$ *and* $\gamma = 0$*, and when it is a damper, we would use* $q = b$ *and* $\gamma = 1$*. Divide both sides by* m*, let* $f(t)$ *be a unit step input, assume zero initial conditions of all orders and take the Laplace transform of both sides, which gives*

$$X(s) = \frac{1}{s\left(s^2 + \frac{q}{m}s^{\gamma}\right)} = \frac{1}{s^{\gamma+1}\left(s^{2-\gamma} + \frac{q}{m}\right)}. \tag{3.3}$$

Referring to Table 2.2, $\alpha = 2 - \gamma$ *and* $\alpha - \beta = -(\gamma + 1)$*, so* $\beta = 3$*, and the time domain response is*

$$x(t) = t^2 \, \mathrm{E}_{2-\gamma,3}\left(-\frac{q}{m}t^{2-\gamma}\right).$$

To validate this, let us plot the solution for $\gamma = 0$*, which would correspond to the connection being a spring, in which case we expect a purly oscillatory response. Additionally, if we increase* q/m *we expect a higher frequency response. Both of these attributes of the solution are illustrated in Figure 3.5. In the case where* $\gamma = 1$*, the connection to the wall is a damper, in which case we would expect the steady-state solution to be a constant velocity where the damper force and applied force are in equilibrium. This case is illustrated in Figure 3.6.*

Finally, the solutions for $\gamma \in \{0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75\}$ *are illustrated in Figure 3.7. We will revisit this problem in the next chapter when we consider numerical methods.*

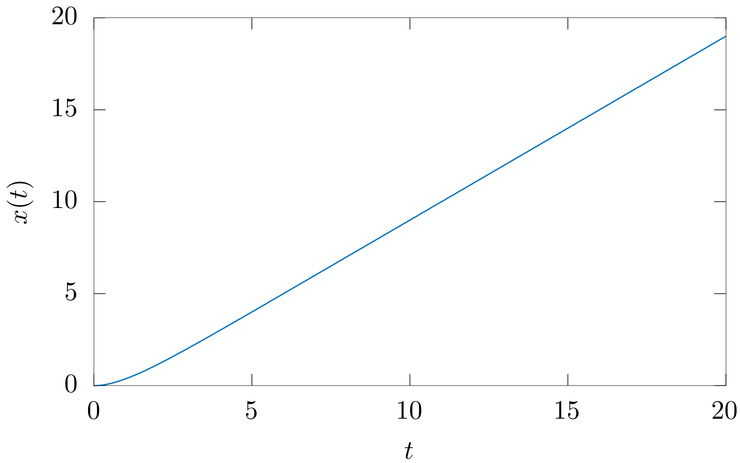

Figure 3.6: Solution to Equation 3.3 in the case where $\gamma = 1$, corresponding to a damper element connecting the mass to the wall.

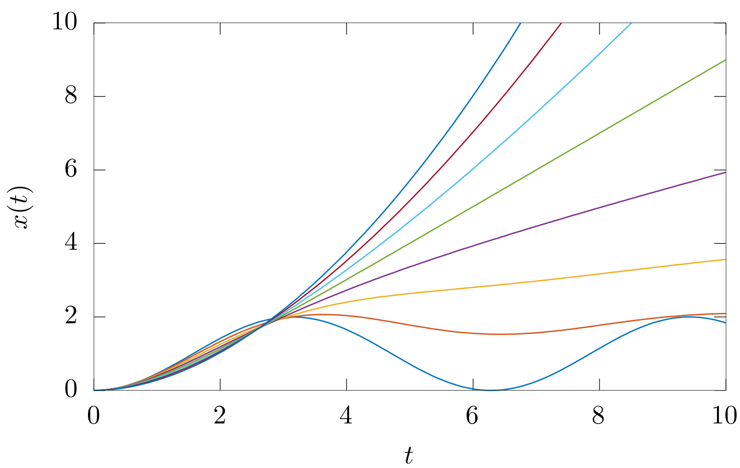

Figure 3.7: Solution to Equation 3.3 for $\gamma = 0, 0.25, 0.5, \ldots, 1.75$.

