Reconfiguration of Colourings and Dominating Sets in Graphs: a Survey

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Abstract

We survey results concerning reconfigurations of colourings and dominating sets in graphs. The vertices of the k-colouring graph $\mathcal{C}_k(G)$ of a graph G correspond to the proper k-colourings of a graph G, with two k-colourings being adjacent whenever they differ in the colour of exactly one vertex. Similarly, the vertices of the k-edge-colouring graph $\mathcal{EC}_k(G)$ of g are the proper k-edge-colourings of G, where two k-edge-colourings are adjacent if one can be obtained from the other by switching two colours along an edge-Kempe chain, i.e., a maximal two-coloured alternating path or cycle of edges.

The vertices of the k-dominating graph $\mathcal{D}_k(G)$ are the (not necessarily minimal) dominating sets of G of cardinality k or less, two dominating sets being adjacent in $\mathcal{D}_k(G)$ if one can be obtained from the other by adding or deleting one vertex. On the other hand, when we restrict the dominating sets to be minimum dominating sets, for example, we obtain different types of domination reconfiguration graphs, depending on whether vertices are exchanged along edges or not.

We consider these and related types of colouring and domination reconfiguration graphs. Conjectures, questions and open problems are stated within the relevant sections.

1 Introduction

In graph theory, reconfiguration is concerned with relationships among solutions to a given problem for a specific graph. The reconfiguration of one solution into another occurs via a

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sequence of steps, defined according to a predetermined rule, such that each step produces an intermediate solution to the problem. The solutions form the vertex set of the associated reconfiguration graph, two vertices being adjacent if one solution can be obtained from the other in a single step. Exact counting of combinatorial structures is seldom possible in polynomial time. Approximate counting of the structures, however, may be possible. When the reconfiguration graph associated with a specific structure is connected, Markov chain simulation can be used to achieve approximate counting. Typical questions about the reconfiguration graph therefore concern its structure (connectedness¹, Hamiltonicity, diameter, planarity), realisability (which graphs can be realised as a specific type of reconfiguration graph), and algorithmic properties (finding shortest paths between solutions quickly).

Reconfiguration graphs can, for example, be used to study combinatorial Gray codes. The term "combinatorial Gray code" refers to a list of combinatorial objects so that successive objects differ in some prescribed minimal way. It generalises Gray codes, which are lists of fixed length binary strings such that successive strings differ by exactly one bit. Since the vertices of a reconfiguration graph are combinatorial objects, with two vertices being adjacent whenever they differ in some small way, a Hamilton path in a reconfiguration graph corresponds to a combinatorial Gray code in the source graph, and a Hamilton cycle to a cyclic combinatorial Gray code.

We restrict our attention to reconfigurations of graph colourings and dominating sets (of several types). Unless stated otherwise, we use n to denote the order of our graphs. As is standard practice we denote the chromatic number of a graph G by $\chi(G)$, its clique number by $\omega(G)$, and its minimum and maximum degrees by $\delta(G)$ and $\Delta(G)$, respectively. We use $\gamma(G)$ and $\Gamma(G)$ to denote the domination and upper domination numbers of G, that is, the cardinality of a minimum dominating set and a maximum minimal dominating set, respectively.

One of the best studied reconfiguration graphs is the k-colouring graph $C_k(G)$, whose vertices correspond to the proper k-colourings of a graph G, with two k-colourings being adjacent whenever they differ in the colour of exactly one vertex. When $C_k(G)$ is connected, a Markov process can be defined on it that leads to an approximation of the number of k-colourings of G; this relationship motivated the study of the connectedness of $C_k(G)$. Some authors consider list colourings with the same adjacency condition, while others consider proper k-edge-colourings, where two k-edge-colourings of G are adjacent in the k-edge-colouring graph $\mathcal{EC}_k(G)$ if one can be obtained from the other by switching two colours along an edge-Kempe chain, i.e., a maximal two-coloured alternating path or cycle of edges.

The domination reconfiguration graph whose definition most resembles that of the k-colouring graph is the k-dominating graph $\mathcal{D}_k(G)$, whose vertices are the (not necessarily minimal) dominating sets of G of cardinality k or less, where two dominating sets are adjacent in $\mathcal{D}_k(G)$ if one can be obtained from the other by adding or deleting one vertex. The k-total-dominating graph $\mathcal{D}_k^t(G)$ is defined similarly using total-dominating sets.

Other types of domination reconfiguration graphs are defined using only sets of cardinalities equal to a given domination parameter π . For example, if π is the domination number γ , then the vertex set of the associated reconfiguration graph, called the γ -graph of G, consists of

¹We use the term *connectedness* instead of *connectivity* when referring to the question of whether a graph is connected or not, as the latter term refers to a specific graph parameter.

the minimum dominating sets of G. There are two types of γ -graphs: $\mathcal{J}(G,\gamma)$ and $\mathcal{S}(G,\gamma)$. In $\mathcal{J}(G,\gamma)$, two minimum dominating sets D_1 and D_2 are adjacent if and only if there exist vertices $x \in D_1$ and $y \in D_2$ such that $D_1 - \{x\} = D_2 - \{y\}$. The γ -graph $\mathcal{J}(G,\gamma)$ is referred to as the γ -graph in the single vertex replacement adjacency model or simply the jump γ -graph. In $\mathcal{S}(G,\gamma)$, two minimum dominating sets D_1 and D_2 are adjacent if and only if there exist adjacent vertices $x \in D_1$ and $y \in D_2$ such that $D_1 - \{x\} = D_2 - \{y\}$. The γ -graph $\mathcal{S}(G,\gamma)$ is referred to as the γ -graph in the slide adjacency model or the slide γ -graph. Note that $\mathcal{S}(G,\gamma)$ is a spanning subgraph of $\mathcal{J}(G,\gamma)$. In general we define the slide π -graph similar to the slide γ -graph and denote it by $\mathcal{S}(G,\pi)$.

We refer the reader to the well-known books [21] and [59] for graph theory concepts not defined here. Lesser known concepts are defined where needed. We only briefly mention algorithmic and complexity results, since a recent and extensive survey of this aspect of reconfiguration is given by Nishimura [53]. We state open problems and conjectures throughout the text where appropriate.

2 Complexity

Many of the published papers on reconfiguration problems address complexity and algorithmic questions. The main focus of much of this work has been to determine the existence of paths between different solutions, that is, to determine which solutions are in the same component of the reconfiguration graph, and if so, how to find a shortest path between two solutions. The questions, therefore, are whether one solution is *reachable* from another according to the rules of adjacency, and if so, to determine or bound the *distance* between them. If all solutions are reachable from one another, the reconfiguration graph is connected and its diameter gives an upper bound on the distance between two solutions.

