§P.3: Rational Exponents & Radicals

Miniboss: $\left(\dfrac{x^3y^{-3}}{5x^{-3}y^9}\right)^{-1/2} = \dfrac{\sqrt{5}\,y^6}{|x|^3}$

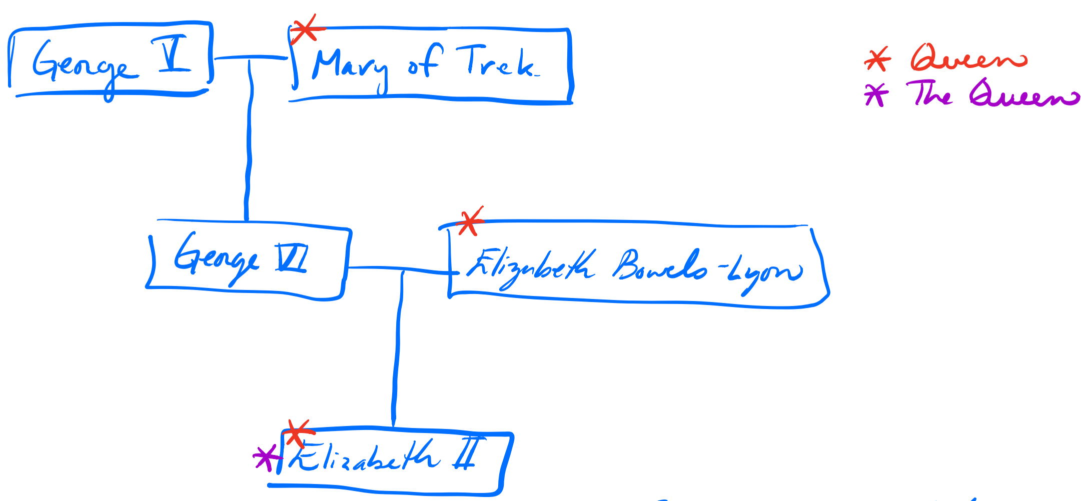
Roots

Def: If n is a positive integer and $a^n = b$, then a is called the <u>nth root</u> of b. If $a^2 = b$, then a is the <u>square root</u> of b and if $a^3 = b$, then a is the <u>cube root</u> of b.

Ex. $3^2 = 9$ and $(-3)^2 = 9$

so both 3 and -3 are both square roots of 9.

However, we consider 3 to be <u>the square root</u>.



-3 is a square root of 9

3 is <u>the</u> square root of 9.

Def: If n is a positive even integer and a is positive, then $a^{\frac{1}{n}}$ denotes the (positive real) nth root of a and is called the principle nth root of a.

#6) $27^{1/3}$

* #7) $64^{1/2}$

#8) $-144^{1/2}$

* Ex) $(-4)^{1/2}$ (P.7)

* Ex) $(-27)^{1/3}$

Rational Exponents

$$3 \cdot 3 \cdot 3 = 3^3 \qquad 3^{1/5} \cdot 3^{1/5} \cdot 3^{1/5} = (3^{1/5}) = 3^{3/5}$$

Def: If m and n are positive integers, then

$$a^{m/n} := \left(a^{\frac{1}{n}}\right)^m$$ provided that $a^{\frac{1}{n}}$ is real.

$A := B$

A is defined to be equal to B.

don't use if $a^{\frac{1}{n}}$ is not real!

Ex) $9^{3/2}$

27

Ex) $8^{2/3}$

4

Ex) 3^{-1}

$\frac{1}{3}$

Rules for Exponent

a, b are real ; r, s are rational
no denominator is zero.

0. $\frac{1}{a^r} = a^{-r}$

1. $a^r a^s = a^{r+s}$

2. $\frac{a^r}{a^s} = a^{r-s}$

3. $(a^r)^s = a^{rs}$

4. $(ab)^r = a^r b^r$

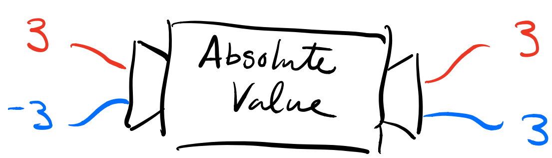
5. $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$

6. $\left(\frac{a}{b}\right)^{-r} = \left(\frac{b}{a}\right)^r$

7. $\frac{a^{-r}}{b^{-s}} = \frac{b^s}{a^r}$

Side Quest: a) Use 0 and 1 to prove 2.
b) Use 0-4 to prove 6.

