Hashing

Plan

- "Classical" hashing
  - hashing by chaining
  - hashing by open addressing
- Universal hashing
- Perfect hashing (quick review)
- Cuckoo hashing
- Bloom filters
- Locality-sensitive hashing

Example 1: path finding

- Assume you want to implement A* shortest path search on a graph of 1000 nodes. You can allocate an array of size 1000 to store distances
- What about search on Rubik's cube graph (order of 10^6 for $2\times2\times2$ cube, 10^{19} for $3\times3\times3$ cube)?

Example 2: data bases

- Maintain a set of employees (students, messenger users, …), each identified by a social security number (student ID, phone number, …)

Example 3: deduplication

- In a programming language compiler, how to store user-declared identifiers?
- Construct an index of a book with all terms pointing to their first occurrence in the book

Hash tables: suppored operations

- A generalization of arrays ("direct addressing")
- *Goal*: maintain a (possibly evolving) set of objects belonging to a large "universe" (e.g. configurations, ID numbers, words, etc.)
- *Applications*: deduplication, indexing, path finding, file integrity test (checksum), etc.

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- INSERT: add a new object
- DELETE: delete existing object
- LOOKUP: check for an object

possibly specified by a *key* "associative array"

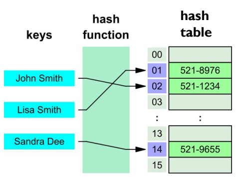
"Dictionary" data structure

Naive solutions

- Bit array (bitmap)
  - still too big for huge applications
  - does not support access to objects
  - BUT … (cf Bloom filters at the end of this lecture)
- Linked list
  - look-up too slow
- Search trees
  - better but still slow and memory demanding

Hash tables

- Notation
  - U : universe of all possible keys (*Ex*: strings, IP addresses, game configurations, …)
  - K : subset of keys (actually stored in the dictionary), $|K| \ll |U|$
  - $|K| = n$
- Use a table of size proportional to $|K|$: hash table
  - we lose the direct-addressing ability
  - hash function maps keys to entries of the hash table (**buckets** or slots)



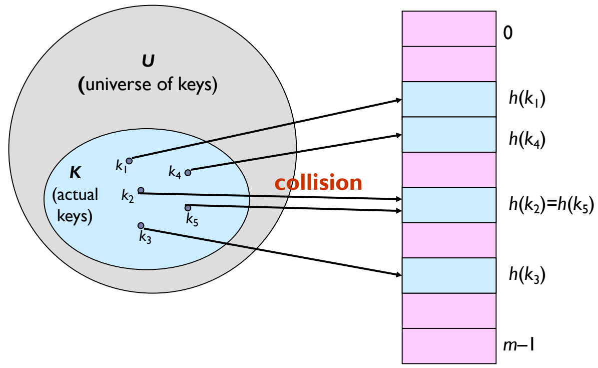
Hash functions

- Hash function h: Mapping from U to the slots of a hash table $T[0..m-1]$.

  $h : U \to \{0,1,\ldots,m-1\}$

- With direct addressing, key k maps to slot $A[k]$
- With hash tables, key k maps or "hashes" to bucket $T[h[k]]$
- $h[k]$ is the *hash value* (or simpy *hash*) of key k

Hashing and collisions



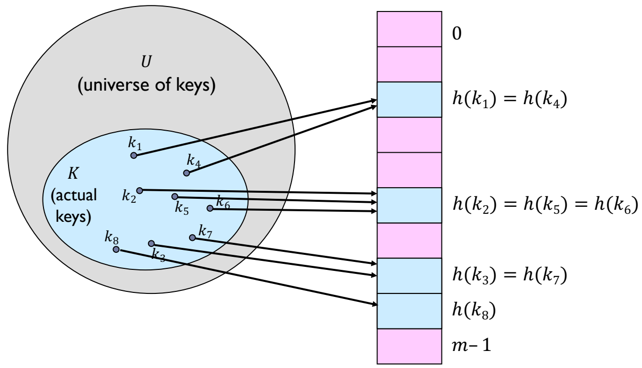
Collisions: birthday "paradox"

- What is the probability that two people from this class (100 students) have their birthday the same day?

