

Insolvability of the Quintic

An unofficial sequel to [An Inquiry-Based Approach to Abstract Algebra](#) by Dana C. Ernst

Joshua Wiscons
California State University, Sacramento

Spring 2019

© 2019 Joshua Wiscons. Some Rights Reserved.

The most up-to-date version of these notes on can be found on GitHub:

<https://github.com/jwiscons/IBL-InsolvabilityOfQuintic>

This work is licensed under the Creative Commons Attribution-Share Alike 4.0 United States License. You may copy, distribute, display, and perform this copyrighted work, but only if you give credit to Joshua Wiscons, and all derivative works based upon it must be published under the Creative Commons Attribution-Share Alike 4.0 International License. Please attribute this work to Joshua Wiscons, Mathematics Faculty at California State University, Sacramento, joshua.wiscons@csus.edu. To view a copy of this license, visit

<https://creativecommons.org/licenses/by-sa/4.0/>

or send a letter to Creative Commons, 171 Second Street, Suite 300, San Francisco, California, 94105, USA.



This work is designed to extend [An Inquiry-Based Approach to Abstract Algebra](#) by Dana C. Ernst. The presentation of the material is heavily influenced by the book [Abstract Algebra: A Concrete Introduction](#) by Robert H. Redfield. Many thanks to both Dana and Bob!

Contents

1	Solving polynomial equations	3
1.1	Solving polynomial equations with formulas	3
A	Hints	5

Chapter 1

Solving polynomial equations

Can you count all of the times in your career that you’ve had to find the zeros of a quadratic polynomial? What about a cubic polynomial? A quartic? Likely your answers are decreasing rapidly, and it’s also likely that you have only solved cubic and quartic in very special situations. Why is this? Is it just that cubic and quartic equations are difficult to solve or could it be that some are impossible to “solve.” Let’s explore this briefly.

Problem 1.1. Determine the roots (i.e. zeros) of each of the following. Try to use tools you’ve accumulated over the years, but you can also use a computer program (e.g. [WolframAlpha](#)) if you need. For each problem, make a note about *how* you found the roots. Try for exact answers, but you can approximate if needed.

(1) $p(x) = x^2 - 5x + 6$

(5) $f(x) = x^4 - 1$

(2) $q(x) = (x - 3)^2 - 2$

(6) $g(x) = x^4 - 5x^2 + 6$

(3) $r(x) = x^2 + x + 1$

(7) $a(x) = x^5 - 1$

(4) $s(x) = x^3 - 4x^2 + 1x + 6$

(8) $b(x) = x^5 + 5x^4 - 5$

Problem 1.2. For each part of Problem 1.1, write down the “smallest” number system needed to express the roots of the given polynomial. Possible answers might be \mathbb{Z} (integers), \mathbb{Q} (rational numbers), \mathbb{Z} together with $\sqrt{3}$, \mathbb{Q} together with $i = \sqrt{-1}$, etc.

1.1 Solving polynomial equations with formulas

Though you may have solved the first three parts of Problem 1.1 different ways, there was one tool that would have solved them all: the quadratic formula. It will be valuable to (re)discover why it’s true.

First, let’s slightly simplify things. Notice that α is a root of $ax^2 + bx + c$ if and only if α is a root of $x^2 + \frac{b}{a}x + \frac{c}{a}$ (assuming $a \neq 0$). This means that an arbitrary quadratic polynomial can always be converted to a quadratic polynomial whose leading coefficient is 1 in such a way that they have the same roots. Thus, if we have a formula that finds the

roots of so-called *monic* quadratic polynomials, we can actually use it to find the roots of *all* quadratic polynomials.

Definition 1.3. A polynomial whose leading coefficient is 1 is called a **monic** polynomial.

Now, let's (re)derive the quadratic formula for monic polynomials. Remember, we are (re)deriving it, so *please don't use the quadratic formula in your proof of the next theorem*.

Theorem 1.4. If $p(x) = x^2 + bx + c$, then the roots of $p(x) = x^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Problem 1.5. Write down the “smallest” number system needed to express the roots of *any* quadratic polynomial.

Well, that takes care of quadratic polynomials. What about finding the roots of cubic polynomials?

Appendix A

Hints

Hint (Theorem [1.4](#)). Try “completing the square” first. Then solve for x .