<u>Absolute Value</u>



$$|x| = \begin{cases} \underline{x} & x \geq 0 \\ \underline{-x} & x < 0 \end{cases}$$

Now that you have that, what is

Ex) $(x^2)^{1/2}$

$|x|$

Ex.) $(x^3 y^6)^{1/2}$

$x^{3/2} |y|^3$

#30) $(a^{1/2} b^{1/3})^2$

$a b^{2/3}$

#38) $\left(\dfrac{x^{1/2} y}{y^{1/2}}\right)^3$

$x^{3/2} y^{3/2}$

work inside out

Radical Notation

__Def:__ If n is a positive integer and a is a number for which $a^{1/n}$ is defined, then the expression $\sqrt[n]{a}$ is called a radical, and

$$\sqrt[n]{a} = a^{1/n}$$

If $n = 2$, we write $\sqrt{a}$ rather than $\sqrt[2]{a}$.

40) $\sqrt{400}$

20

44) $\sqrt[6]{64}$

2

<u>Rules of conversion</u>

If a is a real number and m and n are integers for which $\sqrt[n]{a}$ is real, then

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

54) $-2^{3/4} = -\sqrt[4]{2^3} = -\sqrt[4]{8}$

56) $a(b^4+1)^{-1/2}$

$$\frac{a}{\sqrt{b^4+1}}$$

58) $-4\sqrt{x^3} = -4x^{3/2}$

* # 60) $\sqrt[3]{\underline{x^3+y^3}} = (x^3+y^3)^{1/3} \neq x+y$

block / stuck together

Product and Quotient Rule for Radicals

For any positive integer n and real numbers a, b $(b \neq 0)$

1. $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

provided all the roots are real.

Ex $\sqrt[3]{125 a^6}$

$5a^2$

$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + 5 \cdot 11 = 9^2 + 0$

Ex. $\sqrt{\frac{3}{16}}$

$\sqrt{3}/4$

* Ex. $\sqrt[5]{\frac{-32y^5}{x^{20}}}$

$\frac{-2y}{x^4}$

* Ex. $\sqrt[3]{-8m^9}$

$-2m^3$

Simplified Form

Def: A radical of index n is in simplified form if

1. no perfect nth powers as factors as factors of the radicand. (inside part)
2. no fractions in the radicand
3. no radicals in a denominator.

Removing radicals from the denominator is called <u>rationalizing the denominator</u>.

Ex. $\sqrt{32} = \sqrt{4 \cdot 8}$

$= \sqrt{4} \cdot \sqrt{8}$

$= 2\sqrt{4 \cdot 2}$

$= 2 \cdot (\sqrt{4} \cdot \sqrt{2})$

$= 2 \cdot 2 \cdot \sqrt{2}$

$= 4\sqrt{2}$

Ex. $\sqrt{24x^8y^9} = \sqrt{24} \cdot \sqrt{x^8y^9}$

$= \sqrt{24} \cdot \sqrt{x^8} \cdot \sqrt{y^9}$

$= \sqrt{4 \cdot 6} \sqrt{(x^4)^2} \sqrt{(y^4)^2 y}$

$= \sqrt{2^2} \cdot \sqrt{6} \sqrt{(x^4)^2} \sqrt{(y^4)^2} \sqrt{y}$

$= 2\sqrt{6}\, x^4 y^4 \sqrt{y}$

$= 2x^4y^4\sqrt{6y}$

Ex. $\sqrt[3]{\frac{3}{5a^4}} = \frac{\sqrt[3]{3}}{\sqrt[3]{5}\sqrt[3]{a^4}}$

$= \sqrt[3]{3} \left[\frac{1}{\sqrt[3]{5}}\right]\left[\frac{1}{\sqrt[3]{a^4}}\right]$

Fancy 1!

