Exam 2

Graded

Student

Jacob Hauptman

Total Points

98 / 100 pts

Question 1

True/False **6** / 8 pts

1.1 (no title) **2** / 2 pts

✔ **– 0 pts** Correct: False

– 2 pts Incorrect

1.2 (no title) **0** / 2 pts

– 0 pts Correct: True

✔ **– 2 pts** Incorrect

1.3 (no title) **2** / 2 pts

✔ **– 0 pts** Correct: False

– 2 pts Incorrect

1.4 (no title) **2** / 2 pts

✔ **– 0 pts** Correct: True

– 2 pts Incorrect

Question 2

Find the value of c **6** / 6 pts

✔ **– 0 pts** Correct: $c = 4$

– 6 pts Incorrect

– 3 pts had the correct answer written, but put the wrong answer in the box

Question 3

Normal distribution **6** / 6 pts

✔ **– 0 pts** Correct: .0668

– 2 pts .668 or other small typo

– 6 pts Incorrect

Question 4

Exponential distribution

6 / 6 pts

- **– 0 pts** Correct: A
- **– 6 pts** Incorrect
- **– 3 pts** had the correct answer circled but circled the wrong answer choice

Question 5

Fill out the distribution table

8 / 8 pts

- **– 0 pts** Correct
- **– 4 pts** Wrong distribution of dice
- **– 4 pts** One of the columns incorrect
- **– 8 pts** Incorrect

Question 6

Compute E(X+Y)

6 / 6 pts

- **– 0 pts** Correct: 1.8
- **– 6 pts** Incorrect
- **– 2 pts** Correct but not simplified

Question 7

Continuous random variable

15 / 15 pts

7.1 **Find the probability**

7 / 7 pts

- **– 0 pts** Correct: $\frac{11}{16}$
- **– 3 pts** Incorrect antiderivative, but otherwise correct
- **– 2 pts** Incorrect arithmetic, but otherwise correct
- **– 5 pts** Integral not set up correctly, but computation is correct
- **– 3 pts** Found probability that $X \geq \frac{1}{2}$ instead
- **– 7 pts** Insufficient work to receive credit

7.2 **Find the expected value**

8 / 8 pts

- **– 0 pts** Correct: $\frac{3}{8}$
- **– 2 pts** Small arithmetic error
- **– 4 pts** Incorrect antiderivative (or small error in the integral), but integral was set up correctly
- **– 6 pts** Integral not set up correctly
- **– 8 pts** Insufficient work to receive credit

Question 8

Exponential distribution: tables at a restaurant **15** / 15 pts

8.1 **Find the probability** **5** / 5 pts

✔ **– 0 pts** Correct: $e^{-.5}$

– 2 pts t or x in answer or other small typo

– 2 pts Found probability that X<10 (or other wrong range of outcomes)

– 2 pts $e^{+.5}$

– 3 pts Incorrect antiderivative

– 4 pts Wrong equation (or wrong λ)

– 5 pts Insufficient work to receive credit

– 3 pts Infinity left in answer

8.2 **Conditional probability** **5** / 5 pts

✔ **– 0 pts** Correct: $e^{-.5}$ (or same answer as part a)

– 3 pts Computed the un-conditional probability (or probability for another range of values)

– 4 pts Some relevant work shown, but conditional probability is not used correctly

– 5 pts Insufficient work to receive credit

8.3 **Binomial** **5** / 5 pts

✔ **– 0 pts** Correct: $\binom{15}{5}\,(e^{-.5})^5(1-e^{-.5})^{10}$ or correct using part (a)

– 2 pts Small error in binomial formula

– 3 pts Incorrect use of binomial formula

– 5 pts Insufficient to receive credit

Question 9

Joint distribution discrete **15** / 15 pts

9.1 **Calculate E(XY)** **3** / 3 pts

✔ **– 0 pts** Correct: $.6$

– 1 pt Small error (e.g., missed a negative)

– 2 pts Significant error

– 3 pts Insufficient work to receive credit

9.2 **Calculate E(X)** **3** / 3 pts

✔ **– 0 pts** Correct: 0

– 1 pt Small error (e.g., missed a negative)

– 2 pts Significant error

– 3 pts Insufficient to receive credit

9.3 **Calculate E(Y)** **3** / 3 pts

✔ **– 0 pts** Correct: $-.3$

– 1 pt Small error (e.g., missed a negative)

– 2 pts Significant error

– 3 pts Insufficient to receive credit

9.4 **Calculate Cov(X,Y)** **3** / 3 pts

✔ **– 0 pts** Correct: $E(XY) - E(X)E(Y) = .6$ or correct using (a)-(c)

– 1 pt Small error, e.g., missed a - or calculated Corr(X,Y) instead

– 2 pts Significant error

– 3 pts Insufficient to receive credit

9.5 **Positively associated** **3** / 3 pts

✔ **– 0 pts** Correct: True (or correct using (d))

