Stat 400. Problem Set 7. Due 10/22/24
Joint Distributions

Problem 1. Let $\mathcal{S} = \{(a,b) : a, b \in \{1,2,3,4,5,6\}\}$ be the space used to model two consecutive rolls of a fair die. Consider two random variables defined on S:

$$X(a,b) = a - 2, \text{ and } Y(a,b) = |a - b|$$

(a) What are the distinct values that the random variables $X, Y, X+Y$, and XY can take?
(b) Evaluate $P(X + Y \leq 3)$.
(c) Evaluate the conditional distribution of Y given that $X = 1$.
(d) Evaluate the conditional distribution of $X + Y$ given that $XY = 2$.

Problem 2. Consider a game where we toss an unfair coin with $P(H) = .4$ three times. We win $\$n$ if the first H is on the n^{th} toss, or we lose \$1 if we don't get any H. Let X be the number of Heads, and Y be the net winnings.

(a) Write out the joint distribution table, including the marginals.
(b) Calculate the following expected values:
  (i) $E(X)$
  (ii) $E(Y)$
  (iii) $E(XY)$
(c) Calculate the Covariance.
(d) Calculate the Correlation.

Problem 3. Suppose the joint distribution table of two random variables X and Y is given as follows, where the X values are on the horizontal axis and the Y values are on the vertical axis:

(x,y)	1	2	3	4
-2	.1	0	.1	0
0	0	.2	0	.2
2	.3	0	0	.1

(a) Calculate the marginal pmfs for this joint distribution.
(b) Calculate the conditional pmf for this joint distribution.
(c) Calculate $E(XY^2)$.
(d) Calculate $E(4XY)$.
(e) Calculate $P(|X - Y| > 1)$.
(f) Calculate $P(X + 2Y > 3)$.

Problem 4. Suppose we first randomly choose a real number between -1 and 1. If we get a negative number, we randomly choose another real number between -1 and 0; otherwise we randomly choose another real number between 0 and 1. Let X be the first number, and Y the second number.

(a) Compute the conditional density function of Y given that $X < 0$.
(b) Compute the conditional density function of Y given that $X \geq 0$.
(c) What is the joint pdf of X and Y?
(d) Compute the covariance of X and Y.
(e) Compute the correlation of X and Y.

Problem 5. Suppose the lifetimes of two light bulbs are measured (in 1000s of hours) by random variable X and Y , each having exponential distibution with parameter $\lambda_X = 2$ and $\lambda_Y = 4$. Assume that X and Y are independent of each other.

(a) What is the joint pdf of X and Y?
(b) What is the probability that each light bulb lasts at most 1000 hours (*Hint*: You want to calculate $P(X \leq 1 \text{ and } Y \leq 1)$).
(c) What is the probability that the *total* lifetime of the two bulbs is at most 2000 hours?

(d) What is the probability that the total lifetime of the two bulbs is between 1000 and 2000 hours?