Complexity results concerning the connectedness and diameter of the k-colouring graph $C_k(G)$ are given in [53, Section 6], and those pertaining to domination graphs can be found in [53, Section 7]. We mention complexity results for homomorphism reconfiguration in Section 3.2.

An aspect that has received considerable attention, but has not been fully resolved, is to determine dividing lines between tractable and intractable instances for reachability. Cereceda, Van den Heuvel, and Johnson [19] showed that the problem of recognizing bipartite graphs G such that $C_3(G)$ is connected is coNP-complete, but polynomial when restricted to planar graphs. In [20] they showed that for a 3-colourable graph G of order n, both reachability and the distance between given colourings can be solved in polynomial time. Bonsma and Cereceda [10] showed that when $k \geq 4$, the reachability problem is PSPACE-complete. Indeed, it remains PSPACE-complete for bipartite graphs when $k \geq 4$, for planar graphs when $4 \leq k \leq 6$, and for bipartite planar graphs when k = 4. Moreover, for any integer $k \geq 4$ there exists a family of graphs $G_{N,k}$ of order N such that some component of $C_k(G_{N,k})$ has diameter $\Omega(2^N)$. Bonsma, Mouawad, Nishimura, and Raman [13] showed that when $k \geq 4$, reachability is strongly NP-hard. Bonsma and Mouawad [12] explored how the complexity of deciding whether $C_k(G)$ contains a path of length at most ℓ between two given k-colourings of G depends on k and ℓ , neatly summarizing their results in a table. Other work on the complexity of colouring reconfiguration include [8, 9, 11, 15, 18, 26, 29, 37, 38, 39, 40, 41, 47].

Haddadan, Ito, Mouawad, Nishimura, Ono, Suzuki, and Tebbal [36] showed that determining whether $\mathcal{D}_k(G)$ is connected is PSPACE-complete even for graphs of bounded bandwidth, split graphs, planar graphs, and bipartite graphs, and they developed linear-time algorithms for cographs, trees, and interval graphs. Lokshtanov, Mouawad, Panolan, Ramanujan, and Saurabh [43] showed that, although W[1]-hard when parameterized by k, the problem is fixed-parameter tractable when parameterized by k + d for $K_{d,d}$ -free graphs. For other works in this area see [48, 57].

3 Reconfiguration of Colourings

The set of proper k-colourings of a graph G has been studied extensively via, for example, the Glauber dynamics Markov chain for k-colourings; see e.g. [26, 27, 39, 44, 47]. Algorithms for random sampling of k-colourings and approximating the number of k-colourings arise from these Markov chains. The connectedness of the k-colouring graph is a necessary condition for such a Markov chain to be rapidly mixing, that is, for the number of steps required for the Markov chain to approach its steady state distribution to be at most a polynomial in $\log(n)$, where n = |V(G)|.

3.1 The k-Colouring Graph

Motivated by the Markov chain connection, a graph G is said to be k-mixing if $C_k(G)$ is connected. The minimum integer $m_0(G)$ such that G is k-mixing whenever $k \geq m_0(G)$ is called the mixing number of G. A k-colouring of G is frozen if each vertex of G is adjacent to at least one vertex of every other colour; a frozen k-colouring is an isolated vertex of $C_k(G)$. The colouring number $\operatorname{col}(G)$ of G is the least integer d such that the vertices of G can be ordered as $v_1 \prec \cdots \prec v_n$ so that $|\{v_i : i < j \text{ and } v_i v_j \in E(G)\}| < d$ for all j = 1, ..., n. By colouring the vertices $v_1, ..., v_n$ greedily, in this order, with the first available colour from $\{1, ..., d\}$, we obtain a d-colouring of G; hence $\chi(G) \leq \operatorname{col}(G)$. Here we should mention that some authors define the colouring number to be $\max_{H\subseteq G} \delta(H)$ where the maximum is taken over all subgraphs H of G; this number in fact equals $\operatorname{col}(G) - 1$. Indeed, $\max_{H\subseteq G} \delta(H)$ is often called the degeneracy of G.

The choice of k is important when we consider the connectedness and diameter of $C_k(G)$. Given two colourings c_1 and c_2 , when k is sufficiently large each vertex can be recoloured with a colour not appearing in either c_1 or c_2 and then recoloured to its target colour. Then $C_k(G)$ is connected and has diameter linear in the order of G. This also shows that $m_0(G)$ is defined for each graph G. On the other hand, if k = 2 and G is an even cycle, then no vertex can be recoloured and $C_2(G) = 2K_1$.

Jerrum [39] showed that $m_0(G) \leq \Delta(G) + 2$ for each graph G. Cereceda et al. [18] used the colouring number to bound m_0 . Since $\operatorname{col}(G) \leq \Delta(G) + 1$ and the difference can be arbitrary, their result offers an improvement on Jerrum's bound.²

²Bonsma and Cereceda [10] and Cereceda et al. [18] use the alternative definition of col(G); we have adjusted their statements to conform to the definition given here.

Theorem 3.1 [18] For any graph G, $m_0(G) \leq \operatorname{col}(G) + 1$.

Cereceda et al. [18] used the graph $L_m = K_{m,m} - mK_2$ (the graph obtained from the complete bipartite graph $K_{m,m}$ by deleting a perfect matching) to obtain a graph G and integers $k_1 < k_2$ such that G is k_1 -mixing but not k_2 -mixing: colour the vertices in each partite set of L_m with the colours 1, ..., m, where vertices in different parts that are ends of the same deleted edge receive the same colour. This m-colouring is an isolated vertex in the m-colour graph $C_m(L_m)$. Hence L_m is not m-mixing (there are many m-colourings of L_m). They showed that for $m \geq 3$, the bipartite graph L_m is k-mixing for $3 \leq k \leq m-1$ and $k \geq m+1$ but not k-mixing for k = m. They also showed that there is no expression $\varphi(\chi)$ in terms of the chromatic number χ such that for all graphs G and integers $k \geq \varphi(\chi(G))$, G is k-mixing.

Cereceda et al. [18] also showed that if $\chi(G) \in \{2,3\}$, then G is not $\chi(G)$ -mixing, and that C_4 is the only 3-mixing cycle. In contrast, for $m \geq 4$ they obtained an m-chromatic graph H_m that is k-mixing whenever $k \geq m$: let H_m be the graph obtained from two copies of K_{m-1} with vertex sets $\{v_1, ..., v_{m-1}\}$ and $\{w_1, ..., w_{m-1}\}$ by adding a new vertex u and the edges v_1w_1 and $\{uv_i, uw_i : 2 \leq i \leq m-1\}$. In [19], the same authors characterised 3-mixing connected bipartite graphs as those that are not foldable to C_6 . [If v and w are vertices of a bipartite graph G at distance two, then a fold on v and w is the identification of v and w (remove any resulting multiple edges); G is foldable to H if there exists a sequence of folds that transforms G into H.]

