Comma Category Characterization of an Adjunction

Definition 1. *Let $U : \mathbf{C} \downarrow \mathbf{D} \to \mathbf{C} \times \mathbf{D}$ be the forgetful functor from the comma category into the product category, defined on objects by $U(A, B, f) = (A, B)$ and arrows by $U(i, \beta) = (i, \beta)$.*

Lemma 1. *Let $F : \mathbf{D} \rightleftarrows \mathbf{C} : G$ be to functors and φ, ψ an isomorphism as in the following commuting diagram*

$$\begin{array}{ccc} F \downarrow Id_{\mathbf{C}} & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & Id_{\mathbf{D}} \downarrow G \\ & {\scriptstyle U}\searrow \qquad \swarrow{\scriptstyle U} & \\ & \mathbf{D} \times \mathbf{C} & \end{array}$$

then $F \dashv G$.

Proof. Note, since $U(\varphi(A, B, f)) = U(A, B, f) = (A, B)$, there exists a φ', such that for all A, B, f, $\varphi(A, B, f) = (A, B, \varphi'(f))$. And for arrows it follows $\varphi(i, j) = (i, j)$.

Now, how does φ act on arrows?

$$\begin{array}{ccc} F \downarrow Id_{\mathbf{C}} & & Id_{\mathbf{D}} \downarrow G \\ & & \\ (A, B, f) & & (A, B, \varphi'(f)) \\ \Big\downarrow{\scriptstyle (i,j)} & \overset{\varphi}{\longmapsto} & \Big\downarrow{\scriptstyle (i,j)} \\ (A', B', f') & & (A, B, \varphi'(f')) \end{array} \qquad \text{or} \qquad \begin{array}{ccc} & \mathbf{C} & \\ FA & \overset{f}{\longrightarrow} & B \\ {\scriptstyle Fi}\Big\downarrow & & \Big\downarrow{\scriptstyle j} \\ FA' & \overset{f'}{\longrightarrow} & B' \end{array} \Longrightarrow \begin{array}{ccc} & \mathbf{D} & \\ A & \overset{\varphi'(f)}{\longrightarrow} & GB \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle Gj} \\ A' & \overset{\varphi'(f')}{\longrightarrow} & GB' \end{array}$$

To show: $Hom_{\mathbf{C}}(FA, B) \cong Hom_{\mathbf{D}}(A, GB)$ natural in A and B.

- We define the isomorpism as follows: $\varphi' : Hom_{\mathbf{C}}(FA, B) \rightleftarrows Hom_{\mathbf{D}}(A, GB) : \psi'$. φ', ψ' are an isomorphism because φ, ψ are an isomorphism.

- φ' natural in A, B, i.e.

$$
\begin{array}{ccccccc}
A & & B & & Hom_{\mathbf{C}}(FA, B) & \xrightarrow{\varphi'} & Hom_{\mathbf{D}}(A, GB) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j\,\circ\,-\,\circ\,Fi} & & \downarrow{\scriptstyle Gj\,\circ\,-\,\circ\,i} \\
A' & & B' & & Hom_{\mathbf{C}}(FA', B') & \xrightarrow{\varphi'} & Hom_{\mathbf{D}}(A', GB')
\end{array}
$$

$$
\begin{aligned}
\varphi'(j \circ f \circ Fi) &\overset{\text{I}}{=} \varphi'(j \circ f) \circ i \\
&\overset{\text{II}}{=} G(j \circ f) \circ \varphi'(Fi) \\
&= Gj \circ Gf \circ \varphi'(Fi) \\
&\overset{\text{III}}{=} Gj \circ \varphi'(f) \circ i
\end{aligned}
$$

I $\varphi'(j \circ f \circ Fi) = \varphi'(j \circ f) \circ i$

$$
\begin{array}{ccc}
(A, B', j \circ f \circ Fi) & & (A, B', \varphi'(j \circ f \circ Fi)) \\
\downarrow{\scriptstyle (i,id)} & \xmapsto{\varphi} & \downarrow{\scriptstyle (i,id)} \\
(A', B', j \circ f) & & (A', B', \varphi'(j \circ f))
\end{array}
$$

$$
\begin{array}{ccccccc}
FA & \xrightarrow{j\circ f\circ Fi} & B' & & A & \xrightarrow{\varphi'(j\circ f\circ Fi)} & GB' \\
{\scriptstyle Fi}\downarrow & & \downarrow{\scriptstyle id} & \Longrightarrow & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle G(id)} \\
FA' & \xrightarrow{j\circ f} & B' & & A' & \xrightarrow{\varphi'(j\circ f)} & GB'
\end{array}
$$

II $\varphi'(j \circ f) \circ i = G(j \circ f) \circ \varphi'(Fi)$

$$
\begin{array}{ccc}
(A, FA', Fi) & & (A, FA', \varphi'(Fi)) \\
\downarrow{\scriptstyle (i,j\circ f)} & \xmapsto{\varphi} & \downarrow{\scriptstyle (i,j\circ f)} \\
(A', B', j \circ f) & & (A, B', \varphi'(j \circ f))
\end{array}
$$

$$
\begin{array}{ccccccc}
FA & \xrightarrow{Fi} & FA' & & A & \xrightarrow{\varphi'(Fi)} & GFA' \\
{\scriptstyle Fi}\downarrow & & \downarrow{\scriptstyle j\circ f} & \Longrightarrow & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle G(j\circ f)} \\
FA' & \xrightarrow{j\circ f} & B' & & A' & \xrightarrow{\varphi'(j\circ f)} & GB'
\end{array}
$$

III $Gf \circ \varphi'(i) = \varphi' f \circ i$

$$
\begin{array}{ccc}
(A, FA', Fi) & & (A, FA', \varphi'(Fi)) \\
\big\downarrow{\scriptstyle (i,f)} & \xmapsto{\ \varphi\ } & \big\downarrow{\scriptstyle (i,f)} \\
(A', B', f) & & (A, B', \varphi'(f))
\end{array}
$$

$$
\begin{array}{ccc}
FA & \xrightarrow{Fi} & FA' \\
{\scriptstyle Fi}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
FA' & \xrightarrow{f} & B'
\end{array}
\Longrightarrow
\begin{array}{ccc}
A & \xrightarrow{\varphi'(Fi)} & GFA' \\
\big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle Gf} \\
A' & \xrightarrow{\varphi'(f)} & GB'
\end{array}
$$

- ψ' natural in A, B is similar.

$\square$