Stat 400. Problem Set 1. Due 09/03/24

Introduction to probability

Directions: You must show all work to receive full credit. Turn in your completed assignment to gradescope by 11:59pm on 9/3.

Problem 1. An experiment consists of rolling two fair six-sided dice.

(a) What is the probability that at least one is a 5?
(b) What is the probability that at least one is a 5 *and* that the two numbers are different?
(c) What is the probability that at least one is a 5 *and* the sum of the two numbers is strictly greater 8?
(d) Show that 7 is the most likely sum of the two numbers.

Problem 2. An experiment consists of rolling one fair six-sided die and one unfair six-sided die with probability 0.2 for landing 1, 0.4 for landing 5, and 0.1 for the rest of the 4 numbers.

(a) What is the probability that at least one is a 5?
(b) What is the probability that at least one is a 5 *and* that the two numbers are different?
(c) What is the probability that at least one is a 5 *and* the sum of the two numbers is strictly greater 8?
(d) Is 7 still the most likely sum of the two numbers?

Problem 3. Suppose an experiment consists of tossing a fair coin until a heads shows up. Suppose E is the event: the number of tails before the head shows up is between 1 and 3 (inclusive).

(a) Explain why the sample space of the experiment is infinite.
(b) Calculate $P(E)$.
(c) Suppose instead that the coin is unfair, with $P(H) = 0.9$. What is $P(E)$ now?

Problem 4. Suppose we are working with a circular dart board with 6 concentric regions labelled 1 (outermost) to 6 (disc at the center). Assume that the dart board has radius r and that a dart thrown at the board always hits the board. Then the probability of scoring i points on a single throw is

$$P(i) = \frac{\text{Area of region } i}{\text{Area of the board}}.$$

(a) Show that for each $1 \le i \le 6$,

$$P(i) = \frac{(7-i)^2 - (6-i)^2}{36}.$$

(b) Show that P satisfies each of the probability axioms.
(c) Show that if $i < j$, then $P(i) > P(j)$. That is, P is a decreasing function of i.

Problem 5. Suppose P is a probability function on the pair $(\mathcal{S}, \mathcal{B})$. Using only the three probability axioms, show the following for any $A, B \in \mathcal{B}$.

(1) $P(A^c) = 1 - P(A)$.
(2) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
(3) If $A \subseteq B$, then $P(A) \le P(B)$.

Problem 6. Suppose P is a probability function on the pair $(\mathcal{S}, \mathcal{B})$. Let $A, B \in \mathcal{B}$.
For each of the following, determine if the identity is true or false in general. If it is true, explain why. If it is false, provide an example.

(a) $P(A) < P(A \cup B)$
(b) $P(A) \le P(A \cup B)$
(c) $P(A \cup B) = P(A) + P(B)$
(d) $P(A \setminus B) = P(A) - P(B)$