Exam 3

Graded

Student

Jacob Hauptman

Total Points

92 / 100 pts

Question 1

true/false **8** / 8 pts

1.1 E(2X+3) **2** / 2 pts

- ✔ **− 0 pts** Correct: true
- **− 2 pts** Incorrect

1.2 E(X^2 +3) **2** / 2 pts

- ✔ **− 0 pts** Correct: False
- **− 2 pts** Incorrect

1.3 V(2X+3) **2** / 2 pts

- ✔ **− 0 pts** Correct: False
- **− 2 pts** Incorrect

1.4 V(X^2+3) **2** / 2 pts

- ✔ **− 0 pts** Correct: False
- **− 2 pts** Incorrect

Question 2

Linear rescaling with exponential **6** / 6 pts

- ✔ **− 0 pts** Correct: $E(Y) = -30, V(Y) = 400$
- **− 3 pts** Incorrect $E(Y)$
- **− 3 pts** Incorrect $V(Y)$

Question 3

Central limit theorem usage 3 / 6 pts

- **− 0 pts** Correct: C
- ✔ **− 3 pts** one incorrect answer is circled
- **− 3 pts** C is not circled
- **− 6 pts** more than one incorrect answer circled

Question 4

Central limit theorem multiple choice **6** / 6 pts

✔ **− 0 pts** Correct: A

− 6 pts Incorrect

Question 5

(no title) **8** / 8 pts

5.1 **probability Min = 10** **4** / 4 pts

✔ **− 0 pts** Correct: $.01 + .05 + .05 = .11$

− 4 pts Incorrect

5.2 **expected value of Min** **4** / 4 pts

✔ **− 0 pts** Correct: $10(.11) + 20(.25) + -10(64) = -.3$

− 1 pt Small typo in simplification

− 2 pts Calculated expected value of a different statistic

− 4 pts Incorrect

Question 6

Bias of Min for mu **6** / 6 pts

✔ **− 0 pts** Correct: -7.3 or Answer from 5b - 7

− 6 pts Incorrect

Question 7

Linear combination **13** / 15 pts

7.1 Expected value **5** / 6 pts

– **0 pts** Correct: -3

✔ – **1 pt** small typo

– **3 pts** Incorrect value for E(X)

– **3 pts** Incorrect formula for E(Z)

– **6 pts** Insufficient work to receive credit

7.2 Variance **5** / 6 pts

– **0 pts** Correct: 101

✔ – **1 pt** small typo

– **3 pts** Incorrect variance for X

– **3 pts** Incorrect formula for V(Z)

– **6 pts** Insufficient work to receive credit

7.3 Independence **3** / 3 pts

✔ – **0 pts** Correct: B

– **3 pts** Incorrect

Question 8

Normal distribution **12** / 15 pts

8.1 **probability for X** **5** / 5 pts

✔ **− 0 pts** Correct: .9772

− 2 pts $1 - .9772$

− 3 pts Incorrect z score

− 5 pts Insufficient work to receive credit

8.2 **probability for sample average** **5** / 5 pts

✔ **− 0 pts** Correct: .8413

− 1 pt small typo

− 2 pts $1 - .8413$

− 3 pts Incorrect z score

− 5 pts Insufficient work to receive credit

8.3 **confidence** **2** / 5 pts

− 0 pts Correct: 91

− 3 pts Incorrect z-score (but correct formulas)

✔ **− 3 pts** 109 (z score of 3 instead of -3)

− 4 pts Wrong formulas (e.g., attempt to compute for sample total)

− 6 pts Insufficient work to receive credit

Question 9

Random samples **15** / 15 pts

9.1 **expected value and standard deviation** **5** / 5 pts

✔ **– 0 pts** Correct: $\mu_X = 3$, $\sigma_X = 4$

– 1 pt small typo

– 2 pts variance instead of standard deviation, i.e., $\mu_X = 3$, $\sigma_X = 16$

– 3 pts Incorrect variance

– 3 pts Incorrect expected value

– 5 pts Insufficient work to receive credit

9.2 **sample size 3** **5** / 5 pts

✔ **– 0 pts** Correct: $1 - (.8)^3$

– 1 pt small typo

– 3 pts wrong combinations of samples to achieve sample average less than 2

– 5 pts Insufficient work to receive credit

9.3 **sample size 64** **5** / 5 pts

✔ **– 0 pts** Correct: $.0228$ (or correct given (a))

