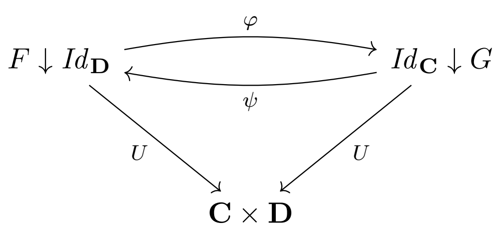
Comma Category Characterization of an Adjunction

Definition 1. *Let $U : \mathbf{C} \downarrow \mathbf{D} \to \mathbf{C} \times \mathbf{D}$ be the forgetful functor from the comma category into the product category, defined on objects by $U(A, B, f) = (A, B)$ and arrows by $U(\alpha, \beta) = (\alpha, \beta)$.*

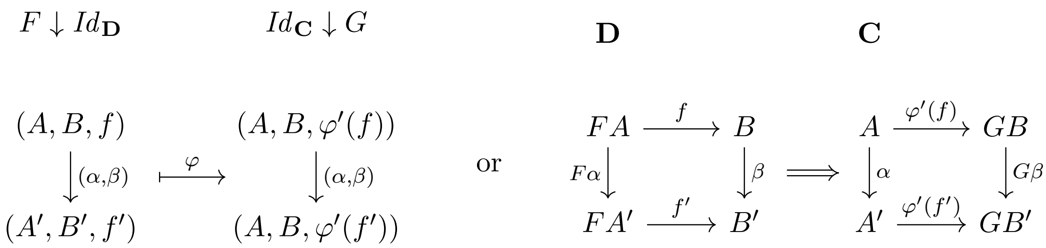
Lemma 1. *Let $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ be to functors and φ, ψ an isomorphism as in the following commuting diagram*



then $F \dashv G$.

Proof. Note, since $U(\varphi(A, B, f)) = U(A, B, f) = (A, B)$, there exists a φ', such that for all A, B, f, $\varphi(A, B, f) = (A, B, \varphi'(f))$. And for arrows it follows $\varphi(\alpha, \beta) = (\alpha, \beta)$.

Now, how does φ act on arrows?



To show: $Hom_{\mathbf{D}}(FA, B) \cong Hom_{\mathbf{C}}(A, GB)$ natural in A and B.

- We define the isomorpism as follows: $\varphi' : Hom_{\mathbf{D}}(FA, B) \rightleftarrows Hom_{\mathbf{C}}(A, GB) : \psi'$. φ', ψ' are an isomorphism because φ, ψ are an isomorphism.

- φ' natural in A, B, i.e.

$$
\begin{array}{ccccc}
A' & B & Hom_{\mathbf{D}}(FA, B) & \xrightarrow{\varphi'} & Hom_{\mathbf{C}}(A, GB) \\
\downarrow \alpha & \downarrow \beta & \downarrow \beta \circ - \circ F\alpha & & \downarrow G\beta \circ - \circ \alpha \\
A & B' & Hom_{\mathbf{D}}(FA', B') & \xrightarrow{\varphi'} & Hom_{\mathbf{C}}(A', GB')
\end{array}
$$

$$
\begin{aligned}
\varphi'(\beta \circ f \circ F\alpha) &\overset{\text{I}}{=} \varphi'(\beta \circ f) \circ \alpha \\
&\overset{\text{II}}{=} G(\beta \circ f) \circ \varphi'(F\alpha) \\
&= G\beta \circ Gf \circ \varphi'(F\alpha) \\
&\overset{\text{III}}{=} G\beta \circ \varphi'(f) \circ \alpha
\end{aligned}
$$

I $\varphi'(\beta \circ f \circ F\alpha) = \varphi'(\beta \circ f) \circ \alpha$

$$
\begin{array}{ccc}
FA & \xrightarrow{\beta \circ f \circ F\alpha} & B \\
{\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle id} \\
FA' & \xrightarrow{\beta \circ f} & B'
\end{array}
\Longrightarrow
\begin{array}{ccc}
A & \xrightarrow{\varphi'(\beta \circ f \circ F\alpha)} & GB \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle G(id)} \\
A' & \xrightarrow{\varphi'(\beta \circ f)} & GB'
\end{array}
$$

II $\varphi'(\beta \circ f) \circ \alpha = G(\beta \circ f) \circ \varphi'(F\alpha)$

$$
\begin{array}{ccc}
FA & \xrightarrow{F\alpha} & B \\
{\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle \beta \circ f} \\
FA' & \xrightarrow{\beta \circ f} & B'
\end{array}
\Longrightarrow
\begin{array}{ccc}
A & \xrightarrow{\varphi'(F\alpha)} & GB \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle G(\beta \circ f)} \\
A' & \xrightarrow{\varphi'(\beta \circ f)} & GB'
\end{array}
$$

III $Gf \circ \varphi'(\alpha) = \varphi' f \circ \alpha$

$$
\begin{array}{ccc}
FA & \xrightarrow{F\alpha} & B \\
{\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle f} \\
FA' & \xrightarrow{f} & B'
\end{array}
\Longrightarrow
\begin{array}{ccc}
A & \xrightarrow{\varphi'(F\alpha)} & GB \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle Gf} \\
A' & \xrightarrow{\varphi'(f)} & GB'
\end{array}
$$

- ψ' natural in A, B is similar.

$\square$