1) $z_1 = a_1 + ib_1 \quad z_2 = a_2 + ib_2$

$$z_1 \cdot z_2 = a_1a_2 + i^2 b_1 b_2 + i a_1 b_2 + i b_1 a_2 = a_1a_2 - b_1b_2 + ia_1b_2 + ib_1a_2 = z_3$$

$$|z_3| = \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2} =$$

$$= \sqrt{a_1^2a_2^2 \cancel{- 2a_1a_2b_1b_2} + b_1^2b_2^2 + a_1^2b_2^2 \cancel{+ 2a_1b_1a_2b_2} + b_1^2a_2^2}$$

$$|z_1| = \sqrt{(a_1)^2 + (b_1)^2} = \sqrt{a_1^2 + b_1^2} \qquad |z_2| = \sqrt{(a_2)^2 + (b_2)^2} = \sqrt{a_2^2 + b_2^2}$$

$$|z_1||z_2| = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + b_1^2a_2^2}$$

$$= |z_3|$$

$$z_3 = \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + b_1^2a_2^2}\,(\cos\theta_3 + i\sin\theta_3)$$

$$z_1 = \sqrt{a_1^2 + b_1^2}\,(\cos\theta_1 + i\sin\theta_1) \quad \sqrt{a_2^2 + b_2^2}\,(\cos\theta_2 + i\sin\theta_2) = z_2$$

$$z_1 z_2 = \sqrt{a_1^2 b_1^2}\sqrt{a_2^2 b_2^2}\,(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= |z_3| \cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2$$

$$= |z_3| \underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + i\underbrace{(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)}_{\sin(\theta_1+\theta_2)}$$

$$z_1 z_2 = |z_3| \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

$z_1 z_2 = z_3$

$$z_3 = |z_3| \cos(\theta_3) + i\sin(\theta_3) \quad \therefore\ \theta_1 + \theta_2 = \theta_3$$

$$\therefore\ \arg(z_3) = \arg(z_1) + \arg(z_2)$$

Mathematica eigen system

2) $\begin{pmatrix} 1 & 1-i & 2i \\ 1+i & 2 & -i \\ -2i & i & 1 \end{pmatrix}$

Eigenvalues

$\det\begin{pmatrix} 1-\lambda & 1-i & 2i \\ 1+i & 2-\lambda & -i \\ -2i & i & 1-\lambda \end{pmatrix}=0$

$(1-\lambda)\left((2-\lambda)(1-\lambda)-(+1)\right)+(1-i)\left((1+i)(1-\lambda)-(-i)(+2i)\right)+2i\left((1+i)(i)-(2-\lambda)(-2i)\right)=0$

$(3+i-\lambda-i\lambda)$ $\quad (-1+5i-2i\lambda)$

$(1-\lambda)\left(\overset{(1-3\lambda+\lambda^2)}{2-3\lambda+\lambda^2-1}\right)-(1-i)(1+i-\lambda-i\lambda+2)+2i(i-1+4i-2i\lambda)=0$

$(1-\cancel{4\lambda}+4\lambda^2-\lambda^3)+(+2\lambda-4+\cancel{2i})+(\cancel{-2i}-10+\cancel{4\lambda})=0$

$-\lambda^3+4\lambda^2+2\lambda-13=0$

$\lambda=-1.696,\ 2.178,\ 3.518$

$\underline{\lambda=-1.696}$

$\begin{pmatrix} 2.696 & 1-i & 2i \\ 1+i & 3.696 & -i \\ -2i & i & 2.696 \end{pmatrix}\begin{pmatrix}1\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad 2.696+(1-i)y+2iz=0$

$1+i+3.696y-iz=0 \qquad 2+2i+7.392y-2iz=0$

$4.696+2i+8.392y-iy=0 \rightarrow \underline{y=-0.524-0.301i}$

$1+i+3.696(-0.524-0.301i)-iz=0 \qquad -0.937-0.112i-iz=0 \qquad -iz=0.937+0.112i$

$z=\frac{-0.937}{i}\frac{(-i)}{(-i)}-0.112=\underline{0.937i-0.112}$

$\left|a\begin{pmatrix}1\\-0.524-0.301i\\0.937i-0.112\end{pmatrix}\right|^2=1 \qquad a^2(1^2+0.524^2+0.301^2+0.937^2+0.112^2)=1 \rightarrow a^2=.443 \quad a=.666$

$\begin{pmatrix}.666\\-.349-.200i\\.624i-.080\end{pmatrix}$

$\lambda = 2.178$

$$\begin{pmatrix} -1.178 & 1-i & 2i \\ 1+i & -0.178 & -i \\ -2i & i & -1.178 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1.178x + (1-i)y + 2iz = 0$$

$$(1+i)x - 0.178y - iz = 0 \rightarrow (1+i)x - 0.178y - i\frac{2ix}{-1.178} + \frac{iy}{1.178} = 0 \rightarrow y = \frac{(1+i)x - \frac{2x}{1.178}}{\frac{1}{1.178} - 0.178}$$

$$-2ix + iy - 1.178z = 0 \rightarrow -1.178z = 2ix - iy \rightarrow z = \frac{2ix}{-1.178} - \frac{iy}{-1.178}$$