– 2 pts $1 - .0228$

– 2 pts Correct standard deviation (given (a)) but incorrect calculations

– 3 pts Incorrect standard deviation

– 5 pts Insufficient work to receive credit

Question 10

Point estimators **15** / 15 pts

10.1 **sample average is unbiased** **3** / 3 pts

✔ **– 0 pts** Correct: True

– 3 pts Incorrect

10.2 **calculate the bias** **6** / 6 pts

✔ **– 0 pts** Correct: $\frac{\sigma^2}{10}$

– 1 pt small typo

– 2 pts Incorrect formula for $V(\overline{X})$

– 2 pts Incorrect formula for $E(\overline{X})$

– 4 pts Incorrect formula for Bias

– 5 pts Correct formula for Bias with respect to ℓ^2, but no other relevant work

– 6 pts Insufficient work to receive credit

10.3 **solve for K** **6** / 6 pts

✔ **– 0 pts** Correct: $\frac{1}{10}$, or correct given (b)

– 2 pts small error

– 4 pts Some relevant work shown

– 6 pts Insufficient work to receive credit

Math 120 Exam 3 November 21, 2024

Name: Jacob Hauptman UID: 120067075

Question	Points	Score
1	8	
2	6	
3	6	
4	6	
5	8	
6	6	
7	15	
8	15	
9	15	
10	15	
Total:	100	

Directions: No notes, text books, calculators, cell phones, or other electronics are allowed. **Unless otherwise specified, you do not need to simplify your answers; however, your answers should not contain the sybmols $\int$, $\sum$, or Φ.** Please sign the University of Maryland honors pledge below.

"I pledge on my honor that I have not given or received any unauthorized assistance on this assessment."

Good luck! This test does not define you :)

Formulas: The following formulas for pmfs and pdfs are provided for your convenience. They may or may not be useful on the exam.

- Binomial: $p(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$, $E(X) = np$, $V(X) = np(1-p)$
- Geometric: $p(x; p) = (1-p)^{x-1} p$, $E(X) = \frac{1}{p}$ $V(X) = \frac{(1-p)}{p^2}$
- Hypergeometric: $p(x; N, M, n) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, $E(X) = n\frac{M}{N}$, $V(X) = n \cdot \frac{N-n}{N-1} \cdot \frac{M}{N} \cdot \left(1 - \frac{M}{N}\right)$
- Uniform continuous: $f(x; A, B) = \frac{1}{B-A}$, $E(X) = \frac{A+B}{2}$, $V(X) = \frac{(B-A)^2}{12}$
- Exponential: $f(x; \lambda) = \lambda e^{-\lambda x}$, $E(X) = \frac{1}{\lambda}$, $V(X) = \frac{1}{\lambda^2}$

CDF for Standard Normal: The following table give some values of the cumulative distribution function for the standard normal distribution.

a	$\Phi(a)$
$-\infty$	0
-3.0	.0013
-2.5	.0062
-2	.0228
-1.5	.0668
-1	.1587
-0.5	.3085
0	.5
.5	.6915
1	.8413
1.5	.9332
2	.9772
2.5	.9938
3.0	.9987
∞	1

Short Answer: Answer the following questions and write your final answer in the box or line when indicated. You do not need to show your work.

1. [8 pts.] Determine whether the following statements are true or false where X and Y are random variables on sample space S.
   (a) $E(2X+3) = 2E(X)+3$.
   True
   (b) $E(X^2+3) = E(X)^2+3$.
   False
   (c) $V(2X+3) = 2V(X)+3$.
   False
   (d) $V(X^2+3) = V(X)^2+3$.
   False

2. [6 pts.] Suppose X has the exponential distribution with parameter $\lambda = \frac{1}{10}$. $E(X) = \frac{1}{\lambda} = \frac{1}{(\frac{1}{10})} = 10$
   Let $Y = -2X - 10$. Find $E(Y)$ and $V(Y)$. $V(X) = \frac{1}{\lambda^2} = \frac{1}{(\frac{1}{10})^2} = 100$

$E(Y) = -2E(X) - 10 = -2(10) - 10 = -20 - 10 = -30$

$V(Y) = (-2)^2 V(X) = 4V(X) = 400$

$E(Y) = -30$

$V(Y) = 400$

3. [6 pts.] Let X be a random variable. Which of the following have approximately normal distributions? **Circle all that apply.**
   A. X itself, if X is a continuous random variable.
   B. $\overline{X}$ (the sample averages), if we take a random sample of size $n = 8$.
   C. T_0 (the sample totals), if we take a random sample of size $n = 80$.
   D. The sample modes, if we take a random sample of size $n = 800$.

Name:

4. [6 pts.] Suppose that the heights of men in the United States are normally distributed with mean 70 inches and standard deviation 3 inches. If we randomly sample 9 men in the united states, what is the probability that their average height is more than 71.5 inches?