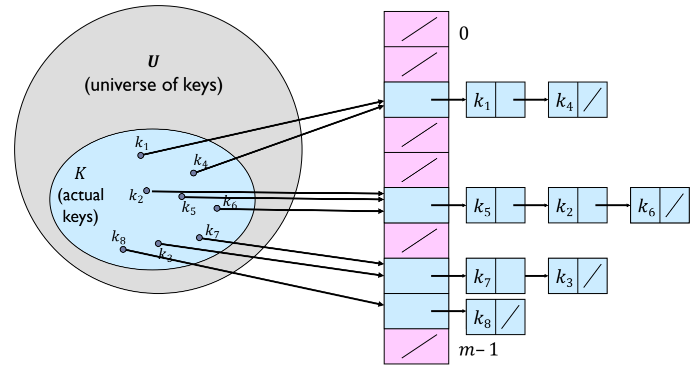
Collisions: birthday "paradox"

- What is the probability that two people from this class (100 students) have their birthday the same day?
- Answer: ≈0.9999997
- Birthday paradox: in a group of 23 people, there is about 50% chance that two people have the same bithday
- 40 people: 89%, 60 people: 99.4%
- *Conclusion*: collisions are frequent

I. Collision Resolution by Chaining



Collision Resolution by Chaining



Hashing with chaining

- INSERT$(T.k)$: $O(1)$
- DELETE$(T.k)$, LOOKUP(k): $O(list\ length)$
- $\Rightarrow$ a good hash function should distribute keys into buckets *as uniformly as possible*
- random hashing $\Rightarrow$ expected list length is $\alpha = n/m$ (*load factor*)
- the *average time* of DELETE and LOOKUP is $O(1+\alpha)$ $\Rightarrow O(1)$ if $n = O(m)$ (practical case)

Good hash functions

- Hash function should be easy to compute
- Desiging good hash functions is tricky. It is easy to design a bad hash function
- *Examples*: Phone numbers. Benford's law (e.g. prices, population sizes, …)
- Keys are usually considered as natural numbers
- *Example*: Interpret a character string as an integer expressed in some radix notation. Suppose the string is CLRS:
  - ASCII values: C=67, L=76, R=82, S=83.
  - There are 128 basic ASCII values.
  - So, CLRS = $67 \cdot 128^3 + 76 \cdot 128^2 + 82 \cdot 128^1 + 83 \cdot 128^0 =$ 141,764,947

Division Method

- Map a key k into one of the m slots by taking the remainder of k divided by m. That is,

  $h(k) = k \bmod m$

- *Example:* $m = 31$ and $k = 78 \Rightarrow h(k) = 16$
- *Advantage:* Fast, since requires just one division operation
- *Disadvantage:* Have to avoid certain values of m.
  - Don't pick certain values, such as $m = 2^p$ (as the hash won't depend on all bits of k)
- *Good choice for m:*
  - Primes, not too close to power of 2 (or 10) are good.

Multiplication Method

- If $0 < A < 1$, $h(k) = \lfloor m\ (kA \bmod 1) \rfloor = \lfloor m(kA - \lfloor kA \rfloor) \rfloor$ where $(kA \bmod 1) = kA - \lfloor kA \rfloor$: the fractional part of kA
- *Disadvantage:* Slower than the division method.
- *Advantage:* Value of m is not critical.
  - Typically chosen as a power of 2, i.e., $m = 2^p$, which makes the implementation easy
- *Example:* $m = 1000, k = 123, A = 0.6180339887\ldots$

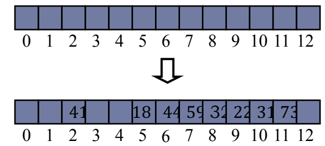
  $h(k) = \lfloor 1000(123 \cdot 0.6180339887 \bmod 1) \rfloor = \lfloor 1000 \cdot 0.018169 \ldots \rfloor = 18$

II. Collision Resolution by Open Addressing

- All elements are stored in the hash table itself
- $\Rightarrow\quad n \le m,$ no pointers
- hash function $h(k, i)$ where $i = 0,1,2, \ldots, m-1$, and $< h(k,0), h(k,1), \ldots, h(k, m-1) >$ is a permutation
- when inserting/looking up k, probe $h(k,0), h(k,1), \ldots$ (*probe sequence*) until
  - we find k, or
  - the bucket contains nil, or
  - m buckets have been unsuccessfully probed
- deletion is complicated, needs a special key "deleted", time may not be dependent on the load factor

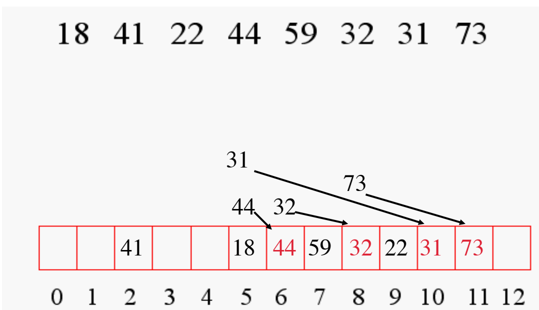
Open Addressing: Linear probing

- The colliding item is placed in a different cell of the table
- **Linear probing** handles collisions by placing the colliding item in the next (circularly) available table cell $h(k,i) = (h'(k) + i) \bmod m$
- Each table cell inspected is referred to as a "probe"
- Colliding items clump together, causing future collisions to cause a longer sequence of probes
- *Example:*
  - $h'(k) = k \bmod 13$
  - Insert keys $18, 41, 22, 44, 59, 32, 31, 73,$ in this order



Example (cont.)

