Efficient algorithms and data structures

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Programming exercises:
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Course

- *What the course is:*
  - a selection of topics on the design and analysis of algorithms
  - with emphasis on rigorous analysis (Ph.Flajolet: "mathematically oriented engineering")
  - dealing with basic data structures (graphs, strings, trees, tables, …)
  - including programming assignments and in-class projects
- *What the course is not:*
  - a programming course
  - a course oriented to a specific programming language (an imperative programming language is assumed, one of Python, C, C++, Java)
  - a course oriented to a specific application area
  - a math course

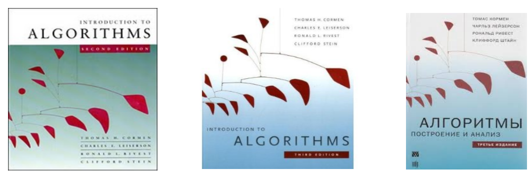
Course

- *Varying level of difficulty*
- *Prerequisites:*
  - imperative programming (C, C++, Java, …)
  - Basic data structures: lists, arrays, stacks, queues
  - Recursion, Big-Oh notation
  - Sorting, …
- *"Free-style" pseudo-code*
- Having a laptop assumed

Grading

- participation in class 10%
  - full attendance is expected
  - in-class projects
- programming exercises 40%
  - one every ~2 weeks
  - plagiarism is not tolerated
- exam 50%

Useful books



CLRS = Cormen & Leiserson & Rivest &Stein

Some other good algorithm textbooks:
- *Steven Skiena*, The Algorithm Design Manual, 2nd Edition, Springer, 2008 [a bit advanced?]
- *Jon Kleinberg* and *Éva Tardos*, Algorithm Design, MIT Press 2005
- *Robert Sedgewick* and *Kevin Wayne*, Algorithms, Addison-Wesley, 4th Edition, 2011 [for beginners, Java-oriented]

How to measure the efficiency of algorithms?

- Efficiency (*in this course*) = TIME and SPACE
  - other possible measure of efficiency: accuracy
- *Classical model*: RAM model of computation
  - all memory accesses have equal cost
  - no parallel execution
  - unit cost ($O(1)$) of basic operations (unless we want to explicitely count individual bits operations)
  - space = # of computer words (unless bit complexity is considered); each computer word contains $\Theta(\log n)$ bits
  - other possible measures can be considered: disk accesses, cache misses, probe model, query complexity …

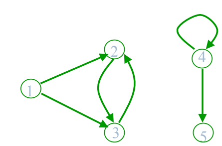
How to measure the efficiency of algorithms?

- Algorithms solve mass problems
  - n: input size (in computer words or bits)
  - time/space as a function of n
- Different *complexity analyses*:
  - ***worst-case*** complexity
  - average-case complexity
  - smoothed analysis
  - query (probe) complexity
  - …

Graphs

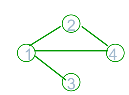
Graphs

Directed graph $G = (V, E)$
V finite set of *nodes (vertices)*
$E \subseteq V \times V$ set of *edges (arcs)*, *i.e.*, a relation on V



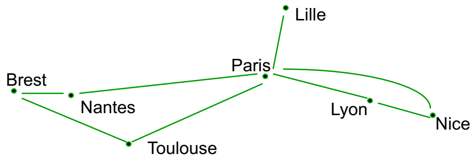
$V = \{ 1, 2, 3, 4, 5 \}$
$E = \{ (1, 2), (1, 3), (2, 3), (3, 2), (4, 4), (4, 5) \}$

Undirected graph $G = (V, E)$
E set of *edges (arcs)*, symmetric relation



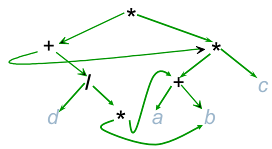
$E = \{ 1, 2, 3, 4 \}$
$V = \{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\} \}$

Graphs are everywhere



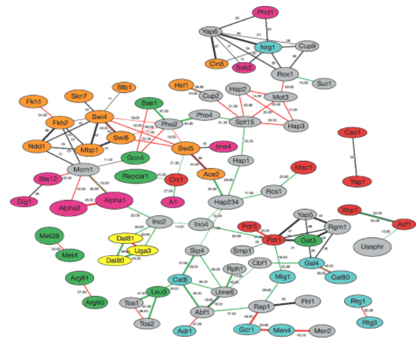
Road Network

Acyclic graph of an expression (DAG)

