§ 2.5 : Inverse Functions

Item	Price
pizza	\$14
hamburger	\$13
cheeseburger	\$14

Item	Price
pizza	\$14
hamburger	\$13
cheeseburger	\$13.50

"undo" functions

"can we undo the function if we don't have any memory of the input"

One-to-One Functions

Def: If a function has no two ordered pairs with different first coordinates and the same second coordinate, the the function is called one-to-one.

* If a function is one-to-one, then if the outputs are the same the the inputs are the same.

Ex 1) Determine whether each function is one-to-one

a) $\{(1,8), (2,9), (3,7), (4,0)\}$

b) $\{(-4,16), (-2,4), (0,0), (2,4), (3,9)\}$

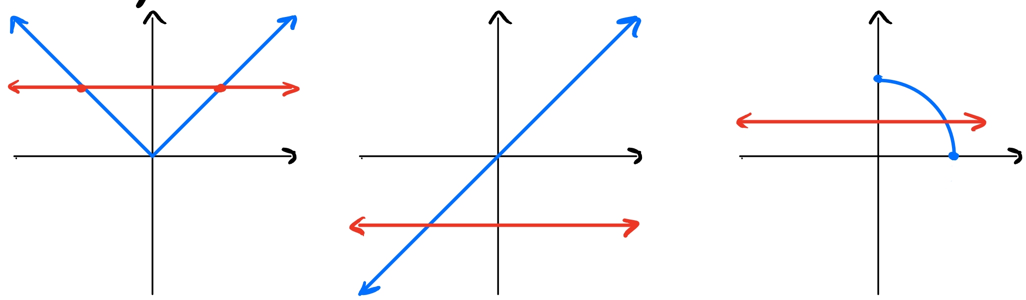
c) $\{(-1,\pi), (0,3.14), (1,\frac{22}{7})\}$

Horizontal Line Test:

If each horizontal line crosses the graph of a function at no more than one point, then the function is one-to-one.

* What did the vertical line test show?

Ex 2) Horizontal line test



Ex 3) Using the definition of one-to-one

a) $f(x) = \dfrac{3x-5}{x+2}$ b) $g(x) = |x|$ c) $h(x) = x^2 + 2x + 5$

$*$ if the outputs are the same, then the inputs are the same. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$?

a) $f(x_1) = f(x_2)$

$$(x_2+2)\cancel{(x_1+2)}\frac{3x_1-5}{\cancel{x_1+2}} = \frac{3x_2-5}{\cancel{x_2+2}}\cancel{(x_2+2)}(x_1+2)$$

$$(x_2+2)(3x_1-5) = (3x_2-5)(x_1+2)$$

$$\begin{array}{rcl} 3x_1x_2 - 5x_2 + 6x_1 - 10 & = & 3x_1x_2 + 6x_2 - 5x_1 - 10 \\ -3x_1x_2 \qquad\qquad\quad +10 & & -3x_1x_2 \qquad\qquad\quad +10 \\ \hline -5x_2 + 6x_1 & = & 6x_2 - 5x_1 \\ +5x_2 + 5x_1 & & +5x_2 + 5x_1 \\ \hline 11x_1 & = & 11x_2 \end{array}$$

$$x_1 = x_2$$

Therefore $f(x)$ is one-to-one.

b) $g(x) = |x|$ need counterexample

* expect one of these on the exam

$|1| = |-1| \Rightarrow g(1) = g(-1)$, but $1 \neq -1$.

Therefore $g(x)$ is not one to one.

c) $h(x) = x^2 + 2x + 5$

$$h(x_1) = h(x_2)$$

$$x_1^2 + 2x_1 + 5 = x_2^2 + 2x_2 + 5$$

$$(x_1^2 + 2x_1 + 1) + 5 - 1 = (x_2^2 + 2x_2 + 1) + 5 - 1$$

$$(x_1 + 1)^2 + 4 = (x_2 + 1)^2 + 4$$

$$-4 \qquad\qquad -4$$

$$(x_1 + 1)^2 = (x_2 + 1)^2$$

$$x_1 + 1 = \pm(x_2 + 1)$$

$$x_1 = \pm(x_2 + 1) - 1$$

$$x_1 = x_2, \ \underline{-x_2 - 2}$$

$$x_2 = 0 \qquad x_1 = -2$$

$h(0) = 0^2 + 2 \cdot 0 + 5$
$= 5$

$h(-2) = (-2)^2 + 2(-2) + 5$
$= 4 - 4 + 5$
$= 5$

* could we have chosen $x_2 = -1$?

Inverse Functions

Def: A function f is said to be <u>invertable</u> if it is one to one.

Def: The <u>inverse</u> of a one-to-one function f is the function f^{-1} (read "f inverse"), where the ordered pairs of f^{-1} are obtained by interchanging the coordinates in each ordered pair of f.

Ex 4) Determine whether each function is invertible. If so, find the inverse

a) $\{(2,1), (3,4), (4,0)\}$

b) $\{(-1,0), (1,0), (5,0)\}$

Ex 5) Let $f = \{(1,2), (2,3), (3,1), (4,4), (5,0)\}$

Find f^{-1}, $f^{-1}(3)$, $f^{-1} \circ f$, and $f \circ f^{-1}$.

$f^{-1} = \{(2,1), (3,2), (1,3), (4,4), (0,5)\}$

$= \{(0,5), (1,3), (2,1), (3,2), (4,4)\}$

$f^{-1}(3) = 2$

$f^{-1} \circ f = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

$f \circ f^{-1} = \{(0,0), (1,1), (2,2), (3,3), (4,4)\}$

$f^{-1} \circ f \stackrel{?}{=} f \circ f^{-1}$ No!