Chapter 4

Numerical Methods for Fractional-Order Differential Equations

Numerical methods are challenging for fractional-order systems because of the non-locality of the fractional derivative.

4.1 Direct Application of the Grünwald-Letnikov Fractional Derivative

Because the Grünwald-Letnikov definition of the fractional derivative contains a limit, we can use it by taking the limiting term to be small as a computational approximation of the fractional derivative. The approach itself is fairly straightforward, but numerical issues arise fairly quickly. We will illustrate both the approach and its limitations with an example.

Recall that the Grünwald-Letnikov definition of the fractional derivative is

$$\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}t^{\alpha}}(t) = \lim_{\Delta t \to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t - k\Delta t)}{(\Delta t)^{\alpha}}$$

where the binomial coefficient is generalized to non-integer arguments as

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)}.$$

In the case where all initial conditions of all orders are zero, the upper limit of the summation can be changed

$$\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}t^{\alpha}}(t) = \lim_{\Delta t \to 0} \frac{\sum_{k=0}^{\left\lfloor \frac{t}{\Delta t} \right\rfloor} (-1)^k \binom{\alpha}{k} f(t - k\Delta t)}{(\Delta t)^{\alpha}}.$$

This, of course, leads to the approximation

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}(t) \approx \frac{\sum_{k=0}^{\lfloor \frac{t}{\Delta t} \rfloor} (-1)^k \binom{\alpha}{k} f\left(t - k\Delta t\right)}{\left(\Delta t\right)^\alpha}, \qquad \Delta t \ll 1.$$

We will return to an example from the Chapter 3 to implement this.

Example 4.1. *Solve*

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + q\frac{\mathrm{d}^\gamma x}{\mathrm{d}t^\gamma}(t) = f(t)$$

assuming all zero initial conditions. Because this is a numerical method, we need to choose numerical values for all the paramters, so let $m = q = 1$ *and* $\gamma = 1/2$ *and let* $f(t)$ *be a unit step input:*

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + \frac{\mathrm{d}^{\frac{1}{2}} x}{\mathrm{d}t^{\frac{1}{2}}} = 1.$$

At this point we can take one of two approaches. The first is to convert this second-order differential equation into two first-order equations and use Euler's method to compute an approximate numerical solution. So, let $x_1 = x$ *and* $x_2 = \dot{x}$ *so that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 1 - \frac{\mathrm{d}^{\frac{1}{2}} x_1}{\mathrm{d}t^{\frac{1}{2}}} \end{bmatrix}$$

which gives the approximation using Euler's method

$$\begin{bmatrix} x_1\left(t + \Delta t\right) \\ x_2\left(t + \Delta t\right) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} x_2(t) \\ 1 - \sum_{k=0}^{\lfloor \frac{t}{\Delta t} \rfloor} \frac{(-1)^k \binom{\gamma}{k} x_1(t - k\Delta t)}{(\Delta t)^\gamma} \end{bmatrix} \Delta t.$$

The second, which we will adopt, is to express the second derivative using finite differences, i.e., basically use the Grünwald-Letnikov definition for the second derivative term,

$$\frac{x(t) - 2x(t - \Delta t) + x(t - 2\Delta t)}{\left(\Delta t\right)^2} + \sum_{k=0}^{\lfloor \frac{t}{\Delta t} \rfloor} \frac{(-1)^\gamma \binom{\gamma}{k} x\left(t - k\Delta t\right)}{\left(\Delta t\right)^\gamma} = 1$$

and solve the resulting expression for $x(t)$

$$x(t) = \frac{\left(\Delta t\right)^{2+\gamma}}{\left(\Delta t\right)^2 + \left(\Delta t\right)^\gamma}\left[1 - \sum_{k=1}^{\lfloor \frac{t}{\Delta t} \rfloor}\left(\frac{(-1)^k \binom{\gamma}{k} x\left(t - k\Delta t\right)}{\left(\Delta t\right)^\gamma}\right) + \frac{2x\left(t - \Delta t\right) - x\left(t - 2\Delta t\right)}{\left(\Delta t\right)^2}\right]$$

Bibliography

[Newton, 1687] Newton, I. (1687). *Philosophiae Naturaliz Principia Mathematica.* J. Societatis Regiae ac Typis J. Streater.