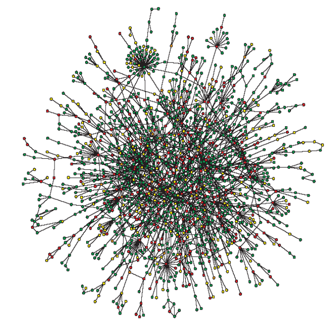


((a+b)*c+d/(b*(a+b))) * (a+b)*c

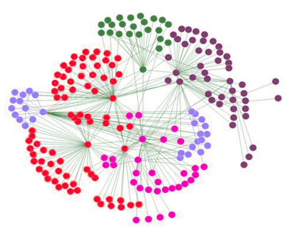
Gene regulation network in biology



Protein-protein interaction network (in yeast)



Social networks



Graph representations

$G = (V, E)$ $V = \{1, 2, ..., n\}$

Adjacency list

reduces the size if $|E| \ll (|V|)^2$

reading time : $O(|V| + |E|)$

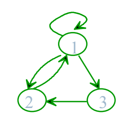
Adjacency matrix

using matrix operations

reading time $O(|V|)^2$

Other representations possible

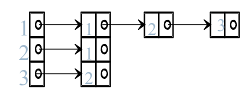
Adjacency lists

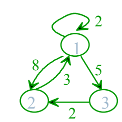


Lists of $A(s)$

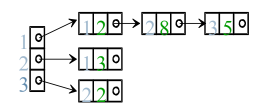
$V = \{ 1, 2, 3 \}$

$E = \{ (1,1), (1, 2), (1, 3), (2, 1), (3, 2)\}$

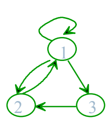




weight: $w : A \longrightarrow X$



Adjacency matrix

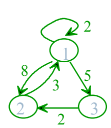


$M[i, j] = 1$ iff j is adjacent to i

$V = \{ 1, 2, 3 \}$

$E = \{ (1,1), (1, 2), (1, 3), (2, 1), (3, 2)\}$

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



weight: $w : A \longrightarrow X$

$$W = \begin{pmatrix} 2 & 8 & 5 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

Graph algorithms

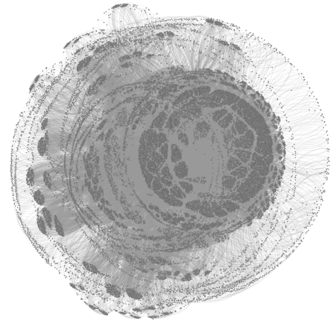
- Exploration
  - Depth-first or breadth-first search
  - Topological sorting
  - Strongly connected components
- Path computation
  - Shortest path
  - Transitive closure
  - Eulerian and Hamiltonian paths
- Minimum spanning trees
  - Kruskal's and Prim's algorithms
- Networks
  - Maximum flow
- Others
  - Graph coloring
  - Planarity testing
  - …

Shortest paths in graphs

Single-source shortest path: unweighted case

- Path length = number of edges
- Distance between two nodes = length of the shortest path
- *Problem*: given a (directed or undirected) graph $G = (V, E)$ and a *source node* $s \in V$, compute the distance from s to each reachable node

Single-source shortest path: unweighted case

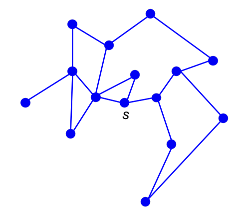


a subgraph (29,160 nodes) of the graph of Rubik's mini cube (2x2x2) configuraitons (3,674,160 nodes)
https://miscellaneouscoder.wordpress.com/2014/07/28/working-with-rubiks-group-cycle-graphs/

Breadth-first search (BFS)

Given a source node s,

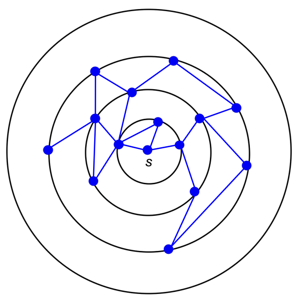
- "Discovers" all nodes reachable from s