$h'(k) = k \bmod 13$



Quadratic probing

- $h(k,j) = (h'(k) + c_1 \cdot j + c_2 \cdot j^2) \bmod m$
- for example, $h(k,j) = (h'(k) + \frac{1}{2} \cdot j + \frac{1}{2} \cdot j^2) \bmod m$

  $h(k,0), h(k,1), \ldots, h(k,m-1)$ is a permutation if m is a power of 2
- quadratic probing works better than linear probing (less clumping)

Double Hashing

- Double hashing uses a secondary hash function $d(k)$ and handles collisions by placing an item in the first available cell of the series

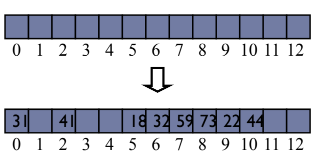
  $h(k,j) = (h(k) + jd(k)) \bmod m$

  for $j = 0,1, \ldots, m-1$
- The secondary hash function $d(k)$ cannot have zero values
- m should be relatively prime to $d(k)$, e.g. $m = 2^q$ and $d(k)$ odd, or m is prime and $d(k) < m$
- Double hashing is usually more efficient than linear and quadratic probing

Example of Double Hashing

- Consider a hash table storing integer keys that handles collision with double hashing
  - $m = 13$
  - $h(k) = k \bmod 13$
  - $d(k) = 7 - k \bmod 7$
- Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

k	$h(k)$	$d(k)$	Probes		
18	5	3	5		
41	2	1	2		
22	9	6	9		
44	5	5	5	10	
59	7	4	7		
32	6	3	6		
31	5	4	5	9	0
73	8	4	8		



Performance of Open Addressing

- Assuming that $< h(k,0), h(k,1), \ldots, h(k, m-1) >$ is a random permutation (uniformly drawn), the expected number of probes in an insertion (or unsuccessful search) with open addressing is

  $1/(1-\alpha)$,

  where $\alpha = n/m$ the load factor
- *Explanation*:

  let $p_i = \mathrm{P}[i \text{ first buckets are full}] = \alpha^i \quad (p_0 = 0)$

  $E[\text{number of probes}] = 1 + \sum_{i=1..m-1} i \cdot P[i \text{ full buckets followed by an empty one}] = 1 + \sum_{i=1..m-1} i \cdot (p_{i-1} - p_i) = 1 + \sum_{i=1..m-1} p_i \approx 1 + \alpha + \alpha^2 + \alpha^3 + \cdots = 1/(1-\alpha)$

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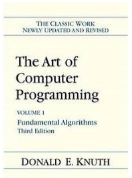
  $1/(1-\alpha)$,

  where $\alpha = n/m$ the load factor
- The expected number of probes for a successful search is

  $(1/\alpha)\log(1/(1-\alpha))$

Historical remarks



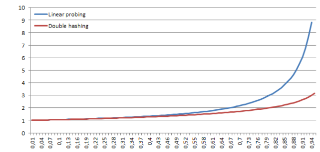
Hashing by open addressing:
Analysed by Donald Knuth in 1962 (invention attributed to Andrei Ershov)

Hashing: some conclusions

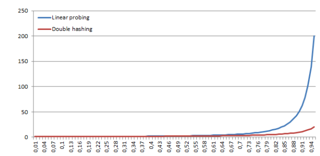
- Chaining:
  - easy implementation
  - fast in practice
  - uses more memory
- Open addressing:
  - uses less memory
  - more complex removals
- Implemented in standard libraries, e.g. `std::unordered_map` in C++

Linear probing vs Double hashing

comparison of average number of operations



successful search



unsuccessful search

Exercise

- Consider inserting the keys $10, 22, 31, 4, 15, 28, 17, 88, 59$ into a hash table of length $m = 11$ using open addressing with the primary hash function $h'(k) = k \bmod m$. Illustrate the result of inserting these keys using linear probing, using quadratic probing with $c_1 = 1$ and $c_2 = 3$, and using double hashing with $h_2(k) = 1 + (k \bmod (m-1))$.