Bonamy and Bousquet [8] used the Grundy number of G to improve Jerrum's bound on $m_0(G)$. A proper k-colouring of G in colours 1, ..., k is called a *Grundy colouring* if, for $1 \le i \le k$, every vertex with colour i is adjacent to vertices of all colours less than i. The *Grundy number* $\chi_g(G)$ of a graph G is the maximum number of colours among all Grundy colourings of G. Note that $\chi_g(G) \le \Delta(G) + 1$ and, as in the case of $\operatorname{col}(G)$, it can be arbitrarily smaller.

Theorem 3.2 [8] For any graph G of order n and any k with $k \geq \chi_g(G)+1$, $C_k(G)$ is connected and diam $(C_k(G)) \leq 4n\chi(G)$.

Since the Grundy number of a cograph (a P_4 -free graph) equals its chromatic number, Theorem 3.2 implies that for $k \geq \chi(G) + 1$, a cograph G is k-mixing and the diameter of $C_k(G)$ is $O(\chi(G) \cdot n)$, (i.e., linear in n). This result does not generalize to P_r -free graphs for $r \geq 5$. Bonamy and Bousquet constructed a family of P_5 -free graphs $\{G_k : k \geq 3\}$ having both a proper (k+1)-colouring and a frozen 2k-colouring. They also showed that the graphs L_m mentioned above are P_6 -free with arbitrary large mixing number and asked the following question.

Question 3.1 [8] Given $r, k \in \mathbb{N}$, does there exist $c_{r,k}$ such that for any P_r -free graph G of order n that is k-mixing, the diameter of $C_k(G)$ is at most $c_{r,k} \cdot n$?

Several other authors also considered the diameter of $C_k(G)$ or of its components when it is disconnected. Cereceda et al. [20] showed that if G is a 3-colourable graph with n vertices, then the diameter of any component of $C_3(G)$ is $O(n^2)$. In contrast, for $k \geq 4$, Bonsma and Cereceda [10] obtained graphs (which may be taken to be bipartite, or planar when $4 \leq k \leq 6$,

or planar and bipartite when k=4) having k-colourings such that the distance between them is superpolynomial in the order and size of the graph. They also showed that if G is a graph of order n and $k \geq 2\operatorname{col}(G) - 1$, then $\operatorname{diam}(\mathcal{C}_k(G)) = O(n^2)$. They stated the following conjecture.

Conjecture 3.1 [10] For a graph G of order n and $k \ge \operatorname{col}(G) + 1$, $\operatorname{diam}(\mathcal{C}_k(G)) = O(n^3)$.

Bonamy, Johnson, Lignos, Patel, and Paulusma [9] determined sufficient conditions for $C_k(G)$ to have a diameter quadratic in the order of G. They showed that k-colourable chordal graphs and chordal bipartite graphs satisfy these conditions and hence have an ℓ -colour diameter that is quadratic in k for $\ell \geq k+1$ and $\ell=3$, respectively. Bonamy and Bousquet [8] proved a similar result for graphs of bounded treewidth. Beier, Fierson, Haas, Russell, and Shavo [4] considered the girth $g(C_k(G))$.

Theorem 3.3 [4] If $k > \chi(G)$, then $g(C_k(G)) \in \{3, 4, 6\}$. In particular, for k > 2, $g(C_k(K_{k-1})) = 6$. If $k > \chi(G) + 1$, or $k = \chi(G) + 1$ and $C_{k-1}(G)$ has an edge, then $g(C_k(G)) = 3$. If $k = \chi(G) + 1$ and $G \neq K_{k-1}$, then $g(C_k(G)) \leq 4$.

The Hamiltonicity of $C_k(G)$ was first considered by Choo [22] in 2002 (also see Choo and MacGillivray [23]). Choo showed that, given a graph G, there is a number $k_0(G)$ such that $C_k(G)$ is Hamiltonian whenever $k \geq k_0(G)$. The number $k_0(G)$ is referred to as the *Gray code number* of G, since a Hamilton cycle in $C_k(G)$ is a (cyclic) combinatorial Gray code for the k-colourings. Clearly, $k_0(G) \geq m_0(G)$. By Theorem 3.1, $m_0(G) \leq \operatorname{col}(G) + 1$. Choo and MacGillivray showed that one additional colour suffices to ensure that $C_k(G)$ is Hamiltonian.

Theorem 3.4 [23] For any graph G and $k \ge \operatorname{col}(G) + 2$, $C_k(G)$ is Hamiltonian.

Choo and MacGillivray also showed that when T is a tree, $k_0(T) = 4$ if and only if T is a nontrivial odd star, and $k_0(T) = 3$ otherwise. They also showed that $k_0(C_n) = 4$ for each $n \geq 3$. Celaya, Choo, MacGillivray, and Seyffarth [17] continued the work of [23] and considered complete bipartite graphs $K_{\ell,r}$. Since $C_2(G)$ is disconnected for bipartite graphs, $k_0(K_{\ell,r}) \geq 3$. They proved that equality holds if and only if ℓ and r are both odd and that $C_k(K_{\ell,r})$ is Hamiltonian when $k \geq 4$. Bard [3] expanded the latter result to complete multipartite graphs.

Theorem 3.5 [3] Fix $a_1, ..., a_t \in \mathbb{N}$. If $k \geq 2t$, then $C_k(K_{a_1,...,a_t})$ is Hamiltonian.

Bard improved this result for special cases by showing that $C_4(K_{a_1,a_2,a_3})$ is Hamiltonian if and only if $a_1 = a_2 = a_3 = 1$, and, for $t \geq 4$, $C_{t+1}(K_{a_1,\ldots,a_t})$ is Hamiltonian if and only if a_1 is odd and $a_i = 1$ for $2 \leq i \leq t$. He showed that for each $k \geq 4$ there exists a graph G such that $C_k(G)$ is connected but not 2-connected.

Question 3.2 [3] (i) Is $K_{2,2,2}$ the only complete 3-partite graph whose 5-colouring graph is non-Hamiltonian?

- (ii) Does there exist a connected 3-colouring graph that is not 2-connected?
- (iii) If $C_k(G)$ is Hamiltonian, is $C_{k+1}(G)$ always Hamiltonian?

Beier et al. [4] considered the problem of determining which graphs are realisable as colouring graphs. That is, given a graph H, when does there exist a graph G and an integer k such that $H \cong \mathcal{C}_k(G)$? To this effect they determined that

- if $C_k(G)$ is a complete graph, then it is K_k , and if k > 1 then $C_k(G) = K_k$ if and only if $G = K_1$;
- K_1 and P_2 are the only trees that are colouring graphs;
- C_3, C_4, C_6 are the only cycles that are colouring graphs;
- every tree is a subgraph of a colouring graph (thus there is no finite forbidden subgraph characterisation of colouring graphs).

