Insolvability of the Quintic

An Inquiry-Based Approach to a Second Course in Abstract Algebra

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Spring 2022 Edition (In Progress)

The most up-to-date version of these notes on can be found on GitHub:

https://github.com/jwiscons/IBL-InsolvabilityOfQuintic

This work is designed to extend An Inquiry-Based Approach to Abstract Algebra by Dana C. Ernst. The presentation of the material is heavily influenced by the book Abstract Algebra: A Concrete Introduction by Robert H. Redfield. Many thanks to Dana and Bob!

I am indebted to every student from my Modern Algebra 2 class in Fall 2019 at California State University, Sacramento for taking part, willingly or otherwise, in this experiment. They are the reason this book exists—each and everyone of them contributed to making this book what it is (though all errors and shortcomings are my own). And with that, a heart-felt thanks to each of: Alejandra Arana, Jack Bailey, Nicholas Banford, Deanna Bosley, Nguyen Cao, Paul Chen, Caitlin Esgana, Arsander Esteban, Andrew Flores, Hera Flores, Arianna Gomez, Gao Hang, Mei Kei Ho, Sarah Kirwan, Greg Kristian, Diego Melgoza, Ian Morrill, Jerod Nooner, Eric Ochoa, Meagan Pham, Kannika Pho, Yong Rong, Hector Ruiz, Ian Smart, Mark Daniel Solomon, Jared Sutton, Brandon Tich, Taylor Underhile, and Erica Yang.

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Chapter 1

Introduction

This course is a story. A repackaging of a famous story, spanning more-or-less 4000 years, about solving polynomial equations. I hope you like it. I also hope enjoy the beautiful sights along the way, many of which were a long time in the making and most of which are still being heavily researched to this day.

1.1 Prerequisites

A first course in abstract algebra, focusing on the basics of group theory, together with exposure to foundational topics like primes and divisibility, functions and relations, differential calculus, and linear algebra form the core prerequisites. Comfort with basic proof techniques, including induction, is also more-or-less required. The following two free and open-source books by Dana C. Ernst will serve you well if you need to review.

- An Inquiry-Based Approach to Abstract Algebra
- An Introduction to Proof via Inquiry-Based Learning

1.2 An Inquiry-Based Approach

This section of the introduction as well as those that follow are (only slightly) modified from the introduction of An Inquiry-Based Approach to Abstract Algebra. The use of "I" below, does indeed refer to me, but mainly just because I also believe the words that Dana originally wrote.

In many courses, math or otherwise, you sit and listen to a lecture. These lectures may be polished and well-delivered. I love lecturing, and I do believe there is value in it. But, I also believe that in reality most students do not learn by simply listening. You must be active in the learning process. Likely, each of you have said to yourselves, "Hmmm, I understood this concept when the professor was going over it, but now that I am alone, I am lost." To promote a more active participation in your learning, we will incorporate ideas from an educational philosophy called inquiry-based learning (IBL)*.

*For more about IBL, check out Dana's blog post, What the Heck is IBL?

Loosely speaking, IBL is a student-centered method of teaching mathematics that engages students in sense-making activities. Students are given tasks requiring them to solve problems, conjecture, experiment, explore, create, communicate. Rather than showing just facts or a clear, smooth path to a solution, the instructor guides and mentors students via well-crafted problems through an adventure in mathematical discovery. Effective IBL courses encourage deep engagement in rich mathematical activities and provide opportunities to collaborate with peers (either through class presentations or group-oriented work). I believe that there are two essential elements to IBL: students should as much as possible be responsible for

(1) guiding the acquisition of knowledge;

(2) validating the ideas presented (so students should not be looking to the instructor as the sole authority).

Much of this course will be devoted to students discussing and proving theorems at the board because I believe that the best way to learn mathematics is by doing mathematics. Someone cannot master a musical instrument or a martial art by simply watching, and in a similar fashion, you cannot master mathematics by simply watching; you must do mathematics! In this class, students will regularly

- read and interact with course notes on their own and with classmates;
- write up their proofs to assigned problems;
- discuss their proofs on the board to the rest of the class;
- participate in discussions centered around a student's presented proof;
- work to respond in flexible, thoughtful, and creative ways to problems that may seem unfamiliar on first glance.

It is important to understand that proving theorems is difficult and takes time. You should not expect to complete a proof in 10 minutes. Sometimes, you might have to stare at the statement for an hour before even understanding how to get started. However, there do exist some hints, collected in Appendix A. If you use the hints, please keep in mind that (1) your own learning will significantly benefit from cognitive struggles (independently and with your peers), so don't turn to the hints too early; and (2) the hints are really just some possible ways to get started. A hint might very well not be the way that makes the most sense to you, so I encourage you to follow you own path.

Lastly, it is highly important to respect learning and to respect other people's ideas. Whether you disagree or agree, please praise and encourage your fellow classmates. Use ideas from others as a starting point rather than something to be judgmental about. Judgement is not the same as being judgmental. Helpfulness, encouragement, and compassion are highly valued.

1.3 Structure of the Notes

As you read the notes, you will be required to digest the material in a meaningful way. It is your responsibility to read and understand new definitions and their related concepts. However, you will be supported in this sometimes difficult endeavor. In addition, you will be asked to complete problems aimed at solidifying your understanding of the material. Most importantly, you will be asked to make conjectures, produce counterexamples, and prove theorems.

The items labeled as **Definition**, **Example**, or **Fact** are meant to be read and digested. However, the items labeled as **Problem**, **Lemma**, **Theorem**, and **Corollary** require action on your part. Items labeled as **Problem** are sort of a mixed bag. Some Problems are computational in nature and aimed at improving your understanding of a particular concept while others ask you to provide a counterexample for a statement if it is false or to provide a proof if the statement is true. Items with the **Lemma**, **Theorem**, and **Corollary** designation are mathematical facts, and the intention is for you to produce a valid proof of the given statement. **Lemma's** are usually stepping stones to the next theorem, though they are often interesting in their own right. **Corollaries** are typically statements that follow quickly from a previous theorem. In general, you should expect corollaries to have very short proofs. However, that doesn't mean that you can't produce a more lengthy yet valid proof of a corollary.

It is important to point out that there are very few examples in the notes. This is intentional. One of the goals of the items labeled as **Problem** is for *you* to produce the examples.

Chapter 2

Solving polynomial equations

The story begins. Can you count all of the times in your career that you've had to find the zeros of a quadratic polynomial? What about a cubic polynomial? A quartic? Likely your answers are decreasing rapidly, and it's also likely that you have only solved cubic and quartic equations in very special situations. Why? Is it just that cubic and quartic equations are difficult to solve or could it be that some are impossible to "solve."

People have been investigating how to solve polynomial equations for about 4000 years. Let's get started.

Problem 2.1. Determine the roots (i.e. zeros) of each polynomial. Try to use tools you've accumulated over the years, but you will need a computer (e.g. WolframAlpha) for some of them. For each problem, make a note about *how* you found the roots; if you used a computation tool, did you get exact answers or only approximate?

(1) $p(x) = x^2 - 5x + 6$

(2) $q(x) = (x-3)^2 - 2$

(3) $a(x) = x^2 + x + 1$

(4) $b(x) = x^3 - 3x - 2$

(5) $f(x) = x^3 + 3x + 2$

(6) $g(x) = x^4 - 2$

(7) $r(x) = x^5 - 1$

(8) $s(x) = x^5 + 5x^4 - 5$

Problem 2.2. For each part of Problem 2.1, write down the "smallest" number system needed to express the roots of the given polynomial. Possible answers might be $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers), $\mathbb{Z}$ together with $\sqrt{3}$, $\mathbb{Q}$ together with $\sqrt{-1}$ and $\sqrt[3]{5}$, etc.

2.1 Solving polynomial equations with formulas

Though you may have solved the first three parts of Problem 2.1 different ways, there was one tool that would have solved them all: the quadratic formula. It will be valuable to (re)discover why it's true.

First, let's slightly simplify things. Notice that α is a root of $ax^2 + bx + c$ if and only if α is a root of $x^2 + \frac{b}{a}x + \frac{c}{a}$ (assuming $a \neq 0$). This means that an arbitrary quadratic polynomial can always be converted to a quadratic polynomial whose leading coefficient is 1 and in

such a way that they have the same roots. Thus, if we have a formula that finds the roots of so-called *monic* quadratic polynomials, we can actually use it to find the roots of *all* quadratic polynomials.

Definition 2.3. A polynomial whose leading coefficient is 1 is called a **monic** polynomial.

Now, let's (re)derive the quadratic formula for monic polynomials. Remember, we are *(re)deriving* it, so *please don't use the quadratic formula in your proof of the next theorem.*

Theorem 2.4. The roots of $p(x) = x^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Problem 2.5. Describe a number system, as small as possible, that is capable of expressing the roots of *any* quadratic polynomial whose coefficients are all rational numbers.

Well, that takes care of quadratic polynomials. What about finding the roots of cubic polynomials? Well, it turns out that there is indeed a cubic formula, though it's decidedly more complicated than the quadratic formula.

A method for deriving a cubic formula, due to Scipione del Ferro and Tartaglia, was published in a book by Cardano in 1545. The starting point is to take a general cubic polynomial and first convert it to a monic polynomial (as we did above) and then convert it to a cubic of the form $x^3 + px + q$, always with the same roots as the original. Then, with work, one arrives at the following formula.

Fact 2.6. The roots of $p(x) = x^3 + px + q$ are

$$\alpha + \beta, \left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)\alpha + \left(-\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)\beta, \left(-\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)\alpha + \left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)\beta$$

where $\alpha = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ and $\beta = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

Problem 2.7. Describe a number system, as small as possible, that is capable of expressing the roots of *any* cubic polynomial of the form $x^3 + px + q$ where $p, q \in \mathbb{Q}$.

Problem 2.8. Use Fact 2.6 to write out the roots of $p(x) = x^3 - 2x - 4$. Are any of the roots integers? Check your answer with WolframAlpha. Does this expose any issues with using the formula?

For more details on solving cubic equations, you can use start with the Wikipedia page about cubic functions. But now, on to quartic functions... perhaps you have a guess...

Problem 2.9. Use the internet and/or library to determine if there is a quartic formula, i.e. a formula that gives the roots of a fourth-degree polynomial in terms of its coefficients. If there is a quartic formula, who are some people that discovered methods to derive it and when did they discover their method?

2.2 The main question(s)

We now turn our attention to finding the roots of polynomials of degree five and higher. Well, what do you think: is there a quintic formula? Actually, there are two questions here: (1) what do we mean by "formula" (e.g. what symbols/functions can we use), and (2) whatever we mean by formula, is there one that works for *all* quintic polynomials?

You investigated two particular quintics in Problem 2.1—what did you find? Did you find exact expressions for the roots? If so, *how* were the roots expressed; that is, what symbols/functions were needed to write out the roots?

Let's first try to tackle the "what do we mean by formula" question. Guided by the quadratic and cubic formulas, let's agree that we are looking for a formula that expresses the roots of an arbitrary quintic polynomial in terms of the coefficients of the polynomial using just the operations of addition, subtraction, multiplication, division, and the extraction of roots (square roots, cube roots,...). This leads to the following intuitive definition that we will work to sharpen later.

Intuitive Definition 2.10. A polynomial is said to be **solvable by radicals** if every root of the polynomial can be expressed in terms of the coefficients of the polynomial using (perhaps repeatedly) the operations of addition, subtraction, multiplication, division, and $\sqrt[n]{\ }$ for any positive integer n.

Problem 2.11. Find the roots of $p(x) = x^4 - 2x^2 - 1$. Explain why $p(x)$ is solvable by radicals according to our intuitive definition.

Problem 2.12. Explain why every quadratic polynomial is solvable by radicals.

Returning to quintic polynomials, our main question is as follows.

Main Question. Does there exist a "quintic formula" that expresses the roots of an arbitrary quintic polynomial, in terms of the coefficients of the polynomial, using just the operations of addition, subtraction, multiplication, division, and the extraction of roots?

Well, if the answer is yes, then it must be that every quintic polynomial is solvable by radicals. *Spoiler Alert!* Our goal (for the rest of the book) is to prove the following theorem, in a rather elegant way.

Main Theorem. Not every quintic polynomial is solvable by radicals.

Main Corollary. There is **no** "quintic formula" that expresses the roots of an arbitrary quintic polynomial in terms of the coefficients of the polynomial using just the operations of addition, subtraction, multiplication, division, and the extraction of roots.

Let's get to work...

Chapter 3

Fields

In Chapter 2, we explored problems about finding and expressing roots of polynomials, finally arriving at the goal of the course: proving that there are quintic polynomials that are *not* solvable by radicals. This chapter serves two main purposes. First, as we look at roots of polynomials and how they can be expressed, it will be convenient to have a common world (i.e. number system) in which they live. For us, this will be the complex numbers, denoted $\mathbb{C}$, which will be reviewed below. Our work with complex numbers will also supply the necessary language to properly talk about n^{th}-roots. Second, we are still in need of a proper definition of what it means for a polynomial to be "solvable by radicals", and this chapter lays the essential groundwork (to be completed in the next chapter). On the way to defining "solvable by radicals", we will be led to study the abstract the structure of $\mathbb{C}$ (and of $\mathbb{Q}$ and $\mathbb{R}$), arriving at the definition of a *field*.

3.1 Complex Numbers

As mentioned above, it will be convenient to work in a world that contains all of the roots of all of the polynomials that we will be studying. Considering the roots of polynomials such as x^2+1, x^2-2, x^2-3, etc., we see that we need to include numbers like $\sqrt{-1}$, $\sqrt{2}$, $\sqrt{3}$, etc., so although there are smaller worlds one could choose, we will opt for the world containing both $\sqrt{-1}$ and $\mathbb{R}$, namely $\mathbb{C}$.

But before we proceed, note that $\sqrt{-1}$ is not really well defined. There are *two* solutions to x^2+1, so when we write $\sqrt{-1}$, we are all agreeing that we mean the same one.

Definition 3.1. Let i (or alternatively $\sqrt{-1}$) denote one particular solution to $x^2+1=0$.

Using i and $\mathbb{R}$, we now build the complex numbers.

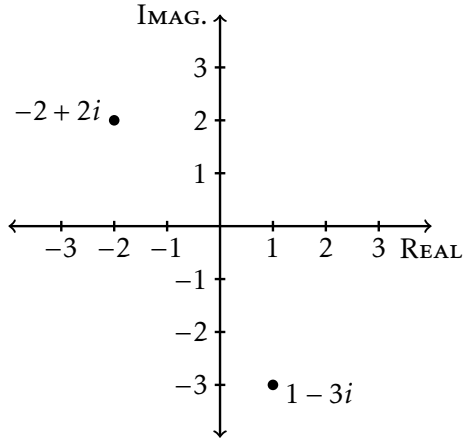
3.1.1 Definition and first principles

Definition 3.2. The **complex numbers** is the set $\mathbb{C} := \{a+bi \mid a,b \in \mathbb{R}\}$. If $z=a+bi$, then a is called the **real part** of z and b is called the **imaginary part** of z.

Note that every complex number $z=a+bi$ is uniquely determined by two numbers: the real and imaginary parts a and b. As such, we often graph complex numbers in

the coordinate plane with the x-axis denoting the real part and the y-axis denoting the imaginary part. This will be called the **complex plane**.

Example 3.3. We graph $-2+2i$ and $1-3i$ below.



We also define some operations on complex numbers.

Definition 3.4. We define the following operations on elements of $\mathbb{C}$.

- **Addition:** $(a+bi)+(c+di) := (a+c)+(b+d)i$
- **Multiplication:** $(a+bi)\cdot(c+di) := (ac-bd)+(ad+bc)i$
- **Complex Conjugation:** $\overline{a+bi} := a-bi$

Notice that in the definition of complex multiplication we are just using the normal distributive law (or FOIL if you like) together with the fact that $i^2 = -1$. Many of the familiar algebraic properties of $\mathbb{R}$ also hold for $\mathbb{C}$, which we will take as a fact.

Fact 3.5. The following are true for $\mathbb{C}$.

- **Addition Laws:** Addition is associative and commutative. There is a unique additive identity, namely $0 = 0+0i$, and every number has a unique additive inverse, denoted $-(a+bi)$.
- **Multiplication Laws:** Multiplication is associative and commutative. There is a unique multiplicative identity, namely $1 = 1+0i$, and every nonzero number has a unique multiplicative inverse, denoted $(a+bi)^{-1}$ or $\frac{1}{a+bi}$.
- **Distributivity Laws:** For all $x, y, z \in \mathbb{C}$, $x(y+z) = xy+xz$ and $(y+z)x = yx+zx$.
- **Conjugation Laws:** For all $x, y \in \mathbb{C}$, $\overline{x+y} = \overline{x}+\overline{y}$ and $\overline{x\cdot y} = \overline{x}\cdot\overline{y}$.

Problem 3.6. Thinking of a complex number $z = a+bi$ as a point in the complex plane, describe *geometrically* what happens when $(c+di)$ is added to z. Also, describe *geometrically* how to find $\overline{z}$ from z.

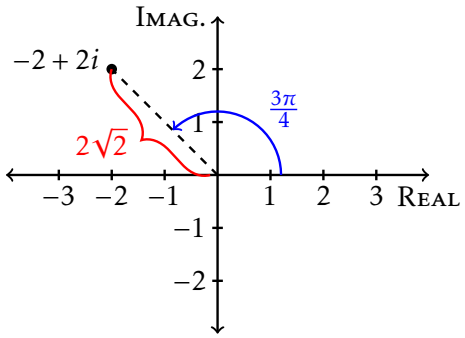
When we plot points, there are different coordinate systems we could use. It turns out that rectangular coordinates are good for adding complex numbers, but polar coordinates are better for multiplication. This lead to the following definition.

Definition 3.7. Let $z = a + bi$.

(1) The **modulus** of z, denoted $|z|$, is the *radius* of the point (a, b) when written in polar coordinates. Thus, $|z| = \sqrt{a^2 + b^2}$.

(2) The **argument** of z, denoted $\text{Arg}(z)$, is the *angle* in the interval $[0, 2\pi)$ of the point (a, b) when written in polar coordinates. Thus, $\cos(\text{Arg}(z)) = \frac{a}{|z|}$ and $\sin(\text{Arg}(z)) = \frac{b}{|z|}$ if $z \neq 0$. The argument of 0 is undefined.

Example 3.8. We have that $|-2 + 2i| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$ and $\text{Arg}(-2 + 2i) = \frac{3\pi}{4}$. (But be careful, $\arctan\left(\frac{2}{-2}\right) = \frac{\pi}{4}$; you must pay attention to which quadrant the number is in.)



Problem 3.9. For each of the following complex numbers,

- write it in the form $a + bi$ (if it is not already),
- plot it in the complex plane,
- find the modulus and argument (if not exact, then a decimal approximation is okay).

(1) $u = -1 - i$

(2) $v = \frac{1}{1+i}$

(3) $w = \frac{(2-i)(1+2i)}{2+3i}$

(4) $z \in \mathbb{C}$ with $|z| = 3$ and $\text{Arg}(z) = \frac{4\pi}{3}$

Theorem 3.10. Let $z \in \mathbb{C}$. If $z \neq 0$, then $z^{-1} = \dfrac{\overline{z}}{|z|^2}$.

The next theorem shows how to find an expression for a complex number given its modulus and argument.

Theorem 3.11. Let $z \in \mathbb{C}$. Then $|z| = r$ and $\text{Arg}(z) = \theta$ if and only if $z = r\cos\theta + ir\sin\theta$ with $0 \leq \theta < 2\pi$.

We now derive some properties of multiplication. The first is quite useful and illustrates how multiplication is rather easy to deal with when numbers are in "polar form".

Theorem 3.12. If $z_1 = r_1\cos\theta_1 + ir_1\sin\theta_1$ and $z_2 = r_2\cos\theta_2 + ir_2\sin\theta_2$, then

$$z_1z_2 = r_1r_2\cos(\theta_1+\theta_2) + ir_1r_2\sin(\theta_1+\theta_2).$$

Corollary 3.13. If $z_1, z_2 \in \mathbb{C}$, then $|z_1z_2|$ is equal to $|z_1||z_2|$ and $\mathrm{Arg}(z_1z_2)$ is equivalent to $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ modulo 2π.

Corollary 3.14 (De Moivre's formula)**.** For each positive $n \in \mathbb{Z}$,

$$(r\cos(\theta) + ir\sin(\theta))^n = r^n\cos(n\theta) + ir^n\sin(n\theta).$$

3.1.2 Roots of unity

We now arrive at an extremely important definition.

Definition 3.15. For each positive $n \in \mathbb{Z}$, define

$$\zeta_n := \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right).$$

Thus, ζ_n (read as "zeta n") is the unique number with magnitude 1 and argument $\frac{2\pi}{n}$.

Problem 3.16. Plot each of the following in the same complex plane: ζ_2, ζ_3, ζ_4, ζ_5.

Problem 3.17. Plot each of the following in the same complex plane: ζ_6, $(\zeta_6)^2$, $(\zeta_6)^3$, $(\zeta_6)^4$, $(\zeta_6)^5$, $(\zeta_6)^6$.

Problem 3.18. Write $\overline{\zeta_8}$ as a power of ζ_8. Conjecture a value for ℓ (in terms of k) such that $\overline{(\zeta_n)^k} = (\zeta_n)^\ell$; then prove that it works. Try starting with $k = 1$.

We now turn our attention back to solving polynomial equations, focusing on those of the form $x^n - a$.

Definition 3.19. Let $a \in \mathbb{C}$. A number $z \in \mathbb{C}$ is called an n^{th} **root of** a if $z^n = a$. In other words, the n^{th} roots of a are the roots of the polynomial $x^n - a$. The n^{th} roots of 1 are also called n^{th} **roots of unity**.

Problem 3.20. Let $w = -1 + i\sqrt{3}$. Write w in polar form; then find a 4^{th} root of w.

Theorem 3.21. For each non-negative $k \in \mathbb{Z}$, $(\zeta_n)^k$ is an n^{th} root of 1.

Lemma 3.22. If z is an n^{th} root of 1, then $z = (\zeta_n)^k$ for some non-negative $k \in \mathbb{Z}$.

Lemma 3.23. For each non-negative $k \in \mathbb{Z}$, $(\zeta_n)^k = (\zeta_n)^r$ for some $0 \le r \le n-1$.

Theorem 3.24. The set

$$\{1, \zeta_n, (\zeta_n)^2, \ldots, (\zeta_n)^{n-1}\}$$

is the set of *all* n^{th} roots of unity. Thus, there are n distinct roots of $x^n - 1$.

Lemma 3.25. Let $a \in \mathbb{C}$ be nonzero, and let b be any one particular n^{th} root of a. Then z is an n^{th} root of a if and only if $\frac{z}{b}$ is an n^{th} root of 1.

Theorem 3.26. Let $a \in \mathbb{C}$ be nonzero, and let b be any one particular n^{th} root of a. The set

$$\{b, b\zeta_n, b(\zeta_n)^2, \ldots, b(\zeta_n)^{n-1}\}$$

is the set of *all* n^{th} roots of a. Thus, there are n distinct roots of $x^n - a$.

Problem 3.27. Find *all* roots of $x^3 - 8$, and also find *all* roots of $x^5 + 7$.

3.1.3 Roots of polynomials over $\mathbb{R}$ and $\mathbb{C}$

We conclude this section with a couple of general results about roots of polynomials.

Theorem 3.28. Suppose that $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ with all $a_i \in \mathbb{R}$. If z is a root of $p(x)$, then $\overline{z}$ is also a root of $p(x)$.

In words, the previous theorem says that if a polynomial has coefficients in $\mathbb{R}$, then the set of roots is "closed under complex conjugation." We end with an extremely important theorem, which will be quite useful for us. However, since its proof is not our main goal (and since it requires sophisticated techniques), we will take it as fact.

Fact 3.29 (Fundamental Theorem of Algebra)**.** If $p(x)$ is a non-constant polynomial with all coefficients in $\mathbb{C}$, then $p(x)$ has a root in $\mathbb{C}$.

In fact, we will see that this implies that *all* roots of such a $p(x)$ lie in $\mathbb{C}$, so in our of study polynomials (often with all coefficients even in $\mathbb{Q}$), $\mathbb{C}$ serves as a uniform world in which we can study the roots.

3.2 An aside: the quaternions

Our construction of the complex numbers creates a structure that contains the real numbers and possesses some nice properties not enjoyed by the real numbers, e.g. every non-constant polynomial with complex coefficients has a complex root. This raises the question: could we further extend the complex numbers to an even larger structure?

Concisely, we built the complex numbers as the set $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ together with the operations of addition and multiplication, which were defined in a natural way from the key identity that $i^2 = -1$. Here, we briefly explore what happens if we build a larger structure in a similar way: $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ where, again, $j^2 = -1$.

Following this path, we formally arrive at $\mathbb{H} = \mathbb{C} + \mathbb{C}j = (\mathbb{R} + \mathbb{R}i) + (\mathbb{R} + \mathbb{R}i)j$, and any definition we give for multiplication of two elements of $\mathbb{H}$ must first define how to multiply i and j (or rather, what properties ij should have). If we set $k = ij$, it turns out that a good route to follow is to decide that k also has the property that it squares to 1, i.e. $k^2 = -1$. There is another important choice one is "forced" to make, namely that $ji = -k$.

Definition 3.30. The **quaternions** are the elements of $\mathbb{H} := \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$, where $i^2 = j^2 = k^2 = -1$. We also define the following operations on elements of $\mathbb{H}$.

- **Addition:** $(a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k) := (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$
- **Multiplication:** use the usual distributive laws together with the identities:

$$\begin{array}{lll} ij = k, & jk = i, & ki = j, \\ ji = -k, & kj = -i, & ik = -j. \end{array}$$

- **Conjugation:** $\overline{a + bi + cj + dk} := a - bi - cj - dk$

Indeed, $\mathbb{H}$ extends the complex numbers, and we have the following containments: $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. It turns out that $\mathbb{H}$ satisfies nearly all of the common algebraic properties of $\mathbb{R}$ and $\mathbb{C}$, with one notable exception, which is highlighted in bold below.

Fact 3.31. The following are true for $\mathbb{H}$.

- **Addition Laws:** Addition is associative and commutative. There is a unique additive identity, namely $0 = 0 + 0i + 0j + 0k$, and every number has a unique additive inverse.
- **Multiplication Laws:** Multiplication is associative but **noncommutative**. There is a unique multiplicative identity, namely $1 = 1+0i+0j+0k$, and every nonzero number has a unique multiplicative inverse.
- **Distributivity Laws:** For all $x, y, z \in \mathbb{H}$, $x(y+z) = xy + xz$ and $(y+z)x = yx + zx$.
- **Conjugation Laws:** For all $x, y \in \mathbb{H}$, $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{x \cdot y} = \overline{x} \cdot \overline{y}$.

We can also define the modulus of a quaternion, analogous to how we defined it for a complex number.

Definition 3.32. The **modulus** of $h = a + bi + cj + dk$, denoted $|h|$, is $|h| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Problem 3.33. Write each of the following quaternions in the form $a + bi + cj + dk$, and find its modulus.

(1) $h = (2i + 4k)(7 - 3j + k)$

(2) k^{-1}

(3) $w = (i + j)^{-1}$

Theorem 3.34. Let $h \in \mathbb{H}$. If $h \neq 0$, then $h^{-1} = \dfrac{\overline{h}}{|h|^2}$.

And as in the complex numbers, the modulus function is multiplicative—we will take this as fact.

Fact 3.35. If $h_1, h_2 \in \mathbb{H}$, then $|h_1 h_2| = |h_1||h_2|$.

We conclude this section by looking at multiplication of quaternions a little closer. As we do, we return to the mathematical notion of a *group*. Please feel free to look over old notes or other books to review the basics. As mentioned in the introduction, our main reference for groups will be An Inquiry-Based Approach to Abstract Algebra.

Problem 3.36. Let G be the subset of $\mathbb{H}$ defined as $G := \{\pm 1, \pm i, \pm j, \pm k\}$. Show that G, together with the operation of quaternion multiplication, is a nonabelian group. If you have encountered this group before, what name (or symbol) did you know it by?

Problem 3.37. Let U be the subset of $\mathbb{H}$ consisting of all quaternions with modulus equal to 1, i.e. $U := \{h \in \mathbb{H} \mid |h| = 1\}$. Show that U, together with the operation of quaternion multiplication, is an infinite, nonabelian group.

It turns out that the group U from the previous problem is isomorphic to the group SU(2) (one of the the so-called special unitary groups), which is quite important in theoretical physics. If you want to learn more, you can start on Wikipedia.

3.3 Abstract fields

Notice that $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ satisfy many common algebraic properties with respect to addition and multiplication. Of course, $\mathbb{H}$ does too, though it lacks commutativity of multiplication. When objects have common properties, it can be extremely valuable to abstract those properties and study them once and for all (as opposed to trying to prove things about each individual structure). This is where we are headed, but first we highlight some related structures (again with algebraic properties similar to $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$) that help to connect this work to our main goal of expressing roots of polynomials.

Problem 3.38. Let $p(x) = x^2 + 3x + 1$. Find the roots of $p(x)$, and show that each root can be written in the form $a + b\sqrt{5}$ with $a, b \in \mathbb{Q}$.

Problem 3.39. Let $S = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$.

(1) Show that S is closed under addition; that is, show that for all $x, y \in S$, $x + y \in S$.

(2) Show that S is closed under multiplication; that is, show that for all $x, y \in S$, $xy \in S$.

