1. Overview

The theme for this conference is connections between curves, modular forms, L-functions, and their computational aspects. Our goals are to provide a setting to

- allow dissemination of recent research results;
- foster collaboration among experts working in areas which are (conjecturally) related;
- aid in the development of number theorists who are early-career or from underrepresented groups; and
- make relevant computational results publicly available.

The format of the event will combine elements of both a conference and a workshop to achieve its goals. Each morning will feature two plenary talks given by invited speakers, starting with background for their area and leading to recent results. Afternoons will be devoted to participants working in small groups. Some groups will work on questions raised by the morning lectures, some will work on computations, and others will work on presenting computational results to the public.

Computational results will be made available through the website *L-Functions and Modular Forms Database* (`http://www.lmfdb.org`), referred to as the LMFDB. This site is an international collaboration dedicated to presenting computational data in number theory and related fields to assist mathematicians conducting research and also to help educate visitors to the site about some of the central objects of modern number theory and their interconnections.

Computational projects provide a good path for young mathematicians to start to get a feel for an area. Most of the senior participants will be well-versed in doing computations in number theory. We will also have experts from the LMFDB project in attendance to aid people who may be contributing to that project for the first time. Since the LMFDB uses Sage, an open-source computational system, we will invite a significant number of participants from the *Women in Sage* community to help broaden the representation of women in this conference.

Theoretical and computational work on elliptic curves over the rationals has enjoyed great advances over the past few decades. Recently there has been surge of work on extending this in two directions, namely to hyperelliptic curves over the rationals and elliptic curves over quadratic fields. In addition, there have been parallel advances on the corresponding automorphic objects, namely Siegel and Hilbert modular forms. A major objective of this conference is to present this recent work to a wider audience and in the process to clarify the connections between the algebraic and automorphic objects. There is a large amount of useful related numerical data which we would like to make available to the mathematics research community.

Recent advances to be covered in workshop lectures:

(1) Elliptic curves over real quadratic fields: one or two lectures
    William Stein and collaborators have been developing the analogue of Cremona's tables for elliptic curves over $\mathbb{Q}(\sqrt{5})$. There are numerous difficulties which are not present in the rational case and also many recent methods for overcoming these difficulties.
(2) Hilbert modular forms: one or two lectures
    John Voight, in collaboration with Steve Donnelly, has made extensive tables of Hilbert modular forms, including over 200,000 forms over totally real fields of degree at most 6 and including many forms over quadratic fields. The basic theory, the computational methods, and the associated L-functions (which provide the connection to elliptic curves) will be described.
(3) Hyperelliptic curves: two lectures
    Kiran Kedlaya, Andrew Sutherland, and their collaborators have undertaken a systematic investigation of the Sato-Tate group of hyperelliptic curves (primarily of genus 2, but also some in genus 3) and consequently have tabulated a wide variety of examples.
(4) Paramodular forms: one lecture
    Siegel modular forms on the paramodular group are (conjecturally) the modular objects associated to genus 2 hyperelliptic curves. Cris Poor and David Yuen have produced the first examples of these objects.

(5) Elliptic curves over non-totally real number fields: two lectures
Elliptic curves over imaginary quadratic fields are fundamentally different than elliptic curves over real quadratic fields, primarily because the associated modular object is a Bianchi modular form rather than a Hilbert modular form, and consequently even the most basic construction of an elliptic curve from a Bianchi form is still unknown. Even with these limitations, John Cremona and his students, and more recently Dan Yasaki, have made tables of elliptic curves over imaginary quadratic fields such as $\mathbb{Q}(i)$.

(6) L-function techniques for hyperelliptic curves: one lecture
Current methods of producing tables of hyperelliptic curves are not capable of proving that the list is complete. David Farmer, Sally Koutsoliotas, and Stefan Lemurell have an L-function approach which (assuming modularity) can provide the missing step in the proof.

2. Scientific content

2.1. **Hyperelliptic curves over $\mathbb{Q}$.** In this workshop we will focus on recent work concerning hyperelliptic curves of genus 2, that is, a smooth curve of the form

$$y^2 = f(x) \tag{2.1}$$

where $f \in k[x]$ is a polynomial of degree 5 or 6 and k is a number field. Our specific focus is on the recent work of Kedlaya and Sutherland and their collaborators on the distribution of Euler factors of the L-function of the curve [9, 10].