Other colouring graphs have also been considered. Haas [32] considered canonical and isomorphic colouring graphs. Two colourings of a graph G are isomorphic if one results from permuting the names of the colours of the other. A proper k-colouring of G with colours 1, ..., kis canonical with respect to an ordering $\pi = v_1, ..., v_n$ of the vertices of G if, for $1 \le c \le k$, whenever colour c is assigned to a vertex v_i , each colour less than c has been assigned to a vertex v_j , j < i. (Thus, a Grundy colouring g becomes a canonical colouring if we order the vertices of G so that $v_i \prec v_j$ whenever $g(v_i) < g(v_j)$.) For an ordering π of the vertices of G, the set of canonical k-colourings of G under π is the set $S_{\operatorname{Can}}(G)$ of pairwise nonisomorphic proper k-colourings of G that are lexicographically least under π . (Given colourings c_1 and c_2 of G and an ordering $v_1, ..., v_n$ of V(G), we say that c_1 is lexicographically less than c_2 if $c_1(v_j) < c_2(v_j)$ for some integer j, $1 \le j \le n$, and $c_1(v_i) = c_2(v_i)$ whenever i < j.) The canonical k-colouring graph $\operatorname{Can}_k^{\pi}(G)$ is the graph with vertex set $S_{\operatorname{Can}}(G)$ in which two colourings are adjacent if they differ at exactly one vertex. Considering only nonisomorphic colourings, Haas defined the isomorphic k-colouring graph $\mathcal{I}_k(G)$ to have an edge between two colourings c and d if some representative of c differs at exactly one vertex from some representative of d. Haas showed that if the connected graph G is not a complete graph, then $\operatorname{Can}_k^{\pi}(G)$ can be disconnected depending on the ordering π and the difference $k - \chi(G)$.

Theorem 3.6 [32] (i) For any connected graph $G \neq K_n$ and any $k \geq \chi(G) + 1$ there exists an ordering π of V(G) such that $\operatorname{Can}_k^{\pi}(G)$ is disconnected.

- (ii) For any tree T of order $n \geq 4$ and any $k \geq 3$ there is an ordering π of V(T) such that $\operatorname{Can}_k^{\pi}(T)$ is Hamiltonian.
- (iii) For any cycle C_n and any $k \geq 4$ there is an ordering π of $V(C_n)$ such that $\operatorname{Can}_k^{\pi}(C_n)$ is connected. Moreover, $\operatorname{Can}_3^{\pi}(C_4)$ and $\operatorname{Can}_3^{\pi}(C_5)$ are connected for some π but for all $n \geq 6$, $\operatorname{Can}_3^{\pi}(C_n)$ is disconnected for all π .

Haas and MacGillivray [33] extended this work and obtained a variety of results on the connectedness and Hamiltonicity of the joins and unions of graphs. They also obtained the following results.

Theorem 3.7 [33] If G is a bipartite graph on n vertices, then there exists an ordering π of V(G) such that $\operatorname{Can}_k^{\pi}(G)$ is connected for $k \geq n/2 + 1$.

Theorem 3.8 [33] Let $G = K_{a_1,...,a_t}$.

- (i) For any $k \geq t$ there exists an ordering π of V(G) such that $\operatorname{Can}_k^{\pi}(G)$ is connected.
- (ii) If $a_i \geq 2$ for each i, then for all vertex orderings π and $k \geq t+1$, $\operatorname{Can}_k^{\pi}(G)$ has a cut vertex and thus is non-Hamiltonian, and if $t \geq 3$, then $\operatorname{Can}_k^{\pi}(G)$ has no Hamiltonian path.
- (iii) For t = 2, K_{a_1,a_2} has a vertex ordering π such that $\operatorname{Can}_k^{\pi}(K_{a_1,a_2})$ has a Hamiltonian path for $a_1, a_2 \geq 2$ and $k \geq 3$.

Thus we see that all bipartite and complete multipartite graphs admit a vertex ordering π such that $\operatorname{Can}_k^{\pi}(G)$ is connected for large enough values of k. Haas and MacGillivray also provided a vertex ordering such that $\operatorname{Can}_k^{\pi}(G)$ is disconnected for all large values of k.

Finbow and MacGillivray [30] studied the k-Bell colour graph and the k-Stirling colour graph. The k-Bell colour graph $\mathcal{B}_k(G)$ of G is the graph whose vertices are the partitions of the vertices of G into at most k independent sets, with different partitions p_1 and p_2 being adjacent if there is a vertex x such that the restrictions of p_1 and p_2 to $V(G) - \{x\}$ are the same partition. The k-Stirling colour graph $\mathcal{S}_k(G)$ of G is the graph whose vertices are the partitions of the vertices of G into exactly k independent sets, with adjacency as defined for $\mathcal{B}_k(G)$. They showed, for example, that $\mathcal{B}_n(G)$ is Hamiltonian whenever G is a graph of order n other than K_n or $K_n - e$. As a consequence of Theorem 3.6(ii), $\mathcal{B}_k(T)$ is Hamiltonian whenever $k \geq 3$ and T is a tree of order at least 4, while $\mathcal{S}_3(T)$ has a Hamiltonian path. In addition, if $\mathcal{C}_k(G)$ is connected, then so is $\mathcal{B}_k(G)$. They extended the result for $\mathcal{S}_3(T)$ to show that $\mathcal{S}_k(T)$ is Hamiltonian for any tree T of order $n \geq k+1$ and $k \geq 4$.

Other variants of vertex colourings for which reconfiguration has been studied include circular colourings [15, 16], acyclic colourings [58] and equitable colourings [58]. Circular colourings and k-colourings are special cases of homomorphisms, which we discuss in the next subsection.

3.2 Reconfiguration of Homomorphisms

For graphs G and H, a homomorphism from G to H is a mapping $\varphi: V(G) \to V(H)$ such that $\varphi(u)\varphi(v) \in E(H)$ whenever $uv \in E(G)$. The collection of homomorphisms from G to H is denoted by Hom(G,H). A k-colouring of G can be viewed as a homomorphism from G to K_k . Thus we also refer to a homomorphism from G to H as an H-colouring of G. The H-colouring graph $\mathcal{C}_H(G)$ of G has vertex set Hom(G,H), and two homomorphisms are adjacent if one can be obtained from the other by changing the colour of one vertex of G. For G, G is called an G-colouring sequence from G to G is called an G-colouring sequence from G to G. For a fixed graph G-colouring problem G-colouring problem G-colouring sequence from G-colouring whether, given G-colouring problem G-colouring sequence from G-colouring sequence from G-colouring whether the transformation can be done in at most G-colouring sequence from the question is whether the transformation can be done in at most G-colouring techniques from topology.