– 3 pts Incorrect

Question 10

Joint distribution continuous **15** / 15 pts

10.1 **Find the value of c** **4** / 4 pts

- ✔ **− 0 pts** Correct: 4
- **− 2 pts** small error, e.g., 1/4 instead of 4
- **− 3 pts** significant error
- **− 4 pts** Insufficient work to receive credit

10.2 **Marginal for X** **4** / 4 pts

- ✔ **− 0 pts** Correct: $\begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$
- **− 1 pt** Missing the range of x
- **− 2 pts** Correct formula
- **− 3 pts** Incorrect formula
- **− 4 pts** Insufficient work to receive credit

10.3 **Marginal for Y** **4** / 4 pts

- ✔ **− 0 pts** Correct: $\begin{cases} 2y & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$
- **− 1 pt** Missing range of y
- **− 2 pts** correct formula
- **− 3 pts** incorrect formula
- **− 4 pts** insufficient work to receive credit

10.4 **Independent** **3** / 3 pts

- ✔ **− 0 pts** Correct: True (or correct using (a)-(c))
- **− 3 pts** Incorrect

Stat 400 Exam 2 October 24, 2024

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Question	Points	Score
1	8	
2	6	
3	6	
4	6	
5	8	
6	6	
7	15	
8	15	
9	15	
10	15	
Total:	100	

Directions: No notes, text books, calculators, cell phones, or other electronics are allowed. **Unless otherwise specified, you do not need to simplify your answers; however, your answers should not contain the sybmols $\int$, $\sum$, or Φ.** Please sign the University of Maryland honors pledge below.

"I pledge on my honor that I have not given or received any unauthorized assistance on this assessment."

Good luck! This test does not define you :)

Formulas: The following formulas for pmfs and pdfs are provided for your convenience. They may or may not be useful on the exam.

- Binomial: $p(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$, $E(X) = np$, $V(X) = np(1-p)$
- Geometric: $p(x; p) = (1-p)^{x-1} p$, $E(X) = \frac{1}{p}$ $V(X) = \frac{(1-p)}{p^2}$
- Hypergeometric: $p(x; N, M, n) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, $E(X) = n\frac{M}{N}$, $V(X) = n \cdot \frac{N-n}{N-1} \cdot \frac{M}{N} \cdot \left(1 - \frac{M}{N}\right)$
- Uniform continuous: $f(x; A, B) = \frac{1}{B-A}$, $E(X) = \frac{A+B}{2}$, $V(X) = \frac{(B-A)^2}{12}$
- Exponential: $f(x; \lambda) = \lambda e^{-\lambda x}$, $E(X) = \frac{1}{\lambda}$, $V(X) = \frac{1}{\lambda^2}$

CDF for Standard Normal: The following table give some values of the cumulative distribution function for the standard normal distribution.

a	$\Phi(a)$
$-\infty$	0
-3.0	.0013
-2.5	.0062
-2	.0228
-1.5	.0668
-1	.1587
-0.5	.3085
0	.5
.5	.6915
1	.8413
1.5	.9332
2	.9772
2.5	.9938
3.0	.9987
∞	1

Short Answer: Answer the following questions and write your final answer in the box or line when indicated. You do not need to show your work.

1. [8 pts.] Determine whether the following statements are true or false where X and Y are random variables on sample space S.

   (a) If X is a continuous random variable, then X has a continuous probability density function.

   False

   (b) If X is a continuous random variable, then X has a continuous cumulative density function.

   ~~True~~ False

   (c) If X and Y are discrete random variables, then $p_{XY}(x, y) = p_X(x) \cdot p_Y(y)$.

   False

   $-\frac{1}{2}$

   $-\frac{1}{2}$

   (d) If X and Y are discrete random variables, then $p_{X|Y}(x, y) = \dfrac{p_{XY}(x, y)}{p_Y(y)}$.

   True

2. [6 pts.] Consider the function

$$f(x) = \begin{cases} x(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

For what value of c is $g(x) = c \cdot f(x)$ a probability density function? **Write your answer as a fraction of two whole numbers.**

$$c\int_0^1 x(1-x^2)dx = c\int_0^1 x - x^3 dx = c\left[\frac{x^2}{2} - \frac{x^4}{4}\Big|_0^1\right] = c\left[\frac{1}{2} - \frac{1}{4}\right] = c\left(\frac{1}{4}\right) = 1$$

$$c = 4$$

$\boxed{c = 4}$

3. [6 pts.] So far this season, the Dallas Cowboys scored an average of 21 points per game, with a standard deviation of 8 points. What is the probability that the Cowboys score more than 33 points next week against the 49ers? Assume a normal distribution.

