Comma Category Characterization of an Adjunction

Definition 1 (Comma Category)**.** *We define the comma category* $\mathbf{F} \downarrow \mathbf{G}$ *of two functors* $F : \mathbf{C} \to \mathbf{E}$ *and* $G : \mathbf{D} \to \mathbf{E}$ *as follows. Its objects are triples* (A, B, f)*, where* A *is an object in* $\mathbf{C}$*,* B *is an object in* $\mathbf{D}$*, and* $f : F(A) \to G(B)$ *is an arrow in* $\mathbf{E}$*. Furthermore, an arrow* $(i, j) : (A, B, f) \to (A', B', g)$ *in* $\mathbf{F} \downarrow \mathbf{G}$ *is a pair of arrows* $i : A \to A'$ *and* $j : B \to B'$*, such that the following diagram commutes:*

$$\begin{array}{ccc} FA & \xrightarrow{f} & GB \\ \Big\downarrow{\scriptstyle Fi} & & \Big\downarrow{\scriptstyle Gj} \\ FA' & \xrightarrow{g} & GB' \end{array}$$

Theorem 1. *Let* $F : \mathbf{D} \rightleftarrows \mathbf{C} : G$ *be two functors and* $\varphi : F \downarrow Id_{\mathbf{C}} \rightleftarrows Id_{\mathbf{D}} \downarrow G : \psi$ *an isomorphism that is the identity on arrows then* $F \dashv G$*.*

Proof. We define the adjunction $F \dashv G$ by an isomorphism between the hom-sets $Hom_{\mathbf{C}}(FA, B)$ and $Hom_{\mathbf{D}}(A, GB)$ that is natural in A and B. We define both sides of the isomorphism $\varphi' : Hom_{\mathbf{C}}(FA, B) \rightleftarrows Hom_{\mathbf{D}}(A, GB) : \psi'$ by $\varphi'(f) = \pi_3(\varphi(A, B, f))$ and $\psi'(g) = \pi_3(\psi(A, B, g))$, where π_3 extracts the arrow f from an object (A, B, f) in the comma category.

- φ' and ψ' are an isomorphism, i.e., $\psi' \circ \varphi' = id$ and $\varphi' \circ \psi' = id$. For all $f : FA \to B$ and $g : A \to GB$, it follows

$$\begin{aligned} \psi'(\varphi'(f)) &= \psi'(\pi_3(\varphi(A, B, f))) \\ &= \pi_3(\psi(A, B, \pi_3(\varphi(A, B, f))) \\ &= \pi_3(\psi(\varphi(A, B, f))) \\ &= \pi_3(A, B, f) \\ &= f \end{aligned} \qquad \begin{aligned} \varphi'(\psi'(g)) &= \varphi'\pi_3(\psi(A, B, f))) \\ &= \pi_3(\varphi(A, B, \pi_3(\psi(A, B, g))) \\ &= \pi_3(\varphi(\psi(A, B, g))) \\ &= \pi_3(A, B, g) \\ &= g. \end{aligned}$$

- φ' is natural in A, B, i.e., for all $i : A \to A'$, $j : B \to B'$, and $f : FA \to B$, $\varphi'(j \circ f \circ Fi) = Gj \circ \varphi'(f) \circ i$ as in the following commuting diagram

$$\begin{CD} Hom_{\mathbf{C}}(FA,B) @>{\varphi'}>> Hom_{\mathbf{D}}(A,GB) \\ @V{j \circ - \circ Fi}VV @VV{Gj \circ - \circ i}V \\ Hom_{\mathbf{C}}(FA',B') @>{\varphi'}>> Hom_{\mathbf{D}}(A',GB'). \end{CD}$$

First, observe how φ acts on arrows. An arrow $(i,j) : (A,B,f) \to (A',B',g)$ in $F \downarrow Id_{\mathbf{C}}$ is mapped to an arrow $(i,j) : (A,B,\varphi'(f)) \to (A',B',\varphi'(g))$ in $Id_{\mathbf{D}} \downarrow G$. This means, for a commuting square in the $\mathbf{C}$, we obtain a commuting square in $\mathbf{D}$:

$$\begin{CD} FA @>{f}>> B \\ @V{Fi}VV @VV{j}V \\ FA' @>{g}>> B' \end{CD} \quad \Longrightarrow \quad \begin{CD} A @>{\varphi'(f)}>> GB \\ @VV{i}V @VV{Gj}V \\ A' @>{\varphi'(g)}>> GB' \end{CD}$$

In other words, we can use φ to do basic rewrite steps on φ' if we provide a suitable commuting squares in $\mathbf{C}$.