Notice $\sqrt[3]{125} = 5$ $\sqrt[3]{a^3} = a$

$$\sqrt[3]{125} = \sqrt[3]{5\cdot 25} = \sqrt[3]{5}\cdot\sqrt[3]{25}$$

$$\sqrt[3]{5} = \frac{\sqrt[3]{125}}{\sqrt[3]{25}} = \frac{5}{\sqrt[3]{25}}$$

$$\frac{1}{\sqrt[3]{5}} = \frac{\sqrt[3]{25}}{5}$$

$$a^2 = \sqrt[3]{a^6} = \sqrt[3]{a^4\cdot a^2}$$

$$= \sqrt[3]{a^4}\cdot\sqrt[3]{a^2}$$

$$\frac{a^2}{\sqrt[3]{a^2}} = \sqrt[3]{a^4}$$

$$\frac{1}{\sqrt[3]{a^4}} = \frac{\sqrt[3]{a^2}}{a^2}$$

$$= \sqrt[3]{3}\left[\frac{\sqrt[3]{25}}{5}\right]\left[\frac{\sqrt[3]{a^2}}{a^2}\right]$$

$$= \frac{\sqrt[3]{3\cdot 25\cdot a^2}}{5a^2}$$

$$= \frac{\sqrt[3]{75a^2}}{5a^2}$$

Another Way...

$$= \sqrt[3]{3}\left[\frac{1}{\sqrt[3]{5}}\cdot\frac{\sqrt[3]{25}}{\sqrt[3]{25}}\right]\left[\frac{1}{\sqrt[3]{a^4}}\cdot\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}\right]$$

$$= \sqrt[3]{3}\left[\frac{\sqrt[3]{25}}{\sqrt[3]{125}}\right]\left[\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^6}}\right]$$

$$= \sqrt[3]{3}\cdot\frac{\sqrt[3]{25}}{5}\cdot\frac{\sqrt[3]{a^2}}{a^2}$$

$$4 = 2^2 = \sqrt[4]{(2^2)^4}$$

$$= \sqrt[4]{2^8}$$

$$4 = 2^2 = \sqrt[4]{2^8}$$

$$= \sqrt[4]{2^5}\cdot\underline{\underline{\sqrt[4]{2^3}}}$$

Ex) $$\frac{1}{\sqrt[4]{32}} = \frac{1}{\sqrt[4]{2^5}}\cdot\frac{\sqrt[4]{2^3}}{\sqrt[4]{2^3}}$$

$$= \frac{\sqrt[4]{2^3}}{4}$$

*Ex) $\dfrac{1}{\sqrt[6]{x^5}}$

Operations with Radical Expressions.

Ex) $\sqrt{20} + \sqrt{5}$

$= \sqrt{4 \cdot 5} + \sqrt{5}$

$= \sqrt{4}\sqrt{5} + \sqrt{5}$

$= 2\sqrt{5} + \sqrt{5}$

$= 3\sqrt{5}$

*#82) $\sqrt{18} - \sqrt{50} + \sqrt{12} - \sqrt{75}$

$-2\sqrt{2} \quad -3\sqrt{3}$

#88) $(3\sqrt{5})^2$

45

#90) $\dfrac{\sqrt{21x^7}}{\sqrt{3x^2}}$

$x^2\sqrt{7x}$

Combining Radicals with Different Indices

Thm. If m and n are positive integers for which all the following roots are real, then

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

write as a single radical

find least common multiple of indices

#96) $\sqrt{5} \cdot \sqrt[3]{4}$

$\sqrt[2]{5} \cdot \sqrt[3]{4}$

$= \sqrt[2]{\sqrt[3]{5^3}} \cdot \sqrt[3]{\sqrt[2]{4^2}}$

$= \sqrt[6]{5^3} \cdot \sqrt[6]{4^2}$

$= \sqrt[6]{5^3 \cdot 4^2}$

$= \sqrt[6]{2000}$

* #98) $\sqrt[3]{3}\,\sqrt[4]{4}$

$\sqrt[6]{72}$

* #100) $\sqrt[3]{2a} \cdot \sqrt{2a}$

$\sqrt[6]{32a^5}$

* #101) $\sqrt[3]{\sqrt{7}}$

$\sqrt[6]{7}$

Miniboss fight:

$$\left(\frac{x^3y^{-3}}{5x^{-3}y^9}\right)^{-1/2} = \frac{\sqrt{5}\,y^6}{|x|^3}$$