$\mu = 21 \quad \sigma = 8 \qquad P(X > 33) = 1 - \Phi\left(\frac{33-21}{8}\right) = 1 - \Phi(1.5)$

$= 1 - .9332$

$33 - 21 = 12$

$12 \div 8 = 1.5$

$0.0668 + 0.9332 = 1.0000$

$\boxed{P(X > 33) = 1 - 0.9332 = 0.0668}$

4. [6 pts.] The time you have to wait for your professor to respond to emails is exponentially distributed, but on average they respond after 10 hours. What is the probability that it takes more than 2 hours for them to respond to your email?

$\lambda = \frac{1}{10}$ $P(X>2)$

A. $e^{-.2}$

B. $1 - e^{-.2}$

C. e^{-20}

D. $1 - e^{-20}$

$\lambda \int_2^\infty e^{-\lambda x} dx = -e^{-\lambda x} \Big|_2^\infty = 0 - (-e^{-\frac{1}{10}(2)}) = e^{-\frac{2}{10}} = e^{-0.2}$

5. [8 pts.] A bag contains two fair dice (Die 1), and one unfair die (Die 2) which has the following probability distribution:

Die 2

x	1	2	3	4	5	6
$P(x)$	0	0	0	.1	.4	.5

Suppose an experiment consists of randomly picking a die from this bag, and then rolling it. Let X be the random variable taking values 1 or 2 depending on which of Die 1 or Die 2 is picked from the bag in Step 1. Let Y be the random variable that keeps track of the outcome of the die roll.

Fill out the joint distribution table of X and Y below.

$\frac{2}{3}\left(\frac{1}{6}\right) = \frac{2}{18} = \frac{1}{9}$

$\frac{1}{3}\left(\frac{1}{10}\right) = \frac{1}{30}$

$\frac{1}{3}\left(\frac{4}{10}\right) = \frac{4}{30}$

$\frac{1}{3}\left(\frac{5}{10}\right) = \frac{5}{30}$

$p(x, y)$	1	2
1	1/9	0
2	1/9	0
3	1/9	0
4	1/9	1/30
5	1/9	4/30
6	1/9	5/30

6. [6 pts.] Using the distribution table below (where X is on the horizontal axis and Y is the vertical axis), compute $E(X + Y)$. **Write your answer as a decimal or a fraction of two whole numbers.**

$E(X) = 2(0.2 + 0.1 + 0.4)$
$= 2(0.7) = 1.4$

$E(Y) = -2(0.2+0.2) + 3(0+0.4)$
$= -2(0.4) + 3(0.4)$
$= -0.8 + 1.2 = 0.4$

X

$p(x, y)$	0	2
-2	.2	.2
0	.1	.1
3	0	.4

Y

$E(X + Y) = 1.4 + 0.4 = 1.8$

Full Response: For full credit, show your work.

7. [15 pts.] Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} \frac{3}{2}(1-x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the probability that $X \le \frac{1}{2}$.

$$\cancel{\frac{3}{2}\int_{\frac{1}{2}}^{1} 1-x^2\,dx = \frac{3}{2}\left[x-\frac{x^3}{3}\Big|_{\frac{1}{2}}^{1}\right] = \frac{3}{2}\left[1-\frac{1}{3}-\left(\frac{1}{2}-\frac{(\frac{1}{2})^3}{3}\right)\right]}$$

$$\cancel{=\frac{3}{2}\left[\frac{2}{3}-\left(\frac{1}{2}-\frac{\frac{1}{8}}{3}\right)\right] = \frac{3}{2}\left[\frac{2}{3}-\left(\frac{1}{2}-\frac{1}{\ }\right.\right.}$$

$$\cancel{\int_{\frac{1}{2}}^{1}\frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\int_{\frac{1}{2}}^{1} 1-x^2\,dx = \frac{3}{2}\left(x-\frac{x^3}{3}\right)\Big|_{\frac{1}{2}}^{1} = \frac{3}{2}\left[1-\frac{1}{3}-\left(\frac{1}{2}-\frac{(\frac{1}{2})^3}{3}\right)\right]}$$

$$\int_0^{\frac{1}{2}}\frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\int_0^{\frac{1}{2}} 1-x^2\,dx = \frac{3}{2}\left(x-\frac{x^3}{3}\right)\Big|_0^{\frac{1}{2}} = \frac{3}{2}\left[\frac{1}{2}-\left(\frac{(\frac{1}{2})^3}{3}\right)-0\right] = \frac{3}{2}\left(\frac{1}{2}-\frac{(\frac{1}{2})^3}{3}\right)$$

$$\boxed{P\left(X \le \frac{1}{2}\right) = \frac{3}{2}\left(\frac{1}{2}-\frac{(\frac{1}{2})^3}{3}\right)}$$

(b) Find the expected value μ_X of X.

$$\mu = \int_0^1 x f_X(x)\,dx = \frac{3}{2}\int_0^1 x(1-x^2)\,dx = \frac{3}{2}\int_0^1 x-x^3\,dx = \frac{3}{2}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2}-\frac{1}{4}\right] = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$

$$\boxed{\mu_X = \frac{3}{8}}$$

8. [15 pts.] Suppose the wait times for tables at a restaurant are exponentially distributed with average wait time 20 minutes. (For this question, you can leave e's and "choose" notation in your answer)

(a) What is the probability that you will wait more than 10 minutes for a table?

