Using a Neural Network Trained Only on Integer Order Systems to Identify Fractional Order Dynamics in Large Scale Systems

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***Abstract*— This paper presents a standard artificial neural network that identifies the order of the dynamics of a unit step response. The system is trained on only first and second order systems, yet identifies fractional order responses with a high degree of accuracy. The details of the structure of the neural network, the training method and the training sets, as well as statistics describing the accuracy of the fractional predictions are presented. Also using the neural network to identify fractional dynamics for a large scale networked system from the authors' prior work is presented as further validation and a demonstration of the applicability of the results. This demonstrates the potential for practicing engineers to use similar machine learning tools trained on "standard" systems with the ability to distinguish when features such as fractional order dynamics are significant and warrant deeper consideration for the design or control of such a system.**

I. INTRODUCTION

Fractional calculus and fractional order dynamics are increasingly important in modern engineered systems. Unlike integer order derivatives, fractional order derivatives, and hence the dynamics that depend on them, are *nonlocal.* As such, many modern, large scale engineered systems may exhibit fractional order dynamics and responses because interactions among various components in the system may be significantly displaced in time or space. In instances where significant fractional order dynamics are present, control algorithms which directly address the fractional nature of the system may be superior. Therefore, tools to readily identify if significant fractional order dymancs are present are needed.

There is a vast literature on fractional calculus. Some textbooks include [1]–[3]. Fractional-order control is also a topical area such as in [4], [5]. An excellent review article illustrating the very broad range of applications of fractional calculus and control in science and engineering is [6].

Our main interets are identifying cases where fractional order models may provide useful "reduced order" models for large scale systems [7], [8] and for exact models for many large scale systems [9]–[13]. While this paper does not build upon it, our closest publication to this would be [14] where we created a symmetric neural network with a sequential set of identical layers. When it was trained on first derivatives of functions, the middle layer could represent the half derivative.

There are many different definitions of the fractional derivative. As will be outlined in the next section, a common feature of these is replacing factorial functions appearing in many integer-order representations of the derivative with gamma functions. The Riemann-Liouville, Caputo and the Grünwald-Letnikov definitions are perhaps the most common examples of fractional derivative definitions, and the reader is referred to the references [15]–[18] for descriptions and definitions of each.

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II. Fractional Calculus

This section gives a very brief overview of only the fractional calclulus topics necessary to implement the methods in this paper and is therefore incomplete. The interested reader is referred to the references above for a complete exposition.

The obvious task for fractional calculus to to define derivatives "in between" the usual integer order derivatives, *e.g.*, the one half derivative of $x(t)$, Consider the first and second order backwards finite difference equations for $\Delta t \ll 1$

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) \approx \frac{x(t) - x(t-\Delta t)}{\Delta t}$$

and

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) \approx \frac{x(t) - 2x(t-\Delta t) + x(t-2\Delta t)}{(\Delta t)^2}.$$

From these, it is clear that the formula for an arbitrary positive integer order derivative is

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n}(t) \approx \frac{\sum_{i=0}^{n} (-1)^i \left(\frac{n!}{i!(n-i)!}\right) x(t - i\Delta t)}{(\Delta t)^n}, \qquad (1)$$

where the fraction with the factorials in the numerator and denominator is the binomial coefficient. The only terms in the preceding equation that must be integers are the factorials. Because the gamma function can be considered a generalization of the factorial because $n! = \Gamma(n+1)$ for integer values of n, a generalized equation results with

$$\frac{\mathrm{d}^\alpha x}{dt^\alpha}(t) \approx \frac{\sum_{i=0}^{\lceil \frac{t}{\Delta t} \rceil} (-1)^i \left(\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}\right) x(t - i\Delta t)}{(\Delta t)^\alpha}. \qquad (2)$$

Unlike the sum in Equation 1 which only ranges from 0 to n because of the range of the definition of the factorials and hence binomial coefficient, the generalized binomial coefficient in Equation 2 is nonzero for all values of i (except when α is an integer), and hence the sum will include times spanning all of history. In the controls context assuming zero initial conditions and zero values for all time prior to zero, the sum will span back to $t = 0$. This highlights an important distinction between integer order derivatives and fractional order derivatives, which is that the latter are *nonlocal*.

