Insolvability of the Quintic

An unofficial sequel to An Inquiry-Based Approach to Abstract Algebra by Dana C. Ernst

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The most up-to-date version of these notes on can be found on GitHub:

`https://github.com/jwiscons/IBL-InsolvabilityOfQuintic`

This work is designed to extend An Inquiry-Based Approach to Abstract Algebra by Dana C. Ernst. The presentation of the material is heavily influenced by the book Abstract Algebra: A Concrete Introduction by Robert H. Redfield. Many thanks to both Dana and Bob!

Contents

Chapter 1

Solving polynomial equations

Can you count all of the times in your career that you've had to find the zeros of a quadratic polynomial? What about a cubic polynomial? A quartic? Likely your answers are decreasing rapidly, and it's also likely that you have only solved cubic and quartic equations in very special situations. Why? Is it just that cubic and quartic equations are difficult to solve or could it be that some are impossible to "solve." Let's explore.

Problem 1.1. Determine the roots (i.e. zeros) of each of the following. Try to use tools you've accumulated over the years, but you can also use a computer program (e.g. WolframAlpha) if you need. For each problem, make a note about *how* you found the roots. Try for exact answers, but you can approximate if needed.

(1) $p(x) = x^2 - 5x + 6$

(2) $q(x) = (x-3)^2 - 2$

(3) $r(x) = x^2 + x + 1$

(4) $s(x) = x^3 - 3x - 2$

(5) $f(x) = x^4 - 1$

(6) $g(x) = x^4 - 5x^2 + 6$

(7) $a(x) = x^5 - 1$

(8) $b(x) = x^5 + 5x^4 - 5$

Problem 1.2. For each part of Problem 1.1, write down the "smallest" number system needed to express the roots of the given polynomial. Possible answers might be $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers), $\mathbb{Z}$ together with $\sqrt{3}$, $\mathbb{Q}$ together with $i = \sqrt{-1}$, etc.

1.1 Solving polynomial equations with formulas

Though you may have solved the first three parts of Problem 1.1 different ways, there was one tool that would have solved them all: the quadratic formula. It will be valuable to (re)discover why it's true.

First, let's slightly simplify things. Notice that α is a root of ax^2+bx+c if and only if α is a root of $x^2+\frac{b}{a}x+\frac{c}{a}$ (assuming $a \neq 0$). This means that an arbitrary quadratic polynomial can always be converted to a quadratic polynomial whose leading coefficient is 1 and in such a way that they have the same roots. Thus, if we have a formula that finds the roots of so-called *monic* quadratic polynomials, we can actually use it to find the roots of *all* quadratic polynomials.

Definition 1.3. A polynomial whose leading coefficient is 1 is called a **monic** polynomial.

Now, let's (re)derive the quadratic formula for monic polynomials. Remember, we are *(re)deriving* it, so *please don't use the quadratic formula in your proof of the next theorem.*

Theorem 1.4. The roots of $p(x) = x^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Problem 1.5. Write down the "smallest" number system needed to express the roots of *any* quadratic polynomial.

Well, that takes care of quadratic polynomials. What about finding the roots of cubic polynomials? Well, it turns out that there is indeed a cubic formula, though it's decidedly more complicated than the quadratic formula.

A method for deriving a cubic formula, due to Scipione del Ferro and Tartaglia, was published in a book by Cardano in 1545. The starting point is to take a general cubic polynomial and first convert it to a monic polynomial (as we did above) and then convert it to a cubic of the form $x^3 + px + q$, always with the same roots as the original. Then, with work, one arrives at the following formula.

Fact 1.6. The roots of $p(x) = x^3 + px + q$ are

$$\alpha + \beta, \left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)\alpha + \left(-\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)\beta, \left(-\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)\alpha + \left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)\beta$$

where $\alpha = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ and $\beta = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

Problem 1.7. Use Fact 1.6 to write out the roots of $p(x) = x^3 - 2x - 4$. Are any of the roots integers? Check your answer with WolframAlpha. Does this expose any issues with using the formula?

For more details on solving cubic equations, you can use start with the Wikipedia page about cubic functions. And now... quartic functions? Perhaps you have a guess.

Problem 1.8. Use the internet and/or library to determine if there is a quartic formula, i.e. a formula to solve fourth-degree polynomials. If there is a quartic formula, who are some people that discovered methods to derive it and when did they discover their method?

Appendix A

Hints

Hint (Theorem 1.4)**.** You are solving $x^2 + bx + c = 0$. Try "completing the square" first; then solve for x.