A graph H has the monochromatic neighbourhood property (MNP), or is an MNP-graph, if for all pairs $a, b \in V(H)$, $|N_H(a) \cap N_H(b)| \leq 1$. Depending on whether H has loops or not, MNP-graphs do not contain C_4 , or K_3 with one loop, or K_2 with both loops; K_3 and graphs with girth at least 5 are all C_4 -free. Note that 3-colourable graphs are MNP-graphs.

Theorem 3.9 [60] If H is an MNP-graph (possibly with loops), then H-RECOLOURING and SHORTEST H-RECOLOURING are in P.

Given positive integers k and q with $k \geq 2q$, the circular clique $G_{k,q}$ has vertex set $\{0,1,...,k-1\}$, with ij an edge whenever $q \leq |i-j| \leq k-q$. A homomorphism $\varphi \in \text{Hom}(G,G_{k,q})$ is called a circular colouring. The circular chromatic number of G is $\chi_c(G) = \inf\{k/q : \text{Hom}(G,G_{k,q}) \neq \emptyset\}$. Brewster, McGuinness, Moore, and Noel [15] considered the complexity of the $G_{k,q}$ -recolouring problem.

Theorem 3.10 [15] If k and q are fixed positive integers with $k \geq 2q$, then $G_{k,q}$ -RECOLOURING is solvable in polynomial time when $2 \leq k/q < 4$ and is PSPACE-complete for $k/q \geq 4$.

The circular mixing number³ of G, written $m_c(G)$, is $\inf\{r \in \mathbb{Q} : r \geq \chi_c(G) \text{ and } \mathcal{C}_{G_{k,q}}(G) \text{ is connected whenever } k/q \geq r\}$. Brewster and Noel [16] obtained bounds for $m_c(G)$ and posed some interesting questions. They characterised graphs G such that $\mathcal{C}_G(G)$ is connected; this result requires a number of definitions and we omit it here.

Theorem 3.11 [16] (i) If G is a graph of order n, then $m_c(G) \leq 2\operatorname{col}(G)$ and $m_c(G) \leq \max\left\{\frac{n+1}{2}, m_0(G)\right\}$. If G has at least one edge, then $m_c(G) \leq 2\Delta(G)$.

- (ii) If G is a tree or a complete bipartite graph and $n \geq 2$, then $m_c(G) = 2$.
- (iii) If G is nonbipartite, then $m_c(G) \ge \max\{4, \omega(G) + 1\}$.

Question 3.3 [16]

- (i) Is $m_c(G)$ always rational? When is it an integer?
- (ii) Does there exist a real number r such that $m_c(G) \leq rm_0(G)$ for every graph G? If so, what is the smallest such r?

3.3 The k-Edge-Colouring Graph

In an attempt to prove the Four Colour Theorem, Alfred Bray Kempe introduced the notion of changing map colourings by switching the colours of regions in a maximal connected section of a map formed by regions coloured with two specific colours, so as to eliminate a colour from regions adjacent to an uncoloured region. (See e.g. [21, Chapter 16].) If we consider proper edge-colourings of a graph G, then the subgraph H of G induced by all edges of two

³For comparison with $m_0(G)$ we deviate slightly from the definition in [16] and adjust the results accordingly.

fixed colours has maximum degree 2; hence it consists of the disjoint union of nontrivial paths and even cycles with edges of alternating colours. These components of H are now called edge-Kempe chains. We say that the proper k-edge-colourings c_1 and c_2 of G are adjacent in the k-edge-colouring graph $\mathcal{EC}_k(G)$ if one can be obtained from the other by switching two colours along an edge-Kempe chain. If a proper k-edge-colouring c_r can be converted to c_s by a (possibly empty) sequence of edge-Kempe switches, that is, if c_r and c_s are in the same component of $\mathcal{EC}_k(G)$, then we say that c_r and c_s are edge-Kempe equivalent and write $c_r \sim c_s$. Note that \sim is an equivalence relation; we may consider its equivalence classes on the set of k-edge-colourings of G. Two edge-colourings that differ only by a permutation of colours are edge-Kempe equivalent, because the symmetric group S_k is generated by transpositions.

Most of the work on edge-Kempe equivalent edge-colourings has focused on the number of equivalence classes of k-edge-colourings, i.e., the number of components of $\mathcal{EC}_k(G)$, which we denote by K'(G,k). In particular, the question of when K'(G,k) = 1 has received considerable attention. In this section we allow our graphs to have multiple edges. We denote the chromatic index (edge-chromatic number) of G by $\chi'(G)$. Vizing (see e.g. [21, Theorem 17.2]) proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for any graph G.

Mohar [46] showed that if $k \geq \chi'(G) + 2$, then $\mathcal{EC}_k(G)$ is connected, i.e., K'(G, k) = 1 for any graph G, while if G is bipartite and $k \geq \Delta(G) + 1$, then K'(G, k) = 1. He stated the characterisation of cubic bipartite graphs G with K'(G,3) = 1 as an open problem, and he conjectured that K'(G,4) = 1 when $\Delta(G) \leq 3$. (By König's Theorem (see e.g. [21, Theorem 17.7]), $\chi'(G) = 3$ for a cubic bipartite graph G.) McDonald, Mohar, and Scheide [45] proved Mohar's conjecture and showed that $K'(K_5,5) = 6$.

Theorem 3.12 [45] (i) If $\Delta(G) \leq 3$, then $K'(G, \Delta(G) + 1) = 1$.

(ii) If
$$\Delta(G) \leq 4$$
, then $K'(G, \Delta(G) + 2) = 1$.

In [5], belcastro and Haas provided partial answers to Mohar's question on cubic bipartite graphs G with K'(G,3) = 1. They showed that all 3-edge-colourings of planar bipartite cubic graphs are edge-Kempe equivalent, and constructed infinite families of simple nonplanar 3-connected bipartite cubic graphs, all of whose 3-edge-colourings are edge-Kempe equivalent. In [6], they investigated $\mathcal{EC}_k(G)$ for k-edge-colourable k-regular graphs, and showed that if such a graph is uniquely k-edge-colourable, then $\mathcal{EC}_k(G)$ is isomorphic to the Cayley graph of the symmetric group S_k with the set of all transpositions as generators.

4 Reconfiguration of Dominating Sets

There are several types of reconfiguration graphs of dominating sets of a graph. Here we consider k-dominating graphs, k-total-dominating graphs, and γ -graphs. In the first two cases, the vertices of the reconfiguration graph correspond to (not necessarily minimal) dominating sets of cardinality k or less, whereas the vertices of γ -graphs correspond to minimum dominating sets, also referred to as γ -sets. A minimal dominating set of maximum cardinality Γ is called a Γ -set.

