Homework 10

● Graded

Student

Jacob Hauptman

Total Points

23 / 24 pts

Question 1

(no title)

5 / 6 pts

– **0 pts** Correct

✔ – **1 pt** small mistake / missing a small step

– **2 pts** missing critical steps

– **3 pts** some steps are correct and some are wrong

– **6 pts** wrong

1 ?

Question 2

(no title)

6 / 6 pts

✔ – **0 pts** Correct

– **1 pt** wrong Var(X) and Var(Y) or other mistake in b

– **1 pt** missing a small step

– **1 pt** small mistake

– **1 pt** c. incomplete

– **2 pts** c. missing

Question 3

(no title)

6 / 6 pts

✔ – **0 pts** Correct

– **1 pt** small mistake

– **3 pts** missing critical steps

– **6 pts** Missing / wrong

Question 4

(no title)

6 / 6 pts

✔ **− 0 pts** Correct

− 1 pt small mistake

− 1 pt b. wrong

− 2 pts missing part b

− 1 pt c. wrong

− 2 pts missing part c

− 6 pts missing

Questions assigned to the following page: 2 and 1

1)

$$E(\hat{p}) = E\left(\frac{r-1}{x+r-1}\right) = \sum_{x=0}^{\infty} \frac{r-1}{x+r-1}\binom{x+r-1}{x} p^r (1-p)^x = \sum_{x=0}^{\infty} \frac{r-1}{x+r-1}\left[\frac{(x+r-1)!}{x!(r-1)!}\right] p^r (1-p)^x$$

$$= \sum_{x=0}^{\infty} \frac{r-1}{x+r-1}\left[\frac{(x+r-1)(x+r-2)!}{x!(r-1)(r-2)!}\right] p^r (1-p)^x = \sum_{x=0}^{\infty} \frac{(x+r-2)!}{x!(r-2)!} p^r (1-p)^x$$

$$= \sum_{x=0}^{\infty} \binom{x+r-2}{x} p^r (1-p)^x$$ Let $r = k+1$, so

$$= \sum_{x=0}^{\infty} \binom{x+k-1}{x} p^{k+1} (1-p)^x \quad \frac{(k+1)(1-p)}{p} = \frac{(r-1+1)(1-p)}{p} = \frac{r(1-p)}{p} \quad \square$$

2)

a $$Bias_{\mu}(\hat{\mu}) = E(\hat{\mu}) - \mu = E(\delta\bar{X} + (1-\delta)\bar{Y}) - \mu = E(\delta\bar{X}) + E(\bar{Y}) - E(\delta\bar{Y}) - \mu$$

$$= \delta\mu + \mu - \delta\mu - \mu = 0$$

b $$V(\hat{\mu}) = \delta^2 V(\bar{X}) + (1-\delta)^2 V(\bar{Y}) = \delta^2\frac{\sigma^2}{m} + (1-\delta)^2\frac{9\sigma^2}{n}$$

c $$\frac{d}{d\delta}\left[\delta^2\frac{\sigma^2}{m} + (1-\delta)^2\frac{9\sigma^2}{n}\right] = 2\delta\frac{\sigma^2}{m} - 2(1-\delta)\frac{9\sigma^2}{n}$$

Set to 0.

$$2\delta\frac{\sigma^2}{m} = 2(1-\delta)\frac{9\sigma^2}{n}$$

$$\frac{\delta}{m} = \frac{9-9\delta}{n} \Rightarrow n\delta = 9m - 9m\delta \Rightarrow n\delta + 9m\delta = 9m \Rightarrow \delta = \frac{9m}{n+9m}$$

$-$ $\qquad$ $+$

$$\frac{9m}{n+9m}$$

So it is a minimum.

Questions assigned to the following page: 3 and 4

3)

• a $(X_i - \bar{X})^2 = X_i^2 - 2X_i\bar{X} + \bar{X}^2$

$$E\left(\sum X_i^2 - 2X_i\bar{X} + \bar{X}^2\right) = E\left(\sum X_i^2 - 2\bar{X}\sum X_i + \bar{X}^2\sum 1\right) = E\left(\sum X_i^2 - 2\bar{X}\sum X_i + n\bar{X}^2\right)$$

$$= E\left(\sum X_i^2 - 2\bar{X}(n\bar{X}) + n\bar{X}^2\right) = E\left(\sum X_i^2 - 2n\bar{X}^2 + n\bar{X}^2\right) = E\left(\sum X_i^2 - n\bar{X}^2\right)$$

$$= \sum E(X_i^2) - E(n\bar{X}^2)$$

$V(X) = E(X^2) - E(X)^2$, so $E(X^2) = \sigma^2 + \mu^2$ and $E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$

$$= \sum(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right) = n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2 = n\sigma^2 - \sigma^2 = \sigma^2(n-1)$$

• b $E(S_{xx}) = \sigma^2(n-1) \Rightarrow \sigma^2 = \dfrac{E(S_{xx})}{n-1}$

Therefore $\frac{S_{xx}}{n-1}$ is an unbiased estimator for σ^2.

$\dfrac{S_{xx}}{n-1} = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$ as well.

4)

• a $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\mu = \ell$ so $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu = \ell$ so

$\bar{X}$ is an unbiased estimator for ℓ since $E(\bar{X}) = \mu = \ell$. The estimator equals the true value

• b $Bias_{\ell^2}(\bar{X}^2) = E(\bar{X}^2) - E(X^2) = V(\bar{X}) + E(\bar{X})^2 - \ell^2 = \frac{\sigma^2}{n} + \ell^2 - \ell^2 = \frac{\sigma^2}{n}$ so

$\bar{X}$ is a bias estimator for ℓ^2 and over estimates by $\frac{\sigma^2}{n}$

• c $Bias_{\ell^2}(\bar{X}^2 - kS^2) = E(\bar{X}^2) - kE(S^2) - \ell^2 = \frac{\sigma^2}{n} + \ell^2 - k\sigma^2 - \ell^2 = \frac{\sigma^2}{n} - k\sigma^2$

$$\frac{\sigma^2}{n} = k\sigma^2 \Rightarrow \frac{1}{n} = k$$