Breadth-first search (BFS)

Given a source node s,

- "Discovers" all nodes reachable from s
- Proceeds by "concentric circles"
- Discovers all nodes at distance d from s before discovering any nodes at distance d+1



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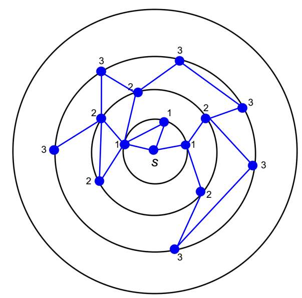
- Computes the distances from s
- Computes a *breadth-first tree* encoding one shortest path for each node



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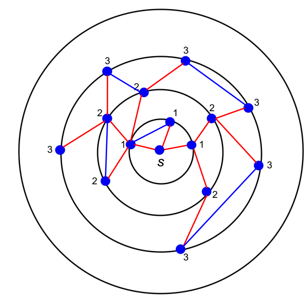
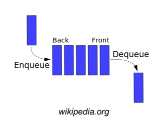


How it works?

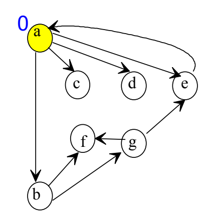
- colors every node *white* (not yet discovered), *yellow* (discovered but may have white adjacent nodes), or *red* (discovered and all adjacent nodes discovered)
- yellow nodes = "active frontier" (nodes under processing)
- when processing a (yellow) node, determine all white neighbors, set their distance to be larger by 1, color them yellow. After that, color the node red.

Breadth-first search (BFS)

```
procedure BFT (s node of V) ;
begin
for each node v of V  do {
        visited[v] = false ; //s is white
        d[v] = ∞ ; π(v) = nil
        }
visited[s]=true ; //s becomes yellow
d[s]=0 ;
Queue = enqueue (empty-queue, s) ;
while not empty (Queue)  do {
        u = dequeue (Queue) ;
        for  t = first to last successor of u do
                if not visited [ t ] then
                        visited[ t ]=true ; //t becomes yellow
                        d[ t ] = d[ u ]+1 ; π(t) = u
                        Queue = enqueue (Queue, t) ;
        //s' becomes red
        }
end
```



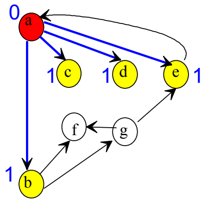
BFS: example



Queue : a

Order of traversal:

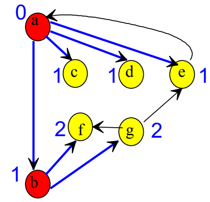
BFS: example



Queue : a b c d e

Order of traversal: a

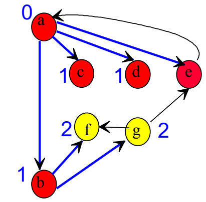
BFS: example



Queue : a b c d e f g

Order of traversal: a b

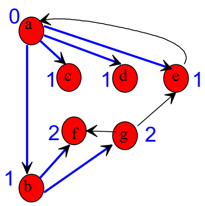
BFS: example



Queue : a b c d e f g

Order of traversal: a b c d e

BFS: example



Queue : a b c d e f g

Order of traversal: a b c d e f g

Questions

- Show that BFS runs in time O(n+m) (assuming the graph is represented by adjacency lists), n=|*V*|, m=|*E*|
- Show that if $(v_1,v_2,\ldots,v_r)$ is the state of the Queue, then $d[v_r] \leq d[v_1]+1$ and $d[v_i] \leq d[v_{i+1}]$ for all *i*
- Show that upon termination $d[v]=\delta(s,v)$, where $\delta(s,v)$ is the length of the shortest path from *s* to *v*