   A. .0668 $\mu=70$ $\sigma=3$ $\sigma_{\bar{x}}=\frac{3}{\sqrt{9}}=\frac{3}{3}=1$

   B. .3085

   C. .6915 $P(X>71.5)=P(Z>\frac{71.5-70}{1})=1-\Phi(1.5)=1-.9332=0.0668$

   D. .9332

5. [8 pts.] Suppose the random variable X has the following distribution:

x	-10	10	20
$P(x)$	.4	.1	.5

Suppose we take a random sample of size two.

(a) What is the probability that the sample minimum is 10?

Min 10

10, 10
20, 10
10, 20

$P(Min=10)=0.1^2+2(0.1)(0.5)$
$=0.01+2(0.05)=.11$
.1

$1-.45=.55$

0.64
.11
.25

$P(\text{Min}=10)=0.1^2+2(0.1)(0.5)=0.11$

(b) Calculate the expected value of the sample minimum.

Min -10   20

-10, -10   20, 20
-10, 10
10, -10
-10, 20
20, -10

$P(Min=-10)=0.4^2+2(0.4)(0.1)+2(0.4)(0.5)$ $P(Min=20)=0.5^2=0.25$
$=0.16+2(0.04)+2(0.2)$
$=0.16+0.08+0.4=0.64$
$.56+0.08=0.64$

0.25
0.34
59

.4/.16
.4/.04

$E(\text{Min})=-10(0.64)+10(0.11)+20(0.25)$
$=-6.4+1.1+5=-6.4+6.1=-0.3$

.25
20
5.00
.6

6. [6 pts.] Using the distribution table from problem 5, calculate the bias of the estimator $\hat{\mu}=$ Min (i.e., sample minimum) for μ if we take a random sample of size two.

$Bias_{\mu}(\hat{\mu})=E(Min)-\mu=-0.3-7=-7.3$

.25
20
5.00

$\mu=E(X)=-10(0.4)+10(0.1)+20(0.5)$
$=-4+1+10=7$

$\text{Bias}(\hat{\mu})=-7.3$

22
.5
10.0

Name:

Full Response: For full credit, show your work.

7. [15 pts.] Let X be the number of times we have to toss a biased coin with $P(H) = .2$ until we get Heads. Let Y have an unknown distribution, with $E(Y) = -1$, $V(Y) = 2$, and $\mathrm{Cov}(X, Y) = -0.25$. Let $Z = -2X + 3Y + 10$.

(a) Calculate the expected value of Z, $E(Z)$.

$E(Z) = -2E(X) + 3E(Y) + 10 =$

X is geometric

$E(X) = \frac{1}{p} = \frac{1}{0.2} = \frac{10}{2} = 5$

$E(Z) = -2(5) + 3(-1) + 10 = 10 - 3 + 10 = 17$

$E(Z) = 17$

(b) Calculate the variance of Z, $V(Z)$.

$V(X) = \frac{1-p}{p^2} = \frac{1-0.2}{0.2^2} = \frac{0.8}{0.04} = \frac{8}{.4} = \frac{80}{4} = 20$

$V(Z) = (-2)^2 V(X) + (3)^2 V(Y) + 2(-2)(3)\mathrm{Cov}(X,Y)$

$= 4(20) + 9(2) - 12(-\frac{1}{4}) = 80 + 18 + 3 = 91$

$V(Z) = 91$

(c) Are X and Y independent?

A. Yes, they are independent.

B. No, they are dependent.

C. We do not have enough information to answer.

Name:

8. [15 pts.] The average time it takes for fully charged cell phone to run out of battery is normally distributed, with mean 100 hours and standard deviation 15 hours.

(a) If you charged your cell phone at 11am on Monday, what is the probability it is still working at 9am on Thursday?

$\mu = 100 \quad \sigma = 15$ 11am to 9am is $2(24)+22$
$= 48 + 22 = 70$

$P(X>70) = P(Z > \frac{70-100}{15}) = 1 - \Phi(-2) = 1 - 0.0228$

$\frac{-30}{15}$

$\boxed{P(X>70) = 1 - 0.0228 = 0.9772}$

(b) Suppose we randomly sample 25 cell phones. What is the probability that their average battery life is longer than 97 hours?

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{5} = 3$

$P(\bar{X} > 97) = P(Z > \frac{97-100}{3}) = 1 - \Phi(-1)$

$\frac{-3}{3}$

$\boxed{P(\bar{X} > 97) = 1 - 0.1587 = 0.8413}$

(c) Suppose we randomly sample 25 cell phones. Find the number N so that we can be 99.87% confident that the average battery life is at least N hours.