Via their Jacobians, this work concerns abelian surfaces; at the workshop we will bring out the connections with other objects. Indeed, it is not the hyperelliptic curve itself but rather its Jacobian A to which we can associate an L-function. There are (up to conjugacy) 52 subgroups of $\mathrm{USp}(4)$ which determine the distribution of Euler factors of the L-function of A. That is, for each abelian surface A there is a closed subgroup of $\mathrm{USp}(4)$, called the Sato-Tate group of A, such that the local factors of the L-function of A have the same limiting distribution as the characteristic polynomials of matrices in the subgroup. Kedlaya and Sutherland exhibit hyperelliptic curves whose Jacobians give examples of all 52 cases, of which 34 occur for $k = \mathbb{Q}$.

Workshop lectures will describe how the Sato-Tate group arises from the Galois action on the Tate module of A and how the subgroups were classified. While some of this material can be described for any number field, we will primarily consider hyperelliptic curves over the rationals. In this case the L-function has degree 4, which makes computation more tractable, and there are connections to objects from other lectures, such as Siegel modular forms. The role of the Sato-Tate group in identifying the associated modular object and determining its properties will be the starting point for discussions on future avenues of research.

2.2. **Elliptic curves over low degree number fields.** An elliptic curve over a number field k has a L-function of degree $2d$, where $d = [k : \mathbb{Q}]$. We will be primarily concerned with quadratic fields, because of their relation to other objects already in the LMFDB.

In the case of real quadratic fields, lectures will focus on the computational challenges faced by Stein and his collaborators as they tabulate elliptic curves over $\mathbb{Q}(\sqrt{5})$ [1]. There are numerous aspects of that work which are hampered by our limited knowledge of this area; this discussion will be the starting point for research project involving both theory and computation. The modular object associated to $E/\mathbb{Q}(\sqrt{5})$ is a Hilbert modular form on GL_2 over $\mathbb{Q}(\sqrt{5})$, which will be the topic of another series of lectures.

In the case of number fields which are not totally real, there has been work led by Cremona on computing elliptic fields over quadratic imaginary fields [3–5]. Here the associated automorphic forms are conjectured to be Bianchi modular forms, although neither direction of the correspondence is known. More recent work by Gunnells, Hajir, Klages-Mundt, and Yasaki [6–8] has started to look at elliptic curves over larger degree fields, including nonnormal extensions of $\mathbb{Q}$.

2.3. **Hilbert modular forms.** Hilbert modular forms are a generalization of classical modular forms (on GL_2) to totally real fields F. Although the associated Hilbert modular variety associated to a field and level can in general have quite large dimension, the Jacquet-Langlands correspondence implies that one can see the same system of Hecke eigenvalues on quaternionic forms of GL_2 and thereby work on either a Shimura curve or with a definite quaternion algebra.

Methods for computing Hilbert modular forms have seen substantial development in the past five years and now there are large databases of forms available. Lectures on Hilbert modular forms would discuss their algorithmic aspects, focusing on concrete examples and future directions of research.

2.4. **Siegel modular forms.** Currently, algorithmic methods for computing Siegel modular forms is quite limited compared to the computation of either classical or Hilbert modular forms. For higher weight there is the possibility of employing cohomological methods, the analogue of modular symbols for classical holomorphic modular forms. However, for the weight 2 case, which is associated to hyperelliptic curves, this is not an option. Specifically, we are interested in weight 2 cusp forms on the level-N paramodular group, defined as

$$K(N) = \left\{ \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} : * \in \mathbb{Z} \right\} \cap \mathrm{Sp}(4, \mathbb{Q}).$$

This group is an an analogue of the Hecke congruence group $\Gamma_0(N)$, in the sense that it is normalized by an analogue of the Fricke involution.

Current methods, primarily due to Poor and Yuen [11, 12], are computationally intensive due to the difficulty of finding a basis for the space of cusp forms. The main technique is via descent from higher weights, and exploiting the "smearing" action of Hecke operators acting on non-cuspforms. Lectures on Siegel modular forms will briefly cover background and computational methods, and then focus on the "paramodular conjecture" [2], which gives a higher rank analogue of modularity for elliptic curves.

2.5. **L-functions.** We consider L-functions of the Selberg class. Each has a Dirichlet series, functional equation, Euler product, and must statisfy few technical conditions. One view of L-functions is that they are the glue which connects related mathematical objects. For example, a geometric object is said to be modular if it has the same L-function as an automorphic form.