$d[v]=\delta(s,v)$: sketch of the proof

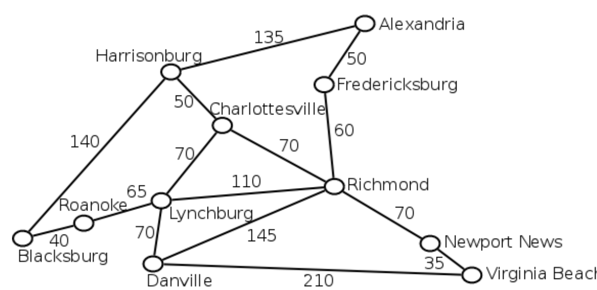
- by contradiction, let v be the closest to s node with $d[v]>\delta(s,v)$
- consider a shortest path from s to *v*, and let *u* be the node preceding *v* in this path
- $\delta(s,v)=\delta(s,u)+1$ (by properties of shortest paths)
- consider the moment when *u* was dequeued ($d[u]=\delta(s,u)$)
- if *v* was white then, we have $d[v]=\delta(s,v)$ $\Rightarrow$ *contradiction*
- if *v* was yellow then, it was visited earlier by exploring the successors of some *w* with $d[w] \leq d[u]$. Then $d[v]=d[w]+1 \leq d[u]+1$ $\Rightarrow$ *contradiction*
- if v was red, then $d[v] \leq d[u]$ $\Rightarrow$ *contradiction*

Space efficient BFS

- BFS stores the queue which (in the worst case) can contain O(n) nodes, i.e. O(n log n) bits
- Can we implement BFS with o(n log n) bits?
- *Example of a result*: There exists an algorithm that outputs vertices in the BFS order in time O(n+m) and uses 2n+o(n) bits
  [N. Banerjee, S. Chakraborty, V. Raman, and S. R. Satti. Space efficient linear time algorithms for BFS, DFS and applications. Theory of Computing Systems, Jan 2018]

Single-source shortest path: weighted case

Single-source shortest path: weighted case



Shortest path problem

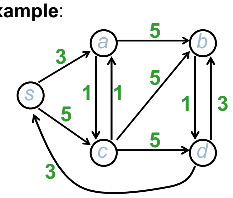
Weighted (directed or undirected) graph: $G = (V, E, w)$ where
$w : E \to \mathbf{R}$ (weight/cost)

Source : $s \in V$

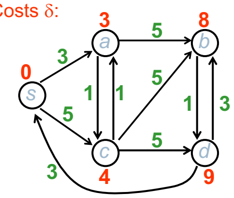
Problem: for all $t \in V$, compute

$\delta(s, t) = \min \{\{ w(c) \; ; \; c \text{ path from } s \text{ to } t \} \cup \{+\infty\}\}$

Example:

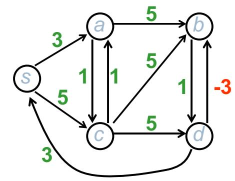


Costs δ:



Properties of the shortest paths

Proposition 1 (existence):
shortest paths are well-defined (i.e. for all $t \in V$, $\delta(s, t) > -\infty$) **iff** the graph does not have a cycle of cost < 0 reachable from s



Proposition 2: if there exists a shortest path from s to t, then there exists one without a cycle

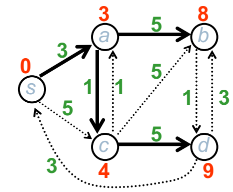
Proposition 3: if there exists a shortest path from s to t, then there exists one with no more than $|V|$-1 edges

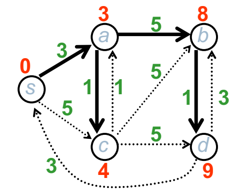
Main properties

Property 1: $G = (V, E, w)$
let c be a shortest path from p to r
and q be the node preceding r in c.
Then $\delta(p, r) = \delta(p, q) + w(q, r)$.

Property 2: A subpath of a shortest path is a shortest path

Shortest path tree: tree rooted at s representing shortest paths





Main properties (cont)

Property 3: $G = (V, E, w)$ let c be a path from p to r and q be the node preceding r in c. Then $\delta(p, r) \leq \delta(p, q) + w(q, r)$.