$\Phi(3) = 0.9987$

$\frac{N-100}{3} = 3 \Rightarrow N - 100 = 9 \Rightarrow N = 109$

$\boxed{N = 109}$

Name:

9. [15 pts.] Suppose X has the following distribution.

x	-5	5
$P(x)$	.2	.8

(a) Calculate μ_X and σ_X.

$\mu_x = E(X) = -5(0.2) + 5(0.8) = -1 + 4 = 3$

$V(X) = E(X^2) - E(X)^2$; $E(X^2) = (-5)^2(0.2) + (5)^2(0.8)$

$= 25(0.2) + 25(0.8)$

$V(X) = 25 - 3^2 = 25 - 9 = 16$ $\quad = 25(0.2 + 0.8) = 25$

$\sigma_X = \sqrt{V(X)} = 4$

$\mu_X = 3$

$\sigma_X = 4$

(b) Suppose we take a random sample of size n and calculate the sample average.

(a) Suppose $n = 3$. What is the probability that the sample average is less than 2?

$\frac{-5-5+5}{3} = \frac{-5}{3} < 2$ $\quad \frac{-5+5+5}{3} = \frac{5}{3} < \frac{6}{3}$ $\quad \frac{-5-5-5}{3} = -5 < 2$

$P(\{-5,-5,5\}) = (0.2)^2(0.8)$

$P(\{-5,5,5\}) = 0.2(0.8)^2$

$P(\{-5,-5,-5\}) = 0.2^3$

$P(\overline{X} < 2) = 0.2^2(0.8) + 0.2(0.8)^2 + (0.2)^3$

(b) Suppose $n = \underline{64}$. What is the probability that the sample average is less than 2?

CLT, $n = 64 \geq 30$

$\sigma = \frac{4}{\sqrt{64}} = \frac{4}{8} = 0.5$

$P(\overline{X} < 2) = P(Z < \frac{2-3}{0.5}) = \Phi(\frac{-1}{1/2}) = \Phi(-2) = 0.0228$

$P(\overline{X} < 2) = 0.0228$

Name:

10. [15 pts.] Using a rod of (unknown) length ℓ, you lay out a square plot of side length ℓ. So the area is ℓ^2. Suppose you take 10 independent measurements $X_1, ..., X_{10}$, each with mean ℓ and variance σ^2.

(a) True or false: $\overline{X}$ is an unbiased estimator for ℓ.

$$Bias_\ell(\bar{X}) = E(\bar{X}) - \ell = \cancel{\mu - \ell} = \ell - \ell = 0$$

$\boxed{\text{True}}$

(b) Calculate the bias of $\overline{X}^2$. (*Hint:* Use a formula for the variance of $\overline{X}$.)

$$Bias_\ell(\bar{X}^2) = E(\bar{X}^2) - \ell^2 \qquad \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} \quad \sigma_{\bar{x}}^2 = \frac{\sigma_x^2}{n} = \frac{\sigma^2}{10}$$

$$V(\bar{X}) = E(\bar{x}^2) - E(\bar{x})^2 \Rightarrow E(\bar{x}^2) = V(\bar{X}) + E(\bar{x})^2 = \frac{\sigma^2}{10} + \mu^2$$

$$E(\bar{x}^2) - \ell^2 = \frac{\sigma^2}{10} + \mu^2 - \ell^2 = \frac{\sigma^2}{10} + \ell^2 - \ell^2 = \frac{\sigma^2}{10}$$

$$\boxed{\text{Bias}_{\ell^2}(\overline{X}^2) = \frac{\sigma^2}{10}}$$

(c) For what value of K is $\overline{X}^2 - KS^2$ an unbiased estimator of ℓ^2? You may use the fact that S^2 (sample standard deviation) is an unbiased estimator for σ^2.

$$Bias_{\ell^2}(\bar{X}^2 - kS^2) = E(\bar{X}^2) - kE(S^2) - \ell^2 = 0$$

Using $E(\bar{X}^2)$ from (b)

$$\frac{\sigma^2}{10} + \mu^2 - kE(S^2) - \ell^2 = 0 \Rightarrow \frac{\sigma^2}{10} + \ell^2 - kE(S^2) - \ell^2 = 0$$

$$\frac{\sigma^2}{10} = kE(S^2)$$

$$k = \frac{\sigma^2}{10}\left(\frac{1}{E(S^2)}\right)$$

$$= \frac{\sigma^2}{10}\left(\frac{1}{\sigma^2}\right) = \frac{1}{10}$$

$$\boxed{K = \frac{1}{10}}$$