Homework 1

● Graded

Student

Jacob Hauptman

Total Points

24 / 24 pts

Question 1

Fair dice

4 / 4 pts

✔ **– 0 pts** Correct

– 1 pt No explanation on part d

Question 2

Unfair dice

4 / 4 pts

✔ **– 0 pts** Correct

– 1 pt Part a is wrong

– 1 pt Part b is wrong

– 1 pt Part c is wrong

– 1 pt Part d is wrong

– 4 pts Missing answer

Question 3

Tossing a coin

4 / 4 pts

✔ **– 0 pts** Correct

– 1 pt Part b is wrong

– 1 pt Part c is wrong.

– 4 pts No attempt.

Question 4

Dart board

4 / 4 pts

✔ **– 0 pts** Correct

– 1 pt No proof on part a

– 1 pt Incomplete part b

– 1 pt Incomplete part c

– 4 pts No attempt.

Question 5

Probability properties

4 / 4 pts

✔ **– 0 pts** Correct

– 4 pts No attempt

Question 6

True or false

4 / 4 pts

✔ **– 0 pts** Correct

– 1 pt Part a is incorrect

– 1 pt Part b is wrong

– 1 pt Part c is wrong.

– 1 pt Part d is incorrect

– 4 pts No attempt

Question assigned to the following page: 1

Jacob Hauptman 09.03.2024

Homework 1

Problem 1:

We know the total number of outcomes is $6 \cdot 6 = 36$.

a) $P = P(\{x\},\{5\}) + P(\{5\},\{x\}) + P(5,5),\ \forall x \in \{1,2,3,4,6\}$

$$= 5\left(\frac{1}{36}\right) + 5\left(\frac{1}{36}\right) + \frac{1}{36} = \boxed{\frac{11}{36}}$$

b) Exclude the $P(5,5)$ from the previous problem.

$$P = \frac{11}{36} - \frac{1}{36} = \boxed{\frac{10}{36}}$$

c) $P = P(\{y\},\{5\}) + P(\{5\},\{y\}) + P(5,5),\ \forall y \in \{4,6\}$.

$$= 2\left(\frac{1}{36}\right) + 2\left(\frac{1}{36}\right) + \frac{1}{36} = \boxed{\frac{5}{36}}$$

d) From the list below we can see 7 has the most options.

1: Impossible $P=0$

2: $(1,1)$ $P=\frac{1}{36}$

3: $(2,1),(1,2)$ $P=\frac{2}{36}$

4: $(3,1),(1,3),(2,2)$ $P=\frac{3}{36}$

5: $(4,1),(1,4),(3,2),(2,3)$ $P=\frac{4}{36}$

6: $(5,1),(1,5),(4,2),(2,4)\ (3,3)$ $P=\frac{5}{36}$

$\boxed{7: (6,1),(1,6),(5,2),(2,5),(4,3),(3,4)}$ $P=\frac{6}{36}$

8: $(6,2),(2,6),(5,3),(3,5),(4,4)$ $P=\frac{5}{36}$

9: $(6,3),(3,6),(5,4),(4,5)$ $P=\frac{4}{36}$

10: $(6,4),(4,6),(5,5)$ $P=\frac{3}{36}$

11: $(6,5),(5,6)$ $P=\frac{2}{36}$

12: $(6,6)$ $P=\frac{1}{36}$

Question assigned to the following page: 2

Problem 2:

Unfair die: $P(1)=\frac{2}{10}$, $P(5)=\frac{4}{10}$, rest are $\frac{1}{10}$ per.

$P(\text{fair},\text{unfair})$ when I write the probabilities.

a) $P = P(\{x\},\{5\}) + P(\{5\},\{x\}) + P(5,5),\ \forall x \in \{1,2,3,4,6\}.$

$= 5\left(\frac{1}{6}\right)\left(\frac{4}{10}\right) + \frac{1}{6}\left[\frac{2}{10} + 4\left(\frac{1}{10}\right)\right] + \frac{1}{6}\left(\frac{4}{10}\right)$

$= \frac{20}{60} + \frac{6}{60} + \frac{4}{60} = \boxed{\frac{1}{2}}$

b) Subtract $P(5,5)$ from part a.