With this technique, we prove the naturality of φ' with three rewrite steps and functoriality of G:

$$\begin{aligned} \varphi'(j \circ f \circ Fi) &\overset{\text{I}}{=} \varphi'(j \circ f) \circ i \\ &\overset{\text{II}}{=} G(j \circ f) \circ \varphi'(Fi) \\ &= Gj \circ Gf \circ \varphi'(Fi) \\ &\overset{\text{III}}{=} Gj \circ \varphi'(f) \circ i \end{aligned}$$

I $\varphi'(j \circ f \circ Fi) = \varphi'(j \circ f) \circ i$

$$\begin{array}{ccc} (A,B',j \circ f \circ Fi) & & (A,B',\varphi'(j \circ f \circ Fi)) \\ \downarrow{\scriptstyle (i,id)} & \xmapsto{\quad\varphi\quad} & \downarrow{\scriptstyle (i,id)} \\ (A',B',j \circ f) & & (A',B',\varphi'(j \circ f)) \end{array}$$

$$\begin{CD} FA @>{j \circ f \circ Fi}>> B' \\ @V{Fi}VV @VV{id}V \\ FA' @>{j \circ f}>> B' \end{CD} \quad \Longrightarrow \quad \begin{CD} A @>{\varphi'(j \circ f \circ Fi)}>> GB' \\ @VV{i}V @VV{G(id)}V \\ A' @>{\varphi'(j \circ f)}>> GB' \end{CD}$$

II $\varphi'(j \circ f) \circ i = G(j \circ f) \circ \varphi'(Fi)$

$$
\begin{array}{ccc}
(A, FA', Fi) & & (A, FA', \varphi'(Fi)) \\
\Big\downarrow {\scriptstyle (i, j\circ f)} & \overset{\varphi}{\longmapsto} & \Big\downarrow {\scriptstyle (i, j\circ f)} \\
(A', B', j \circ f) & & (A, B', \varphi'(j \circ f))
\end{array}
$$

$$
\begin{array}{ccc}
FA & \xrightarrow{Fi} & FA' \\
{\scriptstyle Fi}\Big\downarrow & & \Big\downarrow{\scriptstyle j\circ f} \\
FA' & \xrightarrow{j\circ f} & B'
\end{array}
\Longrightarrow
\begin{array}{ccc}
A & \xrightarrow{\varphi'(Fi)} & GFA' \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle G(j\circ f)} \\
A' & \xrightarrow{\varphi'(j\circ f)} & GB'
\end{array}
$$

III $Gf \circ \varphi'(i) = \varphi' f \circ i$

$$
\begin{array}{ccc}
(A, FA', Fi) & & (A, FA', \varphi'(Fi)) \\
\Big\downarrow {\scriptstyle (i, f)} & \overset{\varphi}{\longmapsto} & \Big\downarrow {\scriptstyle (i, f)} \\
(A', B', f) & & (A, B', \varphi'(f))
\end{array}
$$

$$
\begin{array}{ccc}
FA & \xrightarrow{Fi} & FA' \\
{\scriptstyle Fi}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\
FA' & \xrightarrow{f} & B'
\end{array}
\Longrightarrow
\begin{array}{ccc}
A & \xrightarrow{\varphi'(Fi)} & GFA' \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle Gf} \\
A' & \xrightarrow{\varphi'(f)} & GB'
\end{array}
$$

- ψ' natural in A, B is analogous to naturality of φ'.

□