Stat 400. Problem Set 5. Due 10/08/24
Continuous Random Variables

Problem 1. Consider the function

$$f(x) = \frac{1}{1+x^2},\ x \in (-\infty, \infty).$$

(a) Find a constant c such that $g(x) = c \cdot f(x)$ is a probability density function of some random variable X.
(b) Plot the graph $g(x)$.
(c) Show that $E(X)$ does not exist.

Problem 2. Consider the function

$$f(x) = \begin{cases} 4x & \text{if } x \in [0,1] \\ 4+2x & \text{if } x \in (1,3] \\ 0 & \text{if } x \notin [0,3] \end{cases}$$

(a) Plot the graph of $f(x)$. Is $f(x)$ continuous?
(b) Find a constant c such that $g(x) = c \cdot f(x)$ is a probability density function of some random variable X.
(c) Calculate the cdf $G_X(x)$ of X. Is X a continuous random variable?
(d) True or False: the pdf of a continuous random variable is always a continuous function.
(e) Find the percentiles n_p for each of the following values of p:
  (a) $p = .2$
  (b) $p = .5$

Problem 3. Let α and β be two fixed real numbers. Consider the function

$$f(x) = \begin{cases} \dfrac{\beta\alpha^{\beta}}{x^{\beta+1}} & \text{if } x > \alpha \\ 0 \text{ otherwise} \end{cases}$$

This is known as the Pareto Distribution, and it shows up in many applications, especially in modelling distribution of wealth.

(a) Verify that $f(x)$ is a probability density function of some random variable X.
(b) Calculate the expected value $E(X)$. For which values of β is $E(X)$ a real number?
(c) Calculate the variance $V(X)$. For which values of β is $V(X)$ a real number?
(d) Let $Y = X^3$. Calculate the pdf of Y.

Problem 4. A robot chooses a random number between 0 and $1,000$.

(a) What is the probability density function?
(b) What is the probability that the number is between 2 and 90?
(c) What is the probability that the number has three digits to the left of the decimal point?

Problem 5. The Rockwell hardness of a metal is determined by impressing a hardened point into the surface of the metal and then measuring the depth of penetration of the point. Suppose the Rockwell hardness of a particular alloy is normally distributed with mean 70 and standard deviation 3.

(a) If a specimen is acceptable only if its hardness is between 67 and 75, what is the probability that a randomly chosen specimen has an acceptable hardness?
(b) If the acceptable range of hardness is $(70 - C, 70 + C)$, for what value of C would 95% of all specimens have acceptable hardness?
(c) If the acceptable range is between 67 and 75, and we randomly select (with replacement) 10 specimens, what is the expected number of acceptable specimen among the ten?