(3) Use that $S \subset \mathbb{R}$ to explain why both addition and multiplication of elements of S are associative and commutative and why multiplication distributes over addition.

Problem 3.40. Let $S = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$. Prove or disprove: if $x \in S$ and $x \neq 0$, then x has a multiplicative inverse *in* S (i.e. there is a $y \in S$ such that $xy = 1$).

3.3.1 Definition

We now abstract the common properties of $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ (and also S from Problem 3.39), arriving at the definition of a field.

Definition 3.41. A **field** is a structure $(F, +, \cdot)$ consisting of a set F, containing at least two elements, together with two binary operations $+$ and $\cdot$ (which we call *addition* and *multiplication*) such that for some elements $0, 1 \in F$ the following axioms hold.

- **Addition Axioms:** Addition is associative and commutative; the element 0 is an additive identity; every $x \in F$ has an additive inverse with respect to 0, denoted $-x$.
- **Multiplication Axioms:** Multiplication is associative and commutative; the element 1 is a multiplicative identity; every $x \in F \setminus \{0\}$ has a multiplicative inverse with respect to 1, denoted x^{-1}.
- **Distributivity Axioms:** For all $x, y, z \in F$, $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$.

Recall that "0 is an additive identity" means that "for all $x \in F$, $0 + x = x + 0 = x$," and "$x \in F$ has an additive inverse with respect to 0" means that "there exists some $y \in F$ such that $x + y = y + x = 0$." The meanings of multiplicative identities and inverses are similar to those for addition. Also, recall that $F \setminus \{0\}$ denotes the set obtained by removing the element 0 from F. We introduce some notation for this.

Definition 3.42. If F is a field, then $F \setminus \{0\}$ is denoted by F^*, i.e. F^* is the set of nonzero elements of F.

Using the language of groups, fields can be concisely defined as structures of the form $(F, +, \cdot)$ such that $(F, +)$ is an abelian group with identity 0, $(F^*, \cdot)$ is an abelian group with identity 1, and multiplication distributes over addition.

Now, as with any new definition, we look for examples and basic properties.

3.3.2 Examples and non-examples

It is not hard to verify that $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and S from Problem 3.39 are all fields (with their usual definitions of addition and multiplication). Let's search for more examples and non-examples.

Problem 3.43. Explain why $\mathbb{Z}$ is not a field. Do the same for $\mathbb{H}$.

Problem 3.44. Determine if each of the following is a field. If it is a field, identify an additive and multiplicative identity; if it is not a field, explain why not.

(1) $(F, +, \cdot)$ where $F = \{a, b, c\}$ and $+$ and $\cdot$ are defined as follows:

$+$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$\cdot$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

(2) $(F, +, \cdot)$ where $F = \{0, 1, 2, 3\}$ and $+$ and $\cdot$ are defined as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(3) $(F, +, \cdot)$ where $F = \{0, 1, 2, 3\}$ and $+$ and $\cdot$ are defined as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Problem 3.45. Look back at Problem 3.44. For those that are fields, determine which familiar group each of $(F, +)$ and $(F^*, \cdot)$ is isomorphic to.

To find more examples of fields, Problem 3.44 hints at the fact we may want to look back to modular arithmetic. Following An Inquiry-Based Approach to Abstract Algebra, we define the structures $(\mathbb{Z}_n, +_n, \cdot_n)$ as follows.

Definition 3.46. Let n be a positive integer. The structure $(\mathbb{Z}_n, +_n, \cdot_n)$ consists of the set $\mathbb{Z}_n := \{0, 1, 2, \ldots, n-1\}$ together with the operations $+_n$ and $\cdot_n$ defined as follows.

- **Addition:** $x +_n y$ is the least non-negative number congruent to $x + y$ modulo n.
- **Multiplication:** $x \cdot_n y$ is the least non-negative number congruent to $x \cdot y$ modulo n.

We often refer to the entire structure $(\mathbb{Z}_n, +_n, \cdot_n)$ as simply $\mathbb{Z}_n$. Also, when the context is clear, we may write $+$ and $\cdot$ in place of $+_n$ and $\cdot_n$.

So, in $\mathbb{Z}_5$, we write equations like $3+6=4$, since $3+6=9$ and 9 is congruent to 4 when working modulo 5. If needed, we can highlight that we are working modulo 5 by writing $3+6 \equiv_5 4$. And with respect to multiplication in $\mathbb{Z}_5$, we have equations like $2 \cdot 3 = 1$, which implies that 3 is a multiplicative inverse of 2 (and vice versa) in $\mathbb{Z}_5$.

Fact 3.47. The following are true for $\mathbb{Z}_n$.

- **Addition Laws:** Addition is associative and commutative. There is a unique additive identity, namely 0; each $x \in \mathbb{Z}_n$ has a unique additive inverse, denoted $-x$.
- **Multiplication Laws:** Multiplication is associative and commutative. There is a unique multiplicative identity, namely 1.
- **Distributivity Laws:** For all $x, y, z \in \mathbb{Z}_n$, $x(y+z) = xy + xz$ and $(y+z)x = yx + zx$.

Problem 3.48. Show that $\mathbb{Z}_5$ is a field but $\mathbb{Z}_6$ is not.

Problem 3.49. Make a conjecture as to when $\mathbb{Z}_n$ is a field. That is, try to fill in the blank: "$\mathbb{Z}_n$ is a field provided ____(something about n)____." What evidence do you have?

3.3.3 Basic properties

Let's explore some basic properties of fields. We list some of these as facts since they follow directly from group theory, remembering that $(F, +)$ and $(F^*, \cdot)$ are both groups.

From now on, when we write "let F be a field," we tacitly mean "let $(F, +, \cdot)$ be a field."

Fact 3.50. Let F be a field.

(1) The additive identity and the multiplicative identity are both unique.

(2) Additive inverses and multiplicative inverses are unique.

Theorem 3.51. Let F be a field.

(1) For all $x \in F$, $x \cdot 0 = 0$.

(2) For all $x, y \in F$, $(-x)y = -(xy)$ and $x(-y) = -(xy)$.

(3) For all $x \in F^*$, $-x \in F^*$ and $(-x)^{-1} = -(x^{-1})$.

(4) For all $x, y \in F$, if $xy = 0$, then $x = 0$ or $y = 0$.

(5) The additive and multiplicative identities are different, i.e. $0 \neq 1$.

3.3.4 Another example

We now return to the conjecture you made in Problem 3.49. Combining the next theorem with Theorem 3.51, we see that $\mathbb{Z}_n$ has no hope to be a field unless n is prime.

Theorem 3.52. Let n be a positive integer. If n is not prime, then there exist $a, b \in (\mathbb{Z}_n)^*$ such that $ab = 0$ in $\mathbb{Z}_n$.

And now we completely answer the question. As you explore the next theorem, you can use properties of modular arithmetic that you know from before. For example, you can take for granted that addition and multiplication are both associative and commutative. The crux is in showing that every nonzero element has a multiplicative inverse when n is prime. There are many ways to approach this; one way uses Bézout's lemma from basic number theory. Even if you don't use it now, it's a useful fact to remember.

Fact 3.53 (Bézout's lemma)**.** If $a, b \in \mathbb{Z}$, then there exist $k, l \in \mathbb{Z}$ such that $ka + lb = \gcd(a, b)$.

Theorem 3.54. Let n be a positive integer. Then $\mathbb{Z}_n$ is a field if and only if n is prime.

3.4 Subfields and extension fields

Just as with groups and subgroups, the notion of a subfield is extremely important. Analyzing the subfields of a field F can often yield a better understanding of the whole field F, and vice versa. Also, this will allow us to generate more examples of fields.

Definition 3.55. Let $(E, +, \cdot)$ be a field, and let F be a subset of E. Then F is a **subfield** of E if F is a field in its own right with respect to operations $+$ and $\cdot$ *inherited from* E. When F is a subfield of E, we call E an **extension field** of F.

When checking if a subset of a field is a subfield, it turns out that the subset will automatically satisfy many of the field axioms, leaving only a handful of things to verify.

Theorem 3.56. Let E be a field, and let $F \subseteq E$. Then F is a subfield of E if and only if

(1) F contains at least 2 elements;

(2) for all $x, y \in F$, $x + y \in F$ and $xy \in F$;

(3) for all $x \in F$, $-x \in F$; and

(4) for all $x \in F^*$, $x^{-1} \in F$.

The second item in the above theorem is stating that F is closed under the addition and multiplication inherited from E. The last two items could be read as F being closed under additive and multiplicative inverses.

Theorem 3.57. If F is a subfield of E, then F contains the additive and multiplicative identities of E (namely 0 and 1).

It is not difficult to check that $\mathbb{Q}$ and $\mathbb{R}$ are both subfields of $\mathbb{C}$; S from Problem 3.39 is also a subfield of $\mathbb{C}$ (and of $\mathbb{R}$). Let's look for more that are similar to S.

Problem 3.58. Determine which of the following are subfields of $\mathbb{C}$.

(1) $T_1 = \{a + bi \mid a, b \in \mathbb{Q}\}$

(2) $T_2 = \{a + bi \mid a, b \in \mathbb{Z}\}$

(3) $T_3 = \{a + b\alpha \mid a, b \in \mathbb{Q}\}$ where $\alpha = \sqrt{2} + i$

3.4.1 Generating fields

Paralleling the theory of groups, we now investigate how to generate subfields from subsets of elements. We first need a *definition* of "the subfield generated by a set of elements"; it is essentially the same as for all algebraic structures: take the intersection of all subfields containing the subset.

Theorem 3.59. Suppose S is a subset of a field E. Let K be the intersection of all subfields of E that contain S; that is

$$K := \bigcap \{F \mid F \text{ is a subfield of } E \text{ and } S \subseteq F\}.$$

Then

(1) K is a subfield of E that contains S, and

(2) if F is any subfield of E that contains S, then F also contains K.

In words, the previous theorem says that K is the "smallest" subfield of E containing S, so K is the correct candidate for the subfield generated by S.

Definition 3.60. Suppose S is a subset of a field E. The **subfield of E generated by S**, denoted $\langle S \rangle_{\text{FIELD}}$, is defined to be the intersection of all subfields of E that contain S.

In symbols, $S \subseteq \langle S \rangle_{\text{FIELD}} \subseteq E$, and if F is any subfield of E, then $S \subseteq F \implies \langle S \rangle_{\text{FIELD}} \subseteq F$.

Example 3.61. Let's explore $\langle 1 \rangle_{\text{FIELD}}$ in the field $\mathbb{C}$. By definition, $\langle 1 \rangle_{\text{FIELD}}$ is the intersection of all subfields of $\mathbb{C}$ that contain 1.

Let F be an arbitrary subfield of $\mathbb{C}$ containing 1. By Theorem 3.57, every subfield of $\mathbb{C}$ contains 0 and 1, so F must contain 0 (in addition to 1). Further, F must contain $1 + 1$, $1 + 1 + 1$, etc., because F is closed under addition. So, by induction, F contains the positive integers and 0. Then, since F is closed under additive inverses, F also contains the additive inverse of each positive integer, so in total, we now see that F contains $\mathbb{Z}$. Continuing on, F is closed under multiplicative inverses, so F also contains the multiplicative inverse of every nonzero integer. Thus, $\mathbb{Q} \subseteq F$.

Since F was an *arbitrary* subfield of $\mathbb{C}$ containing 1, everything we said above is true for *every* subfield of $\mathbb{C}$ containing 1; thus it is also true for the intersection of them. Hence $\mathbb{Q} \subseteq \langle 1 \rangle_{\text{FIELD}}$. Now we have $\{1\} \subset \mathbb{Q} \subseteq \langle 1 \rangle_{\text{FIELD}}$, so as $\mathbb{Q}$ is a subfield and $\langle 1 \rangle_{\text{FIELD}}$ is the *smallest* subfield containing 1, it must be that $\mathbb{Q} = \langle 1 \rangle_{\text{FIELD}}$.

Theorem 3.62. If $S \subseteq \mathbb{C}$, then $\mathbb{Q} \subseteq \langle S \rangle_{\text{FIELD}}$.

Problem 3.63. The field defined in Problem 3.44(3) is sometimes denoted $\mathbb{F}_4$. Determine $\langle 1 \rangle_{\text{FIELD}}$ in the field $\mathbb{F}_4$.

Most of the time, we will want to generate fields by adding some elements to an existing field, and we have special notation for this.

Notation 3.64. Let F be a subfield of E, and let $r_1, r_2, \ldots, r_n \in E$. The subfield of E generated by $F \cup \{r_1, r_2, \ldots, r_n\}$ is denoted $F(r_1, r_2, \ldots, r_n)$. In other words,

$$F(r_1, r_2, \ldots, r_n) := \langle F \cup \{r_1, r_2, \ldots, r_n\} \rangle_{\text{FIELD}}.$$

We read $F(r_1, r_2, \ldots, r_n)$ as "F adjoin $r_1, r_2, \ldots, r_n$"; it is the smallest field extension of F that contains $r_1, r_2, \ldots, r_n$.

In the following problems, we are working with subfields of $\mathbb{C}$, even if we don't say it explicitly. Thus, by Theorem 3.62, we are working with field extensions of $\mathbb{Q}$.

Problem 3.65. In Problem 3.58(1), we saw that $\{a + bi \mid a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$. Show that $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$.

Problem 3.66. Show that $\mathbb{R}(i) = \mathbb{C}$.

Problem 3.67. We saw previously that $\{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$. Find some $z \in \mathbb{C}$, such that $\mathbb{Q}(z) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$, and prove that your choice for z works. Do you think there is only one choice for z or might others work?

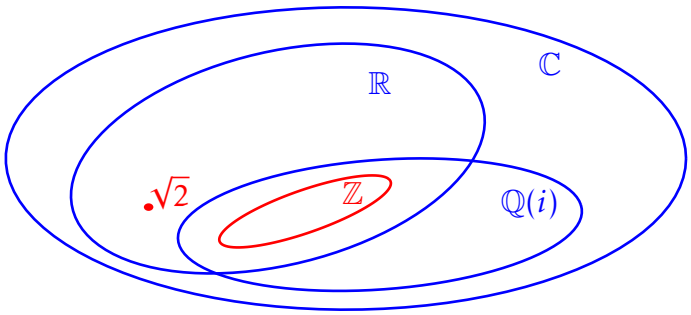
Problem 3.68. Let $\alpha = \sqrt{2} + i$. Show $\{a + b\alpha \mid a, b \in \mathbb{Q}\} \subset \mathbb{Q}(\alpha)$, but $\{a + b\alpha \mid a, b \in \mathbb{Q}\} \neq \mathbb{Q}(\alpha)$.

Theorem 3.69. Let F, L be subfields of E, and let $r_1, r_2, \ldots, r_n \in E$. Then $F(r_1, r_2, \ldots, r_n) \subseteq L$ if and only if $F \subseteq L$ and $r_1, r_2, \ldots, r_n \in L$.

Problem 3.70. Show that $\mathbb{Q}\left(3 - \sqrt{2}, 5 + i\right) = \mathbb{Q}\left(\sqrt{2}, i\right)$.

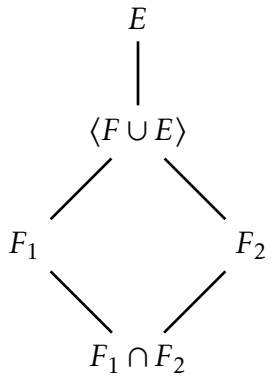
Problem 3.71. Complete the diagram below to illustrate how each of the following sets intersect and where each element is located. Each set that is a field should be drawn in blue; each set that is not a field should be drawn in red. Elements should be illustrated by a dot and then labeled by the name of the element. Some have already been done.

$$\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, 0, 1, \sqrt{2}, i, i\sqrt{2}, \sqrt{2} + i, \mathbb{Q}\left(\sqrt{2}\right), \mathbb{Q}(i), \mathbb{Q}\left(i\sqrt{2}\right) \mathbb{Q}\left(\sqrt{2}, i\right), \{a + bi \mid a, b \in \mathbb{Z}\}$$

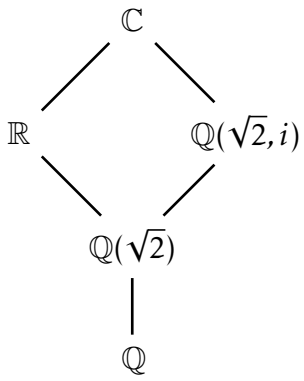


Problem 3.72. Conjecture where $\mathbb{Q}\left(\sqrt{2}+i\right)$ would be in the previous diagram.

Suppose that F_1 and F_2 are subfields of E. Theorem 3.59 tells us that $F_1 \cap F_2$ is again a subfield, and it is the largest subfield contained in both F_1 and F_2. The same theorem, together with Definition 3.60, also tells us that $\langle F_1 \cup F_2 \rangle_{\text{FIELD}}$ is a subfield, and it is the smallest subfield containing both $F_1$and F_2. This implies that the set of all subfields of E forms a lattice. Lattices will not be defined here, but feel free to look them up on your own. We will, however, be interested in illustrating these relationships with a diagram. The situation for F_1 and F_2 described above would be drawn as follows.



For a concrete example, let's draw the portion of the subfield lattice of $\mathbb{C}$ containing $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}\left(\sqrt{2}, i\right)$; this uses some of what you discovered in Problem 3.71.



Problem 3.73. Draw the portion of the subfield lattice of $\mathbb{C}$ that contains the following fields: $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}\left(\sqrt{2}\right)$, $\mathbb{Q}(i)$, $\mathbb{Q}\left(i\sqrt{2}\right)$, and $\mathbb{Q}\left(\sqrt{2}, i\right)$.

Problem 3.74. Draw the portion of the subfield lattice of $\mathbb{C}$ that contains the following fields: $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Q}(\zeta_4)$, $\mathbb{Q}(\zeta_8)$, and $\mathbb{Q}(\zeta_{16})$.

Chapter 4

Solvability by Radicals

Our overarching goal, as laid out in Chapter 2, is to find a polynomial whose roots can **not** be expressed in terms of the coefficients of the polynomial using just the operations of addition, subtraction, multiplication, division, and the extraction of roots. Or, in other words, we are searching for a polynomial that is **not** solvable by radicals, a term that we have only defined informally so far. Laying out a formal definition of solvability by radicals (and trying to wrap our head around it) is the main goal of this chapter.

4.1 Radical extensions

The notion of "solvable by radicals" is about how we may express the roots of a polynomial. We start by formalizing the notion that "a number can be expressed in terms of other numbers using just the operations of addition, subtraction, multiplication, division, and the extraction of roots." In the next section, we apply this to roots of polynomials.

Now, when we define what it means for a number to be built using the various operations listed above, we need to capture the possibility that we may need "iterated roots" to express a number. For example, consider

$$\alpha = \sqrt{2} + \sqrt[3]{-1 + \sqrt{2}}.$$

To see that α can be expressed using addition, subtraction, multiplication, division, and the extraction of roots, we first note that the number $\beta = -1 + \sqrt{2}$ can be built using addition and a square root; we then arrive at α by taking a cube root of β and adding $\sqrt{2}$.

Let's begin to formalize this by introducing fields. Our observations above imply that α can be built using field operations from $\sqrt[3]{\beta}$ and $\sqrt{2}$, and β in turn can be built using field operations from $\sqrt{2}$. Thus, $\alpha \in \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{\beta}\right)$ and $\beta \in \mathbb{Q}\left(\sqrt{2}\right)$. The lattice looks like this.

$$\begin{array}{rc} \alpha \in & \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{\beta}\right) \\ & \Big| \\ \beta \in & \mathbb{Q}\left(\sqrt{2}\right) \\ & \Big| \\ & \mathbb{Q} \end{array}$$

Now, when we talk about extracting roots, we must be careful to avoid ambiguous (not well-defined) notation. For this reason, we usually adopt the point of view of Definition 3.19 where z being an n^{th} root of a means that $z^n = a$ (as opposed to $z = \sqrt[n]{a}$). That said, we do still occasionally use the root symbol when there is no ambiguity. For example, $\sqrt[3]{5}$ and $\sqrt{-1}$ are well-defined: the first is the one and only *real* solution to $x^3 = 5$, and the second is i (which we made a choice about long ago). However, $\sqrt[4]{-1 + i\sqrt{3}}$ is *not* well-defined, as there are 4 equally good choices.

Definition 4.1. We say K is a **radical extension** of a field F if there exist nonzero elements $r_1, r_2, \ldots, r_m \in K$ and positive integers $n_1, n_2 \ldots, n_m$ such that $K = F(r_1, r_2, \ldots, r_m)$, and

$$\begin{aligned} r_1^{n_1} &\in F, \\ r_2^{n_2} &\in F(r_1), \\ r_3^{n_3} &\in F(r_1, r_2), \\ &\vdots \\ r_m^{n_m} &\in F(r_1, \ldots, r_{m-1}). \end{aligned}$$

The definition expresses that each r_i is an n_i^{th}-root of some element in $F(r_1, \ldots, r_{i-1})$, so K may be thought of as being built by iteratively adding in n^{th}-roots of elements. The picture is something like this:

$$\begin{array}{rc} & K = F(r_1, r_2, \ldots, r_m) \\ & \Big| \\ r_m^{n_m} \in & F(r_1, r_2, \ldots, r_{m-1}) \\ & \Big| \\ & \vdots \\ & \Big| \\ r_3^{n_3} \in & F(r_1, r_2) \\ & \Big| \\ r_2^{n_2} \in & F(r_1) \\ & \Big| \\ r_1^{n_1} \in & F \end{array}$$

Example 4.2. Let's show that $K = \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{-1+\sqrt{2}}\right)$ is a radical extension of $\mathbb{Q}$.

If we let $r_1 = \sqrt{2}$ and $r_2 = \sqrt[3]{-1+\sqrt{2}}$, then we see that

- $K = \mathbb{Q}(r_1, r_2)$;
- $r_1^2 = 2 \in \mathbb{Q}$;
- $r_2^3 = -1 + \sqrt{2} \in \mathbb{Q}\left(\sqrt{2}\right) = \mathbb{Q}(r_1)$.

This shows that K is a radical extension of $\mathbb{Q}$ using $r_1 = \sqrt{2}$, $n_1 = 2$, $r_2 = \sqrt[3]{-1+\sqrt{2}}$, and $n_2 = 3$ in the definition of a radical extension. The picture is like this:

$$\begin{array}{ccc} K = \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{-1+\sqrt{2}}\right) & & \\ & | & \\ \left(\sqrt[3]{-1+\sqrt{2}}\right)^3 \in & \mathbb{Q}\left(\sqrt{2}\right) & \\ & | & \\ \left(\sqrt{2}\right)^2 \in & \mathbb{Q} & \end{array}$$

Problem 4.3. Show that $\mathbb{Q}\left(\sqrt{3}, \zeta_5, \sqrt[4]{1-\sqrt{3}}\right)$ is a radical extension of $\mathbb{Q}$. What are you using for r_1, n_1, r_2, n_2, and r_3, n_3 when applying the definition?

Problem 4.4. Show that $\mathbb{C}$ is a radical extension of $\mathbb{R}$.

4.2 Solvability by radicals: the definition

Making precise what it means to express a number using addition, subtraction, multiplication, division, and the extraction of roots was the crux of defining solvability by radicals. However, we are aiming to define what it means to express the roots of a polynomial *in terms of the coefficients* using these operations. Let's establish some notation that allows us to highlight where the coefficients of a polynomial live.

Definition 4.5. Let F be a field. Then $F[x]$ is the set of all polynomials whose coefficients lie in F. This is read "F adjoin x."

For example, consider $a(x) = x^3 - ix^2 - 0.5$. Then $a(x) \notin \mathbb{Q}[x]$, because $i \notin \mathbb{Q}$; however, $a(x) \in \mathbb{C}[x]$ (and, in fact, $a(x) \in \mathbb{Q}(i)[x]$).

Problem 4.6. Give examples of polynomials $a(x)$, $b(x)$, and $c(x)$ such that

(1) $a(x) \in \mathbb{Q}[x]$,

(2) $b(x) \in \mathbb{R}[x]$ but $b(x) \notin \mathbb{Q}[x]$, and

(3) $c(x) \in \mathbb{C}[x]$ but $c(x) \notin \mathbb{R}[x]$.

We now, finally, write down one of our main definitions.

Definition 4.7. Let F be a field, and let $p(x) \in F[x]$. We say that $p(x)$ is **solvable by radicals** over F if all of the roots of $p(x)$ are contained in some radical extension of F.

Problem 4.8. Let $p(x) = x^2 + 3x + 1$. Show that all roots of $p(x)$ lie in $\mathbb{Q}\left(\sqrt{5}\right)$. Use this to explain why $p(x)$ is solvable by radicals over $\mathbb{Q}$.

Problem 4.9. Let $p(x) = x^4 + 2x^2 + 5$. Show that all four roots of $p(x)$ lie in $\mathbb{Q}(i, r, s)$ for some r and s such that $r^2 = -1 - 2i$ and $s^2 = -1 + 2i$. Use this to explain why $p(x)$ is solvable by radicals over $\mathbb{Q}$.

Problem 4.10. Let $p(x) = x^3 - 2$. Use Theorem 3.26 to write out all complex roots of $p(x)$, and then show that $p(x)$ is solvable by radicals over $\mathbb{Q}$.

Theorem 4.11. For each positive $n \in \mathbb{Z}$, $x^n - 1$ is solvable by radicals over $\mathbb{Q}$.

Theorem 4.12. For each positive $n \in \mathbb{Z}$, $x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1$ is solvable by radicals over $\mathbb{Q}$.

Theorem 4.13. Every quadratic polynomial $p(x) \in \mathbb{Q}[x]$ is solvable by radicals over $\mathbb{Q}$.

Problem 4.14. Let $p(x) = x^6 - 3x^3 - 1$. Show that $p(x)$ is solvable by radicals over $\mathbb{Q}$.

Chapter 5

Rings

Our overarching goal, laid out in Chapter 2, is to show that there are quintic polynomials whose roots are *not* expressible in terms of its coefficients using just the operations of addition, subtraction, multiplication, division, and the extraction of roots (thus implying that there is no "quintic formula" that is analogous to the quadratic formula). We decided that we would say that such polynomials are *not* solvable by radicals, and in Chapter 4, we finally were able to write down a formal definition of this term. We also proved there are many polynomials that are solvable by radicals. But, how do we show that a polynomial is *not* solvable by radicals? We start by taking a closer look at polynomials.

5.1 Abstract rings

As we investigate polynomials, it will be useful to harness (and abstract) the algebraic properties that they possesses. For example, if we add two polynomials in $\mathbb{Q}[x]$, we obtain a polynomial that is again in $\mathbb{Q}[x]$, and similarly for multiplication. Let's explore the structure of $F[x]$ in general (where F is any field).

Definition 5.1. Let F be a field. The structure $(F[x], +, \cdot)$ consists of the set $F[x]$ together with the operations $+$ and $\cdot$ defined as follows. Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ and $q(x) = b_0 + b_1 x + \cdots + b_n x^n$ with $m \leq n$.

- **Addition:** $p(x)+q(x) = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$ (where $a_k = 0$ when $k > m$).
- **Multiplication:** $p(x) \cdot q(x) = \displaystyle\sum_{k=0}^{m+n} c_k x^k$ where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0$.

We often refer to the entire structure $(F[x], +, \cdot)$ as simply $F[x]$.

In the definition of polynomial multiplication above, $p(x) \cdot q(x)$ is just the result of applying the distributive law repeatedly to $(a_0 + a_1 x + \cdots + a_m x^m) \cdot (b_0 + b_1 x + \cdots + b_n x^n)$ and then grouping according to the powers of x. We will see that the operations of polynomial addition and multiplication have many familiar properties; let's prove a couple.

Problem 5.2. Using the definitions of polynomial addition and multiplication together with properties of fields, prove that for all fields F, both addition and multiplication in $F[x]$ are commutative.

Problem 5.3. Let F be any field. Find an additive identity for $F[x]$, and prove that it works. Also, if $p(x) = a_0 + a_1x + \cdots + a_mx^m$ is an arbitrary polynomial in $F[x]$, find its additive inverse, and prove that it works.

Problem 5.4. Which elements of $\mathbb{Q}[x]$ have a multiplicative inverse? Which do not? Justify your answer.

As should be becoming clear, many of the properties of F transfer to $F[x]$ (but not all!). Let's record some of those properties in a fact, which we will not prove. The existence of multiplicative inverses is notably absent.

Fact 5.5. Let F be any field. The following are true for $F[x]$.

- **Addition Laws:** Addition is associative and commutative. There is a unique additive identity, namely the constant zero polynomial, and every polynomial $p(x)$ has a unique additive inverse, denoted $-p(x)$.
- **Multiplication Laws:** Multiplication is associative and commutative. There is a unique multiplicative identity, namely the constant polynomial 1.
- **Distributivity Laws:** For all $p(x), q(x), r(x) \in F[x]$, $p(x)(q(x)+r(x)) = p(x)q(x)+p(x)r(x)$ and $(q(x) + r(x))p(x) = q(x)p(x) + r(x)p(x)$.

5.1.1 Definition and first examples

Since $F[x]$ lacks multiplicative inverses for many of its elements, it does not form a field. Nevertheless, motivated by our desire to study polynomials, we will abstract the structure that is present so that we can prove theorems about polynomials over any field, instead of working one field at a time. However, before we do, it's worth noting that there are many other structures that are not fields but do satisfy the laws in Fact 5.5—perhaps the most prominent one is the integers $\mathbb{Z}$. We arrive at the definition of a ring.

