1) (a)

$$N \propto M_l^{-1.35} - M_u^{-1.35}$$

When $M_u >> M_l$, we have $N \propto M_l^{-1.35}$.

(b)

$$M_{total} = \int M\xi(M)dM \propto M_l^{-0.35}.$$

(c)

$$L_{total} = \int L(M)\xi(M)dM = \int M^{\alpha}\xi(M)dM \propto M_u^{\alpha-1.35} - M_l^{\alpha-1.35},$$

so as far as *alpha* > 1.35, the total luminosity is going to depend on bright stars, while the total mass on faint stars.

2) the main point is: there is more volume in region C than region A, even though the width of the shells A, B, C are the same. Therefore, when you have a spread in true luminosity, there will always be more objects scattered into the observed sample from region C, because they are intrinsically brighter than average, than objects scattered out of the observer sample from region A, because they are intrinsically fainter. So on average, you are getting the observed sample to be somewhat intrinsically brighter than the true average over the whole sky. If you are assuming that all objects have the same luminosity, then your sample will reach the distance in region B. But in reality, there are a larger number of more distant stars coming into the sample, so the average distance you find by assuming all objects with the same luminosity is smaller than the true average distance. The true reason behind this bias is even though the error distribution is symmetric for a given objects, they are always more objects at larger distance because of volume difference.

Similarly, average Z in region C will be larger than that in region A, because in a flux limited sample, only more metal rich (and brighter) stars in region C can be seen.

Solution Set #2 : Hints & Answers

3)

Same functional form, just "a" instead of "a_N" and "k" instead of "rho_N"

The density depends on radius r and does not depend on angle: the profile is spherically symmetric. Therefore the equation of continuity of mass $\mathrm{d}M/\mathrm{d}r = 4\pi r^2 \rho$ applies. Substituting for the profile ρ,

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \frac{k}{r(r+a)^2} .$$

Integrating from the centre to radius r,

$$\int_0^{M(r)} \mathrm{d}M = \int_0^r 4\pi r'^2 \frac{k}{r'(r'+a)^2} \, \mathrm{d}r' \qquad \therefore \; M(r) = 4\pi k \int_0^r \frac{r'}{(r'+a)^2} \, \mathrm{d}r' .$$

This integral can be solved using

$$\int \frac{r}{(r+a)^2} \, \mathrm{d}r = \int \frac{r+a-a}{(r+a)^2} \, \mathrm{d}r = \int \frac{r+a}{(r+a)^2} \, \mathrm{d}r - a \int \frac{1}{(r+a)^2} \, \mathrm{d}r$$

$$= \int \frac{1}{(r+a)} \, \mathrm{d}r + a \frac{1}{(r+a)} = \ln(r+a) + \frac{a}{(r+a)} + c .$$

$$\therefore \; M(r) = 4\pi k \left[\ln(r'+a) + \frac{a}{(r'+a)} \right]_0^r$$

$$= 4\pi k \left(\ln(r+a) + \frac{a}{(r+a)} - \ln(a) + \frac{a}{(a)} \right)$$

$$\text{So} \qquad M(r) = 4\pi k \left(\ln\left(\frac{r}{a}+1\right) - \frac{r}{(r+a)} \right) ,$$

which is the mass inside a radius r that the question asks for.
As $r \to \infty$, $M(r) \to \infty$. So the model is not physically realistic at large radii.

For spherical symmetry, $\dfrac{GM(r)}{r^2} = \dfrac{\mathrm{d}\Phi}{\mathrm{d}r}$. Substituting for $M(r)$ and integrating from infinity to radius r,

$$\int_0^{\Phi(r)} \mathrm{d}\Phi = \int_\infty^r \frac{4\pi G k}{r^2} \left(\ln\left(\frac{r}{a}+1\right) - \frac{r}{(r+a)} \right) \mathrm{d}r$$

because the potential is 0 at $r \to \infty$. Therefore,

$$\Phi(r) - 0 = 4\pi G k \int_\infty^r \left(\frac{1}{r^2} \ln\left(\frac{r}{a}+1\right) - \frac{1}{r(r+a)} \right) \mathrm{d}r$$

The integrals can be solved fairly easily.

$$\int \frac{1}{r(r+a)} \, \mathrm{d}r = \frac{1}{a} \int \left(\frac{1}{r} - \frac{1}{r+a} \right) \mathrm{d}r \quad \text{using partial fractions}$$

$$= \frac{1}{a}\Big(\ln r - \ln(r+a) \Big) + c = \frac{1}{a} \ln\left(\frac{r}{r+a}\right) + c$$

Using integration by parts,

$$\int \frac{1}{r^2} \ln\left(\frac{r+a}{a}\right) \mathrm{d}r = -\frac{1}{r}\ln\left(\frac{r+a}{a}\right) + \int \frac{1}{r}\frac{1}{r+a}\,\mathrm{d}r$$

$$= -\frac{1}{r}\ln\left(\frac{r+a}{a}\right) + \frac{1}{a}\ln\left(\frac{r}{r+a}\right) + c \quad \text{using the integral above.}$$

Using these integrals, we get for the potential

$$\Phi(r) = 4\pi Gk\left[-\frac{1}{r}\ln\left(\frac{r+a}{a}\right) + \frac{1}{a}\ln\left(\frac{r}{r+a}\right) - \frac{1}{a}\ln\left(\frac{r}{r+a}\right)\right]_{\infty}^{r}$$

$$= 4\pi Gk\left[-\frac{1}{r}\ln\left(\frac{r+a}{a}\right)\right]_{\infty}^{r} = 4\pi Gk\left(-\frac{1}{r}\ln\left(\frac{r+a}{a}\right) - 0\right)$$

So the potential at a distance r from the centre is

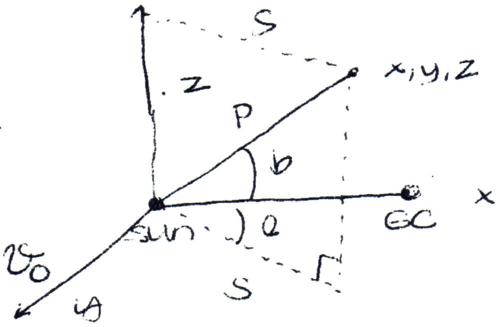
$$\Phi(r) = -\frac{4\pi Gk}{r}\ln\left(\frac{r}{a}+1\right) .$$

The central density is infinite. This appears not to be physically realistic. (However, one of the people who first used this profile to describe galaxies – Carlos Frenk – has argued that the profile might be realistic in the cores of galaxies after all. He was thinking about the possibility that most massive galaxies have black holes – and therefore a singularity – at their cores.)

4)

Spherical to rectangular coordinates

e.g.



$S = \sqrt{x^2+y^2} = p\cos b$

$x = S\cos l$

$x = p\cos l\cos b$

$y = S\sin l$

$\underline{y = p\sin l\cos b}$

$y'\ v_y = v_0\sin l\cos b$