$\lambda = \frac{1}{20}$

$$\int_{10}^{\infty} \lambda e^{-\lambda x} dx = \lambda \int_{10}^{\infty} e^{-\lambda x} dx = -e^{-\lambda x}\Big|_{10}^{\infty} = 0 - (-e^{-\lambda(10)}) = e^{-\frac{1}{20}(10)} = e^{-\frac{1}{2}}$$

$P(X \geq 10) = e^{-\frac{1}{2}}$

(b) Given that you have already waited an hour for a table, what is the probability that you will have to wait more than 10 more minutes?

Memoryless: $P(X \geq 70 \mid X \geq 60) = P(X \geq 10 \mid X \geq 0) = P(X \geq 10) = e^{-\frac{1}{2}}$

$P(X \geq 70 \mid X \geq 60) = e^{-\frac{1}{2}}$

(c) Suppose we poll 15 random restaurant goers (with replacement). What is the probability that exactly 5 of them have to wait more than 10 minutes for a table?

$p = e^{-\frac{1}{2}}$ Binomial

$P(X=5) = \binom{15}{5}(e^{-\frac{1}{2}})^5(1-e^{-\frac{1}{2}})^{10}$

$P = \binom{15}{5}(e^{-\frac{1}{2}})^5(1-e^{-\frac{1}{2}})^{10}$

9. [15 pts.] Suppose the joint distribution table of two random variables X and Y is given as follows, where the X values are on the horizontal axis and the Y values are on the vertical axis:

X

Y

(x, y)	-1	0	1	2
-3	.3	0	.1	0
0	0	.2	0	.1
3	.2	0	0	.1

(a) Calculate $E(XY)$.

$E(XY) = \sum_{\substack{x \in X \\ y \in Y}} xy f_{XY}(x,y)$

$= (-1)(-3)(0.3) + (1)(-3)(0.1) + (-1)(3)(.2) + (2)(3)(0.1)$

$= 3(0.3) - 3(0.1) - 3(0.2) + 6(0.1) = 0.9 - 0.3 - 0.6 + 0.6$

$= 0.6$

$E(XY) = 0.6$

(b) Calculate $E(X)$.

$E(X) = \sum_{\substack{x \in X \\ y \in Y}} x f_{XY}(x,y)$

$= -1(.3 + .2) + 0.1 + 2(0.1 + 0.1) = -(0.5) + 0.1 + 2(0.2)$

$= -0.5 + 0.1 + 0.4 = 0$

$E(X) = 0$

(c) Calculate $E(Y)$.

$E(Y) = \sum_{\substack{x \in X \\ y \in Y}} y f_{XY}(x,y)$

$= -3(0.3 + 0.1) + 3(0.2 + 0.1) = -3(0.4) + 3(0.3) = -1.2 + 0.9$

$= -0.3$

$E(Y) = -0.3$

(d) Calculate $Cov(X, Y)$.

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$= 0.6 - (0)(-0.3) = 0.6$

$Cov(X, Y) = 0.6$

(e) True or False: X and Y are positively associated.

$Cov(X,Y) = 0.6 > 0$

True

10. [15 pts.] Let $f_{XY}(x,y)$ be the function

$$f_{XY}(x,y) = \begin{cases} cxy & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is some constant.

(a) Find the value of c so that f_{XY} is a joint probability density function of random variables X and Y.

$$c\int_0^1\int_0^1 xy\,dx\,dy = c\int_0^1 \frac{x^2y}{2}\Big|_0^1 dy = c\int_0^1 \frac{y}{2}dy = c\left(\frac{y^2}{4}\right)\Big|_0^1 = \frac{c}{4} = 1 \Rightarrow c=4$$

$\boxed{c = 4}$

(b) Compute the marginal distribution function $f_X(x)$.

$$f_X(x) = \int_0^1 4xy\,dy = 4\left(\frac{xy^2}{2}\right)\Big|_0^1 = 4\left(\frac{x}{2}\right) = 2x$$

$$\boxed{f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otw} \end{cases}}$$

(c) Compute the marginal distribution function $f_Y(y)$.

$$f_Y(y) = \int_0^1 4xy\,dx = 2y$$

$$\boxed{f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otw} \end{cases}}$$

(d) True or False: X and Y are independent.

X and Y aren't in the same inequality so don't depend on one another

$\boxed{\text{True}}$