Definition 5.6. A **ring** is a structure $(R, +, \cdot)$ consisting of a set R together with two binary operations $+$ and $\cdot$ (which we call *addition* and *multiplication*) such that for some element $0 \in R$ the following axioms hold.

- **Addition Axioms:** Addition is associative and commutative; the element 0 is an additive identity; every $x \in R$ has an additive inverse with respect to 0, denoted $-x$.
- **Multiplication Axioms:** Multiplication is associative.
- **Distributivity Axioms:** For all $x, y, z \in R$, $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$.

In the case that multiplication is commutative, R is called a **commutative ring**, and in the case that there is a multiplicative identity, R is called a **ring with unity** (or ring with 1).

The notion of a ring is quite general, and the terminology "commutative ring" and "ring with unity" highlight some of the additional properties that $F[x]$ has, but arbitrary rings may not. But notice that fields have all of these properties *and more*. The next definition is meant to highlight this.

Definition 5.7. A **division ring** is a ring with unity such that every nonzero element has a multiplicative inverse.

Problem 5.8. Fill in each box of the table below with **Y**es or **N**o. Assume that + and · are defined "as usual" for each set.*

	ring	commutative ring	ring with unity	division ring	field
$\mathbb{Z}$					
$2\mathbb{Z}$					
$\mathbb{N}$					
$\mathbb{Q}$					
$\mathbb{H}$					
$\mathbb{Z}_6$					
$\mathbb{R}[x]$					
$\{a+bi \mid a,b \in \mathbb{Q}\}$					
$\{a+bi \mid a,b \in \mathbb{Z}\}$					

5.1.2 Basic properties

Many of the basic properties of fields hold also for rings, with essentially the same proofs, so we will just take them as fact.

Fact 5.9 (Compare with Fact 3.50)**.** Let R be a ring.

(1) The additive identity is unique. If there exists a multiplicative identity, it is unique.

(2) Additive inverses are unique. If an element has a multiplicative inverse, it is unique.

Fact 5.10 (Compare with Fact 3.51)**.** Let R be a ring.

(1) For all $x \in R$, $x \cdot 0 = 0 = 0 \cdot x$.

(2) For all $x, y \in R$, $(-x)y = -(xy)$ and $x(-y) = -(xy)$.

(3) If R contains at least two elements and has a multiplicative identity, then the additive and multiplicative identities are different, i.e. $0 \neq 1$.

*$2\mathbb{Z}$ denotes the even integers. The operations are usual integer addition and multiplication.

Let's explore one further property that fields possess but is not listed above: for all x and y in a field, if $xy = 0$, then $x = 0$ or $y = 0$.

Definition 5.11. Let R be a ring. An element $a \in R$ is called a **zero divisor** if a is nonzero and there exists a nonzero $b \in R$ such that $ab = 0$. A ring is called an **integral domain** if it is a commutative ring with unity containing at least two elements but *no zero divisors*.

As remarked above, fields do not have zero divisors, so every field is indeed an integral domain. However, the prototypical integral domain (which explains the choice of name) is $\mathbb{Z}$. Let's look for others.

Problem 5.12. For each of the following rings, determine if there are zero divisors, and if so, find them all. Is the ring an integral domain?

(1) $\mathbb{Z}_5$

(2) $\mathbb{Z}_{10}$

(3) $\mathbb{H}$

(4) $\mathbb{R}[x]$

When working with integral domains, the following property is key.

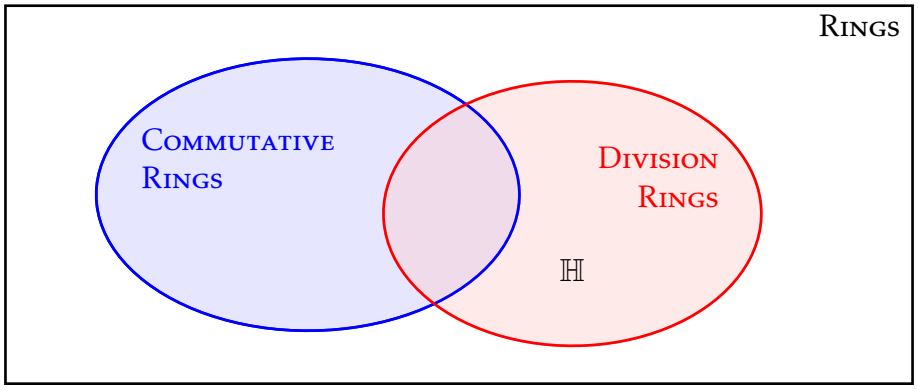
Theorem 5.13 (Cancellation Property)**.** Let R be an integral domain. For all $a, b, c \in R$, if $ab = ac$, then either $a = 0$ or $b = c$.

Problem 5.14. What properties of integral domains did you use in your proof of Theorem 5.13? Can you rewrite the theorem to be more general? Try.

Let's pause to collect and organize all of our new definitions.

Problem 5.15. Complete the following Venn Diagram by adding in shapes for each of the following terms. Try to provide examples that live in each of the gaps, but we have not encountered enough examples (in these notes) to cover all gaps yet.

- Fields
- Commutative Rings
- Division Rings
- Rings
- Rings with unity
- Integral domains



5.1.3 Units

Unless R is actually a division ring, not all elements of R will have a multiplicative inverse. Let's explore those elements that *do* have an inverse.

Definition 5.16. Let R be a ring with unity containing at least two elements. Then, $u \in R$ is called a **unit** if u has a multiplicative inverse. The set of all units in R is denoted $U(R)$.

Problem 5.17. For each of the following rings, find all of the units, i.e. determine $U(R)$.

(1) $\mathbb{Z}$

(2) $\mathbb{Z}_5$

(3) $\mathbb{R}$

(4) $\mathbb{R}[x]$

Problem 5.18. Consider the ring $\mathbb{Z}_{20}$. Find all units of $\mathbb{Z}_{20}$ and also find all zero divisors. What do you notice?

Problem 5.19. Let n be a positive integer. Make a conjecture about $U(\mathbb{Z}_n)$ by filling in the blank: $U(\mathbb{Z}_n) = \{a \in \mathbb{Z}_n \mid$ (fill in the blank) $\}$. What evidence do you have?

Theorem 5.20. Let R be a ring with unity containing at least two elements. If $u \in R$ is a unit, then u is *not* a zero divisor.

Problem 5.21. Either prove or disprove the *converse* of Theorem 5.20.

Theorem 5.22. Let R be a ring with unity containing at least two elements. Then $(U(R), \cdot)$ is a group.

5.2 An aside: matrix rings

Matrix rings are really the prototypical ring with unity. Although you may have only seen matrices with real entries, it turns out that we can do matrix arithmetic with other types of entries, e.g. entries from $\mathbb{C}$ or $\mathbb{Z}$. In fact, the usual matrix addition and multiplication makes sense when the entries come from any ring.

Definition 5.23. Let R be a ring and n a positive integer. Then $M_n(R)$ denotes the set of all $n \times n$ matrices whose entries come from R. The structure $(M_n(R), +, \cdot)$ consists of the set $M_n(R)$ of all $n \times n$ matrices whose entries come from R, together with the operations of usual matrix addition and matrix multiplication.

Problem 5.24. Provide examples of matrices satisfying each of the following conditions.

(1) $A \in M_3(\mathbb{C})$ but $A \notin M_3(\mathbb{R})$

(2) $B \in M_2(\mathbb{H})$ but $B \notin M_2(\mathbb{C})$

(3) $C \in M_2\left(\mathbb{Q}\left(\sqrt{5}\right)\right)$ but $C \notin M_2(\mathbb{Q})$

(4) $D \in M_2(\mathbb{R}[x])$ but $D \notin M_2(\mathbb{R})$

Problem 5.25. Verify that $M_2(\mathbb{Z})$ is closed under matrix multiplication.

The next fact shows that $M_n(R)$ is a ring with unity (for each positive n). Afterward, we will explore some of the other ring properties we discussed above.

Fact 5.26. Let R be any ring. The following are true for $M_n(R)$.

- **Addition Laws:** Addition is associative and commutative. There is a unique additive identity, namely the matrix with all entries equal to 0, and every matrix A has a unique additive inverse, denoted $-A$.
- **Multiplication Laws:** Multiplication is associative. There is a unique multiplicative identity, namely the matrix with 1's on the main diagonal and 0's everywhere else.
- **Distributivity Laws:** For all $A, B, C \in M_n(R)$, $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$.

Problem 5.27. Is $M_2(\mathbb{R})$ commutative? Prove your answer.

Problem 5.28. Does $M_2(\mathbb{R})$ have zero divisors? Prove your answer.

The collection of units in a matrix ring forms a group with respect to matrix multiplication by Theorem 5.22. It is a very important object and even has a special name.

Definition 5.29. Let R be a ring and n a positive integer. The **general linear group** over the ring R, denoted $\mathrm{GL}_n(R)$, is the group of units in the ring $M_n(R)$.

Problem 5.30. Show that $\begin{bmatrix} i & 3 \\ 0 & i \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$ by finding a multiplicative inverse for it. Also, find two different matrices in $M_2(\mathbb{C})$ that are *not* in $\mathrm{GL}_2(\mathbb{C})$.

5.3 Polynomial rings

Our study of rings was motivated by our desire to learn more about polynomials, and we now dive a little deeper into the theory of polynomial rings. Ultimately, we will focus on polynomials rings $F[x]$ where F is a field. In this section, we will see that $F[x]$ behaves in many ways like the integers $\mathbb{Z}$: $F[x]$ is an integral domain, there is a division algorithm for $F[x]$, there exists a greatest common divisor for polynomials, and there is a notion of primes and prime factorizations. Let's start with some important terminology.

Definition 5.31. Let R be a ring, and let $p(x) \in R[x]$ be a *nonzero* polynomial. If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$, then n is called the **degree** of $p(x)$, denoted $\deg p(x)$. In words, $\deg p(x)$ is the highest power of x in $p(x)$ with a nonzero coefficient. The degree of the zero polynomial is undefined.

Problem 5.32. Determine the degree of each of the following polynomials.

(1) $q(x) = 4x^5 + 2x^2 + 5 - 8x^2 + 2x^5$ in the ring $\mathbb{Z}[x]$

(2) $r(x) = 4x^5 + 2x^2 + 5 - 14x^2 + 2x^5$ in the ring $\mathbb{Z}_6[x]$

(3) $p(t) = (3t^2 - \sqrt{2})(-1 + 2t - t^3)$ in the ring $\mathbb{R}[t]$

(4) $s(x) = (5-i)^8 - (5-s)^8$ in the ring $\mathbb{C}[s]$

The degree function is incredibly useful when working with polynomials—let's prove a couple of properties about it.

Theorem 5.33. Let R be a ring. If $p(x)$ and $q(x)$ are nonzero polynomials $R[x]$, then $\deg(p(x)+q(x)) \leq \max(\deg p(x), \deg q(x))$ or $\deg(p(x)+q(x))$ is undefined.

Problem 5.34. Give an example of polynomials $p(x), q(x) \in \mathbb{Q}[x]$ such that $\deg(p(x)+q(x)) < \max(\deg p(x), \deg q(x))$.

Theorem 5.35. Let D be an integral domain. If $p(x)$ and $q(x)$ are nonzero polynomials $D[x]$, then $\deg(p(x)q(x)) = \deg p(x) + \deg q(x)$, and in particular, $\deg(p(x)q(x))$ is defined.

Problem 5.36. Give an example of nonzero polynomials $p(x), q(x) \in \mathbb{Z}_{10}[x]$ such that $\deg(p(x)q(x)) \neq \deg p(x) + \deg q(x)$. Why does this not contradict Theorem 5.35?

Corollary 5.37. If D is an integral domain, then $D[x]$ is an integral domain.

5.3.1 Division algorithm

Here we explore what it means for one polynomial to divide another as well as the idea of a quotient and remainder for division. These should be familiar from previous classes for $\mathbb{R}[x]$, but here we see that they generalize to arbitrary $F[x]$ for F a field.

Definition 5.38. Let R be a ring, and let $a, b \in R$. We say that b **divides** a (or b is a **divisor** of a) if there exists some $q \in R$ such that $a = bq$.

Problem 5.39. Consider the polynomial $p(x) = x^2 - 1$ in $\mathbb{Q}[x]$.

(1) Does $x+1$ divide $p(x)$ in $\mathbb{Q}[x]$? Why or why not?

(2) Does $p(x)$ divide $x+1$ in $\mathbb{Q}[x]$? Why or why not?

(3) Does 3 divide $p(x)$ in $\mathbb{Q}[x]$? Why or why not?

Theorem 5.40. Let $p(x) \in R[x]$ with R a ring. If $c \in R$ and $(x-c)$ divides $p(x)$, then $p(c) = 0$.

Even if $b(x)$ does not divide $a(x)$, it can still be useful to perform the division to obtain a quotient and remainder.

Problem 5.41. Consider the polynomials $a(x) = x^4 + x^3 - 8x + 5$ and $b(x) = x^2 - 3$ in $\mathbb{Q}[x]$. Use polynomial long division to show that $b(x)$ does not divide $a(x)$. What is the quotient and what is the remainder? Write $a(x)$ as $a(x) = b(x)q(x) + r(x)$ for some $q(x), r(x) \in \mathbb{Q}[x]$ with $\deg r(x) < \deg b(x)$.

The "division algorithm" (Theorem 5.43) formalizes what results from long division. And, it turns out that it is true for polynomials over any field (not just $\mathbb{Q}$). The next lemma prepares for the proof of the division algorithm.

Lemma 5.42. Let F be a field, and let $a(x), b(x) \in F[x]$ with $\deg a(x) \geq \deg b(x)$. Assume that $a(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$ and $b(x) = b_0 + b_1 x + \cdots + b_m x^m$ with $b_m \neq 0$. Set $a_1(x) = a(x) - b(x) a_n b_m^{-1} x^{n-m}$. Then $\deg a_1(x) < \deg a(x)$, and $b(x)$ divides $a(x) - a_1(x)$.

Suppose we are trying to divide $b(x)$ into $a(x)$. How do we find the quotient and the remainder? Well, if the degree of $a(x)$ is smaller than the degree of $b(x)$ there is nothing to do (and if $a(x) = 0$, there is also nothing to do). Otherwise, we can use the previous lemma to produce a polynomial $a_1(x)$ such that $\deg a_1(x) < \deg a(x)$ and $b(x)$ divides $a(x) - a_1(x)$, or in other words, $a(x) - a_1(x) = b(x)q_1(x)$ for some $q_1(x)$. Now, suppose we repeat the process and divide $b(x)$ into the resulting $a_1(x)$ to produce $a_2(x)$ and $q_2(x)$. Continuing in this fashion, we produce $a_2, q_2, a_3, q_3 \ldots, a_k, q_k$, stopping once the degree of $a_k(x)$ becomes smaller than the degree of $b(x)$ (or $a_k(x) = 0$). In total, we get something like the following.

$$\begin{aligned} a(x) - a_1(x) &= b(x)q_1(x) \\ a_1(x) - a_2(x) &= b(x)q_2(x) \\ a_2(x) - a_3(x) &= b(x)q_3(x) \\ &\vdots \\ a_{k-1}(x) - a_k(x) &= b(x)q_k(x) \end{aligned}$$

Adding the above equations together and moving things around, we arrive at

$$a(x) = b(x)\left(q_1(x) + q_2(x) + q_3(x) + \cdots + q_k(x)\right) + a_k(x)$$

with the degree of $a_k(x)$ being less than the degree of $b(x)$. Thus, $a_k(x)$ is the remainder and $q_1(x) + \cdots + q_k(x)$ the quotient. This is the rough idea behind the division algorithm.

Theorem 5.43 (Division algorithm for $F[x]$)**.** Let F be a field, and let $a(x), b(x) \in F[x]$ with $b(x) \neq 0$. Then there exist $q(x), r(x) \in F[x]$ such that

$$a(x) = b(x)q(x) + r(x)$$

with $\deg r(x) < \deg b(x)$ or $r(x) = 0$.

The division algorithm is the theoretical analogue of long division. If you want to divide concrete polynomials, use long division, but if you want to prove something about divisibility for arbitrary polynomials, use the division algorithm. It is often used to prove a polynomial $b(x)$ actually divides another polynomial $a(x)$. The strategy is to apply the division algorithm to produce the equation $a(x) = b(x)q(x) + r(x)$ (with $\deg r(x) < \deg b(x)$ or $r(x) = 0$) and then use this to show that, in fact, $r(x) = 0$, implying that $a(x) = b(x)q(x)$ as desired. Let's try using this approach to prove the converse of Theorem 5.40.

Theorem 5.44. Let $p(x) \in F[x]$ for F a field. If $c \in F$ and $p(c) = 0$, then $(x - c)$ divides $p(x)$.

Problem 5.45. Consider the polynomial $p(x) = x^2 + x + 3$ in $\mathbb{Z}_5[x]$. Compute $p(c)$ for each $c \in \mathbb{Z}_5$, and use the results to determine which polynomials of the form $(x - c)$ divide $p(x)$. Factor $p(x)$ into a product of degree 1 polynomials in $\mathbb{Z}_5[x]$, if possible.

Problem 5.46. Consider the polynomial $p(x) = x^2 + x + 1$ in $\mathbb{Z}_5[x]$. Explain why $p(x)$ cannot be factored into a product of degree 1 polynomials in $\mathbb{Z}_5[x]$.

5.3.2 Greatest common divisors

The fact that there is a division algorithm for $F[x]$ (Theorem 5.43) is a rather special property for a ring to possess, and it has several important consequences. The first one we'll explore is the existence of a "greatest common divisor" for two polynomials, and our first order of business is to try to decide on a reasonable definition of this.

Problem 5.47. What are the common divisors of 6 and -9 in $\mathbb{Z}$? Which one is the greatest common divisor?

Problem 5.48. Consider the polynomials $a(x) = 2x^2 - 2$ and $b(x) = 2x^2 + 2x - 4$ in $\mathbb{Q}[x]$.

(1) Show that $x-1$ is a common divisor of $a(x)$ and $b(x)$ by finding $q(x), s(x) \in \mathbb{Q}[x]$ such that $a(x) = (x-1)q(x)$ and $b(x) = (x-1)s(x)$.

(2) Show that $-2(x-1)$ is a common divisor of $a(x)$ and $b(x)$ by finding $q(x), s(x) \in \mathbb{Q}[x]$ such that $a(x) = -2(x-1)q(x)$ and $b(x) = -2(x-1)s(x)$.

(3) Show that $100(x-1)$ is a common divisor of $a(x)$ and $b(x)$ by finding $q(x), s(x) \in \mathbb{Q}[x]$ such that $a(x) = 100(x-1)q(x)$ and $b(x) = 100(x-1)s(x)$.

Which one, if any, would be a good choice as the "greatest common divisor"?

The previous problem highlights that there are several (actually, infinitely many) choices for the "greatest common divisor" of two polynomials. Our choice for which one we call the greatest common divisor is, in some sense, the simplest one.

Definition 5.49. A polynomial $p(x)$ of degree n is called **monic** if the coefficient of x^n (i.e. the leading coefficient) is 1.

For example, $7 - 2x + x^2$ is monic, since the coefficient of x^2 is 1. However, neither $7 - 2x + 3x^2$ nor $7 - 2x - x^2$ are monic.

Definition 5.50. Let F be a field, and let $a(x), b(x) \in F[x]$ be nonzero polynomials. A polynomial $d(x) \in F[x]$ is called a **greatest common divisor** of $a(x)$ and $b(x)$ if

(1) $d(x)$ is monic,

(2) $d(x)$ divides both $a(x)$ and $b(x)$,

(3) if $h(x)$ divides both $a(x)$ and $b(x)$, then $h(x)$ divides $d(x)$.

Thus, in Problem 5.48, the greatest common divisor of the polynomials $a(x)$ and $b(x)$ is $x-1$. That said, we don't yet know that a greatest common divisor always exists, but let's start by showing that if one exists, there is only one.

Lemma 5.51. Let F be a field, and let $a(x), b(x) \in F[x]$ be nonzero polynomials. If $d_1(x)$ and $d_2(x)$ are greatest common divisors of $a(x)$ and $b(x)$, then $d_1(x) = d_2(x)$.

We now work towards the existence of a greatest common divisor for arbitrary polynomials in $F[x]$ (for arbitrary fields). The proof of this result is tightly tied to analyzing certain combinations of the polynomials $a(x)$ and $b(x)$. Let's explore this a bit.

Problem 5.52. Consider the polynomials $a(x) = 2x^2 - 2$ and $b(x) = 2x^2 + 2x - 4$ in $\mathbb{Q}[x]$. Define

$$I = \{f(x)a(x) + g(x)b(x) \mid f(x), g(x) \in \mathbb{Q}[x]\}.$$

(1) Write down 5 different polynomials that are in the set I.

(2) Show that $x - 1$ divides an arbitrary polynomial in I.

The idea behind the second part of Problem 5.52 can be used to prove the following general result about sets like I.

Theorem 5.53. Let F be a field, and let $a(x), b(x) \in F[x]$ be nonzero polynomials. Define

$$I = \{f(x)a(x) + g(x)b(x) \mid f(x), g(x) \in F[x]\}.$$

If $h(x)$ divides both $a(x)$ and $b(x)$, then $h(x)$ divides every $c(x) \in I$.

The existence and uniqueness of greatest common divisors in $F[x]$ is presented in the following fact.

Fact 5.54. Let F be a field, and let $a(x), b(x) \in F[x]$ be nonzero polynomials. There exists a unique greatest common divisor of $a(x)$ and $b(x)$, and if $d(x)$ is the greatest common divisor, then

$$d(x) = f(x)a(x) + g(x)b(x),$$

for some $f(x), g(x) \in F[x]$.

The proof of this fact is interesting, but let's content ourselves to just outline it. The approach is fairly straight forward. Define

$$I = \{f(x)a(x) + g(x)b(x) \mid f(x), g(x) \in F[x]\}.$$

Theorem 5.53 tells us that every common divisor of $a(x)$ and $b(x)$ is a divisor of every polynomial in I. Thus, if I contains a monic, common divisor of $a(x)$ and $b(x)$, it must be the greatest common divisor. So, we look for a common divisor of $a(x)$ and $b(x)$ in I. And to do this, the key idea is to choose a polynomial of smallest degree in I.

Let m be the smallest degree of all nonzero polynomials in I (which exists by the Well-Ordering Property of the natural numbers). Choosing any polynomial of degree m in I, we can divide out the leading coefficient to get a *monic* polynomial $d(x)$, which we can show is still in I. The polynomial $d(x)$ will be the greatest common divisor.

To see that $d(x)$ divides $a(x)$, we use Theorem 5.43 (the division algorithm) to write $a(x) = d(x)q(x) + r(x)$ for $q(x), r(x) \in F[x]$ with $\deg r(x) < \deg d(x)$ or $r(x) = 0$. Towards a contradiction, assume $r(x) \neq 0$. Now, since $d(x) \in I$, there exist $f_d(x), g_d(x) \in F[x]$ such that

$$\begin{aligned} r(x) &= a(x) - d(x)q(x) \\ &= a(x) - [f_d(x)a(x) + g_d(x)b(x)]q(x) \\ &= [1 - f_d(x)q(x)]a(x) + [-g_d(x)q(x)]b(x) \\ &\in I. \end{aligned}$$

Since $\deg r(x) < \deg d(x)$, this contradicts the fact that $d(x)$ had the smallest possible degree of all polynomials in I. Thus, $r(x) = 0$, and $d(x)$ divides $a(x)$. A similar argument shows that $d(x)$ also divides $b(x)$, so $d(x)$ is a monic, common divisor of $a(x)$ and $b(x)$. And, Theorem 5.53 shows that $d(x)$ is a greatest common divisor. But then, it is the unique greatest common divisor by Lemma 5.51.

Using Fact 5.54, we can rewrite the set I is a very nice way.

Corollary 5.55. Let F be a field, and let $a(x), b(x) \in F[x]$ be nonzero polynomials. Define

$$I = \{f(x)a(x) + g(x)b(x) \mid f(x), g(x) \in F[x]\}.$$

If $d(x)$ is the greatest common divisor of $a(x)$ and $b(x)$, then $I = \{p(x)d(x) \mid p(x) \in F[x]\}$.

With similar ideas as in the proof of Fact 5.54, one can prove the following fact that characterizes the greatest common divisor in several different ways.

Fact 5.56. Let F be a field, and let $a(x), b(x) \in F[x]$ be nonzero polynomials. Define

$$I = \{f(x)a(x) + g(x)b(x) \mid f(x), g(x) \in F[x]\}.$$

For any polynomial $d(x) \in F[x]$, the following are equivalent:

(1) $d(x)$ is the greatest common divisor of $a(x)$ and $b(x)$;

(2) $d(x)$ is a monic common divisor of $a(x)$ and $b(x)$, and $d(x) \in I$;

(3) $d(x)$ is a monic, and $I = \{p(x)d(x) \mid p(x) \in F[x]\}$.

So, you may be wondering: how do we compute the greatest common divisor of two polynomials? First think about how you would compute the greatest common divisor of 168 and 180. Really, think about it... Many people will factor both 168 and 180 into primes and then multiply the prime factors they have in common. This works for integers, and in fact, it will also work for polynomials once we develop the notion of a "prime polynomial." However, there is another approach, which in general is way more efficient: the Euclidean Algorithm. We will not develop it here, but you are encouraged to look it up (perhaps starting on Wikipedia).

5.3.3 Irreducible polynomials

We now develop the analogous notion of a prime number for polynomials, which will be an *irreducible* polynomial. The concept of irreducibility makes sense quite generally, so we start by defining it for any integral domain. Recall that, by Corollary 5.37, $F[x]$ is always an integral domain when F is a field.

To motivate the definition, think about how prime integers are defined: $p \in \mathbb{Z}$ is prime if (1) $p > 1$ and (2) $p = ab$ implies that $a = \pm 1$ or $b = \pm 1$. Since the units of $\mathbb{Z}$ are precisely ± 1, the second condition could be rewritten as "$p = ab$ implies that a or b is a unit." Also, since we don't want 1 (or -1) to be considered prime, the first condition is mostly captured by ensuring that "p is not zero and not a unit."

Definition 5.57. Let D be an integral domain. An element $p \in D$ is **irreducible** if

- $p \neq 0$ and $p \notin U(D)$, and
- for all $a, b \in D$, $p = ab$ implies $a \in U(D)$ or $b \in U(D)$.

The element p is **reducible** if it is not irreducible; that is if $p = 0$, $p \in U(D)$, or there exist $a, b \in D$ such that $p = ab$ and $a, b \notin U(D)$.

Problem 5.58. Use Definition 5.57 to show that a field has no irreducible elements.

Problem 5.59. What are the irreducible elements in $\mathbb{Z}$?

In order to investigate irreducibility in an integral domain D, we need to know its units. Our overarching goal is to better understand polynomials, so let's start there.

Theorem 5.60. Let F be a field. Then $p(x)$ is a unit in $F[x]$ if and only if $\deg p(x) = 0$.

Let's rewrite our definition of reducibility in a more useable form for polynomials.

Theorem 5.61. Let F be a field, and let $p(x)$ be a nonconstant polynomial in $F[x]$. Then $p(x)$ is reducible if and only if $\deg p(x) > 0$ and there exist polynomials $a(x), b(x) \in F[x]$ such that $p(x) = a(x)b(x)$ with $\deg a(x) < \deg p(x)$ and $\deg b(x) < \deg p(x)$.

Problem 5.62. Determine if $p(x)$ is reducible or irreducible in the given ring. If it's reducible, write down a factorization.

(1) $p(x) = x^2 + 1$ in $\mathbb{C}[x]$

(2) $p(x) = x^2 + 1$ in $\mathbb{Q}[x]$

(3) $p(x) = x^2 + 1$ in $\mathbb{Z}_2[x]$

(4) $p(x) = x^2 + 1$ in $\mathbb{Z}_3[x]$

Let's catalog a couple of general irreducibility/reducibility results for polynomials of small degree.

Theorem 5.63. Let F be a field. If $\deg p(x) = 1$, then $p(x)$ is irreducible.

Theorem 5.64. Let F be a field. If $\deg p(x) = 2, 3$, then $p(x)$ is reducible if and only if $p(x)$ has a root in F.

Problem 5.65. Determine if $p(x)$ is reducible or irreducible in the given ring. If it's reducible, write down a factorization.

(1) $p(x) = x^3 - 2$ in $\mathbb{Q}[x]$

(2) $p(x) = x^3 - 2$ in $\mathbb{Z}_5[x]$

Problem 5.66. Determine if each of the following polynomials are reducible or irreducible in the given ring.

(1) $p(x) = x^3 - 8$ in $\mathbb{Q}[x]$

(2) $p(x) = x^3 - 8$ in $\mathbb{Z}_5[x]$

(3) $r(x) = x^4 - 8x^2 + 15$ in $\mathbb{Q}[x]$

(4) $r(x) = x^4 - 8x^2 + 15$ in $\mathbb{Z}_5[x]$

To solidify the analogy between irreducible elements and primes, let's prove a factorization theorem.

Theorem 5.67. If F is a field, then any polynomial of positive degree in $F[x]$ can be written as a product of polynomials that are irreducible in $F[x]$.