Relaxation

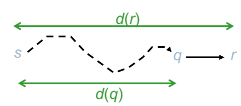
Compute $\delta(s,t)$ by successive approximations

$t \in V$ $d[t]$ = estimate (from above) of $\delta(s, t)$

$\pi[t]$ = predecessor of t on
a path from s to t of cost $d[t]$

Initialization of d and π

```
INIT
    for all  t ∈ V do
    { d[t] = ∞ ;  π[t] = nil }
    d[s] = 0 ;
```



Relaxation of the edge (q, r)

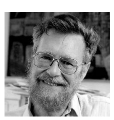
```
RELAX(q, r)
    if d[q] + w(q, r) < d[r]
    then { d[r] = d[q] + w(q, r) ;  π[r] = q }
```

Relaxation (cont)

Proposition :
the following property is an invariant of **`relax`**: for all $t \in V$, $d(t) \geq \delta(s, t)$

Proof: by induction on the number of executions of **`relax`**

Dijkstra's algorithm



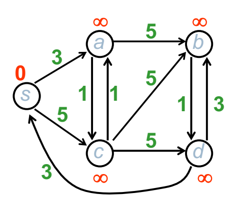
Assumption: $w(p, q) \geq 0$ for all edges (p, q)

```
begin
    INIT;
    S = ∅ ;  Q = V ;
    while Q ≠ ∅  do {
        q = MIN_d(Q) ;  Q = Q \ {q} ;  S = S ∪ {q} ;
        for all  r successor of q  do
            RELAX(q, r) ;
    }
end
```

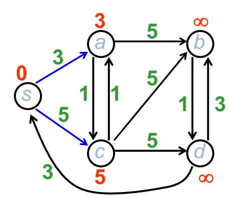
• At each iteration, the algorithm extracts a node from Q that is never returned to Q

• **`RELAX`**(q, r) may change $d[r]$

Example

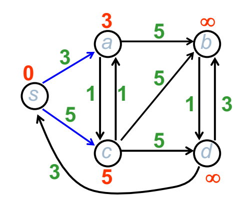


$S = \{\varnothing\}$
$Q = \{s, a, b, c, d\}$
$\pi[s]$ = nil
$\pi[a]$ = nil
$\pi[b]$ = nil
$\pi[c]$ = nil
$\pi[d]$ = nil



$S = \{s\}$
$Q = \{ a, b, c, d \}$
$\pi[s]$ = nil
$\pi[a] = s$
$\pi[b]$ = nil
$\pi[c] = s$
$\pi[d]$ = nil

Example (cont)



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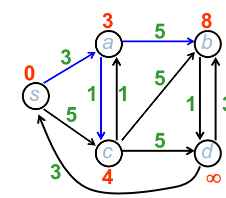

$S = \{s, a\}$
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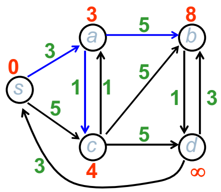
Example (cont)



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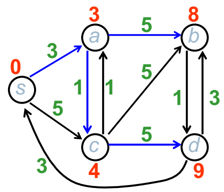
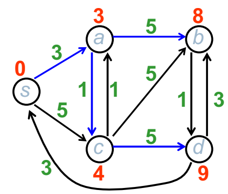
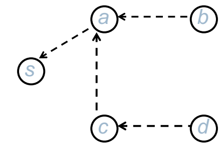
$\pi[d]$ = nil



$S = \{s, a, c\}$
$Q = \{ b, d \}$
$\pi[s]$ = nil
$\pi[a] = s$
$\pi[b] = a$
$\pi[c] = a$
$\pi[d] = c$

Example (cont)



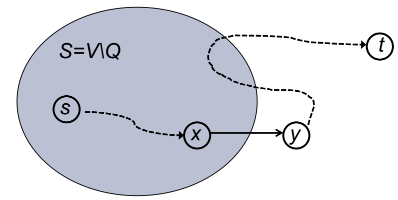


$S = \{s, a, c\}$
$Q = \{ b, d \}$, $Q = \{ d \}$ then $Q = \varnothing$
$\pi[s]$ = nil
$\pi[a] = s$
$\pi[b] = a$
$\pi[c] = a$
$\pi[d] = c$

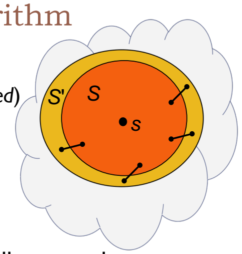
Correctness of Dijkstra's algorithm

Proposition : After the execution of Dijkstra's algorithm on a graph $G = (V, E, w)$, $d[t] = \delta(s, t)$ for all $t \in V$.

