Comparing Path Smoothing Formulations (QP, LP-∞, LP-1) over Various Problem Sizes

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I. Problem Size Effects

There are two primary methods of adjusting the problem size for determining its effects. The first method is to consider the full planning horizon, but to adjust the resolution (step size) used for planning. This method is referred to herein as "slicing" or "sliced". The second method is to only consider a shortened horizon at a fixed resolution or step size. This is referred to as the "short" method.

There are trade-off to the two methods, primarily concerning the resolution and horizon length for planning. A lower resolution (larger step size) can affect the quality of the solution (i.e., higher cost), due to the large times over which each control decision is applied and the potential for greater error introduced by the discretization method. A shorter horizon can lead to greater potential for planning difficulties or even infeasibility, as well as higher final costs (i.e., less optimal total solutions). A longer horizon time (short method) or higher resolution (sliced method), on the other hand, result in more optimization variables, which relates to the solve time of the optimization. The following sections investigate some of these relationships and trade-offs.

A. Solve time

As noted, the solve time of the optimization is related to the problem size in both the number of free variables, and the number of constraints. Due to the nature of this problem, both the number of variables and constraints grow with the increase in the number of discrete steps of the problem, for a given formulation of the problem. There is also a difference in the size of the problem between the methods of formulating the problem, with the QP formulation being the smallest, the LP-∞ formulation the next smallest, and the LP-1 formulation the largest. The solve times for various numbers of discretization steps are compared for the various formulations (QP, LP-∞, and LP-1) and methods (sliced and short).

As shown in Fig. 1 the problem really starts to become burdensome to compute with a path of over 1000 steps. For both a QP (with Dual Simplex) and the LP formulations this is true, although more so for the QP. Note that solving the QP with the Barrier method is much faster (see Fig. 2), even at the largest problem size where the problem is solved in about 1 second compared to the 15 and 100 seconds for the LPs and QP w/Dual Simplex respectively. However, as the Barrier method cannot be used for the QP with mixed integers (i.e., obstacles) in Gurobi, it is more useful to understand the Dual Simplex solve time of the QP.

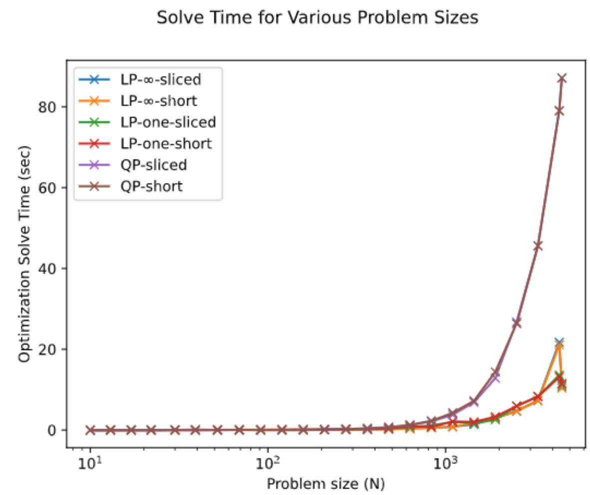


Fig. 1. The solve time of the problem types and formulations over the problem size measured as the number of time steps in the horizon.

Also of note is the fact that the differences between the short and sliced methods are fairly negligible compared to the effect of both problem size and formulation. Additionally, the difference between the two LP formulations is fairly minimal comparative to the differences induced by the differing of the problem size. This indicates that the biggest factors affecting the solve time are the formulation type (i.e., QP vs LP) and the problem size.

B. Objective Value

There is a concern of the resolution (step size) of the sliced method affecting the quality of the solution. To investigate this, the resulting objective values at each resolution (corresponding to number of discretization steps) is compared. This requires some care, however, as the objective value is the sum of the weighted error at each discretization step. Changing the number of steps will naturally increase the objective value as there are more terms in the sum. Thus, to provide a fair comparison the objective value is multiplied by the step size in Fig. 3. The primary take away is that for nearly all step sizes the normalized objective value is nearly constant. Note that the cost differences between the different formulations are due to the different norms that correspond to the formulation, and are expected.