As you know, in the integers every number greater than or equal to 2 can be factored into a product of primes in a way that is *unique up to reordering the factors*. There is a similar uniqueness result for polynomials: any polynomial of positive degree in $F[x]$ can be written as a product of irreducible polynomials in a way that is unique up to reordering the factors and multiplying each factor by a unit.

Problem 5.68. Let $p(x) = 6x^4 - 7x^3 + 15x^2 - 21x - 9$. Then the following are two different factorizations of $p(x)$ into irreducibles in $\mathbb{Q}[x]$:

- $p(x) = (2x-3)(3x+1)(x^2+3)$, and
- $p(x) = (x+\frac{1}{3})(2x^2+6)(3x-\frac{9}{2})$.

Explain why the factorizations are the same after possibly reordering the factors and multiplying each factor by a unit.

5.4 Subrings

We now return to general ring theory. As with groups and fields, the notion of a subring is fundamental.

Definition 5.69. Let $(R, +, \cdot)$ be a ring, and let S be a subset of R. Then S is a **subring** of R if S is a ring in its own right with respect to operations $+$ and $\cdot$ *inherited from* R.

As with fields, many of the properties of the ring R automatically pass to a subset S (e.g. associativity), leaving only a handful of the ring axioms to actually be verified.

Theorem 5.70. Let R be a ring, and let $S \subseteq R$. Then S is a subring of R if and only if

(1) S is nonempty;

(2) for all $x, y \in S$, $x + y \in S$ and $xy \in S$; and

(3) for all $x \in S$, $-x \in S$.

Problem 5.71. Determine if each of the following subsets of $\mathbb{Q}[x]$ are actually subrings.

(1) $A = \{p(x) \mid p(x) = c \text{ for some } c \in \mathbb{Q}\}$ (i.e. the set of constant polynomials)

(2) $B = \{p(x) \mid p(x) = 0 \text{ or } \deg p(x) \leq 1\}$ (i.e. the set of linear polynomials)

(3) $\mathbb{Z}[x]$

(4) $I = \{f(x)x^2 + g(x)(1+x^5) \mid f(x), g(x) \in \mathbb{Q}[x]\}$

Problem 5.72. Explain why $\mathbb{Z}_5$ is *not* a subring of $\mathbb{Z}$.

Examples of subrings of $\mathbb{C}$ include $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{Q}(i)$. These examples can, in turn, be used to create subrings of polynomial rings and matrix rings .

Theorem 5.73. If S is a subring of R, then

(1) $S[x]$ is a subring of $R[x]$, and

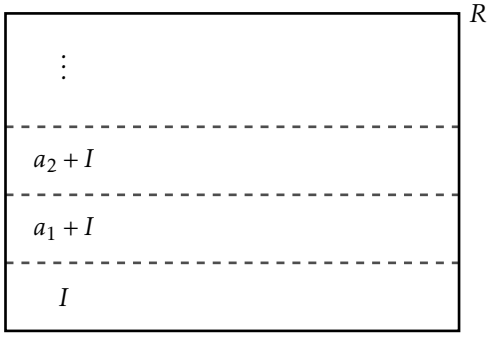
(2) $M_n(S)$ is a subring of $M_n(R)$.

5.5 Ideals and quotients

We now turn our attention to a special class of subrings known as *ideals*. The motivation for studying ideals of a ring is the same as for studying normal subgroups of a group: they give rise to quotients.

Let's explore the extra properties that a subring might need to ensure that the set of cosets can be given the structure of a ring. To start out, let's just assume that I is an *additive subgroup* of the ring R. Since $(R,+)$ is abelian, I is automatically a normal subgroup of $(R,+)$.

Now, let's consider the set of cosets of I in R, which we write as R/I. Recall that $R/I = \{a+I \mid a \in R\}$ where $a+I = \{a+y \mid y \in I\}$. Remember, that the set of cosets will partition R. One way to picture this is given below—it's followed by a couple of important properties about R/I from group theory.



Fact 5.74. Let I be an additive subgroup of a ring R. Then for all $a,b \in R$

(1) $a+I = b+I$ if and only if $a-b \in I$ if and only if $a \in b+I$; and

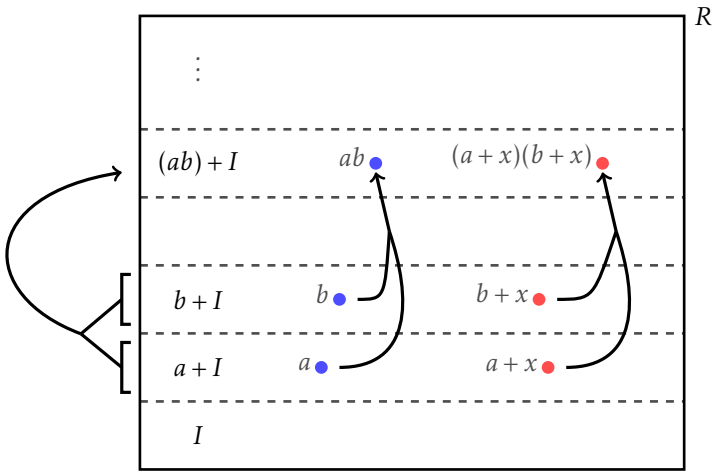
(2) either $(a+I)\cap(b+I) = \emptyset$ or $a+I = b+I$.

The goal is to understand when R/I can be given the structure of a ring. To do this, we need to decide how to add and multiply cosets. We would like to define $(a+I)+(b+I) = (a+b)+I$ and $(a+I)(b+I) = ab+I$, but the worry is that these operations are not well-defined. That is, the coset $a+I$ goes by many names (since $a+I = a'+I$ for every $a' \in a+I$), so we have to make sure that our definitions for the operations do not depend on which names we use for the cosets.

Now, as mentioned before, I is a *normal* subgroup of $(R,+)$, so we know that coset addition is well-defined. Let's see what we need for multiplication. Fix two arbitrary cosets $a+I$ and $b+I$. Then, for all $x,y \in I$, $a+I = (a+x)+I$ and $b+I = (b+y)+I$. Thus, in order for coset multiplication to be well-defined, we need to ensure that

$$ab+I = (a+x)(b+y)+I \text{ for all } a,b \in R \text{ and all } x,y \in I.$$

The desired picture is as follows.



After distributing, we have $ab+I = ab+ay+xb+xy+I$, which simplifies to $I = ay+xb+xy+I$. By Fact 5.74, we see that what we really need to ensure is that

$$ay+xb+xy \in I \text{ for all } a,b \in R \text{ and all } x,y \in I.$$

In particular, this has to be true when $a = b = 0$, which implies that $xy \in I$ for all $x,y \in I$, so I needs to be closed under multiplication, hence a subring. So, let's assume that I is a subring. Then, since $xy \in I$, $ay+xb+xy \in I$ reduces to $ay+xb \in I$. So, assuming that I is a subring, our previous condition becomes

$$ay+xb \in I \text{ for all } a,b \in R \text{ and all } x,y \in I.$$

Now, if a is arbitrary and $b = 0$, then we see that $ay \in I$ for all $a \in R$ and $y \in I$. Similarly, we find that $xb \in I$ for all $b \in R$ and $x \in I$. These are new properties. In words, I must be closed under (left and right) multiplication by elements from R.

In conclusion, if I is a subring that is closed under multiplication by elements from R then the above addition and multiplication for R/I is well-defined, making R/I a ring. The converse is also true. As such, we give these special subrings a special name.

Definition 5.75. Let R be a ring, and let $I \subseteq R$. Then I is an **ideal** of R if

(1) I is a subring; and

(2) for all $r \in R$ and all $a \in I$, both $ra \in I$ and $ar \in I$.

Let's also summarize our work above about defining operations on the quotient R/I.

Fact 5.76. Let R be a ring and let I be an ideal of R. Then R/I is a ring under the binary operations defined as follows. For all $a, b \in R$,

- $(a+I)+(b+I) = (a+b)+I$;
- $(a+I)(b+I) = (ab)+I$.

Problem 5.77. For each subset of the given ring, determine if the subset is an ideal, a subring, or neither.

	ideal	subring	neither
$\mathbb{Z} \subset \mathbb{Q}$			
$2\mathbb{Z} \subset \mathbb{Z}$			
$\{\text{odd integers}\} \subset \mathbb{Z}$			
$\{0,3,6,9\} \subset \mathbb{Z}_{12}$			
$\{p(x) \mid p(0) = 0\} \subset \mathbb{Q}[x]$			
$\{\text{constant polynomials}\} \subset \mathbb{Q}[x]$			

Problem 5.78. Let $I = \{(x^2+1)p(x) \mid p(x) \in \mathbb{Q}[x]\}$.

(1) Show that I is an ideal of $\mathbb{Q}[x]$.

(2) Write out 5 elements of I, each with a different degree.

(3) Explain why I contains polynomials of every degree larger than or equal to 2.

(4) Let $a(x) = x^4 + 3x + 5$. Write out 5 elements of the coset $a(x) + I$.

(5) Find some $b(x)$ in the coset $a(x) + I$ such that $\deg b(x) = 1$.

(6) Explain why $(x+I)^2 = -1 + I$ in the ring $\mathbb{Q}[x]/I$.

Problem 5.79. Let $I = \{(x^2+1)p(x) \mid p(x) \in \mathbb{Q}[x]\}$. Show that every coset $a(x) + I \in \mathbb{Q}[x]/I$ can be represented as $a(x) + I = r(x) + I$ for some $r(x) \in \mathbb{Q}[x]$ where $\deg r(x) < 2$ or $r(x) = 0$.

Problem 5.80. Recall that $6\mathbb{Z} = \{6z \mid z \in \mathbb{Z}\}$.

(1) Show that $6\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

(2) Show that $a + I = b + I$ if and only if $a \equiv_6 b$.

(3) Show that for all $a + I \in \mathbb{Z}/6\mathbb{Z}$, $a + I = r + I$ for some $r \in \mathbb{Z}$ with $0 \leq r < 6$.

Problem 5.81. Let $I = \{3q \mid q \in \mathbb{Q}\}$.

(1) Show that I is an ideal of $\mathbb{Q}$.

(2) Show that $1 \in I$, and use this to explain why $I = \mathbb{Q}$.

Let's record some observations from the previous problems.

Theorem 5.82. Let R be a commutative ring, and let $a \in R$. The set $I = \{ar \mid r \in R\}$ is an ideal of R.

In the previous theorem, the set $\{ar \mid r \in R\}$ should be thought of as the set of all multiples of a, and it is often denoted aR (as in $2\mathbb{Z}$).

Theorem 5.83. Assume R is a ring with unity. Let I be an ideal of R. If I contains a unit of R, then $I = R$.

Theorem 5.84. Let R be a ring. Then $\{0\}$ and R are ideals of R.

Theorem 5.85. Assume R is a commutative ring with $1 \neq 0$. Then R is a field if and only if the only ideals of R are $\{0\}$ and R.

Theorem 5.86. Let I be an ideal of a ring R.

(1) If R is a commutative ring, then R/I is commutative ring.

(2) If R is a ring with unity, then R/I is a ring with unity.

5.5.1 Generating ideals

As with groups and fields, we will want to generate subobjects from subsets. Generating ideals will be more useful for us that subrings, so we will only focus on ideals. Regarding notation, it is common to use (A) for the ideal generated by A instead of $\langle A \rangle$, and we will follow that convention. We begin with intersections.

Theorem 5.87. If I and J are ideals of a ring R, then $I \cap J$ is an ideal of R.

This can be generalized to arbitrary intersections.

Theorem 5.88. If $\mathcal{C}$ is any collection of ideals of a ring R, then the intersection of all ideals from $\mathcal{C}$ is again an ideal of R.

Now, if A is any subset of R, we can let $\mathcal{C}$ be the collection of all ideals containing A, to see that the intersection of all ideals containing A is an ideal, and it must be the smallest ideal containing A. This leads to the following definition.

Definition 5.89. Suppose A is a subset of a ring R. The **ideal of R generated by A**, denoted (A), is defined to be the intersection of all ideals containing A. An ideal generated by one element is a **principal ideal**. If $A = \{a_1, \ldots a_k\}$, we often write $(a_1, \ldots a_k)$ in place of (A).

Problem 5.90. Consider the ring $\mathbb{Z}$.

(1) Recall that $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$. Now use the definition of (2) to explain why $(2) \subseteq 2\mathbb{Z}$.

(2) Use that (2) is an ideal containing 2 to explain why $2\mathbb{Z} \subseteq (2)$. Conclude that $(2) = 2\mathbb{Z}$.

(3) Use that $(6, 10)$ is an ideal containing 6 and 10, to write down 5 elements of $(6, 10)$.

(4) Use the definition of $(6,10)$ to explain why $(6,10) \subseteq (2)$.

(5) Use Bézout's lemma (Fact 3.53) to show that $(2) \subseteq (6,10)$. Conclude that $(6,10) = (2)$.

(6) For arbitrary $m, n \in \mathbb{Z}$, do you think $(m,n) = (a)$ for some $a \in \mathbb{Z}$? Why or why not?

Let's work to abstract some of what we discovered in this problem.

Theorem 5.91. If R is a commutative ring with unity, then $(a) = \{ar \mid r \in R\}$.

Theorem 5.91 says that (a) is precisely the set of all multiples of a. Or, in other words, (a) is the set of all elements that are divisible by a. In particular, $(n) = n\mathbb{Z}$ in the ring $\mathbb{Z}$. But what about (m,n)?

Theorem 5.92. If $m, n \in \mathbb{Z}$ are nonzero, then $(m,n) = (d)$ where $d = \gcd(m,n)$.

In words, an ideal of $\mathbb{Z}$ that is generated by two elements can actually be generated by a single element. But more is true. The method for constructing the greatest common divisor for two elements can be easily adapted to show that *any* ideal of $\mathbb{Z}$ can be generated by a single element, which yields the following fact.

Fact 5.93. If I is any ideal of $\mathbb{Z}$, then I is a principle ideal. Moreover, if I is not the zero ideal, then $I = (d)$ if and only if d has the smallest possible absolute value among all nonzero elements of I.

In fact, a similar result holds in any ring with a division algorithm. Importantly for us, this applies to polynomial rings over fields. Let's prove the result for ideals generated by two elements and leave the general case as a fact.

Theorem 5.94. Let F be a field. If $a(x), b(x) \in F[x]$ are nonzero, then $(a(x), b(x)) = (d(x))$ where $d(x) = \gcd(a(x), b(x))$.

Fact 5.95. Let F be a field. If I is any ideal of $F[x]$, then I is a principle ideal. Moreover, if I is not the zero ideal, then $I = (d(x))$ if and only if $d(x)$ has the smallest possible degree among all nonzero elements of I.

Problem 5.96. Consider $a(x) = -x^2 - 3x + 10$, $b(x) = 2x^2 + 8x - 10$, and $c(x) = x^3 - 2$ in $\mathbb{Q}[x]$.

(1) Find a $d(x) \in \mathbb{Q}[x]$ such that $(a(x), b(x)) = (d(x))$. Is $(a(x), b(x)) = \mathbb{Q}[x]$? Explain.

(2) Find a $d'(x) \in \mathbb{Q}[x]$ such that $(a(x), c(x)) = (d'(x))$. Is $(a(x), c(x)) = \mathbb{Q}[x]$? Explain.

We saw that $\mathbb{Z}$ and $F[x]$ have a special property: every ideal is a principal ideal. This does not happen in every ring, as we'll see, so rings with this property get a special name.

Definition 5.97. An integral domain is called a **principal ideal domain (PID)** if every ideal is a principal ideal.

Thus, $\mathbb{Z}$ and $F[x]$ (for F a field) are examples of PIDs. Let's show that $\mathbb{Z}[x]$ is not.

Problem 5.98. Consider the set $I := \{2f(x) + xg(x) \mid f(x), g(x) \in \mathbb{Z}[x]\}$ in the ring $\mathbb{Z}[x]$. The set I is an ideal of $\mathbb{Z}[x]$; you do *not* need to prove this. Let's show that I is not principal. Towards a contradiction, assume that $I = (d(x))$ for some $d(x) \in \mathbb{Z}[x]$.

(1) Use the fact that $2 \in I = (d(x))$ to show that $d(x)$ is a constant polynomial and, moreover, that $d(x) = \pm 1, \pm 2$.

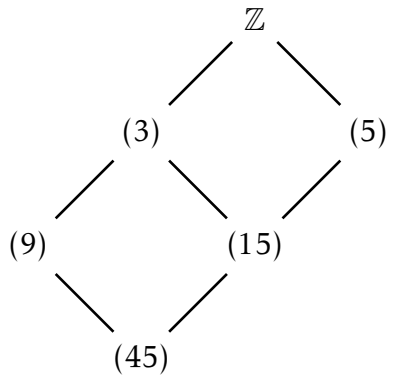
(2) Explain why every polynomial in I has a constant term that is even, and use this to show that in fact $d(x) = \pm 2$.

(3) Now, we also know that $x \in I = (d(x))$. Why is this a contradiction?

Let's return to $\mathbb{Z}$, and put together what we have learned about its ideals. First, by Fact 5.93, every ideal of $\mathbb{Z}$ is a principal ideal, so every ideal of $\mathbb{Z}$ is of the form (n) for some $n \in \mathbb{Z}$. Moreover, Theorem 5.91 tell us that (n) is just the set of all multiple of n, so $(n) = n\mathbb{Z}$. Thus, we know all of the ideals of $\mathbb{Z}$, and we know that they have a nice form. But how do they fit together? Let's explore this.

Theorem 5.99. Let R be a commutative ring with unity, and let $a, b \in R$. Then $(a) \subseteq (b)$ if and only if b divides a.

This theorem can be used to quickly draw portions of the lattice of ideals of $\mathbb{Z}$. For example, suppose we want to draw all of the ideals that contain the ideal (45). Every ideal is principal; suppose (n) is an ideal containing (45). By Theorem 5.99, n must divide 45. So, looking at all divisors of 45 (and noticing that $(-n) = (n)$), we get the following lattice.



Problem 5.100. Draw the the lattice of ideals of $\mathbb{Z}$ that contain the ideal (60).

Let's focus on the ideals of $\mathbb{Z}$ that are at the top of the lattice but below $\mathbb{Z}$.

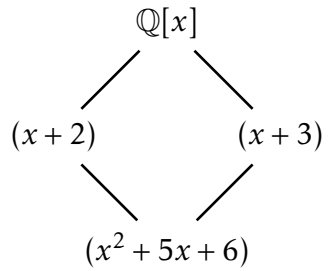
Definition 5.101. An ideal M of a ring R is called a **maximal ideal** if $M \neq R$ and the only ideals containing M are M and R.

Theorem 5.102. An ideal I of $\mathbb{Z}$ is maximal if and only if $I = (p)$ for some prime $p \in \mathbb{Z}$.

Since $F[x]$ (for F a field) is also PID, it's relatively easy to study the ideals of $F[x]$ too. As with $\mathbb{Z}$, every ideal is a principal ideal, and we can use Theorem 5.99 to see how they fit together. There is one preliminary result that will help us avoid redundancies.

Theorem 5.103. Let R be a commutative ring with unity. If $a \in R$ and $u \in U(R)$, then $(a) = (ua)$.

Now, suppose we want to find all of the ideals of $\mathbb{Q}[x]$ that contain the ideal (x^2+5x+6). As before, Theorem 5.99 tells us that we should look at divisors of $x^2 + 5x + 6$ in $\mathbb{Q}[x]$. Noting that $x^2 + 5x + 6 = (x + 2)(x + 3)$, we get the following.



Problem 5.104. Draw the the lattice of ideals of $\mathbb{Q}[x]$ that contain the ideal $(x^4 + x^2)$.

Theorem 5.105. Let F be a field. An ideal I of $F[x]$ is maximal if and only if $I = (p(x))$ for some irreducible polynomial $p(x) \in F[x]$.

5.6 Homomorphisms

As with groups, we use homomorphisms (and isomorphisms) to compare rings and fields.

Definition 5.106. Let R and S be rings. A map $\phi : R \to S$ is called a **ring homomorphism** if the following are true for all $a, b \in R$:

(1) $\phi(a + b) = \phi(a) + \phi(b)$;

(2) $\phi(ab) = \phi(a)\phi(b)$.

If ϕ is a bijection, then ϕ is called an **isomorphism**, in which case, we say that R and S are **isomorphic rings** and write $R \cong S$.

Problem 5.107. Determine which of the following are ring homomorphisms. Explain.

(1) $\phi : \mathbb{Z} \to 3\mathbb{Z}$ defined by $\phi(n) = 3n$

(2) $\alpha : \mathbb{C} \to \mathbb{C}$ defined by $\alpha(a + bi) = a - bi$

(3) $\beta : \mathbb{Z} \to \mathbb{Z}$ defined by $\beta(a) = a^3$

(4) $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ defined by $f(a) = a^3$

(5) $g : \mathbb{C} \to D_2(\mathbb{R})$ defined by $g(a + bi) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $D_2(\mathbb{R}) = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(6) $h : \mathbb{Q}[x] \to \mathbb{C}$ defined by $h(p(x)) = p(0)$

Problem 5.108. Which of the homomorphisms in Problem 5.107 were isomorphisms?

In Problem 5.107(6), we saw that "evaluating at zero" was a homomorphism from $\mathbb{Q}[x]$ to $\mathbb{C}$. At its core, this fact simply rests on how we add and multiply polynomials and does not require us to evaluate specifically at zero. For example, if $f(x), g(x) \in R[x]$ (for a ring R) and $c \in R$, then writing out $f(x) = a_0 + \cdots + a_m x^m$ and $g(x) = b_0 + \cdots + b_m x^m$ (with some a_i and b_j possible zero), we can show that $(f+g)(c) = f(c) + g(c)$. The analogous statement holds for multiplication too. We'll add in the details in the next theorem to see that "evaluating at c" is a homomorphism (whether or not $c = 0$).

Theorem 5.109 (Evaluation homomorphism)**.** Let R be a ring, and let $c \in R$. Define a function $\mathcal{E}_c : R[x] \to R$ by the rule $\mathcal{E}_c(p(x)) = p(c)$. Then $\mathcal{E}_c$ is a homomorphism.

Let's prove some general properties about homomorphisms.

Theorem 5.110. Let $\phi : R \to S$ be a ring homomorphism. Then

(1) $\phi(0) = 0$; and

(2) for all $a \in R$, $\phi(-a) = -\phi(a)$.

Theorem 5.111. Suppose that R is a ring with unity. Let $\phi : R \to S$ be a surjective ring homomorphism. Then

(1) $\phi(1) = 1$; and

(2) for all $a \in U(R)$, $\phi(a^{-1}) = (\phi(a))^{-1}$.

Theorem 5.112 (Composition of homomorphisms)**.** Let $\phi : R_1 \to R_2$ and $\psi : R_2 \to R_3$ be ring homomorphisms. Then $\psi \circ \phi : R_1 \to R_3$ is also a ring homomorphism.

There are two important sets attached to homomorphisms: the kernel and the image.

Definition 5.113. Let $\phi : R \to S$ be a ring homomorphism.

- The **kernel** of ϕ, denote $\ker \phi$, is $\ker \phi = \{a \in R \mid \phi(a) = 0\}$.
- The **image** of ϕ, denoted $\operatorname{im} \phi$ or $\phi(R)$, is $\operatorname{im} \phi = \{b \in S \mid b = \phi(a) \text{ for some } a \in R\}$.

Problem 5.114. Determine the kernel and image of each homomorphism.

(1) $\alpha : \mathbb{Q}[x] \to \mathbb{C}$ defined by $\alpha(p(x)) = p(0)$

(2) $\beta : \mathbb{Z} \to \mathbb{Z}_5$ defined by $\beta(n) = n \pmod 5$

Problem 5.115. Let I be an ideal of a ring R. Define a function $\pi : R \to R/I$ by $\pi(r) = r + I$.

(1) Show that π is a homomorphism.

(2) What is the kernel of π?

(3) What is the image of π?

As with groups, the kernel and image of a homomorphism have special properties and can be used to detect if a function is injective (one-to-one) or surjective (onto).

Theorem 5.116. Let $\phi : R \to S$ be a ring homomorphism. Then $\ker(\phi)$ is an ideal of R, and $\operatorname{im}\phi$ is a subring of S.

Theorem 5.117. Let $\phi : R \to S$ be a ring homomorphism. Then

(1) ϕ is injective if and only if $\ker\phi = \{0\}$, and

(2) ϕ is surjective if and only if $\operatorname{im}\phi = S$.

Problem 5.118. By Theorem 5.109, the "evaluation at i" function from $\mathbb{C}[x]$ to $\mathbb{C}$ is a homomorphism. We can restrict the domain to $\mathbb{Q}[x]$ to obtain a homomorphism from $\mathcal{E}_i : \mathbb{Q}[x] \to \mathbb{C}$ defined by $\mathcal{E}_i(p(x)) = p(i)$, where i denotes the usual complex number.

(1) Show that $x^2 + 1$ is in $\ker\mathcal{E}_i$.

(2) By Fact 5.95, $\ker\mathcal{E}_i = (m(x))$ for some $m(x) \in \mathbb{Q}[x]$. Use the fact that $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$, to show that $m(x) = a(x^2 + 1)$ for some nonzero constant $a \in \mathbb{Q}$.

(3) Explain why $\ker\mathcal{E}_i = (x^2 + 1)$ (where $(x^2 + 1)$ is the ideal generated by $x^2 + 1$).

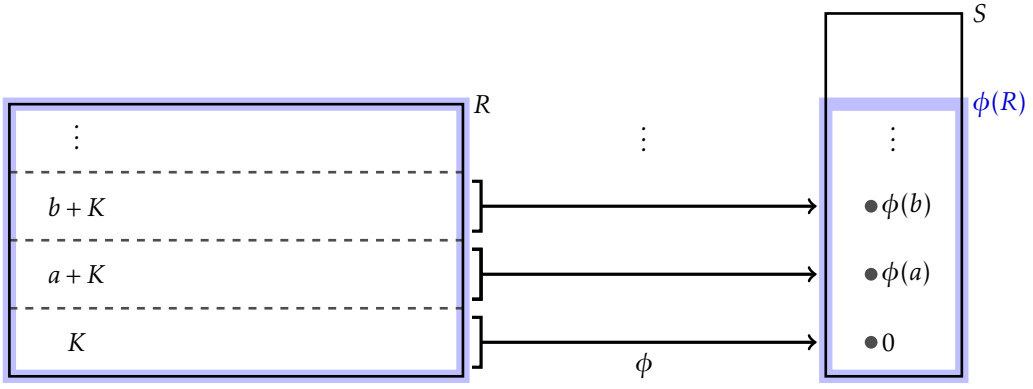
(4) Explain why all of $\mathbb{Q}$ as well as i are contained in the image of $\mathcal{E}_i$.

(5) Explain why the image of $\mathcal{E}_i$ is contained in $\mathbb{Q}(i)$.

5.6.1 First Isomorphism Theorem

The proof of the First Isomorphism Theorem for rings is essentially the same as for groups. Suppose we have a ring homomorphism $\phi : R \to S$. In order for ϕ to be an isomorphism, it must also be one-to-one and onto, but it may very well not be. However, with some slight adjustments, we can modify it to be both one-to-one and onto.

First, to make ϕ surjective, we change the codomain from S to $\phi(R)$, i.e. the image of ϕ. Technically, we have a different function now, but we will still call it ϕ. Anyway, $\phi : R \to \phi(R)$ is now surjective (by the definition of the image). But how do we make it one-to-one? By Theorem 5.117, we need the kernel to be trivial. To accomplish, we work in quotient ring $R/\ker\phi$. This has the effect of collapsing all elements in the kernel to a single element (the coset $\ker\phi$) that maps to 0. Setting $K := \ker\phi$, the picture is like this.



Theorem 5.119 (First Isomorphism Theorem for Rings)**.** If $\phi : R \to S$ is a ring homomorphism, then $R/\ker\phi \cong \operatorname{im}\phi$.

Problem 5.120. Use the First Isomorphism Theorem together Problem 5.114 to prove each of the following.

(1) $\mathbb{Q}[x]/(x) \cong \mathbb{Q}$ (where (x) is the ideal generated by x)

(2) $\mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}_5$

We now work towards a very nice and useable criterion to determine when a quotient ring is actually a field. Our approach will rely on Theorem 5.85, so we need to be able determine the relationship between ideals of R and ideals of the quotient R/I. We will investigate this via homomorphisms, which we can then apply to R/I using Problem 5.115.

Theorem 5.121. Let $\phi : R \to S$ be a surjective ring homomorphism. If I is an ideal of R, then $\phi(I) = \{b \in S \mid b = \phi(a) \text{ for some } a \in I\}$ (called the **image** of I) is an ideal of S.

Theorem 5.122. Let $\phi : R \to S$ be a ring homomorphism. If J is an ideal of S, then $\phi^{-1}(J) := \{a \in R \mid \phi(a) \in J\}$ (called the **inverse image** of J) is an ideal of R and, moreover, $\ker\phi \subseteq \phi^{-1}(J)$.

Combining Theorems 5.121 and 5.122 with Theorem 5.85 yields the following result.

Theorem 5.123. Let R be a commutative ring with $1 \neq 0$. Suppose $\phi : R \to S$ is a surjective ring homomorphism. Then $\ker\phi$ is maximal ideal of R if and only if S is field.

In light of Problem 5.115, our desired criterion for when a quotient ring is a field is a relatively quick consequence of the previous result.

Theorem 5.124. Let R be a commutative ring with $1 \neq 0$. Then I is maximal ideal of R if and only if R/I is field.