Proof by contradiction: let $d[t] \neq \delta(s, t)$



Properties of Dijkstra's algorithm



- Algorithm maintains three sets:
  - *S*: *finished* nodes, for which $d[t] = \delta(s, t)$ *(red)*
  - S': nodes of *Q* with $d[t]<\infty$ *(yellow)*
  - nodes of *Q* with $d[t]=\infty$ *(white)*
- Algorithm can be seen as expanding a ball centered at s following a *greedy strategy*

Implementation

With adjacency matrix
time $O(n^2)$ (where $n=|V|$)

With adjacency lists
depends on the data structure for *Q*

we need to support operations:
- insert an element to *Q*
- extract an element with minimum *d* value
- modify (decrease) the *d* value of an element (when relaxing)

⇒ (min-)priority queue

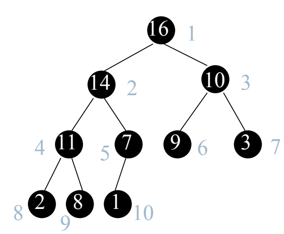
Priority Queues

- (max-)Priority Queue is a data structure that supports operations
  - INSERT(S,*x*)
  - MAX(S)
  - EXTRACT-MAX(S)
  - INCREASE-KEY(S,*x*,*k*): increase the key of *x* to *k*
- Priority Queues are used in
  - Dijkstra's algorithm for shortest paths
  - Prim's algorithm for minimum spanning tree
  - other greedy algorithms
- Implemented using heaps

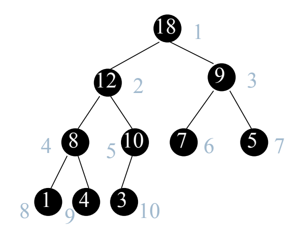
Binary Heaps

- Binary heap:
  - a **binary tree** that is
  - **complete**: every level except possibly the bottom one is completely filled and the leaves in the bottom level are as far left as possible
  - satisfies the **(max-)heap property**: the key stored in every node is greater than or equal to the keys stored in its children
    - If the key at each node is smaller than or equal to the keys of its children, then we have a **min-heap**

Binary (max-)heap: example



Binary heaps stored in arrays



Due to their regular structure, binary heaps are easily stored in arrays

Given index i of a node,
- the index of its parent is $\lfloor i / 2 \rfloor$
- the indices of its children are $2i$ and $2i+1$

18	12	9	8	10	7	5	1	4	3	*element key A*[i]
1	2	3	4	5	6	7	8	9	10	*index* i

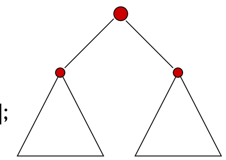
Binary heaps: some properties

- The height of a heap is $\lfloor \log(n) \rfloor$
- Not every array represents a heap
- In a max-heap, the largest element is at the root and the smallest element is in a leaf

Heapify

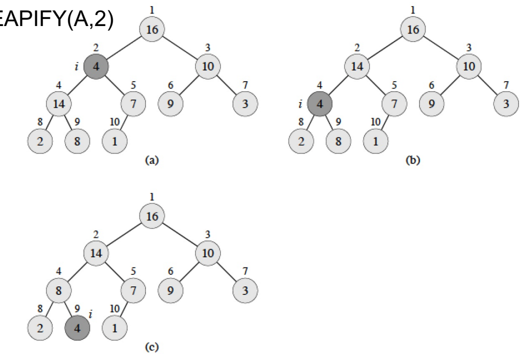
- Assume that node i violates the heap property, but the children nodes 2i and 2i+1 (if exist) are heaps.