Ex6) Find the inverse function by reversing composition.

a) $F = \frac{9}{5}C + 32$

$C = 5 \cdot \frac{1}{9}(F-32) = \frac{5}{9}(F-32)$

Given the temp in °F,

	inverse	
1) divide by 5	$\longrightarrow$	3) multiply by 5
2) multiply by 9	$\longrightarrow$	2) divide by 9
3) add 32	$\longrightarrow$	1) subtract 32

Given the Temp in °F

b) $g(x) = \frac{x^3+5}{2}$

$g^{-1}(x) = \sqrt[3]{2x-5}$

1) cube	$\longrightarrow$	3) cube root
2) add 5	$\longrightarrow$	2) subtract 5
3) divide by 2	$\longrightarrow$	1) multiply by 2

Ex 7) Graph a function and its inverse

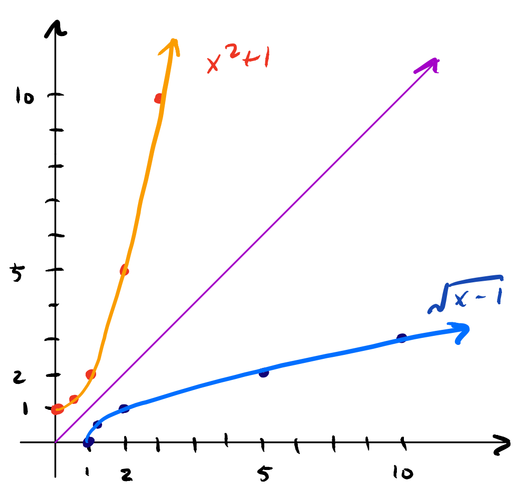
$f(x) = \sqrt{x-1}$

$f^{-1}(x) = x^2 + 1, \ x \geq 0$

1) subtract 1	$\longrightarrow$	2) add 1
2) square root	$\longrightarrow$	1) square

x	$f(x)$
1	0
5/4	1/2
2	1
5	2
10	3

x	$f^{-1}(x)$
0	1
1/2	5/4
1	2
2	5
3	10



$D(f) = [1, \infty)$ → $R(f^{-1}) = [1, \infty)$

$R(f) = [0, \infty)$ → $D(f^{-1}) = [0, \infty)$

Switch and Solve Method

Procedure: Finding $f^{-1}(x)$

To find the inverse of a one-to-one function

1. Replace $f(x)$ by y
2. Interchange x and y
3. Solve the equation for y.
4. Replace y by $f^{-1}(x)$
5. Check that the domain of f is the range of f^{-1} and the range of f^{-1} is the domain of f.

Ex 8) Find the inverses of each function

a) $f(x) = 4x - 1$

$y = 4x - 1 \Rightarrow x = 4y - 1$

$x + 1 = 4y$

$\boxed{f^{-1}(x) = \frac{x+1}{4}} \Leftarrow \frac{x+1}{4} = y$

b) $f(x) = \frac{2x+1}{x-3}$

$y = \frac{2x+1}{x-3} \Rightarrow x = \frac{2y+1}{y-3}$

$x(y-3) = 2y + 1$ *collect y

$xy - 3x = 2y + 1$

$-xy - 1 \quad -xy - 1$

$-3x - 1 = 2y - xy$

$-(3x+1) = y(2-x)$

$-\frac{3x+1}{2-x} = y$

$f^{-1}(x) = -\frac{3x+1}{2-x}$

$\boxed{f^{-1}(x) = \frac{3x+1}{x-2}}$

c) $f(x) = \sqrt{x+2} - 3$

$y = \sqrt{x+2} - 3$

$x = \sqrt{y+2} - 3$

$x+3 = \sqrt{y+2}$

$(x+3)^2 = y+2$

$(x+3)^2 - 2 = y$

$f^{-1}(x) = (x+3)^2 - 2$

$= x^2 + 6x + 9 - 2$

$\boxed{f^{-1}(x) = x^2 + 6x + 7}$

Verifying inverses

Thm: The functions f and g are inverses of each other if and only if

1. $g(f(x)) = x$ for all x in the domain of f and
2. $f(g(x)) = x$ for all x in the domain of g.

Ex9) Using composition to verify inverse functions.

* expect one of these on the exam.

a) $f(x) = x^3 - 1 \qquad g(x) = \sqrt[3]{x+1}$

$g(f(x)) = g(x^3 - 1)$

$= \sqrt[3]{(x^3-1)+1}$

$= \sqrt[3]{x^3-1+1}$

$= \sqrt[3]{x^3}$

$= x$

$f(g(x)) = f(\sqrt[3]{x+1})$

$= (\sqrt[3]{x+1})^3 - 1$

$= (x+1) - 1$

$= x$

$D(f) = (-\infty, \infty)$

$D(g) = (-\infty, \infty)$

Therefore $f = g^{-1}$ and $g = f^{-1}$

b) $f(x) = x^2$ $\qquad$ $g(x) = \sqrt{x}$

$$g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$$

$$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

$D(f) = (-\infty, \infty)$ $\qquad$ $D(g) = [0, \infty)$

$\Rightarrow$ $f(x) = x^2,\ x \geq 0$ and $g(x) = \sqrt{x}$ are inverses of one another.

f is not one to one! So f is not inventable to begin with

c) $f(x) = \sqrt{x}$ $\qquad$ $g(x) = x^2 + 1$

$$g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1 \neq x$$

So f and g are not inverses of one another