Problem 5.125. Let's revisit Problem 5.118.

(1) Use Theorems 5.105 and 5.124 to explain why $\mathbb{Q}[x]/(x^2+1)$ is a field.

(2) Use Theorem 5.119 (and Problem 5.118) to explain why the image of $\mathcal{E}_i : \mathbb{Q}[x] \to \mathbb{C}$ is a field that contains $\mathbb{Q}$ and i.

(3) Explain why the image of $\mathcal{E}_i$ must be $\mathbb{Q}(i)$, and conclude that $\mathbb{Q}[x]/(x^2+1) \cong \mathbb{Q}(i)$.

Let's end this section (as well as this decidedly long chapter!) by mentioning that there is an analog of Theorem 5.124 characterizing when R/I is an integral domain. The condition is that I is a so-called *prime* ideal. The result is tangential to our story, so we'll leave it for another time (see Wikipedia or most any other book on abstract algebra).

Chapter 6

Algebraic extension fields

Our goal remains: to show that there exist polynomials that are *not* solvable by radicals over $\mathbb{Q}$. In Chapter 4, we finally succeeded in properly formulating the notion of solvability by radicals, which we did in the language of field extensions. Additionally, we were able to catalog many polynomials that we are certain are solvable by radicals. Then, in Chapter 5, we took a much closer look a polynomials, ultimately building a significant amount of language and theory to analyze polynomial rings over fields. The conclusion of the chapter hinted at the tight connection between polynomial rings and field extensions where we saw that $\mathbb{Q}[x]/(x^2+1) \cong \mathbb{Q}(i)$. In this chapter, we will clarify this connection and exploit it to significantly deepen our understanding of extension fields of the form $\mathbb{Q}(\alpha)$ where α is a root of some polynomial in $\mathbb{Q}[x]$.

6.1 Algebraic elements

Recall that a polynomial in $F[x]$ is solvable by radicals over F if all of the roots of the polynomial lie in some radical extension of F. Our focus is on roots of polynomials, and the next definition gives us some language to highlight this.

Definition 6.1. Let F be a subfield of E. An element $\alpha \in E$ is **algebraic** over F if $p(\alpha) = 0$ for some nonzero $p(x) \in F[x]$. If α is not algebraic over F, then it is said to be **transendental** over F.

To show an element α is algebraic over F, we need only produce a polynomial *with coefficients in* F for which α is a root. For example, the complex number $\sqrt{2}$ is algebraic over $\mathbb{Q}$ because $\sqrt{2}$ is a root of $p(x) = x^2 - 2$ and $p(x) \in \mathbb{Q}[x]$. Also, π is algebraic over $\mathbb{R}$ because π is a root of $q(x) = x - \pi$ and $q(x) \in \mathbb{R}[x]$. However, it is *much* harder to show that π is *not* algebraic over $\mathbb{Q}$ (so π is transcendental over $\mathbb{Q}$). Incidentally, a set-theoretic argument shows that almost all elements of $\mathbb{C}$ are transcendental over $\mathbb{Q}$; nevertheless, we will focus on algebraic elements.

Problem 6.2. Show that each of the following are algebraic over $\mathbb{Q}$: 11, $\sqrt[3]{11}$, ζ_{11}, and i.

Problem 6.3. Suppose that $\gamma \in \mathbb{C}$ and $\gamma^5 = 2\gamma^2 - 7$. Explain why γ is algebraic over $\mathbb{Q}$.

Problem 6.4. Let $\alpha = \sqrt{2} + i$. Show that α is algebraic over $\mathbb{Q}$.

Now, an algebraic element over F is a root of *some* nonzero polynomial over F, but such an element will be a root of lots of polynomials. For example, since $\sqrt{2}$ is a root of $p(x) = x^2 - 2$, we find that for *every* $q(x) \in \mathbb{Q}[x]$, $\sqrt{2}$ is a root of $q(x)p(x)$ because $q(\sqrt{2})p(\sqrt{2}) = q(\sqrt{2})p(0) = 0$. The polynomial $x^2 - 2$ is special because it is a polynomial *of smallest degree* for which $\sqrt{2}$ is a root. In order to formalize this observation, we need to weave together several results from the last chapter.

Let F be a subfield of E, and suppose that $\alpha \in E$ is algebraic over F. Then α is the root of *some* nonzero $p(x) \in F[x]$. Let's look at the set of *all* polynomials for which α is a root: $I = \{p(x) \in F[x] \mid p(\alpha) = 0\}$. By Theorem 5.109, "evaluation at α" gives rise to a homomorphism $\mathcal{E}_\alpha : F[x] \to E[x]$, and notice that I is precisely the kernel of $\mathcal{E}_\alpha$. Thus, by Theorem 5.116, I is an ideal of $F[x]$, and since $F[x]$ is a PID, it must be that $I = (m(x))$ for some $m(x) \in F[x]$.

Now, $m(x)$ is nonzero since I contains some nonzero polynomial, and this also implies that $m(x)$ is not a constant polynomial since the only constant polynomial that has roots is the zero polynomial. Moreover, if $m(x)$ is not monic, we can make it monic by multiplying by the inverse of the leading coefficient and the result will still generate I by Theorem 5.103. So, we may assume that $I = (m(x))$ with $m(x)$ nonconstant and monic.

Also, notice that by Theorem 5.91, $m(x)$ divides every polynomial in I. So if $I = (n(x))$ for some other monic polynomial $n(x)$, then $m(x)$ and $n(x)$ would divide each other. In other words, $m(x) = a(x)n(x) = a(x)b(x)m(x)$ for some polynomials $a(x), b(x) \in F[x]$. Considering the degree of both sides of $m(x) = a(x)b(x)m(x)$, we see that $a(x)$ and $b(x)$ must be constant polynomials. But since $m(x) = a(x)n(x)$ with $m(x)$ and $n(x)$ both monic, the only conclusion is that $a(x) = 1$, so $m(x) = n(x)$.

In summary, $I = \{p(x) \in F[x] \mid p(\alpha) = 0\}$ is an ideal, and there is a unique nonconstant monic polynomial $m(x)$ that generates I. In fact, more is true.

Lemma 6.5. Let F be a subfield of E, and suppose that $\alpha \in E$ is algebraic over F. Let $I = \{p(x) \in F[x] \mid p(\alpha) = 0\}$, and suppose that $I = (m(x))$ for some nonconstant $m(x) \in F[x]$. Then $m(x)$ is irreducible.

Combining Lemma 6.5 with our previous discussion, we arrive at the following fact.

Fact 6.6. Let F be a subfield of E, and suppose that $\alpha \in E$ is algebraic over F. Then there is a unique irreducible monic polynomial $m(x) \in F[x]$ such that $m(\alpha) = 0$. Moreover, if $p(x) \in F[x]$ and $p(\alpha) = 0$, then $m(x)$ divides $p(x)$.

The polynomial $m(x)$ from Fact 6.6 gets a special name.

Definition 6.7. Suppose that $\alpha \in E$ is algebraic over F. The unique irreducible monic polynomial $m(x) \in F[x]$ such that $m(\alpha) = 0$ is called the **minimal polynomial** of α over F. The **degree** of α over F is defined to be the degree of the minimal polynomial of α over F.

Theorem 6.8. Let F be a subfield of E. Suppose that $\alpha \in E$ is algebraic over F, and let $m(x)$ be the minimal polynomial of α over F. If $V = \{p(x) \in F[x] \mid p(\alpha) = 0\}$ (i.e the set of all polynomials that vanish at α), then $V = (m(x))$.

We will see shortly that the minimal polynomial of α over F is key to understanding the field extension $F(\alpha)$. But how do we find the minimal polynomial of α over F? The first step is to find *any* monic polynomial $p(x) \in F[x]$ for which $p(\alpha) = 0$ (which also verifies that α is algebraic over F). If we can show that $p(x)$ is irreducible, then $p(x)$ is the minimal polynomial, and we're done. Otherwise, we factor $p(x)$ into irreducible polynomials in $F[x]$, and by Fact 6.6, the minimal polynomial will be whichever one of the factors has α as a root. Let's test this out with some examples.

Problem 6.9. Explain why the minimal polynomial for ζ_3 over $\mathbb{Q}$ is *not* $x^3 - 1$. Find the minimal polynomial ζ_3 over $\mathbb{Q}$, and determine the degree of ζ_3 over $\mathbb{Q}$?

Problem 6.10. The polynomial $p(x) = x^3 - 11$ has three roots in $\mathbb{C}$.

(1) Find the degree over $\mathbb{Q}$ of each of the three roots.

(2) Find the degree over $\mathbb{R}$ of each of the three roots.

Problem 6.11. The polynomial $p(x) = x^5 - 2x^3 - 3x$ has five roots in $\mathbb{C}$. Find the minimal polynomial over $\mathbb{Q}$ of each of the five roots.

Problem 6.12. Let $z = a + bi$ with $a, b \in \mathbb{Q}$, and define $p(x) = (x - z)(x - \overline{z})$. Prove that $p(x) \in \mathbb{Q}[x]$, and then use this to find the minimal polynomial of $2 + i$ over $\mathbb{Q}$.

Problem 6.13. Find the minimal polynomial of $3 - \sqrt{5}$ over $\mathbb{Q}$.

6.1.1 Describing elements of $F(\alpha)$

In Chapter 3, we explored extension fields of the form $F(\alpha)$ where α was chosen from some larger field E. However, the definition of $F(\alpha)$ was abstract and often hard to work with. For example, $\mathbb{Q}(i)$ is defined to be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and i, but this doesn't tell us much about what the elements of $\mathbb{Q}(i)$ actually look like. Nevertheless, in Problem 3.65, we were able show that $\mathbb{Q}(i)$ is precisely the set of elements of the form $a + bi$ with $a, b \in \mathbb{Q}$, and we also succeeded in showing that $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$.

However, we noticed that describing $\mathbb{Q}(\alpha)$ is not always so easy since, for example, if $\alpha = \sqrt{2} + i$, then $\mathbb{Q}(\alpha) \neq \{a + b\alpha \mid a, b \in \mathbb{Q}\}$. That said, we've learned a lot since Chapter 3, so let's take another go at trying to describe fields like $\mathbb{Q}(\alpha)$.

Remember that (for $\alpha = \sqrt{2} + i$) we were able to show that $\{a + b\alpha \mid a, b \in \mathbb{Q}\} \subseteq \mathbb{Q}(\alpha)$, but the reverse containment did not hold. And, the reason the reverse containment didn't hold was because $\{a + b\alpha \mid a, b \in \mathbb{Q}\}$ is not a field, which can be seen fairly easily since $\alpha^2 \notin \{a + b\alpha \mid a, b \in \mathbb{Q}\}$ (so it's not closed under multiplication).

So why not add in α^2 and consider something like $\{a+b\alpha+c\alpha^2 \mid a, b, c \in \mathbb{Q}\}$? Remembering that $\mathbb{Q}(\alpha)$ is a field containg $\mathbb{Q}$ and α (and is closed under addition and multiplication), we see again that $\{a+b\alpha+c\alpha^2 \mid a, b, c \in \mathbb{Q}\} \subseteq \mathbb{Q}(\alpha)$. So maybe $\{a+b\alpha+c\alpha^2 \mid a, b, c \in \mathbb{Q}\} = \mathbb{Q}(\alpha)$. Or, maybe we need to look at $\{a + b\alpha + c\alpha^2 + d\alpha^3 \mid a, b, c, d \in \mathbb{Q}\}$. These are all good ideas since $a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots + a_n\alpha^n \in \mathbb{Q}(\alpha)$ whenever $a_0, a_1, a_2, a_3, \ldots, a_n \in \mathbb{Q}$. Let's formalize this.

Theorem 6.14. Let F be a subfield of E, and let $\alpha \in E$. If $p(x) \in F[x]$, then $p(\alpha) \in F(\alpha)$.

Problem 6.15. Let $p(x) = 2x^3 + 7x^2 - \frac{1}{2}$. We know that $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$ by Problem 3.67. Verify directly that $p(\sqrt{5}) \in \mathbb{Q}(\sqrt{5})$ by writing $p(\sqrt{5})$ in the form $a + b\sqrt{5}$ for some $a, b \in \mathbb{Q}$.

We are studying the elements we get when we plug α into polynomials. Let's frame this in terms of the evaluation homomorphism introduced in Theorem 5.109, like we did at the beginning of this chapter. The next theorem generalizes Problem 5.118.

Theorem 6.16. Let F be a subfield of E. Suppose that $\alpha \in E$ is algebraic over F, and let $m(x)$ be the minimal polynomial of α over F. If $\mathcal{E}_\alpha : F[x] \to E$ is the homomorphism defined by $\mathcal{E}_\alpha(p(x)) = p(\alpha)$, then

(1) $\ker(\mathcal{E}_\alpha) = (m(x))$, and

(2) $\operatorname{im}(\mathcal{E}_\alpha) \subseteq F(\alpha)$ and $F \cup \{\alpha\} \subset \operatorname{im}(\mathcal{E}_\alpha)$.

Using Theorems 5.105 and 5.124 (and the First Isomorphism Theorem for Rings), we can deduce that the image of $\mathcal{E}_\alpha$ is a field. But then, if we can verify that the image of $\mathcal{E}_\alpha$ contains F and α, we can conclude that the image of $\mathcal{E}_\alpha$ is actually *equal* to $F(\alpha)$ (by the definition of $F(\alpha)$). Let's put this together.

Theorem 6.17. Let F be a subfield of E. Suppose that $\alpha \in E$ is algebraic over F, and let $m(x)$ be the minimal polynomial of α over F. Then $F(\alpha) \cong F[x]/(m(x))$.

Notice that we still haven't succeeded in providing a nice description $F(\alpha)$, but we will if we can find a nice description of $F[x]/(m(x))$.

Lemma 6.18. Let F be a subfield of E, and suppose that $\alpha \in E$ is algebraic over F. Let $m(x)$ be the minimal polynomial of α over F, and let n be the degree of α over F. If $a(x) + (m(x)) \in F[x]/(m(x))$, then $a(x) + (m(x)) = r(x) + (m(x))$ where $r(x)$ is the remainder obtained when dividing $a(x)$ by $m(x)$. Consequently,

$$F[x]/(m(x)) = \{r(x) + (m(x)) \mid \deg r(x) < n \text{ or } r(x) = 0\}.$$

Tying together all of our work, we finally arrive at our desired description of $F(\alpha)$.

Theorem 6.19. Let F be a subfield of E. Suppose that $\alpha \in E$ is algebraic over F, and let n be the degree of α over F. Then

$$F(\alpha) = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in F\}.$$

We had to work hard for Theorem 6.19, but using it is fairly easy. For example, suppose we want to describe $\mathbb{Q}(\sqrt[3]{2})$. We first need to know the degree of $\sqrt[3]{2}$ over $\mathbb{Q}$. Of course, $\sqrt[3]{2}$ is a root of $m(x) = x^3 - 2$, and $m(x)$ is irreducible by Theorems 3.26 and 5.64. Thus, $m(x) = x^3 - 2$ is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$, so $\sqrt[3]{2}$ has degree 3 over $\mathbb{Q}$. Now we apply Theorem 6.19 to find that $\mathbb{Q}(\sqrt[3]{2}) = \{a_0 + a_1\sqrt[3]{2} + a_2(\sqrt[3]{2})^2 \mid a_0, a_1, a_2 \in \mathbb{Q}\}$.

Problem 6.20. Use Theorem 6.19 to describe $\mathbb{Q}(\zeta_3)$.

Problem 6.21. Let $p(x) = x^3 - x + 1 \in \mathbb{Z}_3[x]$, and let r be a root of $p(x)$.

(1) Prove that $p(x)$ is irreducible in $\mathbb{Z}_3[x]$.

(2) Use Theorem 6.19 to describe $\mathbb{Z}_3(r)$.

(3) How many elements are in the field $\mathbb{Z}_3(r)$?

Problem 6.22. Let $\alpha = \sqrt{2} + i$. In Problem 6.4, we saw that α is algebraic over $\mathbb{Q}$ since α is a root of $x^4 - 2x^2 + 9 \in \mathbb{Q}[x]$. Let $m(x)$ denote the minimal polynomial for α over $\mathbb{Q}$.

(1) Use Theorem 6.19 (and Problem 3.65) to explain why the degree of $m(x)$ isn't 1 or 2.

(2) Use Fact 6.6 to explain why $m(x)$ is a factor of $x^4 - 2x^2 + 9$.

(3) Explain why the degree of $m(x)$ isn't 3.

(4) Explain why $m(x) = x^4 - 2x^2 + 9$.

(5) Use Theorem 6.19 to describe $\mathbb{Q}(\sqrt{2} + i)$.

The next problem highlights how we use the minimal polynomial for α to compute (or rather, simplify) in $F(\alpha)$.

Problem 6.23. The polynomial $p(x) = x^5 + 2x + 2$ is irreducible in $\mathbb{Q}[x]$ (you do not need to prove this). Let s be a root of $p(x)$. By Theorem 6.19, every element of $\mathbb{Q}(s)$ can be written in the form $a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4$ for some $a_0, a_1, a_2, a_3, a_4 \in \mathbb{Q}$.

(1) Use the fact that $p(s) = 0$ to write s^5 in the form $a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4$.

(2) Rewrite each of the following elements of $\mathbb{Q}(\alpha)$ in the form $a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4$.

   (a) $(s^3 + 2)(s^3 + 3s)$

   (b) $3s^4(2 + s^3)(5 - s + s^2)$

Problem 6.24. Let's try to describe $\mathbb{Q}(\sqrt{2}, i)$. Notice that $\mathbb{Q}(\sqrt{2}, i) = \big(\mathbb{Q}(\sqrt{2})\big)(i)$. In words, the field obtained by adjoining $\sqrt{2}$ and i at the same time is equal to the field obtained by first adjoining $\sqrt{2}$ and then adjoining i to the result.

(1) Use Theorem 6.19 to describe $\mathbb{Q}(\sqrt{2})$.

(2) Use Theorem 5.64 to explain why $x^2 + 1$ is the minimal polynomial of i over $\mathbb{Q}(\sqrt{2})$.

(3) Use Theorem 6.19 to describe $\big(\mathbb{Q}(\sqrt{2})\big)(i)$, which is equal to $\mathbb{Q}(\sqrt{2}, i)$.

(4) How does your description of $\mathbb{Q}(\sqrt{2}, i)$ compare to that for $\mathbb{Q}(\sqrt{2}+i)$ in Problem 6.22?

6.1.2 Eisenstein's irreducibility criterion

Theorem 6.19 provides a nice description of $F(\alpha)$ when α is algebraic over F. However, the description rests on us knowing the minimal polynomial of α over F (or at least its degree), which in turn rests on us being able to determine when polynomials are irreducible. We now introduce a useful irreducibility criterion for polynomials in $\mathbb{Q}[x]$.

Fact 6.25 (Eisenstein's Irreducibility Criterion (EIC))**.** Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ with all $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Suppose that there is some *prime* number $p \in \mathbb{Z}$ such that all of the following conditions are met:

(1) p does *not* divide a_n,

(2) p does divide a_i for all $i < n$, and

(3) p^2 does *not* divide a_0.

Then $p(x)$ is irreducible in $\mathbb{Q}[x]$.

The proof of the EIC is interesting and not too difficult, but as it is more number-theoretic than algebraic, we will leave it for another time. The idea is to take the polynomial $p(x)$ whose coefficients are all integers and consider it as a polynomial in $\mathbb{Z}_p[x]$ by reducing all of the coefficients modulo p. Details can be found in other books or on Wikipedia.

Problem 6.26. Use the EIC to show that each of the following polynomials are irreducible in $\mathbb{Q}[x]$. What did you choose as your prime p? Were there other choices?

(1) $f(x) = 7x^4 + 6x^3 + 12x - 30$

(2) $g(x) = x^8 - 6x^5 - 30x^3 + 12$

Problem 6.27. Let α be a root of $p(x) = x^5 + 5x^4 - 5$. (You don't need to compute α!)

(1) Prove that $p(x)$ is irreducible over $\mathbb{Q}$.

(2) Use Theorem 6.19 to describe $\mathbb{Q}(\alpha)$.

Problem 6.28. Consider the polynomial $f(x) = x^7 - \frac{5}{2}$.

(1) Explain why the EIC does *not* apply to $f(x)$.

(2) Prove that if $f(x)$ is reducible in $\mathbb{Q}[x]$, then so is $g(x) = 2x^7 - 5$.

(3) Use the EIC to show that $g(x)$ is irreducible, and conclude that $f(x)$ is irreducible.

Let's try to use the EIC to determine the minimal polynomial of some special elements, namely the ζ_n. We know that ζ_n is algebraic over $\mathbb{Q}$ because it is a root of $x^n - 1$. However, $x^n - 1$ can not be the minimal polynomial since it has $x - 1$ as a factor. But, we've observed a few times now that

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1).$$

Set $\Psi_n(x) = x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1$. Since $(\zeta_n)^n - 1 = 0$, it must be $\zeta_n - 1 = 0$ or $\Psi_n(\zeta_n) = 0$. Of course, $\zeta_n - 1 \neq 0$, so ζ_n is a root of $\Psi_n(x)$. Could $\Psi_n(x)$ be the minimal polynomial?

Problem 6.29. Show that $\Psi_4(x) = x^3 + x^2 + x + 1$ is *not* the minimal polynomial for ζ_4 over $\mathbb{Q}$. What is the minimal polynomial?

So, we see that $\Psi_n(x)$ is not always the minimal polynomial for ζ_n, as it fails for $n = 4$. But what about for $n = 5$? Could $\Psi_5(x) = x^4 + x^3 + x^2 + x + 1$ be the minimal polynomial for ζ_5? We need to determine if $\Psi_5(x)$ is irreducible, but the EIC does not apply because there is no prime that divides the non-leading coefficients. Let's see if we can transform $\Psi_5(x)$ into a related polynomial for which the EIC will apply.

Theorem 6.30. Let F be a field, and let $p(x) \in F[x]$. If $p(x)$ is reducible in $F[x]$, then $p(x+1)$ is also reducible in $F[x]$.

Problem 6.31. Consider the polynomial $\Psi_5(x) = x^4 + x^3 + x^2 + x + 1$.

(1) Compute $\Psi_5(x+1)$, and write it in the form $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. Consider using the fact that $x^5 - 1 = (x-1)\Psi_5(x)$ to help with the computation.

(2) Use the EIC to show that $\Psi_5(x+1)$ is irreducible.

(3) Explain why $\Psi_5(x)$ is the minimal polynomial for ζ_5 over $\mathbb{Q}$.

(4) Use Theorem 6.19 to describe $\mathbb{Q}(\zeta_5)$.

So, why was $\Psi_5(x)$ irreducible while $\Psi_4(x)$ was not? To address the general case, we could use a similar approach as in Problem 6.31 to analyze $\Psi_n(x+1)$ (using the Binomial Theorem to simplify the expression).

Problem 6.32. Make a conjecture as to when $\Psi_n(x) = x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1$ is the minimal polynomial for ζ_n over $\mathbb{Q}$. That is, try to fill in the blank: "$\Psi_n(x)$ is the minimal polynomial for ζ_n over $\mathbb{Q}$ if and only if ___(something about n)___." If you have the time, try to prove your conjecture.

To learn more about the minimal polynomial for ζ_n (for any n), try looking up "cyclotomic polynomials" on Wikipedia.

6.2 Extension fields as vector spaces

When we work with complex numbers, we often write them in the form $a + bi$ for $a, b \in \mathbb{R}$. So, every complex number can be described using two real numbers: a and b. Moreover, each complex number is described by a *unique* choice of a and b. This allows us to associate a complex number $a + bi$ with a vector in $\mathbb{R}^2$ via

$$a + bi \mapsto \begin{bmatrix} a \\ b \end{bmatrix}.$$

Additionally, adding two complex numbers $a+bi$ and $c+di$ corresponds to adding the two associated vectors $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$, and multiplying $a + bi$ by a *real* number r corresponds to multiplying the vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ by the scalar r. But, we should be careful with multiplication

because multiplying $a+bi$ and $c+di$ does not correspond to multiplying the entries in the corresponding vectors—do you see why? Nevertheless, we find that $\mathbb{C}$ is a vector space over $\mathbb{R}$, and the fact that every complex number can be described by a unique pair of real numbers is expressing that $\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$.

Can we do this for other fields? In Problem 6.27, we saw that if α is a root of x^5+5x^4-5, then each element $y \in \mathbb{Q}(\alpha)$ is of the form $y = a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4$ for $a, b, c, d, e \in \mathbb{Q}$. If we knew that there was a *unique* choice of a, b, c, d, e for each y, then like before we could associate each element of $\mathbb{Q}(\alpha)$ with a vector in $\mathbb{Q}^5$ via

$$a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4 \mapsto \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}.$$

We'll see that these observations generalize to every extension field E of a field F. Let's start by properly defining vector spaces. Note that, in the formal definition below, scalar multiplication by a number c is being written as $\lambda_c v$, instead of just cv, but as we continue on, we will return to writing simply cv.

Definition 6.33. Let F be a field. A **vector space over** F is a structure $(V, +, \{\lambda_c \mid c \in F\})$ consisting of a set V together with a binary operation $+$ and a unary operation λ_c for each $c \in F$ (which we call *addition* and *scalar multiplication by* c) such that for some element $0 \in V$ the following axioms hold.

- **Addition Axioms:** Addition is associative and commutative; the element 0 is an additive identity; every $x \in F$ has an additive inverse with respect to 0, denoted $-x$.
- **Distributivity Axioms:** For all $u, v \in V$ and all $c \in F$, $\lambda_c(u+v) = \lambda_c u + \lambda_c v$.
- **Compatibility Axioms:** For all $v \in V$ and all $c, d \in F$,
  (1) $\lambda_{c+d} v = \lambda_c v + \lambda_d v$,
  (2) $\lambda_c(\lambda_d v) = \lambda_{cd} v$, and
  (3) $\lambda_1 v = v$.

Notice that the distributivity axiom can also be expressed by saying that each λ_c is a homomorphism from $(V, +)$ to $(V, +)$. The compatibility axioms can also be framed in terms of a homomorphism, but we will not explore that here.

As mentioned above, we tend to omit writing the λ, so for example, the distributivity axiom will be often written as $c(u+v) = cu + cv$.

Theorem 6.34. Let E be an extension field of F. Then E is a vector space over F where vector addition for E is just the usual field addition for E and scalar multiplication by an element $c \in F$ is just the usual field multiplication by c.

6.2.1 Degree of a field extension

We can now explore core concepts of linear algebra like linear independence, spanning sets, bases, dimension, and linear transformations. Basic results from a first course on linear algebra (over $\mathbb{R}$) transfer to our more general setting, and you should feel free to use them here.

Definition 6.35. Let V be a vector space over a field F, and let $v_1, \ldots, v_n \in V$. Then

- $v_1, \ldots, v_n$ are **linearly independent** if for all $c_1, \ldots, c_n \in F$, $c_1 v_1 + \cdots + c_n v_n = 0$ implies that $c_1 = \cdots = c_n = 0$;
- $v_1, \ldots, v_n$ **span** V if for all $w \in V$, there exist $c_1, \ldots, c_n \in F$ such that $c_1 v_1 + \cdots + c_n v_n = w$;
- $v_1, \ldots, v_n$ form a **basis** for V if they are linearly independent and span V.

Notice that we have defined linear independence and span only for finite sets of vectors, but the concepts can also be defined for infinite sets of vectors in a similar way. The importance of bases is more-or-less summarized in the following fact.

Fact 6.36. If V is a vector space, then all bases have the same cardinality ("size"). If $\mathcal{B}$ is any basis for V, then every element of V can be expressed as a linear combination of vectors in $\mathcal{B}$ in one and only one way.

This leads to the notion of dimension, which when considering field extensions (as in Theorem 6.34) we will refer to as the degree of the extension.

Definition 6.37. The **dimension** of a vector space V over a field F, denoted $\dim V$, is the cardinality of any basis for V.

Definition 6.38. If E is an extension field of F, then the dimension of E as a vector space over F is called the **degree of** E **over** F, denoted $[E : F]$. If $[E : F]$ is finite, we say that E is **finite dimensional** over F.

Problem 6.39. Let α be a root of $p(x) = x^5 + 5x^4 - 5$. In Problem 6.27, we saw that $p(x)$ is the minimal polynomial for α over $\mathbb{Q}$ and that $\mathbb{Q}(\alpha) = \{a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4 \mid a, b, c, d, e \in \mathbb{Q}\}$. By Theorem 6.34, $\mathbb{Q}(\alpha)$ is a vector space over $\mathbb{Q}$.

(1) Explain why the elements $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ span $\mathbb{Q}(\alpha)$ as a vector space over $\mathbb{Q}$.

(2) Assume that $c_0 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + c_4\alpha^4 = 0$ for some $c_0, \ldots, c_4 \in \mathbb{Q}$. Show that if at least one of $c_0, \ldots, c_4$ is nonzero, then α is a root of some *nonzero* polynomial that has degree at most 4.

(3) Explain why the elements $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ are linearly independent over $\mathbb{Q}$.

(4) What is the degree of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$? That is, find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

Generalizing our work in Problem 6.39, we obtain a crucial theorem.

Theorem 6.40. Let F be a subfield of E. Suppose that $\alpha \in E$ is algebraic over F, and let n be the degree of α over F. Then $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ is a basis for $F(\alpha)$ over F, and $[F(\alpha) : F] = n$.