In this workshop the general theory of L-functions will be introduced, and this framework will be used to provide a high-level description of the connections between the other objects which have been introduced. There are two topics of current research which will be presented and then will form the basis of later discussions. The first is the recent discovery by Farmer, Koutsoliotas, and Lemurell that it is possible in many cases to generate the L-function of an object without first finding the object itself. This method has been successful for a variety of degree 4 L-functions, but it does not work in all cases. Since the objects being described in the other lectures are associated with degree 4 L-functions, this topic is timely and the methods and current limitations will be described carefully.

The second topic is the fact that high degree L-functions (degree 3 or larger) are computationally expensive to evaluate. This limits our ability to make detailed studies of, for example, the distribution of zeros of these functions. The underlying cause of these computational problems will be described, leading to discussions for possible ways to improve the calculations. One possibility, which is speculative but worth considering, is that properties of the underlying objects could be exploited in some way. This will take advantage of the fact that both the L-functions and their underlying objects will be discussed in detail at the workshop.

3. Logistics

The conference will be held on the main campus of Arizona State University in Tempe, AZ. Tempe can be easily accessed from Phoenix Sky Harbor international airport, which is a hub for a major airline, so is easily accessible.

ASU will provide meeting space and wifi access for conference activities, and the School of Mathematical and Statistical Sciences at ASU will provide secretarial and clerical support for the conference.

4. Personnel

The organizing committee for the conference is David Farmer, Paul Gunnells, John Jones, and Holly Swisher. We have tentative agreements from the following mathematicians to give plenary talks: John Cremona, Kiran Kedlaya, Sally Koutsoliotas, Stefan Lemurell, Cris Poor, William Stein, Andrew Sutherland, John Voight, and David Yuen.

5. Budget

All requested funds are for participant support. We plan to invite 30 mathematicians who we feel will be able to contribute in significant ways to the conference. We are requesting funds to pay for travel, food, and lodging for these people.

We will invite mathematicians all different stages in their careers: graduate students, post-docs and other recent Ph.D.s, junior faculty, and senior faculty (in the last case, primarily to give some of the plenary talks). We will also focus on inviting mathematicians from underrepresented groups. One such group where we expect to have the most success is with female mathematicians. There are natural connections between the LMFDB project and SAGE development. One of the conference organizers has been involved in Women in Sage. We plan to devote at least a fourth of the invitations to members of underrepresented groups.

Attendence at the conference will be open to all mathematicians. We are requesting funding to defray travel costs of a limited number of other participants. Priority for this funding follows the reverse seniority rule: highest priority goes to graduate students, then post-docs and junior faculty, and finally to established faculty who do not have external funding to support their travel. Mathematicians from underrepresented groups are treated one level higher in terms of priority.

The School of Mathematical and Statistical Sciences at ASU has pledged to contribute funds to pay for refreshments during the workshop.

Since we hope to fund almost all participants, ASU is providing meeting space and refreshments, we will not charge participants a registration fee.

6. Dissemination

Dissemination of results from this workshop will be through the LMFDB. This web site is the result of an ongoing international collaboration, with contributions from roughly 50 number theorists, thus far. It serves the mathematical community in two ways.

First, it provides results of rigorous computations of mathematical objects of interest in modern number theory organized around the theme of L-functions. The objects include elliptic curves over the rationals, number fields, and assorted types of automorphic forms (classical, Hilbert, Siegel). Computational data of this sort has proven useful to number theory researchers in formulating and testing conjectures, and in some cases can play a role in the proof of theorems.

Second, the LMFDB plays an educational role for people interested in number theory. It has an overview of number theoretic objects, contains brief explanations regarding the objects it contains, and shows interconnections between them.

For example, when viewing the web page for a specific elliptic curve over $\mathbb{Q}$, users see the important invariants of the curve such as its conductor, rank, and Mordell-Weil generators. They also have links to pages for its associated modular form, isogenous curves, its L-function, and symmetric powers of its L-function.

During the afternoons, we will have partipants extending and improving the LMFDB in ways related to the themes of this workshop. We expect to expand the LMFDB to include sections for hyperelliptic curves over $\mathbb{Q}$, elliptic curves over various quadratic fields, and their associated L-functions. We also expect to improve the areas on Hilbert and Siegel modular forms, and to improve the presentation/expository aspects of the site.

7. Results from current and prior NSF support

PI Jones is currently supported by NSF grant DUE-1226081, "Collaborative Research: Updating the WeBWorK National Problem Library". The project is in its first year. Thus far, PI Jones has made changes to the WeBWorK system to facilitate improvements to the National Problem Library as proposed, and these have been accepted by the WeBWorK developers. The project also involves organizing workshops to have faculty collaborate on the organization of the WeBWorK National Problem Library; the first workshop is scheduled for June, 2013.

References

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