The "weirdness" of the end values is due to the fact that for a trajectory with an original number of 4516 steps and a step size of 0.01 seconds the only way to achieve a number of

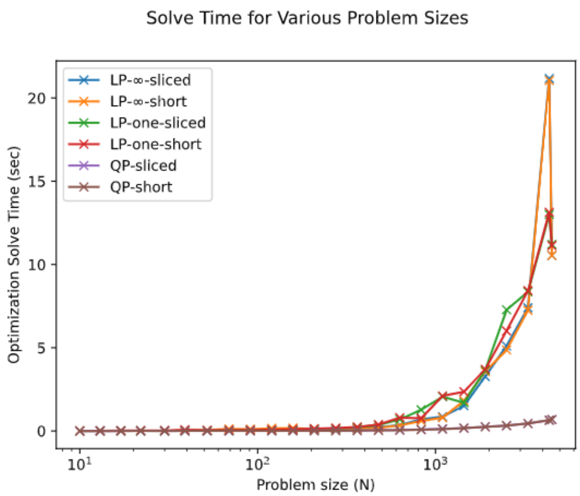


Fig. 2. The solve time where the barrier method is used for the QP rather than the dual-simplex method.

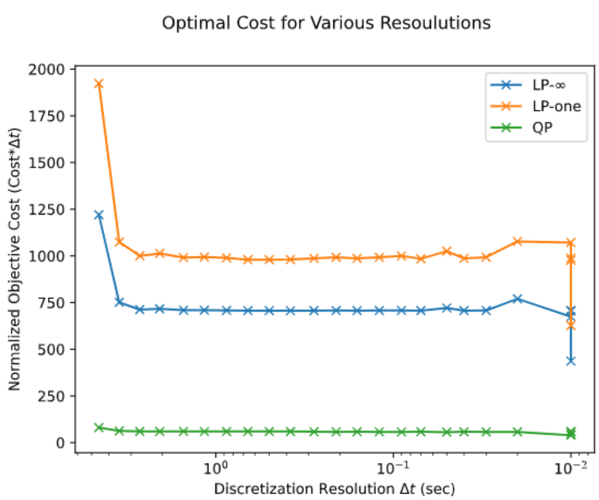


Fig. 3. The step size versus the normalized objective value.

steps between $4516 \div 2 = 2258$ and 4516 by using the points from the original trajectory is to just use that many points from the trajectory along with the original step size. Thus, the trajectory for the last few points of the plot have the same step size but different number of steps in the trajectory. Thus, they have a different number of steps in the sum, but are being "normalized" with the same step size (0.01).

The figure also shows that for low resolution (large step sizes) the value begins to be affected, and highly so in the cases of the LPs. It is unclear why the LPs are so highly affected when the QP formulation is not, and the effects for even lower resolutions were not investigated (the lowest resolution is at a step size of ≈4.5 seconds).

C. Take-away

The most relevant results from investigating the effects of the problem size indicate that there is a range of problem sizes that constitute a sort of "sweet spot". Using a resolution that is too low results in the solution quality being negatively affected, but using a larger problem size (potentially corresponding to a higher resolution) and the solve time begins to be too burdensome. While acceptable solve times vary by application, it is clear that much more that a problem size of $N = 1000$ and the computational burden is likely to be too much, or very nearly so. A problem with a discretization step size greater than about one to two seconds, on the other hand, is likely to provide a solution that is of a low quality.

While the problem size and the resolution are not directly linked, they are implicitly connected and so the problem size and resolution limits provide upper and lower bounds of sorts. For example, in an MPC application where a fixed horizon length (in seconds, not number of steps) is desired, the resolution does directly correlate to problem size, and so there are upper and lower bounds on the resolution (problem size) based on the desired solve time, and the need for resulting solutions to be quality.

II. Obstacle Effects

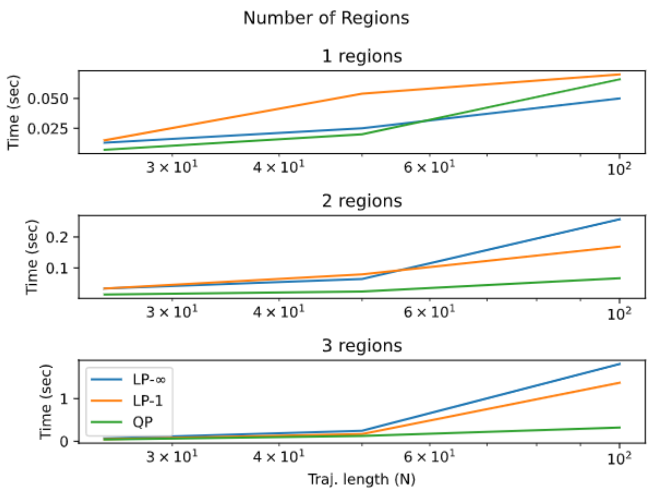
In addition to knowing how the "internal" parameters of the problem (resolution and problem size) affect solution time, it is also important to understand how aspects of the environment can affect solution times. For this reason, various aspects of the obstacle representation and their relationship to the trajectory location are investigated in terms of the effect on solve time.