Problem 6.41. Use Theorem 6.40 and the results of Problem 6.10 to find a basis for $\mathbb{Q}(\sqrt[3]{11})$ over $\mathbb{Q}$ and compute $[\mathbb{Q}(\sqrt[3]{11}) : \mathbb{Q}]$.

Problem 6.42. Use Theorem 6.40 and the results of Problem 6.21 to find a basis for $\mathbb{Z}_3(r)$ over $\mathbb{Z}_3$ and compute $[\mathbb{Z}_3(r) : \mathbb{Z}_3]$ where r is a root of $p(x) = x^3 - x + 1 \in \mathbb{Z}_3[x]$.

Problem 6.43. Use Theorem 6.40 and the results of Problem 6.31 to find a basis for $\mathbb{Q}(\zeta_5)$ over $\mathbb{Q}$ and compute $[\mathbb{Q}(\zeta_5) : \mathbb{Q}]$.

Theorem 6.40 (which built off of Theorem 6.19) completes our goal of describing $F(\alpha)$, but we may want to adjoin more than one element to a field. For example, when we defined solvability by radicals in Chapter 4, we needed to ensure that all roots of the polynomial lived in some radical extension, often built by adjoining several elements.

Our approach to this will be as in Problem 6.24. Suppose we want a basis for $F(\alpha, \beta)$ over F. We can first find a basis for $F(\alpha)$ over F, and then find a basis for $F(\alpha, \beta)$ over $F(\alpha)$. If $1, \alpha, \ldots, \alpha^{m-1}$ is a basis for $F(\alpha)$ over F and $1, \beta, \ldots, \beta^{n-1}$ is a basis for $F(\alpha, \beta)$ over $F(\alpha)$, then we have that

$$F(\alpha) = \{a_0 + a_1\alpha + \cdots + a_{m-1}\alpha^{m-1} \mid a_0, \ldots, a_{m-1} \in F\}; \text{ and}$$
$$F(\alpha, \beta) = \{b_0 + b_1\beta + \cdots + b_{n-1}\beta^{n-1} \mid b_0, \ldots, b_{n-1} \in F(\alpha)\}.$$

But what we want is a basis for $F(\alpha, \beta)$ over F, so we want to express elements of $F(\alpha, \beta)$ using coefficients from F (not $F(\alpha)$). However, notice that since each b_i is in $F(\alpha)$, we can write each b_i in terms of $1, \alpha, \ldots, \alpha^{m-1}$, using only coefficients from F. Doing this for each b_i and simplifying, we see that every element of $F(\alpha, \beta)$ can be written as a linear combination of the elements

$$\begin{aligned} &1, \alpha, \ldots, \alpha^{m-1}, \\ &\beta, \alpha\beta, \ldots, \alpha^{m-1}\beta, \\ &\vdots \\ &\beta^{n-1}, \alpha\beta^{n-1}, \ldots, \alpha^{m-1}\beta^{n-1} \end{aligned}$$

using only coefficients from F. And moreover, it can be shown that these are linearly independent, so we found a basis for $F(\alpha, \beta)$ over F. The basis has size mn.

In fact, this process generalizes in a straightforward way to any chain of field extensions $F \subseteq K \subseteq L$. In words, a basis for L as a vector space over F can be found by multiplying a basis for K over F by the elements of a basis for L over K. This also yields an extremely useful multiplicative property for the degrees in a chain of field extensions, namely that $[L : F] = [L : K][K : F]$. The next fact summarizes our findings.

Fact 6.44. Let $F \subseteq K \subseteq L$ be fields. If $\{u_1, \ldots, u_m\}$ is basis for K over F and $\{w_1, \ldots, w_n\}$ is basis for L over K, then

$$\{u_1w_1, \ldots, u_mw_1, u_1w_2, \ldots, u_mw_2, \ldots, u_1w_n, \ldots, u_mw_n\}$$

is a basis for L over F. In particular, $[L : F] = [L : K][K : F]$.

Problem 6.45. Let's find a basis for $\mathbb{Q}(\sqrt[4]{2}, \zeta_3)$ over $\mathbb{Q}$.

(1) Use the EIC to show $x^4 - 2$ is the minimal polynomial of $\sqrt[4]{2}$ over $\mathbb{Q}$.

(2) Use Theorem 6.40 to find a basis for $\mathbb{Q}(\sqrt[4]{2})$ over $\mathbb{Q}$.

(3) Use Theorem 5.64 to show $x^2 + x + 1$ is the minimal polynomial of ζ_3 over $\mathbb{Q}(\sqrt[4]{2})$.

(4) Use Theorem 6.40 to find a basis for $\mathbb{Q}(\sqrt[4]{2}, \zeta_3)$ over $\mathbb{Q}(\sqrt[4]{2})$.

(5) Use Fact 6.44 to find a basis for $\mathbb{Q}(\sqrt[4]{2}, \zeta_3)$ over $\mathbb{Q}$ and determine $[\mathbb{Q}(\sqrt[4]{2}, \zeta_3) : \mathbb{Q}]$.

When exploring degrees of extensions, the following fact from linear algebra often comes up: if W is a subspace of V, then $\dim W = \dim V$ if and only if $W = V$. Applying this to field extensions yields the following.

Fact 6.46. Let $F \subseteq K \subseteq L$ be fields. Then $[K : F] = [L : F]$ if and only if $K = L$.

Problem 6.47. Let's revisit the fields $\mathbb{Q}(\sqrt{2} + i)$ and $\mathbb{Q}(\sqrt{2}, i)$.

(1) Explain why $\mathbb{Q}(\sqrt{2} + i) \subseteq \mathbb{Q}(\sqrt{2}, i)$.

(2) Use Theorem 6.40 and Problem 6.22 to determine $[\mathbb{Q}(\sqrt{2} + i) : \mathbb{Q}]$.

(3) Use Fact 6.44 and Problem 6.24 to determine $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}]$.

(4) Use the previous parts to show that $\mathbb{Q}(\sqrt{2} + i) = \mathbb{Q}(\sqrt{2}, i)$.

The next couple of problems illustrate the power of the multiplicative property of field degrees for a chain of fields.

Problem 6.48. Let's take a look at $\mathbb{Q}(\sqrt[4]{2}, \zeta_3)$ over $\mathbb{Q}(\zeta_3)$.

(1) Explain why the EIC can *not* be used to show that $x^4 - 2$ is irreducible in $\mathbb{Q}(\zeta_3)[x]$.

(2) Use that $\mathbb{Q} \subset \mathbb{Q}(\zeta_3) \subset \mathbb{Q}(\sqrt[4]{2}, \zeta_3)$ together with the multiplicative property of field degrees (from Fact 6.44) to determine $[\mathbb{Q}(\sqrt[4]{2}, \zeta_3) : \mathbb{Q}(\zeta_3)]$. Remember that you computed $[\mathbb{Q}(\sqrt[4]{2}, \zeta_3) : \mathbb{Q}]$ in Problem 6.45.

(3) Use the fact that $[\mathbb{Q}(\sqrt[4]{2}, \zeta_3) : \mathbb{Q}(\zeta_3)]$ equals the degree of $\sqrt[4]{2}$ over $\mathbb{Q}(\zeta_3)$ (by Theorem 6.40) to explain why $x^4 - 2$ is irreducible in $\mathbb{Q}(\zeta_3)[x]$.

Problem 6.49. Let's show that $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{2})$.

(1) Explain why $\sqrt[3]{2} \in \mathbb{Q}(\sqrt{2})$ would imply that $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}(\sqrt{2})$.

(2) Use the multiplicative property of field degrees to show that $\sqrt[3]{2} \in \mathbb{Q}(\sqrt{2})$ would imply that $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$ is a divisor of $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$.

(3) What is $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$? What is $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$? Prove that $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt{2})$.

Theorem 6.50. Let $m, n \in \mathbb{Z}$ with $2 \leq m < n$. If $p, q \in \mathbb{Z}$ are prime, then $\sqrt[n]{p} \notin \mathbb{Q}(\sqrt[m]{q})$.

Problem 6.51. Let's find the degree of $\mathbb{Q}(\sqrt[5]{3}, \zeta_5)$ over $\mathbb{Q}$.

(1) Show that $[\mathbb{Q}(\sqrt[5]{3}) : \mathbb{Q}] = 5$.

(2) Use that ζ_5 is a root of $x^4 + x^3 + x^2 + x + 1$ to explain why $[\mathbb{Q}(\sqrt[5]{3}, \zeta_5) : \mathbb{Q}(\sqrt[5]{3})] \leq 4$. Then Use Fact 6.44 to show that $[\mathbb{Q}(\sqrt[5]{3}, \zeta_5) : \mathbb{Q}] \leq 20$.

(3) Use Problem 6.31 and Fact 6.44 to show $[\mathbb{Q}(\sqrt[5]{3}, \zeta_5) : \mathbb{Q}]$ is divisible by 4.

(4) Explain why $[\mathbb{Q}(\sqrt[5]{3}, \zeta_5) : \mathbb{Q}] = 20$.

Let's wrap up this section with a couple more results. The first says that if E is a *finite dimensional* extension of F, then every element of E is in fact algebraic over F. To prove this, we need to take an arbitrary $r \in E$, and show it is a root of some nonzero polynomial $F[x]$. But how do we find such a polynomial? To explore this, let's let $n = [E : F]$ (which we are assuming is finite). This means that every basis for E over F consists of n elements. And by a result from linear algebra, a set of $n + 1$ vectors must be linearly dependent. If we apply this to the set $\{1, r, r^2, \ldots, r^n\}$, we see that there are elements $a_0, a_1, a_2, \ldots, a_n \in F$ that are *not* all zero such that

$$a_0 + a_1 r + a_2 r^2 + \cdots + a_n r^n = 0.$$

This implies that r is a root of the *nonzero* polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, so r is indeed algebraic over F. Also, $r \in E$ implies that $F(r) \subseteq E$, so we now have that $[E : F] = [E : F(r)][F(r) : F]$. Thus $[F(r) : F]$ divides $[E : F]$, so as $[F(r) : F]$ equals the degree of r over F (by Theorem 6.40), we also get that the degree of r over F divides $[E : F]$. Here is the summary.

Fact 6.52. Let E be an extension field of F. Assume that $[E : F]$ is finite. If $r \in E$, then r is algebraic over F, and the degree of r over F divides $[E : F]$.

Combining this with Theorem 6.40, we obtain the following characterization of algebraic elements in terms of the fields they generate.

Corollary 6.53. Let E be an extension field of F, and let $r \in E$. Then r is algebraic over F if and only if $[F(r) : F]$ is finite.

6.2.2 Linear transformations

Let's briefly explore linear transformations in the context of field extensions.

Definition 6.54. Let V and W be vector spaces over a field F. A map $\phi : V \to W$ is called an F-**linear transformation** (or **homomorphism of** F-**vector spaces**) if the following are true for all $u, v \in V$ and all $c \in F$:

(1) $\phi(u + v) = \phi(u) + \phi(v)$;

(2) $\phi(cu) = c\phi(u)$.

Problem 6.55. Show that $\phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ defined via $\phi(a+b\sqrt{2}) = a-b\sqrt{2}$ is a $\mathbb{Q}$-linear transformation.

Problem 6.56. Show that $\gamma : \mathbb{C} \to \mathbb{C}$ defined via $\gamma(z) = \overline{z}$ is an $\mathbb{R}$-linear transformation.

Problem 6.57. The results of Problem 6.24 show that

$$\mathbb{Q}(\sqrt{2}, i) = \left\{ a_0 + a_1\sqrt{2} + a_2 i + a_3 i\sqrt{2} \mid a_0, a_1, a_2, a_3 \in \mathbb{Q} \right\}.$$

Consider $\phi : \mathbb{Q}(\sqrt{2}, i) \to \mathbb{Q}(\sqrt{2}, i)$ defined by $\phi(z) = \overline{z}$. As $\mathbb{Q}(\sqrt{2}, i)$ is an extension of $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{2}, i)$ is a vector space over $\mathbb{Q}(\sqrt{2})$. Show that ϕ is a $\mathbb{Q}(\sqrt{2})$-linear transformation.

6.3 Isomorphisms of fields

We are closing in on our goal of understanding when a polynomial is solvable by radicals or not. Recall that $p(x) \in F[x]$ is solvable by radicals if all of the roots of $p(x)$ are contained in some radical extension of F. This implies that if $r_1, \ldots, r_n$ are the roots of $p(x)$, then $F(r_1, \ldots, r_n)$ must also be contained in a radical extension. We've been working hard to understand fields like $F(r_1, \ldots, r_n)$, and with Fact 6.44, we are now able to explicitly describe the elements of $F(r_1, \ldots, r_n)$ as linear combinations of a particular basis, which involves certain powers of $r_1, \ldots, r_n$. But, we still need tools for analyzing $F(r_1, \ldots, r_n)$ in order to understand if it could be contained a radical extension or not.

It turns out that the key idea is to study certain functions from $F(r_1, \ldots, r_n)$ to itself. Remembering that $F(r_1, \ldots, r_n)$ is a field (hence a ring) and also a vector space over F, we look at functions that preserve both structures, namely ring homomorphism that are also F-linear transformations. The next theorem provides a convenient characterization of F-linear transformation for maps that are already known to be ring homomorphisms.

Theorem 6.58. Let K and L be extension fields of F, and let $\phi : K \to L$ be a surjective ring homomorphism. Then ϕ is an F-linear transformation if and only if $\phi(c) = c$ for all $c \in F$.

In Theorem 6.58, the property that "$\phi(c) = c$ for all $c \in F$" will be written as ϕ *fixes* F or ϕ *leaves* F *fixed*.

Definition 6.59. Let $\phi : X \to Y$ be a function.

- Let $A \subseteq X$. We say that ϕ **fixes** A if $\phi(a) = a$ for all $a \in A$.
- Define $\mathrm{Fix}(\phi)$ to be the set of *all* $x \in X$ such that $\phi(x) = x$.

In fact, we've already seen many examples of morphisms that fix a field; the next problem highlights two of them.

Problem 6.60. Let's explore a couple familiar maps and show that they fix certain fields.

(1) Define $\phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ via $\phi(a + b\sqrt{2}) = a - b\sqrt{2}$. Show that ϕ fixes $\mathbb{Q}$.

(2) Define $\gamma : \mathbb{C} \to \mathbb{C}$ via $\gamma(z) = \overline{z}$. Show that γ fixes $\mathbb{Q}$. What is $\mathrm{Fix}(\gamma)$?

We now start to expose the implications of a morphism fixing a field. The next theorem is quite important.

Theorem 6.61. Let K and L be extension fields of F, and let $p(x) \in F[x]$. Suppose that $\phi : K \to L$ is a ring homomorphism that fixes F. If $\alpha \in K$ is a root of $p(x)$, then $\phi(\alpha)$ is also a root of $p(x)$.

Problem 6.62. Suppose that $\phi : \mathbb{Q}(\sqrt[3]{7}) \to \mathbb{C}$ is a ring homomorphism that fixes $\mathbb{Q}$. Use the fact that $\sqrt[3]{7}$ is a root of $x^3 - 7$ together with Theorems 3.26 and 6.61 to list the possible values of $\phi(\sqrt[3]{7})$.

Problem 6.63. Suppose that $\psi : \mathbb{C} \to \mathbb{C}$ is an isomorphism that fixes $\mathbb{Q}$. Use Theorem 6.61 (and the idea of Problem 6.62) to list the possible values of $\psi(\sqrt{5})$ and $\psi(\zeta_5)$. Try to make each list as short as possible, and explain your reasoning.

Problem 6.64. Show that each map below fixes $\mathbb{Q}$, and then use Theorem 6.61 to explain why neither map is a homomorphism.

(1) $\delta : \mathbb{Q}(\sqrt{5}) \to \mathbb{Q}(i)$ defined by $\delta(a + b\sqrt{5}) = a + bi$

(2) $\sigma : \mathbb{Q}(\sqrt[3]{7}) \to \mathbb{Q}(\sqrt[3]{7})$ defined by $\sigma(a + b\sqrt[3]{7} + c(\sqrt[3]{7})^2 = a - b\sqrt[3]{7} + c(\sqrt[3]{7})^2$

Let's use what we've learned so far to explore which subfields L of $\mathbb{C}$ could be isomorphic to $\mathbb{Q}(\sqrt[3]{11})$ (just as an example). Let $\phi : \mathbb{Q}(\sqrt[3]{11}) \to L$ be an isomorphism—we'll try to determine the possibilities for L. Notice that Theorem 6.61 may be helpful, but only if we know that ϕ fixes the coefficients of a polynomial that has $\sqrt[3]{11}$ as a root. Let's first show that ϕ must fix all of $\mathbb{Q}$.

Problem 6.65. Let K and L be extension fields of $\mathbb{Q}$. Suppose that $\phi : K \to L$ is a surjective ring homomorphism. We'll show that ϕ fixes $\mathbb{Q}$.

(1) Use Theorem 5.111 (and properties of homomorphisms) to show that ϕ fixes every positive integer n. Remember that $2 = 1 + 1$, $3 = 1 + 1 + 1$, etc.

(2) Use Theorem 5.110 to conclude that ϕ fixes all integers.

(3) Let $\frac{a}{b} \in \mathbb{Q}$ be any rational number (with $a, b \in \mathbb{Z}$). Use Theorem 5.111, to show that ϕ fixes $\frac{a}{b}$.

Problem 6.66. Assume that $\phi : \mathbb{Q}(\sqrt[3]{11}) \to L$ is an isomorphism for some subfield L of $\mathbb{C}$. Recall from Problem 6.41 that $1, \sqrt[3]{11}, (\sqrt[3]{11})^2$ is a basis for $\mathbb{Q}(\sqrt[3]{11})$, so

$$\mathbb{Q}(\sqrt[3]{11}) = \{a + b\sqrt[3]{11} + c(\sqrt[3]{11})^2 \mid a, b, c \in \mathbb{Q}\}.$$

(1) Use Theorem 6.61 and Problem 6.65 to list the three possible values of $\phi(\sqrt[3]{11})$.

(2) Use Problem 6.65 to describe the possible values of $\phi(a + b\sqrt[3]{11} + c(\sqrt[3]{11})^2)$.

(3) What does this imply are the possibilities for L?

Problem 6.66 raises an important question: can we always create an isomorphism taking $\sqrt[3]{11}$ to β whenever β is a root of the same minimal polynomial as $\sqrt[3]{11}$? The next fact answers the quesion—it will be *extremely important* for us. The proof is not too difficult, but we will take it as fact. The main ingredients are some of the linear algebra that we've developed and (perhaps not surprisingly) the division algorithm.

Fact 6.67. Let F be a subfield of E. Suppose that $\alpha \in E$ is algebraic over F. Let $m(x)$ be the minimal polynomial of α over F, and let n be the degree of α over F. Suppose that β is also a root of $m(x)$, and define $\phi : F(\alpha) \to F(\beta)$ by

$$\phi(a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}) = a_0 + a_1\beta + \cdots + a_{n-1}\beta^{n-1}$$

for all $a_0, a_1, \ldots, a_{n-1} \in F$. Then ϕ is an isomorphism, and ϕ fixes F.

Problem 6.68. Use Theorem 6.61 (together with Problem 6.65) and Fact 6.67 to determine if each pair of fields are isomorphic or not. If they are, write down a formula for an isomorphism; if they are not, explain why not.

(1) $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$

(2) $\mathbb{Q}(\sqrt[4]{11})$ and $\mathbb{Q}(i\sqrt[4]{11})$

6.3.1 Automorphisms

As mentioned earlier, we really want to study maps from a field to itself. We now define one of the keys ingredients in our eventual solution to the insolvability of the quintic.

Definition 6.69. Let K be a field. An isomorphism from K to K is called an **automorphism** of K. We define $\mathrm{Aut}(K)$ to be the set of all automorphisms of K. If F is a subfield of K, we define $\mathrm{Aut}(K/F)$ to be the set of all automorphisms of K that fix F.

Unpacking the definition, we see that automorphisms of K are bijections from K to K (i.e permutations of K) that are also homomorphisms. Since we know that the composition of two bijections is a bijections and the composition of two homomorphisms is a homomorphism (see Theorem 5.112), we see that $\mathrm{Aut}(K)$ is closed under function composition. Moreover, each element of $\mathrm{Aut}(K)$ is a bijection, hence has an inverse, and it is not too difficult to show that the inverse is also a homomorphism. Thus, $\mathrm{Aut}(K)$ is closed under taking inverses. Of course, $\mathrm{Aut}(K)$ contains the identity function from K to K, so we conclude that $\mathrm{Aut}(K)$ is a group with respect to function composition.

Theorem 6.70. Let K be a field. Then $\mathrm{Aut}(K)$ is a group with respect to function composition. The identity is the identity function, which will be denoted id.

Let's show that $\mathrm{Aut}(K/F)$ is also a group—to do that we need to show that if two automorphisms of K fix F, then their composition does too.

Theorem 6.71. If F is a subfield of K, then $\mathrm{Aut}(K/F)$ is a subgroup of $\mathrm{Aut}(K)$.

In light of Theorems 6.70 and 6.71, $\mathrm{Aut}(K)$ will be referred to as the **automorphism group** of K and $\mathrm{Aut}(K/F)$ will be called the **automorphism group of K over F**. While we're at it, take Theorem 6.71 a bit further.

Theorem 6.72. If $F \subseteq K \subseteq L$ is a chain of fields, then $\mathrm{Aut}(L/K) \subseteq \mathrm{Aut}(L/F)$.

Theorem 6.72 is starting to build a correspondence from subfields of L to subgroups of $\mathrm{Aut}(L/F)$. However, notice that the correspondence is inclusion reversing.

$$\begin{array}{ccc} L & \longrightarrow & \mathrm{Aut}(L/L) \\ \cup\mathrel{|} & & \mathrel{|}\cap \\ K & \longrightarrow & \mathrm{Aut}(L/K) \\ \cup\mathrel{|} & & \mathrel{|}\cap \\ F & \longrightarrow & \mathrm{Aut}(L/F) \end{array}$$

In the picture above we listed the group $\mathrm{Aut}(L/L)$. These are the automorphism of L that fix all of L—there is only one such automorphism: the identity. Thus, $\mathrm{Aut}(L/L) = \{\mathrm{id}\}$.

Let's further explore these automorphism groups with some examples.

Example 6.73. Let's try to compute $\mathrm{Aut}(\mathbb{Q}(\zeta_5/\mathbb{Q})$. First, we know that $x^4+x^3+x^2+x+1$ is the minimum polynomial for ζ_5 over $\mathbb{Q}$, so by Theorem 6.40

- $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ is a basis for $\mathbb{Q}(\zeta_5)$ over $\mathbb{Q}$, and
- $\mathbb{Q}(\zeta_5) = \{a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3 \mid a, b, c, d \in \mathbb{Q}\}$.

Thus, every function $\phi \in \mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ can be expressed by a formula of the form

$$\phi(a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3) = \text{ ???}$$

Now, if $\phi \in \mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$, then ϕ fixes $\mathbb{Q}$. By Theorem 6.58, ϕ is a $\mathbb{Q}$-linear transformation from $\mathbb{Q}(\zeta_5)$ to itself, and by a result from linear algebra, ϕ is completely determined by its values on a basis. That is, once we determine the values of $\phi(1)$, $\phi(\zeta_5)$, $\phi(\zeta_5^2)$, and $\phi(\zeta_5^3)$, we will know a formula for ϕ. In fact, this is easy to see directly:

$$\begin{aligned} \phi(a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3) &= \phi(a) + \phi(b\zeta_5) + \phi(c\zeta_5^2) + \phi(d\zeta_5^3) \\ &= \phi(a) + \phi(b)\phi(\zeta_5) + \phi(c)\phi(\zeta_5^2) + \phi(d)\phi(\zeta_5^3) \\ &= a + b\phi(\zeta_5) + c\phi(\zeta_5^2) + d\phi(\zeta_5^3). \end{aligned}$$

In fact, we can take this further:

$$\begin{aligned} \phi(a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3) &= a + b\phi(\zeta_5) + c\phi(\zeta_5^2) + d\phi(\zeta_5^3) \\ &= a + b\phi(\zeta_5) + c\phi(\zeta_5)^2 + d\phi(\zeta_5)^3. \end{aligned}$$

So, to find a formula for ϕ we just need to determine the value for $\phi(\zeta_5)$; it can then be plugged into the above formula to find the value of ϕ on an arbitrary element of $\mathbb{Q}(\zeta_5)$.

Now, since ζ_5 is a root of $x^4+x^3+x^2+x+1$, Theorem 6.61 says that $\phi(\zeta_5)$ must be one of the roots of $x^4+x^3+x^2+x+1$, which are ζ_5, ζ_5^2, ζ_5^3, and ζ_5^4. The possibilities are named below. We will use repeatedly that $\zeta_5^5 = 1$; if desired, we could also use $\zeta_5^4 = -\zeta_5^3 - \zeta_5^2 - \zeta_5 - 1$.

- ϕ_1 sends $\zeta_5 \mapsto \zeta_5$, which implies $\phi_1(a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3) = a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3$;
- ϕ_2 sends $\zeta_5 \mapsto \zeta_5^2$, which implies $\phi_2(a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3) = a + b\zeta_5^2 + c\zeta_5^4 + d\zeta_5$;
- ϕ_3 sends $\zeta_5 \mapsto \zeta_5^3$, which implies $\phi_3(a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3) = a + b\zeta_5^3 + c\zeta_5 + d\zeta_5^4$;
- ϕ_4 sends $\zeta_5 \mapsto \zeta_5^4$, which implies $\phi_4(a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3) = a + b\zeta_5^4 + c\zeta_5^3 + d\zeta_5^2$.

Here's another way to organize the possibilities.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$\zeta_5 \mapsto$	ζ_5	ζ_5^2	ζ_5^3	ζ_5^4

We now have to determine if each ϕ_i is in $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ or not. As ϕ_1 is just the identity, $\phi_1 \in \mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. To check the other possibilities, we can use Fact 6.67. For example, Fact 6.67 says that ϕ_2 is an isomorphism from $\mathbb{Q}(\zeta_5)$ to $\mathbb{Q}(\zeta_5^2)$. Since $\mathbb{Q}(\zeta_5^2) = \mathbb{Q}(\zeta_5)$ (because $\zeta_5^2 \in \mathbb{Q}(\zeta_5)$ and $\zeta_5 \in \mathbb{Q}(\zeta_5^2)$), ϕ_2 is indeed an automorphism of $\mathbb{Q}(\zeta_5)$, so $\phi_2 \in \mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. Similarly, $\phi_3, \phi_4 \in \mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. These are all possibilities, so

$$\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) = \{\phi_1, \phi_2, \phi_3, \phi_4\} = \{\mathrm{id}, \phi_2, \phi_3, \phi_4\}.$$

We now know that $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ is a group of order 4. What group is it?

Problem 6.74. Let's determine what group $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ is isomorphic to.

(1) Which element of $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ is $(\phi_2)^2$ equal to? (Remember $(\phi_2)^2$ means $\phi_2 \circ \phi_2$.)

(2) Do the same for $(\phi_2)^3$ and $(\phi_2)^4$. Are either $(\phi_2)^3$ or $(\phi_2)^4$ equal to id? What is the order of ϕ_2? (The order of ϕ_2 will be the smallest positive k such that $(\phi_2)^k = \mathrm{id}$.)

(3) What are the orders of ϕ_3 and ϕ_4?

(4) Is $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ cyclic or not? Is $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ isomorphic to $\mathbb{Z}_4$ or V_4?

An important observation in Example 6.73 was that the possibilities for ϕ are determined simply by the possible values of $\phi(\zeta_5)$. This is true in general.

Fact 6.75. Let F be a subfield of E. Suppose that $\alpha \in E$ is algebraic over F, and let n be the degree of α over F. Then each $\phi \in \mathrm{Aut}(F(\alpha)/F)$ is completely determined by the value $\phi(\alpha)$. Consequently, $|\mathrm{Aut}(F(\alpha)/F| \leq n = [F(\alpha) : F]$.

Problem 6.76. Follow Example 6.73 to determine $\mathrm{Aut}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$. What familiar group is it isomorphic to?

Problem 6.77. Let's determine $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$.

(1) First explain why $\mathbb{Q}(\sqrt[3]{2}) \neq \mathbb{Q}(\sqrt[3]{2}\zeta_3)$ and why $\mathbb{Q}(\sqrt[3]{2}) \neq \mathbb{Q}(\sqrt[3]{2}\zeta_3^2)$.

(2) Follow Example 6.73 to show that $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{\mathrm{id}\}$.

What happens if we adjoin more than one element? Can we compute $\mathrm{Aut}(F(\alpha_1,\alpha_2)/F)$ in a similar way as to how we computed $\mathrm{Aut}(F(\alpha)/F)$? The answer is yes, and the starting point is the following analog of Fact 6.75.

Fact 6.78. Let F be a subfield of E. Suppose that $\alpha_1,\ldots,\alpha_k \in E$ are algebraic over F. Then each $\phi \in \mathrm{Aut}(F(\alpha_1,\ldots,\alpha_k)/F)$ is completely determined by the values of $\phi(\alpha_1),\ldots,\phi(\alpha_k)$. Consequently, $|\mathrm{Aut}(F(\alpha_1,\ldots,\alpha_k)/F| \leq [F(\alpha_1,\ldots,\alpha_k) : F]$.

Problem 6.79. Let's determine $\mathrm{Aut}(\mathbb{Q}(\sqrt{2},i)/\mathbb{Q})$.