A graph G is well-covered if all its maximal independent sets have cardinality $\alpha(G)$. A set $X \subseteq V(G)$ is irredundant if each vertex in X dominates a vertex of G (perhaps itself) that is not dominated by any other vertex in X. An irredundant set is maximal irredundant if it has no irredundant proper superset. The lower and upper irredundant numbers $\operatorname{ir}(G)$ and $\operatorname{IR}(G)$ of G are, respectively, the smallest and largest cardinalities of a maximal irredundant set of G. If X is a maximal irredundant set of cardinality $\operatorname{ir}(G)$, we call X an ir-set; an IR-set is defined similarly.

A graph G is *irredundant perfect* if $\alpha(H) = \operatorname{IR}(H)$ for all induced subgraphs H of G. Given a positive integer k, the family \mathcal{L}_k consists of all graphs G containing vertices x_1, \ldots, x_k such that for each i, the subgraph induced by $N[x_i]$ is complete, and $\{N[x_i]: 1 \leq i \leq k\}$ partitions V(G). Let $\mathcal{L} = \bigcup_{k>1} \mathcal{L}_k$. We use the graphs defined here in the next section.

4.1 The k-Dominating Graph

The concept of k-dominating graphs was introduced by Haas and Seyffarth [34] in 2014. This paper stimulated the work of Alikhani, Fatehi, and Klavžar [1], Mynhardt, Roux, and Teshima [51], Suzuki, Mouawad, and Nishimura [56], and their own follow-up paper [35].

As is the case for k-colouring graphs, we seek to determine conditions for the k-dominating graph $\mathcal{D}_k(G)$ to be connected. Haas and Seyffarth [34] showed that any Γ -set S of G is an isolated vertex of $\mathcal{D}_{\Gamma}(G)$ (because no proper subset of S is dominating). Therefore, $\mathcal{D}_{\Gamma}(G)$ is disconnected whenever G has at least one edge (and thus at least two minimal dominating sets). In particular, $\mathcal{D}_{n-1}(K_{1,n-1})$ is disconnected, but $\mathcal{D}_k(K_{1,n-1})$ is connected for all $k \in \{1,...,n\} - \{n-1\}$. This example demonstrates that $\mathcal{D}_k(G)$ being connected does not imply that $\mathcal{D}_{k+1}(G)$ is connected. However, Haas and Seyffarth showed that if $k > \Gamma(G)$ and $\mathcal{D}_k(G)$ is connected, then $\mathcal{D}_{k+1}(G)$ is connected. They defined $d_0(G)$ to be the smallest integer ℓ such that $\mathcal{D}_k(G)$ is connected for all $k \geq \ell$, and noted that, for all graphs G, $d_0(G)$ exists because $\mathcal{D}_n(G)$ is connected. They bounded $d_0(G)$ as follows.

Theorem 4.1 [34] For any graph G with at least one edge, $d_0(G) \ge \Gamma(G) + 1$. If G has at least two disjoint edges, then $d_0(G) \le \min\{n-1, \Gamma(G) + \gamma(G)\}$.

Haas and Seyffarth [35] showed that all independent dominating sets of G are in the same component of $D_{\Gamma(G)+1}(G)$ and established the following upper bound for $d_0(G)$; for a graph with $\gamma = \alpha$ it improves the bound in Theorem 4.1.

Theorem 4.2 [35] For any graph G, $d_0(G) \leq \Gamma(G) + \alpha(G) - 1$. Furthermore, if G is triangle-free, then $d_0(G) \leq \Gamma(G) + \alpha(G) - 2$.

Graphs for which equality holds in the lower bound in Theorem 4.1 (provided they are connected and nontrivial) include bipartite graphs, chordal graphs [34], graphs with $\alpha \leq 2$, graphs that are perfect and irredundant perfect, well-covered graphs with neither C_4 nor C_5 as subgraph, well-covered graphs with girth at least five, well-covered claw-free graphs without 4-cycles, well-covered plane triangulations, and graphs in the class \mathcal{L} [35].

Suzuki et al. [56] were first to exhibit graphs for which $d_0 > \Gamma + 1$. They constructed an infinite class of graphs $G_{(d,b)}$ (of tree-width 2b-1) for which $d_0(G_{(d,b)}) = \Gamma(G_{(d,b)}) + 2$; the smallest of these is $G_{(2,3)} \cong P_3 \square K_3$, which is planar. Haas and Seyffarth [35] also found a graph G_4 such that $d_0(G_4) = \Gamma(G_4) + 2$, and they mentioned that they did not know of the existence of any graphs with $d_0 > \Gamma + 2$. Mynhardt et al. [51] constructed classes of graphs that demonstrate (a) the existence of graphs with arbitrary upper domination number $\Gamma \geq 3$, arbitrary domination number in the range $1 \leq \gamma \leq \Gamma$, and $1 \leq \gamma \leq \Gamma$ (see Figure 1 for an example), and (b) the existence of graphs with arbitrary upper domination number $1 \leq 3$, arbitrary domination number in the range $1 \leq \gamma \leq \Gamma - 1$, and $1 \leq \gamma \leq \Gamma - 1$, and $1 \leq \gamma \leq \Gamma = 1$ (see Figure 2 for an example). For $1 \leq \gamma \leq 1$ this was the first construction of graphs with $1 \leq 1 \leq 1$ and $1 \leq 1 \leq 1 \leq 1$. These results are best possible in both cases, since it follows from Theorems 4.1 and 4.2 that $1 \leq 1 \leq 1 \leq 1 \leq 1$. For $1 \leq 1 \leq 1 \leq 1 \leq 1$ and $1 \leq 1 \leq 1 \leq 1 \leq 1 \leq 1 \leq 1$.

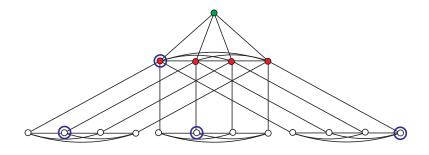


Figure 1: A graph G with $\gamma(G) = \Gamma(G) = 4$ and $d_0(G) = 7 = \Gamma(G) + \gamma(G) - 1$

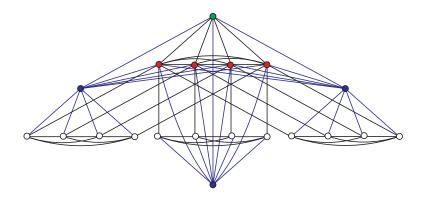


Figure 2: A graph Q with $\gamma(Q) = 3$, $\Gamma(Q) = 4$ and $d_0(Q) = 7 = \Gamma(Q) + \gamma(Q)$

Suzuki et al. [56] related the connectedness of $\mathcal{D}_k(G)$ to matchings in G by showing that if G has a matching of size (at least) $\mu + 1$, then $\mathcal{D}_{n-\mu}(G)$ is connected. This result is best possible with respect to the size of a maximum matching, since the path P_{2k} has matching number $\mu = k = \Gamma(P_{2k}) = n - \mu$, hence $\mathcal{D}_{n-\mu}(P_{2k})$ is disconnected. It also follows that the diameter of $\mathcal{D}_{n-\mu}(G)$ is in O(n) for a graph G with a matching of size $\mu + 1$. On the other hand, they constructed an infinite family of graphs G_n of order 63n - 6 such that $\mathcal{D}_{\gamma(G)+1}(G_n)$ has exponential diameter $\Omega(2^n)$.