The obstacle avoidance constraint is formulated as a Mixed-Integer Linear constraint by constraining every point of the trajectory to be within one of several obstacle free polyhedral regions. An obvious avenue of investigation is to explore the effect of the number of polyhedral regions on the solve time. Another less obvious avenue is the effect of the relative position of an obstacle (encapsulated by the relative position of the obstacle free regions) to the trajectory, and specifically to the beginning and end points of the trajectory as the solution is constrained to be at both points.

A. Number of Regions

While the effects of problem size were more extensively studied without the presence of obstacles, the presence of the obstacle avoidance constraints can affect the solve time, and so it is useful to have a few points of reference with regards to problem size for a more in depth study. The results of this investigation are shown in Fig. 4.

Of primary note is that the QP formulation performs the best in these cases with these problem sizes. Based on the previous results with problem size, however, it is expected that the QP will not scale as effectively to larger problem sizes. While the best solve times are with only 25 steps over the 10 second horizon, this results in a step size of 0.4 seconds. This is still reasonable, as demonstrated by the resolution comparison in Subsection I-B, but the increase in required solve time is not substantial when using 50 steps while the resulting step size is half the size (0.2 seconds). This indicates that a problem size of 50 steps may be a good compromise point.



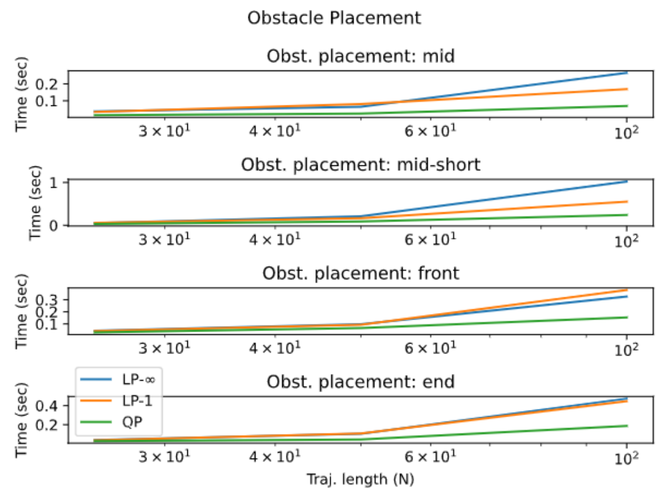


Fig. 4. The effects on the number of regions for a selection of problem sizes. The resolution is scaled by the problem size in order to maintain a 10 second trajectory, with the step size being 0.1 seconds for the problem with 100 steps. Note that the solve time of the QP with three regions and 50 steps is about 125 ms.

Fig. 5. The effect on the solve time of the placement of the obstacle in relation to the trajectory.

B. Obstacle Placement

In addition to the number of regions, the physical relationship between these regions and the trajectory also can have an effect on the solve time of the optimization. As may be expected, if the input trajectory already lies completely within the obstacle free regions the solve time is quite fast. Similarly, if there is a "significant" amount of space in between the constrained points of the trajectory (in this case the beginning and ending points of the trajectory) and the location of the obstacle (or where the trajectory is outside the obstacle free regions) the optimization runs relatively quickly.

However, when one (or more) constrained points of the trajectory are too near an obstacle the solve time can be greatly affected. This is shown in Fig. 5 where four cases are shown, where the obstacle is in the middle of the trajectory, in the middle of a short trajectory, near the beginning of the trajectory, and near the end of the trajectory. This results in the obstacle being "far" from both of the constrained points (beginning and end), "close" to both, "close" to the beginning, and "close" to the end, respectively. The resulting trajectories can be seen in Figs. 6 - 9.