(1) Let $\phi \in \mathrm{Aut}(\mathbb{Q}(\sqrt{2},i)/\mathbb{Q})$. Use Theorem 6.61 to explain why there are only two choices for $\phi(\sqrt{2})$ and only two choices for $\phi(i)$. What are they?

(2) Combine the different possibilities for $\phi(\sqrt{2})$ and $\phi(i)$ to complete the table below.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$\sqrt{2} \mapsto$	$\sqrt{2}$			
$i \mapsto$	i			

(3) Follow Example 6.73 to determine $\mathrm{Aut}(\mathbb{Q}(\sqrt{2},i)/\mathbb{Q})$.

(4) What familiar group is $\mathrm{Aut}(\mathbb{Q}(\sqrt{2},i)/\mathbb{Q})$ isomorphic to?

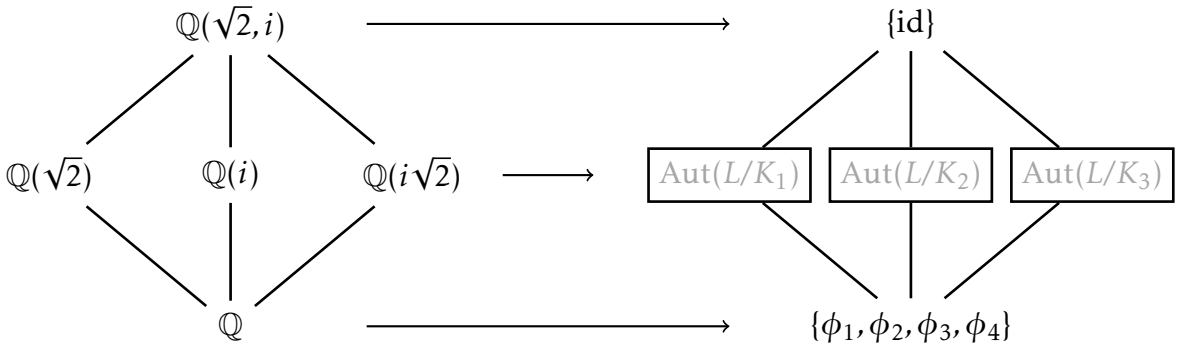
Problem 6.80. Set $L = \mathbb{Q}(\sqrt{2},i)$. In Problem 6.79 we determined $\mathrm{Aut}(L/\mathbb{Q})$. Let's connect the subfields of L with subgroups of $\mathrm{Aut}(L/\mathbb{Q})$ using Theorem 6.72.

(1) Let $K_1 = \mathbb{Q}(\sqrt{2})$. Find $\mathrm{Aut}(L/K_1)$ by determining which of $\phi_1,\phi_2,\phi_3,\phi_4$ are in $\mathrm{Aut}(L/K_1)$.

(2) Repeat for $K_2 = \mathbb{Q}(i)$. Find $\mathrm{Aut}(L/K_2)$.

(3) Repeat for $K_3 = \mathbb{Q}(i\sqrt{2})$. Find $\mathrm{Aut}(L/K_3)$.

(4) Use Theorem 6.72 to organize your findings by writing the appropriate elements in the boxes in the subgroup lattice of $\mathrm{Aut}(L/\mathbb{Q})$.



Problem 6.80 highlights quite well the tight connection between subfields of an extension field L of F and subgroups of $\mathrm{Aut}(L/F)$. So far, we've seen how each subfield K gives rise to a subgroup $\mathrm{Aut}(L/K)$. The next theorem indicates how we might reverse this.

Theorem 6.81. Let F be a subfield of L and H a subgroup of $\mathrm{Aut}(L/F)$. Define

$$\mathrm{Fix}_L(H) = \{k \in L \mid k \text{ is fixed by every } \phi \in H\}.$$

Then $\mathrm{Fix}_L(H)$ is a subfield of L, and $F \subseteq \mathrm{Fix}_L(H) \subseteq L$.

The picture is as follows.

$$\begin{array}{ccc} L & & \{\mathrm{id}\} \\ \cup\mathrel{|} & & \mathrel{|}\cap \\ \mathrm{Fix}_L(H) & \longleftarrow & H \\ \cup\mathrel{|} & & \mathrel{|}\cap \\ F & & \mathrm{Aut}(L/F) \end{array}$$

Taking a closer look at Problem 6.80, we can see that the maps $K \mapsto \mathrm{Aut}(L/K)$ and $H \mapsto \mathrm{Fix}_L(H)$ are actually inverses of each other. For example, $\mathrm{Fix}_L(\mathrm{Aut}(L/K_1)) = K_1$, so the composition of the maps looks like $K_1 \mapsto \mathrm{Aut}(L/K_1) \mapsto \mathrm{Fix}_L(\mathrm{Aut}(L/K_1)) = K_1$. However, this is not true for all fields, and Problem 6.77 gives an example. In the next chapter we'll study an important collection of fields (in fact, *the* collection of fields) for which $K \mapsto \mathrm{Aut}(L/K)$ and $H \mapsto \mathrm{Fix}_L(H)$ are always inverses.

Chapter 7

Galois theory

We finished Chapter 6 by computing automorphism groups of field extensions. We also began to connect the subfields of an extension field L of F to subgroups of $\mathrm{Aut}(L/F)$. We now narrow our focus on which types of extension fields we consider, and in doing so, we significantly sharpen what we can say about this connection. It will be lynchpin of our argument showing that not all polynomials over $\mathbb{Q}$ are solvable by radicals over $\mathbb{Q}$.

Also, from here on, we will exclusively focus on subfields of $\mathbb{C}$. This will streamline (and simplify) our work, but it will also slightly obscure the general theory. Which is to say, this is more the beginning of the story than the end.

7.1 Galois extensions and Galois groups

Definition 7.1. Let F be a subfield of $\mathbb{C}$, and let $p(x) \in F[x]$. Define $F^{p(x)}$ to be the subfield of $\mathbb{C}$ generated by F and all roots of $p(x)$; thus, $F^{p(x)} = F(r_1, \ldots, r_n)$ where $r_1, \ldots, r_n$ are all of the roots of $p(x)$ in $\mathbb{C}$.

For example, if $p(x) = x^5 - 1$, then by Theorem 3.24, the roots of $p(x)$ are $1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$, so $\mathbb{Q}^{p(x)} = \mathbb{Q}(1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$.

Problem 7.2. Let $p(x) = x^5 - 1$. Use Theorem 3.69 to explain why $\mathbb{Q}^{p(x)} = \mathbb{Q}(\zeta_5)$.

Problem 7.3. Let $p(x) = x^3 - 2$. Explain why $\mathbb{Q}^{p(x)} \neq \mathbb{Q}(\sqrt[3]{2})$.

Problem 7.4. For each field F below, find a polynomial $p(x) \in \mathbb{Q}[x]$ such that $F = \mathbb{Q}^{p(x)}$.

(1) $F = \mathbb{Q}(\sqrt{2})$

(2) $F = \mathbb{Q}(\sqrt{2}, i)$

(3) $F = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$

(4) $F = \mathbb{Q}(\zeta_{12})$

Definition 7.5. Let $F \subseteq K$ be subfields of $\mathbb{C}$.

(1) We say that K is a **Galois extension** of F if $K = F^{p(x)}$ for some $p(x) \in F[x]$.

(2) If K is a Galois extension of F, then $\mathrm{Aut}(K/F)$ is called the **Galois group** of K over F.

Revisiting Problem 7.4 with this new terminology, we see that each of $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{2}, i)$, $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$, and $\mathbb{Q}(\zeta_{12})$ are Galois extensions of $\mathbb{Q}$. Also, Problem 7.3 hints at the fact that $\mathbb{Q}(\sqrt[3]{2})$ might not be a Galois extension of $\mathbb{Q}$ (but there is more to prove to establish that).

Let's generalize parts of Problem 7.4 and record some types of extensions that are always Galois.

Theorem 7.6. Let $a \in \mathbb{Q}$. Then $\mathbb{Q}(\sqrt{a})$ is a Galois extension of $\mathbb{Q}$.

Theorem 7.7. Let n be a positive integer. Then $\mathbb{Q}(\zeta_n)$ is a Galois extension of $\mathbb{Q}$.

As mentioned above, $\mathbb{Q}(\sqrt[3]{2})$ might not be a Galois extension of $\mathbb{Q}$, but it is true that $F(\sqrt[3]{2})$ a Galois extension of F provided F contains ζ_3. The next theorem addresses this.

Theorem 7.8. Let F be a subfield of $\mathbb{C}$. Suppose that $r \in \mathbb{C}$ and $r^n \in F$ for some positive integer n. If $\zeta_n \in F$, then $F(r)$ is a Galois extension of F.

7.1.1 Size of Galois groups

The next fact highlights the importance of Galois extensions. The point is roughly that the automorphism group of a Galois extension has the "expected" number of automorphisms; whereas, automorphism groups of non-Galois extension will necessarily have fewer.

Fact 7.9. Let $F \subseteq K$ be subfields of $\mathbb{C}$. If K is a Galois extension of F, $|\mathrm{Aut}(K/F)| = [K : F]$.

Fact 7.9 is extremely powerful. Let's start by seeing how it can help streamline the computation of certain automorphism groups.

Problem 7.10. Let $L = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. Let's determine $\mathrm{Aut}(L/\mathbb{Q})$. Recall from Problem 7.4 that L is a Galois extension of $\mathbb{Q}$.

(1) What is minimal polynomial for $\sqrt[3]{2}$ over $\mathbb{Q}$? Why?

(2) What is minimal polynomial for ζ_3 over $\mathbb{Q}(\sqrt[3]{2})$? Why?

(3) Use Fact 6.44 to explain why $[L : \mathbb{Q}] = 6$.

(4) Let $\phi \in \mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q})$. Use Theorem 6.61 to explain why there are only 3 choices for $\phi(\sqrt[3]{2})$ and only two choices for ζ_3. What are they?

(5) Complete the table of possible elements of $\mathrm{Aut}(L/\mathbb{Q})$.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
$\sqrt[3]{2} \mapsto$	$\sqrt[3]{2}$					
$\zeta_3 \mapsto$	ζ_3					

(6) Use Fact 7.9 to explain why every function in the table above must be in $\mathrm{Aut}(L/\mathbb{Q})$.

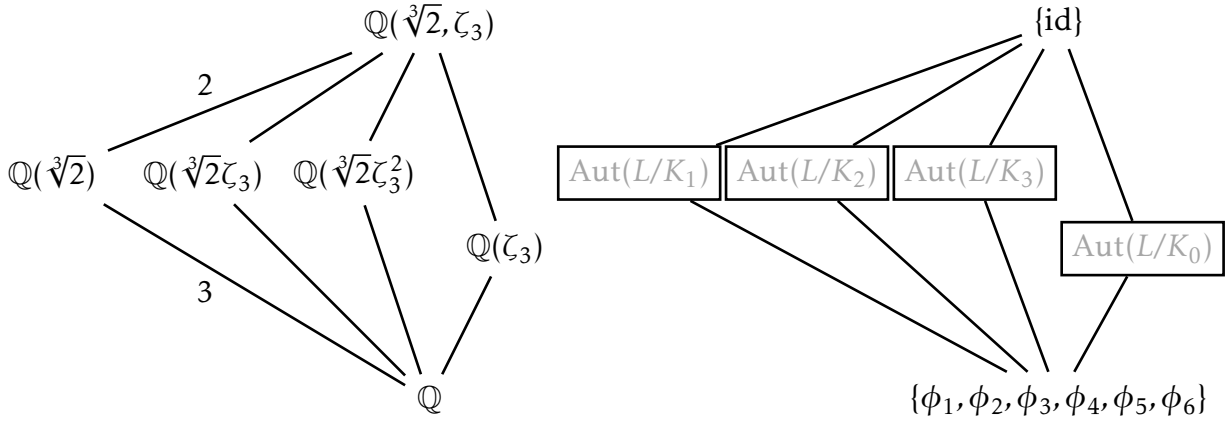
Problem 7.11. Let's revisit $L = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ from Problem 7.10 and connect subfields of L with subgroups of $\mathrm{Aut}(L/\mathbb{Q})$ using Theorem 6.72. The following are subfields of L.

$$K_0 = \mathbb{Q}(\zeta_3) \quad K_1 = \mathbb{Q}(\sqrt[3]{2}) \quad K_2 = \mathbb{Q}(\sqrt[3]{2}\zeta_3) \qquad K_3 = \mathbb{Q}(\sqrt[3]{2}\zeta_3^2)$$

(1) Compute $\mathrm{Aut}(L/K_0)$, $\mathrm{Aut}(L/K_1)$, $\mathrm{Aut}(L/K_2)$, and $\mathrm{Aut}(L/K_3)$ by determining which of $\phi_1, \ldots, \phi_6$ are in each one.

(2) Use Theorem 6.72 to organize your findings by writing the appropriate elements in the boxes in the subgroup lattice of $\mathrm{Aut}(L/\mathbb{Q})$.

- Label the degree of each field extension on the lines of the lattice on the left. Fact 6.44 should help. A couple have been done for you.
- Label the order and index of each subgroup on the lines of the lattice on the right.



(3) What familiar group is $\mathrm{Aut}(L/\mathbb{Q})$ isomorphic to?

Problem 7.12. Use Fact 7.9 and Problem 6.77 to argue that $\mathbb{Q}(\sqrt[3]{2})$ is *not* a Galois extension of $\mathbb{Q}$

7.1.2 Galois groups as permutation groups

We now explore how to look at Galois groups as groups of permutations. The key, yet again, is Theorem 6.61. We begin by recalling a some definitions from group theory.

Definition 7.13. Let X be a set. A bijection from X to X is called a **permutation** of X. The set of all permutations of X is denoted $\mathrm{Sym}(X)$. The set of all permutations of $\{1, \ldots, n\}$ is usually denoted by S_n (instead of $\mathrm{Sym}(\{1, \ldots, n\})$).

Recall that, for any set X, $\mathrm{Sym}(X)$ is a group with respect to function composition. The identity is the identity function, denoted id.

Theorem 7.14. Let F be a subfield of $\mathbb{C}$. Let $p(x) \in F(x)$ be a polynomial of degree n, and let $R = \{r_1, \ldots, r_n\}$ be the set of all of roots of $p(x)$ in $\mathbb{C}$. Then

(1) for all $\phi \in \mathrm{Aut}(F^{p(x)}/F)$, restricting the domain of ϕ to R yields a permutation of R;

(2) the map $\mathrm{Aut}(F^{p(x)}/F) \to \mathrm{Sym}(R)$ that restricts the domain of each automorphism to R is an injective homomorphism.

Consequently, $\mathrm{Aut}(F^{p(x)}/F)$ is isomorphic to a subgroup of $\mathrm{Sym}(R)$.

Corollary 7.15. Let F be a subfield of $\mathbb{C}$. Let $p(x) \in F(x)$ be a polynomial of degree n. Then $\mathrm{Aut}(F^{p(x)}/F)$ is isomorphic to a subgroup of S_n.

To view $\mathrm{Aut}(F^{p(x)}/F)$ as a subgroup of S_n, we just need to label the roots of $p(x)$ by $1,\ldots,n$ in some way and then record how each element of $\mathrm{Aut}(F^{p(x)}/F)$ permutes the roots. Let's take a look at an example of this.

Example 7.16. Similar to Problem 7.2, we can see that $\mathbb{Q}(\zeta_5) = \mathbb{Q}^{p(x)}$ for $p(x) = x^4 + x^3 + x^2 + x + 1$. We know that the set of roots of $p(x)$ is $R = \{\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$.

By Corollary 7.15, $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ is isomorphic to a subgroup of S_4 because $p(x)$ has degree 4 (hence 4 roots to permute). Let's find an explicit isomorphism. Recall from Example 6.73 that the elements of $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ are defined by the following table.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$\zeta_5 \mapsto$	ζ_5	ζ_5^2	ζ_5^3	ζ_5^4

Now let's expand the table to see how the automorphisms operate on all roots of $p(x)$.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$\zeta_5 \mapsto$	ζ_5	ζ_5^2	ζ_5^3	ζ_5^4
$\zeta_5^2 \mapsto$	ζ_5^2	ζ_5^4	ζ_5	ζ_5^3
$\zeta_5^3 \mapsto$	ζ_5^3	ζ_5	ζ_5^4	ζ_5^2
$\zeta_5^4 \mapsto$	ζ_5^4	ζ_5^3	ζ_5^2	ζ_5

Next, let's identify the roots with the numbers 1 up to 4 as follows.

$$\zeta_5 \leftrightarrow 1 \quad \zeta_5^2 \leftrightarrow 2 \quad \zeta_5^3 \leftrightarrow 3 \quad \zeta_5^4 \leftrightarrow 4.$$

Then the previous table becomes as follows.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$1 \mapsto$	1	2	3	4
$2 \mapsto$	2	4	1	3
$3 \mapsto$	3	1	4	2
$4 \mapsto$	4	3	2	1

So, using our labeling of the four roots, we can view $\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ as a subgroup of S_4. If we write the permutation using cycle notation, we have

$$\phi_1 = \mathrm{id} \quad \phi_2 = (1243) \quad \phi_2 = (1342) \quad \phi_4 = (14)(23).$$

Problem 7.17. Let's look at Problem 6.79 again. Notice that $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}^{p(x)}$ for $p(x) = (x^2-2)(x^2+1)$, and the set of roots of $p(x)$ are $R = \{\sqrt{2}, -\sqrt{2}, i, -i\}$.

(1) Fill in the extended table below to list how the elements of $\mathrm{Aut}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ operate on the elements of R. Two of the lines were completed for you.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$\sqrt{2} \mapsto$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$
$-\sqrt{2} \mapsto$				
$i \mapsto$	i	$-i$	i	$-i$
$-i \mapsto$				

(2) Label the roots of $p(x)$ via: $\sqrt{2} \leftrightarrow 1$, $-\sqrt{2} \leftrightarrow 2$, $i \leftrightarrow 3$, and $-i \leftrightarrow 4$. Write each element of $\mathrm{Aut}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ as a permutation in S_4 using cycle notation as in Example 7.16.

Problem 7.18. Let's revisit Problem 7.10. Set $L = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. We've seen that $L = \mathbb{Q}^{p(x)}$ for $p(x) = x^3 - 2$, and the set of roots of $p(x)$ are $R = \{\sqrt[3]{2}, \sqrt[3]{2}\zeta_3, \sqrt[3]{2}\zeta_3^2\}$.

(1) Fill in the extended table below to list how the elements of $\mathrm{Aut}(L/\mathbb{Q})$ operate on the elements of R. For the first two lines, use what you wrote in Problem 7.10.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
$\sqrt[3]{2} \mapsto$	$\sqrt[3]{2}$					
$\zeta_3 \mapsto$	ζ_3					
$\sqrt[3]{2}\zeta_3 \mapsto$						
$\sqrt[3]{2}\zeta_3^2 \mapsto$						

(2) Label the elements of R as follows: $\sqrt[3]{2} \leftrightarrow 1$, $\sqrt[3]{2}\zeta_3 \leftrightarrow 2$, and $\sqrt[3]{2}\zeta_3^2 \leftrightarrow 3$. Write each element of $\mathrm{Aut}(L/\mathbb{Q})$ as a permutation in S_3 using cycle notation as in Example 7.16.

Let's apply the many things that we've learned to a very specific map: complex conjugation (which sends $z \mapsto \overline{z}$). Remember that we know a lot about this map. In Chapter 5, we noted that complex conjugation yields an isomorphism from $\mathbb{C}$ to $\mathbb{C}$, and in Problem 6.60, we saw that it fixes every real number.

We'll investigate complex conjugation when we restrict the domain to a subfield K of $\mathbb{C}$. Let γ denote complex conjugation. As γ is a homomorphism and is injective (so $\ker\gamma = \{0\}$), the First Isomorphism Theorem tells us that γ gives an isomorphism of K with its image under γ (i.e. $K \cong \gamma(K)$). Additionally, if $K = \gamma(K)$, then γ will be an *automorphism* of K, and the next theorem identifies one situation where this always happens.

Theorem 7.19. Let $p(x) \in \mathbb{Q}[x]$, and let $R = \{r_1, \ldots, r_n\}$ be the set of all of roots of $p(x)$ in $\mathbb{C}$. If γ is the complex conjugation map defined via $\gamma(z) = \overline{z}$, then

(1) restricting the domain of γ to R yields a permutation of R, and

(2) $\gamma \in \mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$.

Problem 7.20. Let $p(x) = (x^2-2)(x^2+1)$. Theorem 7.19 says that the complex conjugation map is in $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$. Look back at Problem 7.17 and determine which element of $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ corresponds to complex conjugation. Write your answer in cycle notation as in Problem 7.17.

Problem 7.21. Repeat Problem 7.20 for $p(x) = x^3 - 2$. Use Problem 7.18 to determine which element of $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ corresponds to complex conjugation. Write your answer in cycle notation as in Problem 7.18.

Problem 7.22. Repeat Problem 7.20 for $p(x) = x^4 + x^3 + x^2 + x + 1$. Use Example 7.16 to determine which element of $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ corresponds to complex conjugation. Write your answer in cycle notation as in Example 7.16.

The similarities and difference between our answers to Problems 7.20–7.22 hint at the following theorem.

Theorem 7.23. Let $p(x) \in \mathbb{Q}[x]$, and suppose that $p(x)$ has exactly two roots in $\mathbb{C}$ that are not in $\mathbb{R}$. Then when viewing $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ as permutations of the roots of $p(x)$, $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ contains a transposition.

Problem 7.24. Consider $p(x) = x^5 + 5x^4 - 5 \in \mathbb{Q}[x]$. Graph $p(x)$ or use calculus to show that $p(x)$ has exactly 3 roots in $\mathbb{R}$, and use Theorem 7.23 to conclude that when $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ is viewed as permutations of the 5 roots of $p(x)$, $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ contains a transposition.

7.2 Fundamental theorem of Galois theory

We finally arrive at the main course. Looking back at Problems 6.80 and 7.11, we see that there is a tight connection between subfields of an extension field L of F to subgroups of $\mathrm{Aut}(L/F)$. However, the extensions we considered in those problems where not just any extensions of $\mathbb{Q}$, they were *Galois extensions*. And in fact, the connection broke down for $\mathbb{Q}(\sqrt[3]{2}$ in Problem 6.77 where we saw that $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{\mathrm{id}\}$, but as noted in Problem 7.12, this is *not* a Galois extension of $\mathbb{Q}$.

As it turns out, the connection we observed between subfields and subgroups holds for all Galois extensions, and the Fundamental Theorem of Galois Theory makes this explicit. Let's quickly establish some notation.

Notation 7.25. Let F be a subfield of L, and let G be a group. Define

- $\textsc{Sub}(L/F)$ to be the set of all subfields K such that $F \subseteq K \subseteq L$, and
- $\textsc{Sub}(G)$ to be the set of all subgroups of G.

The set $\textsc{Sub}(L/F)$ can be concisely read as "the set of subfields of L containing F"; for example, $\mathbb{Q}(\zeta_3) \in \textsc{Sub}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q})$. Also, recall that we have drawn the lattices associated to $\textsc{Sub}(L/F)$ and $\textsc{Sub}(\mathrm{Aut}(L/F))$ (the latter upside down) in several problems before.

There is a lot it digest when reading the Fundamental Theorem of Galois Theory, but remember that we have observed almost all of it in previous problems. It may be valuable to look back at Problem 7.11 while reading the theorem. Anyway, here we go...

Fact 7.26 (Fundamental Theorem of Galois Theory (for $\mathbb{C}$))**.** Let $F \subseteq L$ be subfields of $\mathbb{C}$. Assume that L is a Galois extension of F.

(1) The following maps are bijections and inverses of each other.

- $\textsc{Sub}(L/F) \to \textsc{Sub}(\mathrm{Aut}(L/F))$ defined by $K \mapsto \mathrm{Aut}(L/K)$,
- $\textsc{Sub}(\mathrm{Aut}(L/F)) \to \textsc{Sub}(L/F)$ defined by $H \mapsto \mathrm{Fix}_L(H)$

(2) The map $K \mapsto \mathrm{Aut}(L/K)$

- reverses inclusions: $K_1 \subseteq K_2$ if and only if $\mathrm{Aut}(L/K_2) \subseteq \mathrm{Aut}(L/K_1)$ and
- sends Galois extensions to normal subgroups: K is a Galois extensions of F if and only if $\mathrm{Aut}(L/K) \trianglelefteq \mathrm{Aut}(L/F)$.

Moreover, if K is a Galois extension of F, then $\mathrm{Aut}(K/F) \cong \mathrm{Aut}(L/F)/\mathrm{Aut}(L/K)$.

(3) For all $K \in \textsc{Sub}(L/F)$,

- $[L : K] = |\mathrm{Aut}(L/K)|$,
- $[K : F] = |\mathrm{Aut}(L/F) : \mathrm{Aut}(L/K)|$, and

Note that since $H \mapsto \mathrm{Fix}_L(H)$ is the inverse of $K \mapsto \mathrm{Aut}(L/K)$, $H \mapsto \mathrm{Fix}_L(H)$ is also inclusion reversing and it sends normal subgroups to Galois extensions.

Let's revisit Problems 6.80 and 7.11 yet again and highlight the connection between Galois extensions and normal subgroups in part (2) of Fact 7.26.

Problem 7.27. Look back at Problem 6.80. Use what you learned in group theory to determine which subgroups of $\mathrm{Aut}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ are normal subgroups. Then use Fact 7.26(2) to determine which subfields of $\mathbb{Q}(\sqrt{2}, i)$ are Galois extensions of $\mathbb{Q}$. You can check you answers by directly verifying which extensions are Galois using the definition.

Problem 7.28. Repeat Problem 7.27 for $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}$. Look at Problem 7.11, and determine which subgroups of $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q})$ are normal subgroups. Then use Fact 7.26(2) to determine which subfields of $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ are Galois extensions of $\mathbb{Q}$.

7.3 A criterion for solvability by radicals

Let's try to apply Galois theory to the problem of determining if a polynomial is solvable by radicals or not. Recall that $p(x) \in \mathbb{Q}[x]$ is *solvable by radicals* over $\mathbb{Q}$ if all of the roots of $p(x)$ are contained in some radical extension of $\mathbb{Q}$; notice that this is the same as requiring that $\mathbb{Q}^{p(x)}$ is contained in some radical extension of $\mathbb{Q}$.

We'll first take a closer look at radical extensions of $\mathbb{Q}$ and then we'll investigate the implications for $\mathbb{Q}^{p(x)}$. Of course, our goal is to find a criterion that we can use to show that some $p(x)$ is *not* solvable by radicals.

7.3.1 Radical extensions, take 2

Let K be any radical extension of $\mathbb{Q}$. Thus, there exist nonzero elements $r_1, r_2, \ldots, r_m \in \mathbb{C}$ and positive integers $n_1, n_2 \ldots, n_m$ such that $K = \mathbb{Q}(r_1, r_2, \ldots, r_m)$, and

$$r_1^{n_1} \in \mathbb{Q}, r_2^{n_2} \in \mathbb{Q}(r_1), r_3^{n_3} \in \mathbb{Q}(r_1, r_2), \ldots, r_m^{n_m} \in \mathbb{Q}(r_1, \ldots, r_{m-1}).$$

Now, K might not be a Galois extension of $\mathbb{Q}$, so we'll try to expand K to a possibly larger radical extension L in such a way that L is a Galois extension of $\mathbb{Q}$ *and* that, as we iteratively add in elements, each field in the sequence is a Galois extension of the one that comes before it. Let's consider

$$L = \mathbb{Q}(\zeta_{n_1}, r_1, \zeta_{n_2}, r_2, \ldots, \zeta_{n_m}, r_m).$$

Lemma 7.29. The field L is a radical extension of $\mathbb{Q}$ and $K \subseteq L$.

Let's now look at L as a series of extensions. Note that our definitions of L_i and F_i below imply that $L_i = F_i(r_i)$ and $F_i = L_{i-1}(\zeta_{n_i})$.

$$\begin{array}{c} L = L_m = \mathbb{Q}(\zeta_{n_1}, r_1, \zeta_{n_2}, r_2, \ldots, \zeta_{n_{m-1}}, r_{m-1}, \zeta_{n_m}, r_m) \\ \cup| \\ F_m = \mathbb{Q}(\zeta_{n_1}, r_1, \zeta_{n_2}, r_2, \ldots, \zeta_{n_{m-1}}, r_{m-1}, \zeta_{n_m}) \\ \cup| \\ \vdots \\ \cup| \\ L_2 = \mathbb{Q}(\zeta_{n_1}, r_1, \zeta_{n_2}, r_2) \\ \cup| \\ F_2 = \mathbb{Q}(\zeta_{n_1}, r_1, \zeta_{n_2}) \\ \cup| \\ L_1 = \mathbb{Q}(\zeta_{n_1}, r_1) \\ \cup| \\ F_1 = \mathbb{Q}(\zeta_{n_1}) \\ \cup| \\ L_0 = \mathbb{Q} \end{array}$$

The next task to to show that each field in the sequence is a Galois extension of the one below it—Theorems 7.7 and 7.8 do most of the work.

Lemma 7.30. Each L_i is a Galois extension of F_i, and each F_i is a Galois extension of L_{i-1}.