Question 4.1

- (i) [34] Characterise graphs for which $d_0 = \Gamma + 1$.
- (ii) [51] Is it true that $d_0(G) = \Gamma(G) + 1$ when G is triangle-free?
- (iii) [34] When is $\mathcal{D}_k(G)$ Hamiltonian?
- (iv) [51] Suppose $\mathcal{D}_i(G)$ and $\mathcal{D}_j(G)$ are connected and i < j. How are diam($\mathcal{D}_i(G)$) and diam($\mathcal{D}_i(G)$) related? (If $i > \Gamma(G)$, then diam($\mathcal{D}_i(G)$) \geq diam($\mathcal{D}_i(G)$).)

Haas and Seyffarth [34] considered the question of which graphs are realisable as k-dominating graphs and observed that for $n \geq 4$, $\mathcal{D}_2(K_{1,n-1}) = K_{1,n-1}$. Alikhani et al. [1] proved that these stars are the only graphs with this property, i.e. if G is a graph of order n with no isolated vertices such that $n \geq 2$, $\delta \geq 1$, and $G \cong \mathcal{D}_k(G)$, then k = 2 and $G \cong K_{1,n-1}$ for some $n \geq 4$. They also showed that C_6, C_8, P_1 and P_3 are the only cycles or paths that are dominating graphs of connected graphs $(\mathcal{D}_2(K_3) = C_6, \mathcal{D}_3(P_4) = C_8, \mathcal{D}_1(K_1) = P_1$ and $\mathcal{D}_2(K_2) = P_3)$. They remarked that $\mathcal{D}_n(G)$ has odd order for every graph G (since G has an odd number of dominating sets [14]), and showed that if m is odd and $0 < m < 2^n$, then there exists a graph X of order n such that $\mathcal{D}_n(X)$ has order m.

It is obvious that $\mathcal{D}_k(G)$ is bipartite for any graph G of order n and any k such that $\gamma(G) \leq k \leq n$; in fact, $\mathcal{D}_k(G)$ is an induced subgraph of $Q_n - v$, a hypercube with one vertex deleted [1].

Question 4.2 Which induced subgraphs of Q_n occur as $\mathcal{D}_k(G)$ for some n-vertex graph G and some integer k?

4.2 The k-Total-Dominating Graph

For a graph G without isolated vertices, a set $S \subseteq V(G)$ is a total-dominating set (TDS) if every vertex of G is adjacent to a vertex in S. We denote the minimum (maximum, respectively) cardinality of a minimal TDS by $\gamma_t(G)$ ($\Gamma_t(G)$, respectively). Alikhani, Fatehi, and Mynhardt [2] initiated the study of k-total-dominating graphs (see Section 1). Since any TDS is a dominating set, $\mathcal{D}_k^t(G)$ is an induced subgraph of $\mathcal{D}_k(G)$ for any isolate-free graph G and any integer $k \geq \gamma_t(G)$. However, since Γ and Γ_t are not comparable (for n large enough, $\Gamma_t(K_{1,n}) = 2 < \Gamma(K_{1,n}) = n$ but $\Gamma(P_n) < \Gamma_t(P_n)$), the two graphs $\mathcal{D}_k(G)$ and $\mathcal{D}_k^t(G)$ can be different.

To study the connectedness of $\mathcal{D}_k^t(G)$, we define $d_0^t(G)$ similar to $d_0(G)$ (Section 4.1). Unlike $\mathcal{D}_{\Gamma}(G)$, there are nontrivial connected graphs G such that $\mathcal{D}_{\Gamma_t}(G)$ is connected and $d_0^t(G) = \Gamma_t(G)$, as shown below. The unique neighbour of a vertex of degree one is called a *stem*. Denote the set of stems of G by S(G).

Theorem 4.3 [2] If G is a connected graph of order $n \geq 3$, then

(i) $\mathcal{D}_{\Gamma_t}^t(G)$ is connected if and only if S(G) is a TDS of G,

- (ii) $\Gamma_t(G) \leq d_0^t(G) \leq n$,
- (iii) any isolate-free graph H is an induced subgraph of a graph G such that $\mathcal{D}_{\Gamma_t}^t(G)$ is connected (G is the corona of H),
- (iv) if G is a connected graph of order $n \geq 3$ such that S(G) is a TDS, then $\mathcal{D}_{\gamma_t}^t(G)$ is connected (S(G)) is the unique TDS.

The lower bound in Theorem 4.3(ii) is realised if and only if G has exactly one minimal TDS, i.e. if and only if S(G) is a TDS. The upper bound is realised if and only if $\Gamma_t(G) = n - 1$, i.e. if and only if n is odd and G is obtained from $\frac{n-1}{2}K_2$ by joining a new vertex to at least one vertex of each K_2 .

For specific graph classes, Alikhani et al. [2] showed that $d_0^t(C_n) = \Gamma_t(C_n) + 1$ if $n \neq 8$, while if n = 8, then $d_0^t(C_8) = \Gamma_t(C_8) + 2$. Hence $\mathcal{D}_{\Gamma_t+1}^t(C_8)$ is disconnected, making C_8 the only known graph with this property. For paths, $d_0^t(P_2) = \Gamma_t(P_2) = d_0^t(P_4) = \Gamma_t(P_4) = 2$ and $d_0^t(P_n) = \Gamma_t(P_n) + 1$ if n = 3 or $n \geq 5$.

As shown in [2], Q_n and $K_{1,n}$, $n \geq 2$, are realisable as total-dominating graphs, and $C_4, C_6, C_8, C_{10}, P_1, P_3$ are the only realisable cycles and paths.

Question 4.3 [2]

- (i) Construct classes of graphs G_r such that $d_0^t(G_r) \Gamma_t(G_r) \ge r \ge 2$.
- (ii) Find more classes of graphs that can/cannot be realised as k-total-domination graphs.
- (iii) Note that $\mathcal{D}_3^t(P_3) \cong P_3$. Characterise graphs G such that $\mathcal{D}_k^t(G) \cong G$ for some k.