Again, it can be seen in the figure that the QP formulation is the fastest for solving the optimization for these situations. Also similar to the previous results, the problem size of 50 steps strikes a balance between solve time and step size, although one that is not as convincing as previously. Given the results of both investigations, it would seem that using a QP formulation with a 50 step horizon is a good choice for an MPC formulation. This choice allows for rapid online updates with a horizon that allows for quality solutions. Actual MPC validations must be performed as there may be additional factors not accounted for in these tests, but these tests provide a good starting point and intuition for investigating the MPC performance under its various hyper-parameters.

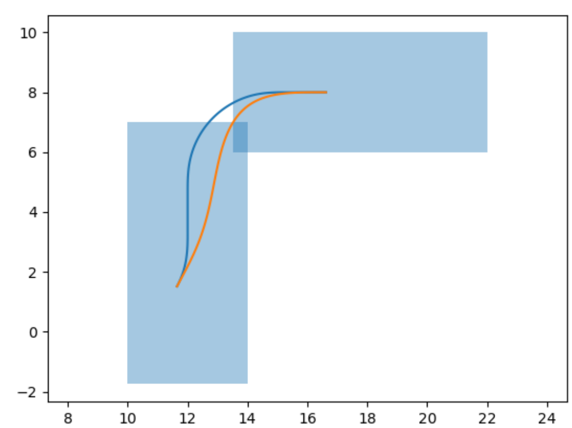


Fig. 6. The obstacle is in the middle of the trajectory, i.e., far from the beginning and end points that are constrained in place.

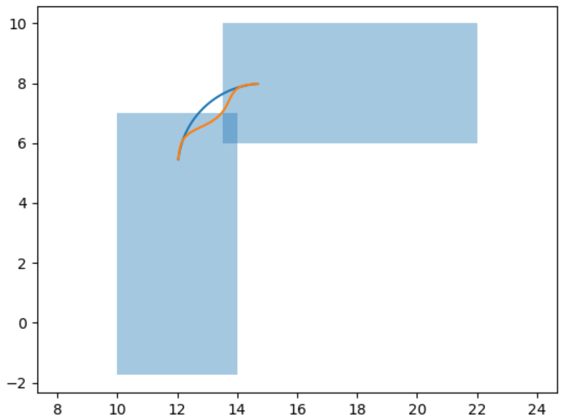


Fig. 7. The obstacle is in the middle of a shorter trajectory, i.e., close to both the beginning and end points that are constrained in place.

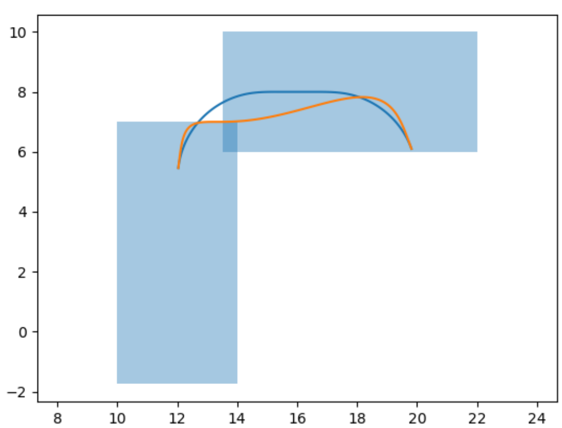


Fig. 8. The obstacle is in the beginning of the trajectory, i.e., close to the beginning point, but far from the end point.

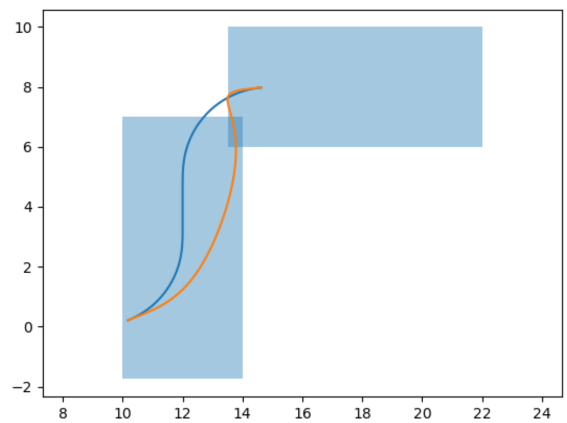


Fig. 9. The obstacle is in the end of the trajectory, i.e., far from the beginning point and close to the end point.