HEAPIFY(A,i)
  if A[2i]>A[i] **or** A[2i+1]>A[i] **then**
    if A[2i+1]>A[2i] **then**
      exchange A[i] and A[2i+1];
      HEAPIFY(A, 2i+1)
    else
      exchange A[i] and A[2i];
      HEAPIFY(A, 2i)
    end
  end



Heapify: example

HEAPIFY(A,2)



Building a binary heap

- Given an array A[1..n], build a binary heap for array elements

BUILD-HEAP(A,n)
  for i=$\lfloor n/2 \rfloor$ **downto** 1 **do** HEAPIFY(A,i);

- *Exercise*: build the heap for A=[4,1,3,2,16,9,10,14,8,7]

BUILD-HEAP: complexity

- Straightforward estimation O(n · log(n))
- Refined analysis:
  - Cost of a call to HEAPIFY at a node depends on the height, *h*, of the node – *O*(*h*).
  - Height of most nodes smaller $\lfloor \log(n) \rfloor$
  - Height of nodes *h* ranges from 0 to $\lfloor \log(n) \rfloor$
  - number of nodes of height *h* is at most $\lceil n / 2^{h+1} \rceil$?

Heap Characteristics

- Height = $\lfloor \log n \rfloor$
- Number of leaves = $\lceil n/2 \rceil$
- Number of nodes of height $h \leq \lceil n/2^{h+1} \rceil$
- *Proof by induction*:
  - remove all leaves from the heap
  - there remains n - $\lceil n/2 \rceil$ =$\lfloor n/2 \rfloor$ nodes
  - the height of each node is decremented by 1
  - nb of nodes of height h-1 is (by induction) $\lceil \lfloor n/2 \rfloor / 2^h \rceil \leq \lceil n/2^{h+1} \rceil$

Tighter bound for BUILD-HEAP: $O(n)$

time of BUILD-HEAP is $\sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h}\right)$

note that $\sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h} \leq \sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$

therefore the time is $O(n)$

Priority Queue

- MAX(A): return the heap root
- EXTRACT-MAX(A):

Priority Queue

- MAX(A): return the heap root
- EXTRACT-MAX(A): exchange A[1] and A[n], discard element n, and apply HEAPIFY(A,1)
- INCREASE-KEY(A,i,k):

Priority Queue

- MAX(A): return the heap root
- EXTRACT-MAX(A): exchange A[1] and A[n], discard element n, and apply HEAPIFY(A,1)
- INCREASE-KEY(A,i,k):

  A[i]$\leftarrow k$;

  while A[$\lfloor i/2 \rfloor$]<A[i] **do**

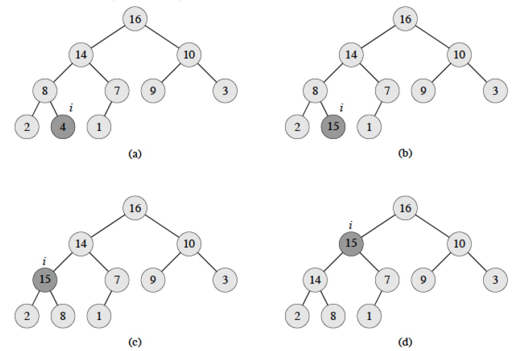
  exchange A[$\lfloor i/2 \rfloor$] and A[i];

  i$\leftarrow$parent(i)

  end
- INSERT(A,i): insert a new leaf n+1 with key $-\infty$; call INCREASE-KEY(A,n+1,k)

INCREASE-KEY: example

INCREASE-KEY(A,9,15)



Priority Queues: time bounds

- MAX: $O(1)$
- EXTRACT-MAX, INCREASE-KEY, INSERT: $O(\log(n))$

Priority Queues: time bounds

- MAX: $O(1)$
- EXTRACT-MAX, INCREASE-KEY, INSERT: $O(\log(n))$

Various improvements have been proposed

- *Fibonacci heaps* take $O(1)$ *amortized* time for INSERT and INCREASE-KEY
- if keys are integers bounded by C, van Emde Boas trees support INSERT, DELETE, MAX, MIN, SUCC, PRED in time O(log log(C))

Back to Dijkstra's algorithm

With adjacency matrix

time $O(n^2)$

With adjacency lists

Q : priority queue

if implemented by binary heaps:

n building a heap of n elements : O(n)

n operations $\mathbf{MIN}_d$: $O(n \cdot \log n)$

m operations **RELAX** : $O(m \cdot \log n)$

total time $O((n+m) \cdot \log n)$:

improves over $O(n^2)$ if $m=o(n^2/\log n)$

time can be improved to $O(n \cdot \log n + m)$ using *Fibonacci heaps,*

as decreasing the key takes O(1) amortized