The method to compute the fractional step responses that the neural network will identify was computed by a different method than what is expressed in Equation 2, but is similar in spirit. Specifically, we use the `numfracpy` python library to numerically compute fractional step responses, and from its documentation[1] the step response is computed using a fractional order Adams predictor corrector method given by

$$x^P_{n+1} = \sum_{j=0}^{m-1} \frac{t^j_{n+1}}{j!} x^j_0 + (\Delta t)^\alpha \sum_{j=0}^{n} b_{j,n+1} f\left(t_j, x_j\right)$$

$$x_{n+1} = \sum_{j=0}^{m-1} \frac{t^j_{n+1}}{j!} x^j_0 + \sum_{j=0}^{n} a_{j,n+1} f\left(t_j, x_j\right) + a_{n+1,n+1} f\left(t_{n+1}, x^P_{n+1}\right)$$

where

$$b_{j,n+1} = \frac{1}{\Gamma(\alpha+1)} \left[(n-j+1)^\alpha - (n-j)^\alpha\right]$$

and

$$\alpha_{0,n+1} = \frac{(\Delta t)^\alpha \left[n^{\alpha+1} - (n-\alpha)(n+1)^\alpha\right]}{\Gamma\left(\alpha+2\right)},$$

$$a_{j,n+1} = \frac{(\Delta t)^\alpha}{\Gamma\left(\alpha+2\right)} \left[(n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}\right]$$

for $1 \leq j \leq n$ and

$$\alpha_{n+1,n+1} = \frac{(\Delta t)^\alpha}{\Gamma\left(\alpha+2\right)}.$$

The summations in the expressions for x_{n+1} are the parts of the expressions that include the nonlocal terms.

III. Neural Network and Training

The neural network presented in this paper is illustrated in Figure 1. There is an input layer with 101 nodes, a hidden layer with 64 nodes, a subsequent hidden layer with 16 nodes, another hidden layer with 16 nodes, and an output layer with 1 node. Each hidden layer has the ReLU activation function. The values input to the network is the unit step response of a linear system in the time range of $0 \leq t \leq 10$ discretized into time steps of $\Delta t = 0.1$ [s], which gives 101 input nodes. The single output of the network is the order of the system that produced the input step response.

[1]http://tinyurl.com/yyytpv8d

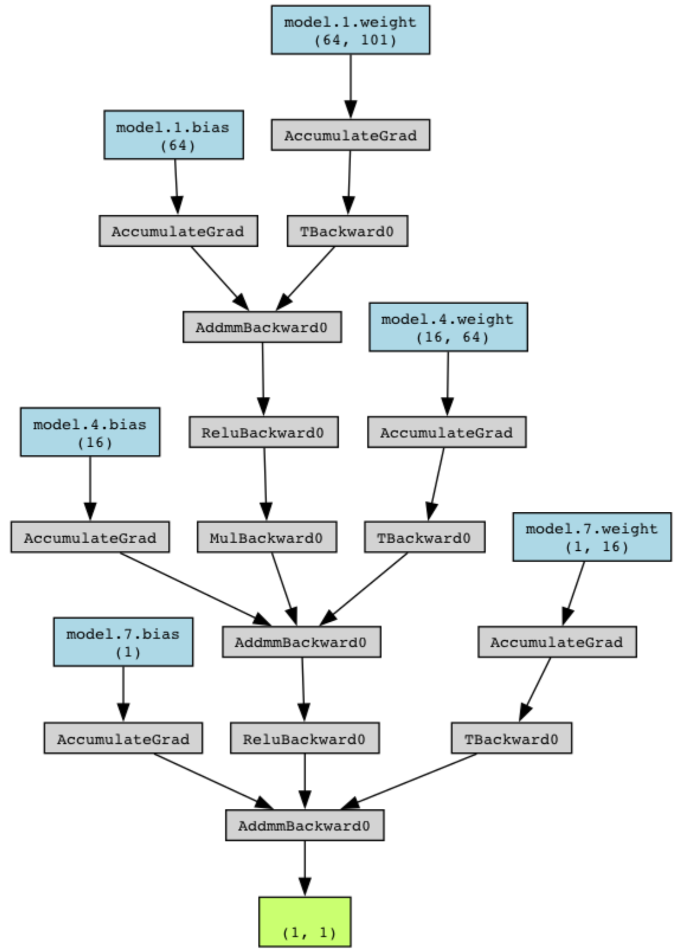

Fig. 1. The neural network.