We now apply the Fundamental Theorem of Galois Theory to our chain of extensions. Importantly, Fact 7.26(2), implies that $\mathrm{Aut}(L/L_i) \trianglelefteq \mathrm{Aut}(L/F_i)$ and $\mathrm{Aut}(L/L_i)/\mathrm{Aut}(L/F_i) \cong \mathrm{Aut}(L_i/F_i)$. A similar statement holds for each extension F_i over L_{i-1}, and we get the following picture.

$$
\begin{array}{ll}
L = L_m & \{\mathrm{id}\} \\
\cup\mid & \mid\triangle \quad \Big\} \cong \mathrm{Aut}(L/F_m) \\
F_m & \mathrm{Aut}(L/F_m) \\
\cup\mid & \mid\triangle \\
\vdots & \vdots \qquad\qquad \vdots \\
\cup\mid & \mid\triangle \\
L_2 & \mathrm{Aut}(L/L_2) \\
\cup\mid & \mid\triangle \quad \Big\} \cong \mathrm{Aut}(L_2/F_2) \\
F_2 & \mathrm{Aut}(L/F_2) \\
\cup\mid & \mid\triangle \quad \Big\} \cong \mathrm{Aut}(F_2/L_1) \\
L_1 & \mathrm{Aut}(L/L_1) \\
\cup\mid & \mid\triangle \quad \Big\} \cong \mathrm{Aut}(L_1/F_1) \\
F_1 & \mathrm{Aut}(L/F_1) \\
\cup\mid & \mid\triangle \quad \Big\} \cong \mathrm{Aut}(F_1/\mathbb{Q}) \\
L_0 = \mathbb{Q} & \mathrm{Aut}(L/\mathbb{Q})
\end{array}
$$

We'll now investigate the structure of each of the corresponding Galois groups, starting with $\mathrm{Aut}(L_i/F_i)$.

Lemma 7.31. Consider the field $L_i = F_i(r_i)$. The minimal polynomial of r_i over F_i is a factor of $x^{n_i} - r^{n_i}$, so the possible elements of $\mathrm{Aut}(L_i/F_i)$ are described by the following table.

	id	ϕ_1	ϕ_2	ϕ_3	$\cdots$	ϕ_{m-1}
$r_i \mapsto$	r_i	$r_i\zeta_{n_i}$	$r_i\zeta_{n_i}^2$	$r_i\zeta_{n_i}^3$	$\cdots$	$r_i\zeta_{n_i}^{m-1}$

Corollary 7.32. The group $\mathrm{Aut}(L_i/F_i)$ is abelian.

We now investigate $\mathrm{Aut}(F_i/L_{i-1})$ and obtain a similar result.

Lemma 7.33. Consider the field $F_i = L_{i-1}(\zeta_{n_i})$. The minimal polynomial of ζ_{n_i} over L_{i-1} is a factor of $x^{n_i} - 1$, and the possible elements of $\mathrm{Aut}(F_i/L_{i-1})$ are described by the following table.

	id	ϕ_2	ϕ_3	$\cdots$	ϕ_{n_i-1}
$\zeta_{n_i} \mapsto$	ζ_{n_i}	$\zeta_{n_i}^2$	$\zeta_{n_i}^3$	$\cdots$	$\zeta_{n_i}^{n_i-1}$

Corollary 7.34. The group $\mathrm{Aut}(F_i/L_{i-1})$ is abelian.

We've learned a lot about L, or, more specifically, about $\mathrm{Aut}(L\mathbb{Q})$. We see now that $\mathrm{Aut}(L/\mathbb{Q})$ has a chain of subgroups, each normal in the next, such that the corresponding quotient groups are abelian. Let's name this property.

Definition 7.35. Let G be a group with identity 1. We say that G is a **solvable** group if there exists a chain of subgroups

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_k = G$$

such that for all $1 \leq i \leq k$, the quotient group H_i/H_{i-1} is abelian.

Using this new language, we can summarize our findings above as follows.

Fact 7.36. If $p(x) \in \mathbb{Q}[x]$ is solvable by radicals over $\mathbb{Q}$, then $\mathbb{Q}^{p(x)}$ is contained in a subfield L of $\mathbb{C}$ for which

(1) L is a Galois extension of $\mathbb{Q}$, and

(2) $\mathrm{Aut}(L/\mathbb{Q})$ a solvable group.

7.3.2 The criterion

Notice that Fact 7.36 tells us a lot about L, but on the surface, it doesn't seem to address $\mathbb{Q}^{p(x)}$. However, by the definition of $\mathbb{Q}^{p(x)}$, we know that $\mathbb{Q}^{p(x)}$ is a Galois extension of $\mathbb{Q}$.

Applying the Fundamental Theorem of Galois Theory (specifically Fact 7.26(2)) to the sequence $\mathbb{Q} \subseteq \mathbb{Q}^{p(x)} \subseteq L$, we find that $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q}) \cong \mathrm{Aut}(L/\mathbb{Q})/\mathrm{Aut}(L/\mathbb{Q}^{p(x)})$. Thus, $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ is isomorphic to a quotient group of $\mathrm{Aut}(L/\mathbb{Q})$. The following fact from group theory now applies.

Fact 7.37. Suppose that G is a solvable. Then every subgroup of G and every quotient group of G is also a solvable group.

The implication is that if $\mathrm{Aut}(L/\mathbb{Q})$ is a solvable group, then $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ is also a solvable group. Putting everything together, we get the following lovely (and quite useful) test to determine if a polynomial is solvable by radicals.

Fact 7.38 (Solvability by Radicals Criterion)**.** If $p(x) \in \mathbb{Q}[x]$ is solvable by radicals over $\mathbb{Q}$, then $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ a solvable group.

So, if we can find some $p(x) \in \mathbb{Q}[x]$ for which $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ is *not* a solvable group, then we will be able to conclude that $p(x)$ is *not* solvable by radicals over $\mathbb{Q}$.

Incidentally, the converse of Fact 7.38 is also true! This means that $p(x) \in \mathbb{Q}[x]$ is solvable by radicals over $\mathbb{Q}$ *if and only if* $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ a solvable group.

Chapter 8

End game

Chapter 7 finished with a criterion, given as Fact 7.38, that can be used to show that a polynomial $p(x)$ is not solvable by radicals over $\mathbb{Q}$. It says that if $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ is *not* a solvable group, then $p(x)$ is not solvable by radicals over $\mathbb{Q}$. We are extremely close to our goal. Here begins the end game.

8.1 Solvable groups

To better understand how we might apply Fact 7.38, let's try to collect some examples of solvable groups as well as some examples of groups that are not solvable.

We'll initially focus on groups that have arisen as we studied $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ for various polynomials $p(x)$; here are the groups that we encountered.

- If $p(x) = (x^2-2)(x^2+1)$, then $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q}) \cong V_4$. See Problems 6.79 and 7.17.
- If $p(x) = x^4+x^3+x^2+x+1$, then $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q}) \cong \mathbb{Z}_4$. See Examples 6.73 and 7.16 and Problem 6.74.
- If $p(x) = x^3-2$, then $\mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q}) \cong S_3$. See Problems 7.10 and 7.18.

We know that each of the polynomials listed above are solvable by radicals, so by Fact 7.38, each of the associated automorphism groups must be solvable. However, let's see this directly from the definition of a solvable group (Definition 7.35). Since V_4 and $\mathbb{Z}_4$ are both abelian groups, the next theorem confirms that both groups are solvable.

Theorem 8.1. Every abelian group is a solvable group.

Let's address S_3 next. In showing S_3 is solvable, there are various theorems from group theory that are helpful. The next fact highlights one of them. Recall that $[G:H]$ denotes the *index* of H in G, which is the number of left cosets of H in G. In practice, $[G:H]$ is often computed using Lagrange's Theorem, which is also given below.

Fact 8.2. Suppose that G is a group, and $H \leq G$. If $[G:H] = 2$, then then $H \trianglelefteq G$.

Fact 8.3 (Lagrange's Theorem)**.** If G is a finite group and $H \leq G$, then $|G| = [G:H] \cdot |H|$.

Problem 8.4. Let's show that S_3 is a solvable group. Recall that $R = \{\text{id}, (123), (132)\}$ is a subgroup of S_3. Consider the chain of subgroups

$$\{\text{id}\} \leq R \leq S_3.$$

(1) Briefly explain why $\{\text{id}\} \trianglelefteq R$.

(2) Use Fact 8.2 to explain why $R \trianglelefteq S_3$.

(3) Prove that R is abelian.

(4) Compute $|S_3/R|$. What familiar group is S_3/R isomorphic to? Conclude that S_3/R is abelian.

(5) Use Definition 7.35 (and the previous parts) to show that S_3 is a solvable group.

Let's continue looking at the symmetric groups. Is S_4 solvable? What about S_5? Note that these questions are highly relevant to determining if a polynomial is solvable by radicals since Corollary 7.15 tells us that $\text{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$ is isomorphic to a subgroup of S_n where $n = \deg p(x)$.

Problem 8.5. Let's now show that S_4 is a solvable group. We'll focus on two special subgroups: the alternating group A_4, and the group $V = \{id, (12)(34), (13)(24), (14)(23)\}$. Recall that A_4 consists of those permutations that can be written as a product of an *even* number of transpositions. In particular, $V \leq A_4$. Also, exactly half of the elements of S_4 lie in A_4, so $|A_4| = 12$. Consider the chain of subgroups

$$\{\text{id}\} \leq V \leq A_4 \leq S_4.$$

(1) Prove that $V \trianglelefteq A_4$. (In fact, something stronger holds: $V \trianglelefteq S_4$.)

(2) Use Fact 8.2 to explain why $A_4 \trianglelefteq S_4$.

(3) Prove that V is abelian.

(4) Compute $|A_4/V|$. What familiar group is A_4/V isomorphic to? Conclude that A_4/V is abelian.

(5) Compute $|S_4/A_4|$. What familiar group is S_4/A_4 isomorphic to? Conclude that S_4/A_4 is abelian.

(6) Use Definition 7.35 (and the previous parts) to show that S_4 is a solvable group.

Okay, so what about S_5? As we'll see, S_n is not solvable when $n \geq 5$, and this is *precisely* why there are degree 5 polynomials that are not solvable by radicals. Whoa. Let's start with a couple of lemmas.

Lemma 8.6. Let H be a group, and let $N \trianglelefteq H$. Suppose that H/N is abelian. Then for all $x, y \in H$, we have that $x^{-1}y^{-1}xy \in N$.

Lemma 8.7. Let $n \geq 5$. Let $(a,b,c) \in S_n$ be any 3-cycle. If d and e are such that $1 \leq d,e \leq n$ and a,b,c,d,e are all distinct, then $(a,b,c) = x^{-1}y^{-1}xy$ for $x = (a,d,b)$ and $y = (a,e,c)$.

We're now ready to show that S_n is not solvable when $n \geq 5$. The strategy is to argue by contradiction. If S_n is solvable, then there is a chain of subgroups $\{\mathrm{id}\} = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = S_n$ with each successive quotient an abelian group.

Working with the group $S_n/H_{k-1} = H_k/H_{k-1}$ and using Lemmas 8.6 and 8.7, we can try to show that all of the 3-cycles must be contained in H_{k-1}. Then, if H_{k-1} contains all of the 3-cycles, we can repeat the argument in the group H_{k-1}/H_{k-2} to show that all of the 3-cycles must be contained in H_{k-2}. Continuing on, we'll eventually arrive at a contradiction.

Theorem 8.8. If $n \geq 5$, then S_n is not a solvable group.

8.2 Checkmate

Let's take another look at a polynomial that's come up several times before:

$$s(x) = x^5 + 5x^4 - 5.$$

We know a little about $\mathrm{Aut}(\mathbb{Q}^{s(x)}/\mathbb{Q})$, but seemingly not so much. Let's set $A = \mathrm{Aut}(\mathbb{Q}^{s(x)}/\mathbb{Q})$, and review what we know about A.

I. By Corollary 7.15, A can be viewed as a subgroup of S_5.

II. By Problem 7.24 (which relied on Theorem 7.23), A contains a transposition.

But in fact, we know a bit more.

Problem 8.9. Let α be a root of $s(x)$. Recall that $s(x)$ is irreducible by EIC, so $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 5$ by Theorem 6.40.

(1) Use Fact 6.44 to explain why $[\mathbb{Q}^{s(x)} : \mathbb{Q}]$ is divisible by 5.

(2) Use Fact 7.9 to explain why $|A|$ is divisible by 5.

We'll now invoke an important result from group theory: Cauchy's Theorem. Note that Lagrange's Theorem (Fact 8.3) implies that the order of any element of a finite group divides the order of the group—Cauchy's Theorem can be viewed as a partial converse.

Fact 8.10 (Cauchy's Theorem)**.** Let p be a prime. If G is any finite group such that $|G|$ is divisible by p, then G contains an element of order p.

Applying Cauchy's Theorem to A (in light of Problem 8.9), we see that A has an element of order 5. Let's add this to our list of observations from above.

III. A has an element of order 5.

It turns out that our list is quite restrictive. We know that A contains a transposition and an element of order 5, but then A also contains everything that those two elements generate. To explore what these elements generate, let's start by recalling a basic fact from group theory.

Fact 8.11. The set of transpositions is a generating set for S_n.

Fact 8.11 is a launching point for finding other generating sets for S_n. Here's another.

Fact 8.12. If $(a_1, a_2, \ldots, a_n)$ is any n-cycle in S_n, then $(a_1, a_2, \ldots, a_n)$ together with the transposition (a_1, a_2) generate S_n.

Fact 8.12 has an extremely important implication for us.

Theorem 8.13. The group S_5 is generated by any element of order 5 together with any transposition. Consequently, if a subgroup of S_5 contains both an element of order 5 and a transposition, then the subgroup is all of S_5.

So here we are. It's time to tie everything together to prove the Main Theorem. Combining our three observations above with Theorem 8.13, we see that A is isomorphic to S_5. Then Theorem 8.8 applies, and we find that A is *not* a solvable group. Finally, we invoke Fact 7.38.

Theorem 8.14. The polynomial $s(x) = x^5 + 5x^4 - 5$ is *not* solvable by radicals over $\mathbb{Q}$.

The End

Appendix A

Hints

Below are some hints, which should be interpreted as possible (but not the only!) ways to get started.

Hint (Theorem 2.4)**.** You are solving $x^2 + bx + c = 0$. Try "completing the square" first; then solve for x.

Hint (Problem 3.9)**.** Multiplying a fraction by the complex conjugate of the denominator can be an effective way to simplify an expression.

Hint (Theorem 3.11)**.** Think back to changing from polar to rectangular coordinates (or parametrizing circles or solving triangles).

Hint (Theorem 3.12)**.** Try using Theorem 3.11 + trigonometric identities.

Hint (Problem 3.20)**.** You want to find a z such that $z^4 = w$. You are working with powers, so try writing z in the form $z = r\cos\theta + ir\sin\theta$. Now you can use Corollary 3.14 to simplify z^4 and compare with the polar form of w. What can you deduce about r and θ?

Hint (Lemma 3.22)**.** Similar to Problem 3.20, try writing z in the form $z = r\cos\theta + ir\sin\theta$. Now, what does $z^n = 1$ imply about r and θ?

Hint (Lemma 3.23)**.** It may be helpful to draw some pictures first. Try plotting ζ_8, $(\zeta_8)^2$, $(\zeta_8)^3$, ..., $(\zeta_8)^8$, $(\zeta_8)^{14}$, $(\zeta_8)^{85}$. Now, you know by a previous problem that $(\zeta_n)^n = 1$, so also $(\zeta_n)^{2n} = 1$ and so on. Try (using the division algorithm) to write $k = qn + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r \leq n-1$ and plug that into $(\zeta_n)^k$.

Hint (Theorem 3.24)**.** You may want to view this as the following "if and only if" statement: z is an n^{th} root of $1 \iff z = (\zeta_n)^k$ for some $0 \leq k \leq n-1$. Now make use of the previous lemma and theorems you proved. Don't forget to explain why each of $1, \zeta_n, (\zeta_n)^2, \ldots, (\zeta_n)^{n-1}$ are all different.

Hint (Theorem 3.28)**.** Suppose that z is a root of $p(x)$. Then $p(z) = 0$, so $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0$. This last equation is is just comparing two complex numbers—try taking the conjugate of both sides. Fact 3.5 is helpful.

Hint (Problem 3.40). You are trying to find $(a+b\sqrt{5})^{-1} = \frac{1}{a+b\sqrt{5}}$. Try multiplying top and bottom by the conjugate: $a-b\sqrt{5}$.

Hint (Theorem 3.51). For the first part, notice that $x \cdot 0 = x(0+0)$. For the last part, remember that the definition of a field ensures that F has at least two elements, so there is some $a \in F$ with $a \neq 0$. Now, what happens if $0 = 1$?

Hint (Theorem 3.54). The crux is to show that every nonzero element has a multiplicative inverse when n is prime. Let $a \in (\mathbb{Z}_n)^*$. You need to find some integer b such that $ab = 1$ modulo n. Now, since $a \in (\mathbb{Z}_n)^*$ and n is prime, $\gcd(a,n) = 1$. By Bézout's Lemma, there exist $k,l \in \mathbb{Z}$ such that $1 = ka + ln$. What happens when you consider the equation $1 = ka + ln$ modulo n?

Hint (Problem 3.58). If T_3 is a subfield, then, in particular, it is closed under multiplication, so it must be that $\alpha^2 \in T_3$. That means that $\alpha^2 = a + b\alpha$ for some $a,b \in \mathbb{Q}$. What does this imply?

Hint (Problem 3.65). Try following the approach in Example 3.61. First show $\{a+bi \mid a,b \in \mathbb{Q}\} \subseteq \mathbb{Q}(i)$ by showing that every subfield that contains $\mathbb{Q}$ and i must also contain $\{a+bi \mid a,b \in \mathbb{Q}\}$. To show the reverse containment, use the fact that $\{a+bi \mid a,b \in \mathbb{Q}\}$ is a subfield, by a previous problem.

Hint (Problem 3.68). Remember, in Problem 3.58(3), we saw that $\{a+b\alpha \mid a,b \in \mathbb{Q}\}$ is *not* a subfield of $\mathbb{C}$.

Hint (Problem 3.70). Use the previous theorem. To show $\mathbb{Q}\left(3-\sqrt{2}, 5+i\right) \subseteq \mathbb{Q}\left(\sqrt{2}, i\right)$, you need to show that $\mathbb{Q} \subset \mathbb{Q}\left(\sqrt{2}, i\right)$ and that $3-\sqrt{2}, 5+i \in \mathbb{Q}\left(\sqrt{2}, i\right)$. Then show the reverse containment in a similar way.

Hint (Theorem 4.12). Note that $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1)$. Now use Theorem 3.24; note that $x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1$ should only have $n-1$ roots.

Hint (Problem 4.14). First find the roots of $z^2 - 3z - 1$. Then, for each of those roots, use Theorem 3.26 to solve for z. You should have 6 different roots in the end.

Hint (Theorem 5.20). Try a proof by contradiction. Assume that u is a unit and that u is a zero divisor. Now, what does the definition of being a zero divisor tell you about u?

Hint (Theorem 5.33). To get started, let $n = \deg p(x)$ and $m = \deg q(x)$, and then write $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$ and $q(x) = b_0 + b_1 x + \cdots + b_m x^m$ with $b_m \neq 0$. You want to understand the degree of $p(x) + q(x)$, so you need to determine the largest power of x in the sum $p(x) + q(x)$.

Hint (Theorem 5.35). As with the previous theorem, let $n = \deg p(x)$ and $m = \deg q(x)$, and then write $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$ and $q(x) = b_0 + b_1 x + \cdots + b_m x^m$ with $b_m \neq 0$. You need to determine the largest power of x in the product $p(x)q(x)$. What do you think is the largest power of x in the product $p(x)q(x)$? What is its coefficient, and how do you know it's not zero?

Hint (Corollary 5.37)**.** There are several things to verify to ensure that $D[x]$ is an integral domain, but we've talked about most of them already. The main thing that remains is to prove that $D[x]$ has no zero divisors—try a proof by contradiction. This is a corollary of Theorem 5.35, which means that it should be "not too hard" to prove using Theorem 5.35.

Hint (Theorem 5.43)**.** One approach is to polish up and fill in the gaps of the outline presented in the notes right before the statement of Theorem 5.43. A related, but slightly different, approach is to try using induction on the degree of $a(x)$.

Hint (Theorem 5.44)**.** Try using the division algorithm to write $a(x) = (x-c)q(x) + r(x)$ for some $q(x), r(x) \in F[x]$ with $\deg r(x) < \deg(x-c)$ or $r(x) = 0$. Now show that $r(x)$ must be the zero polynomial.

Hint (Lemma 5.51)**.** First, explain why $d_1(x)$ must divide $d_2(x)$ and why $d_2(x)$ must divide $d_1(x)$. Now return to the definition of "to divide" and see what you can write down.

Hint (Theorem 5.53)**.** Follow the definitions. Since $c(x) \in I$, it can be written a particular way. Then write down what it means for $h(x)$ to divide both $a(x)$ and $b(x)$. Combine.

Hint (Theorem 5.60)**.** For the forward direction, start with the definition of a unit and apply the degree function. For the reverse direction, what does $\deg p(x) = 0$ imply about $p(x)$? Can you explicitly write down the a multiplicative inverse for $p(x)$?

Hint (Theorem 5.64)**.** Consider using Theorem 5.40.

Hint (Theorem 5.67)**.** Consider using using strong induction on the degree of the polynomial. Let $\varphi(n)$ be the statement "every polynomial in $F[x]$ of degree n can be written as a product of polynomials that are irreducible in $F[x]$."

For the base case, you want to show that $\varphi(1)$ is true. Assume that $p(x) \in F[x]$ has degree 1. Then what?

Next, assume that $\varphi(k)$ is true for all $1 \leq k \leq n$. We need to show that $\varphi(n+1)$ is true. Assume that $p(x) \in F[x]$ has degree $n+1$. There are two cases to consider: $p(x)$ is irreducible or $p(x)$ is reducible. Keep going...

Hint (Problem 5.79)**.** Use the division algorithm to write $a(x) = (x^2+1)q(x) + r(x)$. What does this tell you?

Hint (Theorem 5.86)**.** Using Fact 5.76, you know that R/I is ring. So, for the first part, assume R is commutative, and use this to show R/I is commutative. The starting point is to choose two arbitrary elements of R/I, which would be something like $a+I$ and $b+I$ for $a, b \in R$. Now show that $(a+I)(b+I) = (b+I)(a+I)$ using the definition of multiplication in Fact 5.76.

Hint (Problem 5.80)**.** For the second part, remember that $a \equiv_6 b \iff a-b$ is a multiple of 6. For the last, use the division algorithm to write $a = 6q + r$. What does this imply?

Hint (Theorem 5.83)**.** By definition of an ideal, $I \subseteq R$, so what we really need to show is that $R \subseteq I$. Remember that I is closed under multiplication by elements of R. So, if $a \in I$, then $ra \in R$. Try to first show that $1 \in R$.

Hint (Theorem 5.85)**.** Theorem 5.83 should help with the forward direction. For the backward direction, let $a \in R^*$; you need to show a has an inverse. Try using Theorem 5.82: the set $I = \{ar \mid r \in R\}$ is an ideal. By assumption, $I = \{0\}$ or $I = R$. Which is it? Notice that if $I = R$, then $1 \in I$.

Hint (Problem 5.96)**.** Use Theorem 5.94. Theorem 5.83 may also be helpful.

Hint (Theorem 5.99)**.** Try using Theorem 5.91.

Hint (Theorem 5.119)**.** Assume $\phi : R \to S$ is a ring homomorphism. We need to define a suitable homomorphism from $R/\ker\phi$ to $\phi(R)$, and then check that it is bijective. Let's let $K := \ker\phi$. Try defining $\hat{\phi} : R/K \to \phi(R)$ via $\hat{\phi}(a+K) = \phi(a)$. A very important point, is that we don't actually know that $\hat{\phi}$ is a well-defined function. We know that a coset $a+K$ might be equal to $a'+K$, so we'd better make sure that if $a+K = a'+K$ then $\hat{\phi}(a+K) = \hat{\phi}(a'+K)$. Do that first. Then, verify that $\hat{\phi}$ is a homomorphism that is also surjective and injective. For injectivity, it may be useful to use Theorem 5.117 and instead show that $\ker\hat{\phi} = \{0+K\}$.

Hint (Theorem 5.121)**.** We want to show that $\phi(I)$ is an ideal of S. Elements of $\phi(I)$ look like $\phi(a)$ for some $a \in I$. To show that $\phi(I)$ is a subring of S, let $\phi(a_1), \phi(a_2) \in \phi(I)$ for some $a_1, a_2 \in I$. Now explain why $\phi(a_1) + \phi(a_2)$, $\phi(a_1)\phi(a_2)$, and $-\phi(a_1)$ are all in $\phi(I)$. You also should say why $\phi(I)$ is nonempty. Finally, you also need to show that for all $s \in S$, $s\phi(a_1)$ is in $\phi(I)$. Remember that ϕ maps *onto* S, so $s = \phi(r)$ for some $r \in R$. Now keep going.

Hint (Theorem 5.122)**.** We want to show that $\phi^{-1}(J)$ is an ideal of R. Let $a_1, a_2 \in \phi^{-1}(J)$. This means that $\phi(a_1), \phi(a_2) \in J$. To show that $a_1 + a_2$, $a_1 a_2$, and $-a_1$ are in $\phi^{-1}(J)$, you just need to show that $\phi(a_1) + \phi(a_2)$, $\phi(a_1)\phi(a_2)$, and $-\phi(a_1)$ are all in J (using that $a_1, a_2 \in \phi^{-1}(J)$ and J is an ideal). You also need to show that $ra_1 \in \phi^{-1}(J)$, and to do that, you need to show that $\phi(ra_1) \in J$.

Hint (Problem 6.4)**.** Notice that $\alpha^2 = 2 + 2\sqrt{2}i - 1$, so $\alpha^2 - 1 = 2\sqrt{2}i$. What happens if you square both sides?

Hint (Lemma 6.5)**.** Towards a contradiction, assume that $m(x)$ is reducible. By Theorem 5.61, $m(x) = a(x)b(x)$ for some $a(x), b(x) \in F[x]$ with $\deg a(x)$ and $\deg b(x)$ both smaller than $\deg m(x)$. Now, $m(x) \in I$, so $0 = m(\alpha) = a(\alpha)b(\alpha)$. Explain why this implies that $a(x)$ or $b(x)$ is in I. But $I = (m(x))$, so by Theorem 5.91, $m(x)$ divides $a(x)$ or $b(x)$. What's the contradiction?

Hint (Problem 6.9)**.** Theorem 4.12 might provide some inspiration.

Hint (Problem 6.22)**.** To see why the degree of $m(x)$ can not be 3, suppose it is. Then $x^4 - 2x^2 + 9 = m(x)q(x)$ for some $q(x) \in \mathbb{Q}[x]$ with $\deg q(x) = 1$. Explain why $q(x)$ has a root that lies in $\mathbb{Q}$. But the root of $q(x)$ is a root of $x^4 - 2x^2 + 9$, so find the roots of $x^4 - 2x^2 + 9$ (and thus a contradiction).

Hint (Theorem 6.58)**.** Consider using Theorem 5.111 and remember that $c = c \cdot 1$.

Hint (Theorem 7.8)**.** Let $p(x) = x^n - r^n$. Why is $p(x) \in F[x]$? Can you show that $F(r) = F^{p(x)}$?

Hint (Theorem 7.19)**.** First use Theorem 6.61 to show that γ maps R to itself, then make use of the fact that γ is an injective function.

Next, use results from Chapter 5 (including the First Isomorphism Theorem for rings) to show that γ is an isomorphism from $\mathbb{Q}^{p(x)}$ to $\gamma(\mathbb{Q}^{p(x)})$ and that γ fixes $\mathbb{Q}$. Then, to show that $\gamma \in \mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$, it only remains to show that $\mathbb{Q}^{p(x)} = \gamma(\mathbb{Q}^{p(x)})$. Let $r_1, \ldots, r_n$ be the roots of $p(x)$, so $\mathbb{Q}^{p(x)} = \mathbb{Q}(r_1, \ldots, r_n)$. Use the definition of $\mathbb{Q}(r_1, \ldots, r_n)$ to show that $\gamma(\mathbb{Q}(r_1, \ldots, r_n)) = \gamma(\mathbb{Q})(\gamma(r_1), \ldots, \gamma(r_n))$ then explain why $\gamma(\mathbb{Q})(\gamma(r_1), \ldots, \gamma(r_n)) = \mathbb{Q}(r_1, \ldots, r_n)$.

Hint (Theorem 7.23)**.** Let γ be complex conjugation. By Theorem 7.19, $\gamma \in \mathrm{Aut}(\mathbb{Q}^{p(x)}/\mathbb{Q})$. What does γ do to the real roots of $p(x)$? What about to those that are not real?

Hint (Lemma 8.6)**.** You want to show that $x^{-1}y^{-1}xy \in N$. Look back at properties of cosets to see that this is equivalent to showing that $(x^{-1}y^{-1}xy)N = N$ in the quotient group H/N. Work in H/N, and compute $(x^{-1}N)(y^{-1}N)(xN)(yN)$. Don't forget that H/N is abelian.

Hint (Theorem 8.13)**.** Let σ be an element of order 5 and τ a transposition. First explain why σ must be a 5-cycle. Then notice that we can write $\sigma = (a, b, c, d, e)$ and $\tau = (a, x)$ where $x \in \{b, c, d, e\}$. Try to use Fact 8.12. If $x = b$, you can directly apply Fact 8.12; if not, consider σ^2, σ^3,...