4.3 Jump γ -Graphs

Sridharan and Subramanaian [54] introduced jump γ -graphs $\mathcal{J}(G,\gamma)$ in 2008; they used the notation $\gamma \cdot G$ instead of $\mathcal{J}(G,\gamma)$. The γ -graphs $\mathcal{J}(G,\gamma)$ for $G \in \{P_n, C_n\}$ were determined in [54], as were the graphs $\mathcal{J}(H_{k,n},\gamma)$ for some values of k and n, where $H_{k,n}$ is a Harary graph, i.e. a k-connected graph of order n and minimum possible size $\lceil kn/2 \rceil$. The authors of [54] showed that if T is a tree, then $\mathcal{J}(T,\gamma)$ is connected. Haas and Seyffarth [34] showed that if $\mathcal{D}_{\gamma(G)+1}(G)$ is connected, then $\mathcal{J}(G,\gamma)$ is connected, thus relating k-dominating graphs to γ -graphs.

Sridharan and Subramanaian [55] showed that trees and unicyclic graphs can be realised as jump γ -graphs. Denoting the graph obtained by joining the two vertices of $K_{2,3}$ of degree 3 by Δ_3 , they showed that if H contains Δ_3 as an induced subgraph, then H is not realisable as a γ -graph $\mathcal{J}(G,\gamma)$. Following the same line of enquiry, Lakshmanan and Vijayakumar [42] proved that if H is a γ -graph, then H contains none of $K_{2,3}, K_2 \vee P_3, (K_1 \cup K_2) \vee 2K_1$ as an induced subgraph. They showed that the collection of γ -graphs is closed under the Cartesian product and that a disconnected graph is realisable if and only if all its components are realisable. They also proved that if G is a connected cograph, then $\dim(\mathcal{J}(G,\gamma)) \leq 2$, where $\dim(\mathcal{J}(G,\gamma)) = 1$

if and only if G has a universal vertex. Bień [7] studied $\mathcal{J}(T,\gamma)$ for trees of diameter at most 5 and for certain caterpillars.

In his Master's thesis [25], Dyck illustrated a connection between γ -graphs and Johnson graphs. The *Johnson graph* J(n,k) is the graph whose vertex set consists of all k-subsets of $\{1,...,n\}$, where two vertices are adjacent whenever their corresponding sets intersect in exactly k-1 elements.

Theorem 4.4 [25] A graph H is realisable as $\mathcal{J}(G,\gamma)$, where G is an n-vertex graph with $\gamma(G) = k$, if and only if H is isomorphic to an induced subgraph of J(n,k).

Edwards, MacGillivray, and Nasserasr [28] obtained results which hold for jump and slide γ -graphs; we report their results in Theorem 4.6.

4.4 Slide γ -Graphs

Fricke, Hedetniemi, Hedetniemi, and Hutson [31] introduced slide γ -graphs $\mathcal{S}(G, \gamma)$ in 2011; they used the notation $G(\gamma)$ instead of $\mathcal{S}(G, \gamma)$. They showed that every tree is realisable as a slide γ -graph, that $\mathcal{S}(T, \gamma)$ is connected and bipartite if T is a tree, and that $\mathcal{S}(G, \gamma)$ is triangle-free if G is triangle-free. They determined $\mathcal{S}(G, \gamma)$ for a number of graph classes, including complete and complete bipartite graphs, paths and cycles.

Connelly, Hedetniemi, and Hutson [24] extended the realisability result obtained in [31].

Theorem 4.5 [24] Every graph is realisable as a γ -graph $S(G, \gamma)$ of infinitely many graphs G.

Connelly et al. [24] also showed that the γ -graphs of all graphs of order at most 5 are connected and characterised graphs of order 6 with disconnected γ -graphs.

Edwards et al. [28] investigated the order, diameter, and maximum degree of jump and slide γ -graphs of trees, providing answers to questions posed in [31].

Theorem 4.6 [28] If T is a tree of order n having s stems, then

- (i) $\Delta(S(T,\gamma)) \leq n \gamma(T)$ and $\Delta(J(T,\gamma)) \leq n \gamma(T)$,
- (ii) $\operatorname{diam}(\mathcal{S}(T,\gamma)) \le 2(2\gamma(T)-s)$ and $\operatorname{diam}(\mathcal{J}(T,\gamma)) \le 2\gamma(T)$,
- $(iii) |V(S(T,\gamma))| = |V(J(T,\gamma))| \le ((1+\sqrt{13})/2)^{\gamma(T)}.$

It follows that the maximum degree and diameter of γ -graphs of trees are linear in n. Edwards et al. exhibited an infinite family of trees to demonstrate that the bounds in Theorem 4.6(i) are sharp and mentioned that there are no known trees for which $\dim(\mathcal{S}(T,\gamma))$ or $\dim(\mathcal{J}(T,\gamma))$ exceeds half the bound given in Theorem 4.6(ii). They also demonstrated that $|V(\mathcal{S}(T,\gamma))| > 2^{\gamma(T)}$ for infinitely many trees.

Question 4.4

- (i) [31] Which graphs are γ -graphs of trees?
- (ii) [52] Is every bipartite graph the γ -graph of a **bipartite** graph?

4.5 Irredundance

Mynhardt and Teshima [52] studied slide reconfiguration graphs for other domination parameters. In particular, for an arbitrary given graph H they constructed a graph G_H to show that H is realisable as the slide Γ -graph $S(G_H, \Gamma)$ of G_H . Although G_H satisfies $\Gamma(G_H) = \operatorname{IR}(G_H)$, it has more IR-sets than Γ -sets. Hence H is not an IR-graph of G_H . They left the problem of whether all graphs are IR-graphs open. Mynhardt and Roux [50] responded as follows.

Theorem 4.7 [50] (i) All disconnected graphs can be realised as IR-graphs.

(ii) Stars $K_{1,k}$ for $k \geq 2$, the cycles C_5 , C_6 , C_7 , and the paths P_3 , P_4 , P_5 are not IR-graphs.

Mynhardt and Roux also showed that the double star S(2,2) (obtained by joining the central vertices of two copies of P_3), and the tree obtained by joining a new leaf to a leaf of S(2,2), are the unique smallest IR-trees with diameters 3 and 4, respectively. The only connected IR-graphs of order 4 are K_4 and C_4 . We close with one of their questions and a conjecture.

Conjecture 4.1 [50] P_n is not an IR-graph for each $n \geq 3$, and C_n is not an IR-graph for each n > 5.

Question 4.5 [50] Are complete graphs and C_4 the only claw-free IR-graphs?

Acknowledgement This survey was published as [49].

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