§P.7: Complex Numbers

Def: The imaginary number i is defined by
$$i^2 = -1.$$
We may also write $i = \sqrt{-1}$.

Def: The set of complex numbers is the set of all numbers of the form $a+bi$ where a and b are real numbers.

- For the complex number $a+bi$,
  a is called the real part
  b is called the imaginary part
- $a+bi = c+di$ if and only if $a=c$ and $b=d$.
- If $b=0$, then $a+bi$ is real.
- If $a=0$, then $a+bi$ is imaginary.
- The form $a+bi$ is standard form; however, if either the real or imaginary part is zero, then that part can be omitted.

ex. $0+5i = 5i$, $3+0i = 3$, $0+0i = 0$

Adding

#14) $(-3+2i)+(5-6i)$

$= -3+5 \quad +2i-6i$

$= 2 \quad -4i$

Ex) $(a+bi)+(c+di)$ $\qquad$ a, b, c, d are real

$= (a+c)+(b+d)i$

Subtracting

#16) $(6-7i)-(3-4i)$

$= 6-7i-3+4i$

$= (6-3)+(-7i+4i)$

$= 3+3i$

Ex) $(a+bi)-(c+di)$ $\qquad$ a, b, c, d, real

$= a+bi - c-di$

$= (a-c)+(bi-di)$

$= (a-c)+(b-d)i$

Multiplying

#23) $(4-5i)(6+2i)$

$= 24+8i-30i-10i^2$

$= 24-22i+10$

$= 34-22i$

Ex.) $(a+bi)(c+di)$

$$= ac + a(di) + (bi)c + (bi)(di)$$
$$= ac + adi + bci + bdi^2$$
$$= (ac - bd) + (ad + bc)i$$

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

$$(a+bi) - (c+di) = (a-c) + (b-d)i$$

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

Simplifying a power of i

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
i^n				1	i	-1	$-i$	1	i	-1	$-i$	1	i	-1

i^n repeats with a period of 4.

Ex. $i^{25} = i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^1$

$$= i^{6 \cdot 4} \cdot i$$
$$= (i^4)^6 \cdot i$$
$$= 1^6 \cdot i$$
$$= i$$

Notice that 1 is the remainder, when 25 is divided by 4.

Ex) i^{2019}

$$= i^{4\cdot 54+3}$$

$$= i^{4\cdot 54}\cdot i^3$$

$$= (i^4)^{54}\cdot i^3$$

$$= i^3$$

$$= -i$$

$$\begin{array}{r} 54 \text{ r}3 \\ 4\overline{)2019} \\ -20 \\ \hline 19 \\ 16 \\ \hline 3 \end{array}$$

$2019 = 4\cdot 54+3$

Ex) $\dfrac{1}{i^{23785}}$

$$= \frac{1}{(i^4)^{5946}\cdot i}$$

$$= \frac{1}{i}\cdot\frac{-i}{-i}$$

$$= \frac{-i}{i(-i)} = -i$$

$$\begin{array}{r} 5946 \text{ r}1 \\ 4\overline{)23785} \\ 20 \\ \hline 3785 \\ 36 \\ \hline 185 \\ 16 \\ \hline 25 \\ 24 \\ \hline 1 \end{array}$$

Def: If a, b are real, then $a+bi$ and $a-bi$ are said to be conjugates of each other.

Ex.) $(2+3i)(2-3i)$

$$= 4 \cancel{(-6i+6i)} - 9i^2$$

$$= 4+9$$

$$= 13$$

Ex) $(a+bi)(a-bi)$

$$= a^2 \cancel{(-abi+abi)} - b^2 i^2$$

$$= a^2+b^2$$

Thm. If a and b are real numbers, then the product of $a+bi$ and its conjugate $a-bi$ is the real number a^2+b^2. In symbols

$$(a+bi)(a-bi) = a^2+b^2$$

Dividing

#56)
$$\frac{1}{5+2i} \cdot \frac{5-2i}{5-2i}$$
$$= \frac{5-2i}{5^2+2^2}$$
$$= \frac{5-2i}{25+4}$$
$$= \frac{5-2i}{29}$$
$$= \frac{5}{29} - \frac{2}{29}i$$

#64)
$$\frac{4+2i}{5-3i} \cdot \frac{5+3i}{5+3i}$$
$$= \frac{(4+2i)(5+3i)}{5^2+3^2}$$
$$= \frac{20 + 12i + 10i + 6i^2}{34}$$
$$= \frac{16 + 22i}{34}$$
$$= \frac{16}{34} + \frac{22}{34}i$$
$$= \frac{8}{17} + \frac{11}{17}i$$

Ex) $\dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di}$

$$= \frac{(a+bi)(c-di)}{c^2+d^2}$$

$$= \frac{ac - adi + bci - bd\,i^2}{c^2+d^2}$$

$$= \frac{(ac+bd) - (ad-bc)i}{c^2+d^2}$$