$\frac{30}{60} - \frac{4}{60} = \frac{26}{60} = \boxed{\frac{13}{30}}$

c) $P = P(\{y\},\{5\}) + P(\{5\},\{y\}) + P(5,5),\ \forall y \in \{4,6\}.$

$= \frac{2}{6}\left(\frac{4}{10}\right) + \frac{1}{6}\left[2\left(\frac{1}{10}\right)\right] + \frac{1}{6}\left(\frac{4}{10}\right)$

$= \frac{8}{60} + \frac{2}{60} + \frac{4}{60} = \boxed{\frac{7}{30}}$

d) Yes.

1: Impossible $P=0$

2: $(1,1)$ $P=\frac{1}{6}\left(\frac{2}{10}\right)=\frac{2}{60}$

3: $(2,1),(1,2)$ $P=\frac{1}{6}\left(\frac{2}{10}\right)+\frac{1}{6}\left(\frac{1}{10}\right)=\frac{3}{60}$

4: $(3,1),(1,3),(2,2)$ $P=\frac{1}{6}\left(\frac{2}{10}\right)+2\left[\frac{1}{6}\left(\frac{1}{10}\right)\right]=\frac{4}{60}$

5: $(4,1),(1,4),(3,2),(2,3)$ $P=\frac{1}{6}\left(\frac{2}{10}\right)+3\left[\frac{1}{6}\left(\frac{1}{10}\right)\right]=\frac{5}{60}$

6: $(5,1),(1,5),(4,2),(2,4)\ (3,3)$ $P=\frac{1}{6}\left(\frac{2}{10}\right)+\frac{1}{6}\left(\frac{4}{10}\right)+3\left[\frac{1}{6}\left(\frac{1}{10}\right)\right]=\frac{9}{60}$

$\boxed{7: (6,1),(1,6),(5,2),(2,5),(4,3),(3,4)}$ $P=\frac{1}{6}\left(\frac{2}{10}\right)+\frac{1}{6}\left(\frac{4}{10}\right)+4\left[\frac{1}{6}\left(\frac{1}{10}\right)\right]=\frac{10}{60}$

8: $(6,2),(2,6),(5,3),(3,5),(4,4)$ $P=\frac{1}{6}\left(\frac{4}{10}\right)+4\left[\frac{1}{6}\left(\frac{1}{10}\right)\right]=\frac{8}{60}$

9: $(6,3),(3,6),(5,4),(4,5)$ $P=\frac{1}{6}\left(\frac{4}{10}\right)+3\left[\frac{1}{6}\left(\frac{1}{10}\right)\right]=\frac{7}{60}$

10: $(6,4),(4,6),(5,5)$ $P=\frac{1}{6}\left(\frac{4}{10}\right)+2\left[\frac{1}{6}\left(\frac{1}{10}\right)\right]=\frac{6}{60}$

11: $(6,5),(5,6)$ $P=\frac{1}{6}\left(\frac{4}{10}\right)+\frac{1}{6}\left(\frac{1}{10}\right)=\frac{5}{60}$

12: $(6,6)$ $P=\frac{1}{60}$

Questions assigned to the following page: 3 and 4

Problem 3:

a) Infinite outcomes since we it's until a heads shows up without specifying when it shows up. e.g. $\{H, TH, TTH, TTTH, \ldots\}$.

b) $E = \{TH, TTH, TTTH\}$.

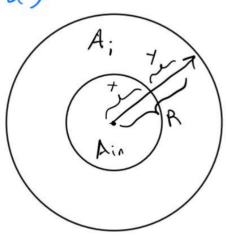
$$P(E) = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \boxed{\tfrac{7}{16}}$$

c) $P(H) = \frac{9}{10}$ $\therefore$ $P(T) = \frac{1}{10}$

$$P(E) = \tfrac{1}{10}\left(\tfrac{9}{10}\right) + \left(\tfrac{1}{10}\right)^2\left(\tfrac{9}{10}\right) + \left(\tfrac{1}{10}\right)^3\left(\tfrac{9}{10}\right) = \tfrac{9}{100} + \tfrac{9}{1000} + \tfrac{9}{10{,}000} = \boxed{\tfrac{999}{10{,}000}}$$

Problem 4:

a)



$$P(i) = \frac{A_i - A_{in}}{A} = \frac{\pi y^2 - \pi x^2}{\pi R^2} = \frac{y^2 - x^2}{R^2}$$

We know $R = 6$ because there are 6 rings and $1 \le i \le 6$.

$$P(i) = \frac{y^2 - x^2}{36}$$

$y = x + 1 \Rightarrow x = y - 1$ because that is one ring smaller.

$$P(i) = \frac{y^2 - (y-1)^2}{36}$$

when $i = 6$, x must be 1 because it's the outside and needs a thickness of 1.