The network was implemented in python using the `torch` library and `pytorch_lightning` tools. This network is not trained as a classifier because we want it to be able to generalize first and second order systems to fractional orders between them, so the loss function is the mean squared error function, `mse_loss()`, and the optimization method adopted was Adam optimizer, `torch.optim.Adam()` with a learning rate of 0.001. A branch of our github repositiry that shoudl repeatably replicate the results presented in this paper is at **ZZZZZZ**.

A. Integer Order Training, Validation and Testing Data

An individual element of the training, validation and training sets is the step response for a first or second order system. The manner in which they are generated is:

- Select a value from a uniform random distribution between 0 and 1, and if the value is less than 0.5, then the step response will be for a second order system, and if not, then it will be for a first order system.

- Select two numbers, c_1 and c_2 from a uniform distribution with values between 0.01 and 4.
  - If the response is for a first order system, then the transfer function is
$$G(s) = \frac{c_2}{c_1 s + c_2}.$$
  - If the response is for a second order system, then the selected transfer function is
$$G(s) = \frac{c_2}{c_1 s^2 + c_2}.$$
- The unit step response from 0 to 10 seconds for the transfer function is generated using the `control.step_response()` function from the python control system library. It is sampled every 0.1 second so that the length of the response vector is 101.

Using this method we generate a set of 100,000 first or second order step responses with approximately the same number of first and second order responses. The training set is split into three subsets: 60,000 training elements, 20,000 validation elements and 20,000 testing elements. For training, the training set is shuffled at the beginning of each epoch and the optimization method is applied to change the weights in the network. At the end of the epoch, the network is run on the validation set to compute an error for data points the network was not trained on. Evidence of overtraining would be if the validation set error decreases and then increases. At the end of all the training, the error for the testing set is computed.

Figure 2 illustrates the error on the validation set for 10 training runs for the network versus epoch. It appears that if the validation error increases, it tends to start to do so around 1000 epochs; otherwise it tends to stop changing around 1000 epochs (there seems to be one exception). As such, we will fix the number of training epochs at 1000. Additionally, various alternative configurations to the network described above, including more hidden nodes and different activation functions were investigated, and the configuration presented above generally gave the best results.

IV. Using the Integer Trained Network on Fractional Order Step Responses

In order to determine how well the trained network will generalize from integer order training data to fractional order step responses,

more text

V. CONCLUSIONS

A conclusion section is not required. Although a conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

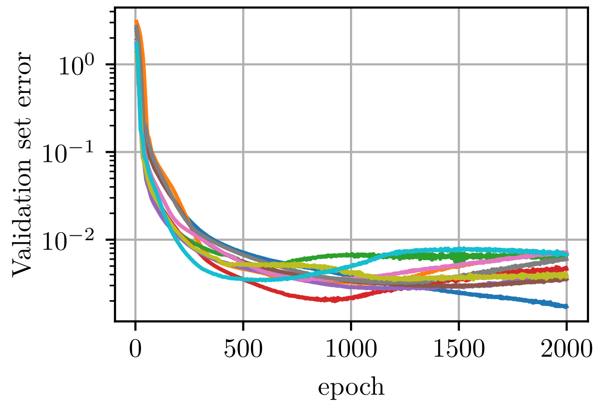

Fig. 2. Neural network output error on validation (not training) set verus training epoch.

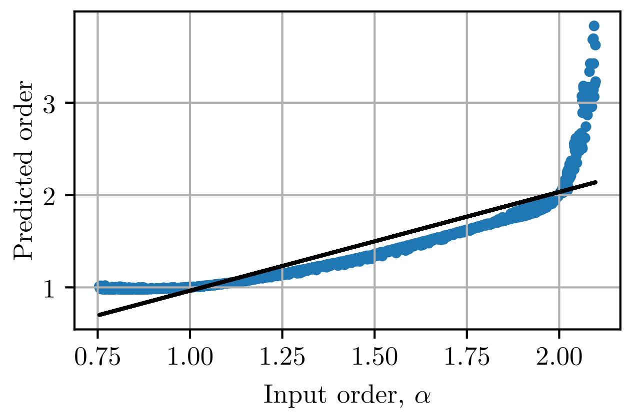

Fig. 3. Comparison of actual fractional orders and predicted orders.

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