$\therefore$ $x = 7 - i$

$$P(i) = \frac{(7-i)^2 - (7-i-1)^2}{36} = \boxed{\frac{(7-i)^2 - (6-i)^2}{36}}$$

Question assigned to the following page: 4

b)

<u>Normilization</u>

$$\sum_{i=1}^{6} P(i) = \sum_{i=1}^{6} \frac{(7-i)^2-(6-i)^2}{36} = 1$$

$$\frac{1}{36}\left[6^2-5^2+5^2-4^2+4^2-3^2+3^2-2^2+2^2-1^2+1^2-0\right] = \frac{36}{36} = 1$$

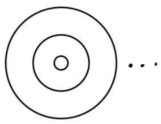
<u>Positivity</u>

b.c. $(7-i)^2 > (6-i)^2 \quad 1 \le i \le 6$

$$\therefore \frac{(7-i)^2-(6-i)^2}{36} > 0$$

<u>Countable Additivity</u>

The radius of the rings are disjoint (don't overlap)



$$\therefore P(i \cup j) = P(i) + P(j), \quad 1 \le j \le 6$$

c)

$$\frac{dP(i)}{di} = \left(\frac{1}{36}\right)\frac{d}{di}\left[(7-i)^2-(6-i)^2\right]$$

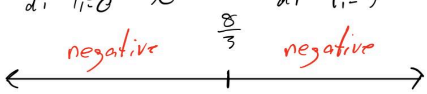
$$= \frac{1}{36}\left[-2(7-i)+(6-i)\right] = \frac{1}{36}(-8-3i)$$

$$\frac{dP(i)}{di} = \frac{1}{36}(-8-3i) = 0$$

$$-8-3i = 0$$

$$i = \frac{8}{3}$$

$$\left.\frac{dP(i)}{di}\right|_{i=0} = -\frac{8}{36} \qquad \left.\frac{dP(i)}{di}\right|_{i=3} = -\frac{17}{36}$$



$\therefore$ When $i<j$ then $P(i) > P(j)$, so P is decreasing.

Questions assigned to the following page: 5 and 6

Problem 5:

a) Using normilization $P(S)=1$, therefore $P(A)+P(A^c)=1$ which can be rewritten as $P(A^c)=1-P(A)$.

b) Using normilization $P(S)=1$, therefore $P(A)+P(B)=1$.

Clearly $P(A)+P(B)>1$ when $A \cap B \neq \emptyset$.

When we add it includes the intersection twice so we must subtract it once.

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

c) Since $A \subseteq B$, $A \leq B$.

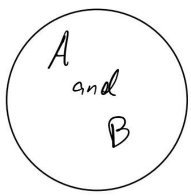
$\therefore B = A + (B \setminus A)$ (Can rewrite as $A \cup (B \setminus A)$ but it doesn't matter because $B \setminus A$ gets rid of overlap)

This shows $P(A) \leq P(B)$.

Problem 6:

a) False. If $A=B$, then $P(A)=P(A \cup B)$.

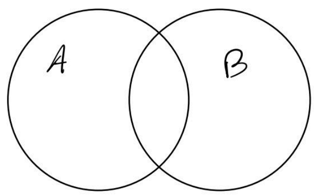
b) True. Since A and B are elements of the same set, the least they can be is equal.



$A=B$

$\therefore$

$P(A)=P(A \cup B)$



$P(A)<P(A \cup B)$

Question assigned to the following page: 6

c) False. If $A = B$, $P(A \cup B) \neq P(A) + P(B)$.

It also doesn't take into account the intersection being counted twice and needs a "$-P(A \cap B)$".

d) False. If $P(A) < P(B)$ then $P(A) - P(B) < 0$ failing positivity.