$$-1.178x + (1-i)\left(\frac{(1+i)x - \frac{2x}{1.178}}{\frac{1}{1.178} - 0.178}\right) + 2i\left(\frac{2ix}{-1.178} - \frac{i}{-1.178}\right)\left(\frac{(1+i)x - \frac{2x}{1.178}}{\frac{1}{1.178} - 0.178}\right) = 0$$

$$-1.178x + (1-i)\frac{(1+i)x}{.671} - 2.531x + 2i\left(-1.698ix + \frac{i}{1.178}\right)\left(\frac{(1+i)x}{.671} - 2.531x\right) = 0$$

$$-1.178x + 2.981x - 2.531x + (3.396x - 1.698)(-1.041x + 1.490ix)$$

$$-1.178x + 2.981x - 2.531x - 3.534x^2 + 1.768x + 5.060ix^2 - 2.530$$

$$0.54x + 1.526x^2 - 2.53 = 0 \qquad x = 1.123$$

$$y = \frac{(1+i)(1.123) - \frac{2(1.123)}{1.178}}{0.384} = -2.041 + 2.924i$$

$$z = \left(-1.698i(1.123) + \frac{i}{1.178}\right)\left(\frac{(1+i)1.123}{.671} - 2.531 \cdot 1.123\right) = (1.058i)(-1.169 + 1.674i) = 1.237i + 1.771$$

$$(1.123a)^2 + |-2.041a + 2.924ia|^2 + |1.237ia + 1.771a|^2 = 1$$

$$1.261a^2 + 4.166a^2 + 8.550a^2 + 1.530a^2 + 3.136a^2 = 1$$

$$a = .232$$

$$.232\begin{pmatrix} 1.123 \\ -2.041 + 2.924i \\ 1.237i + 1.771 \end{pmatrix}$$

$\lambda = 3.518$

$$\begin{pmatrix} -2.518 & 1-i & 2i \\ 1+i & -1.518 & -i \\ -2i & i & -2.518 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2.518x + y - iy + 2i = 0 \rightarrow -2.518\left(\frac{1}{2}y + 1.259i\right) + y - iy + 2i = 0$

$x + ix - 1.518y - i = 0$

$-2ix + iy - 2.518 = 0 \rightarrow x = \dfrac{2.518 - iy}{-2i} = \frac{1}{2}y + 1.259i$

$-1.259y - 3.170i + y - iy + 2i = 0 \rightarrow -0.259y - 1.17i - iy = 0 \rightarrow y(-0.259 - i) = 1.17i$

$y = \dfrac{1.17i}{-0.259-i}\left(\dfrac{-0.259+i}{-.259+i}\right) = \dfrac{-.303i - 1.17}{.067+1} = \underline{-1.096 - 0.284i}$

$x = \frac{1}{2}(-1.096 - .284i) + 1.259i = \underline{-0.548 + 1.117i}$

$(-.548a)^2 + (1.117a)^2 + (1.096a)^2 + (.284a)^2 + a^2 = 1$

$.300a^2 + 1.248a^2 + 1.201a^2 + .081a^2 + a^2 = 1 \qquad 3.83a^2 = 1 \qquad a = .511$

$$\begin{pmatrix} -.280 + .571i \\ -.560 - .145i \\ .511 \end{pmatrix}$$

3) $|\psi_1\rangle = \frac{1}{\sqrt3}|+\rangle + i\frac{\sqrt2}{\sqrt3}|-\rangle = \begin{pmatrix}\frac{1}{\sqrt3}\\ i\sqrt2/\sqrt3\end{pmatrix}$

$$P_{+z} = \left|\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt3\\ i\sqrt2/\sqrt3\end{pmatrix}\right|^2 = \left|1/\sqrt3\right|^2 = \frac{1}{3}$$

$$P_{-z} = \left|\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt3\\ i\sqrt2/\sqrt3\end{pmatrix}\right|^2 = \left|i\sqrt2/\sqrt3\right|^2 = \frac{2}{3}$$

$$P_{+x} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt3\\ i\sqrt2/\sqrt3\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt3} + \frac{1}{\sqrt2}i\frac{\sqrt2}{\sqrt3}\right|^2 = \left|\frac{1}{\sqrt6} + i\frac{\sqrt2}{\sqrt6}\right|^2 = \left(\frac{1}{\sqrt6}\right)^2 + \left(\frac{\sqrt2}{\sqrt6}\right)^2 = \frac16 + \frac26 = \frac12$$

$$P_{-x} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt3\\ i\sqrt2/\sqrt3\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt3} - \frac{1}{\sqrt2}i\frac{\sqrt2}{\sqrt3}\right|^2 = \left|\frac{1}{\sqrt6} - \frac{\sqrt2}{\sqrt6}i\right|^2 = \left(\frac{1}{\sqrt6}\right)^2 + \left(\frac{-\sqrt2}{\sqrt6}\right)^2 = \frac16 + \frac26 = \frac12$$

$$P_{+y} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1\\i\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt3\\ i\sqrt2/\sqrt3\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt3} + i\frac{1}{\sqrt2}i\frac{\sqrt2}{\sqrt3}\right|^2 = \left|\frac{1}{\sqrt6} - \frac{\sqrt2}{\sqrt6}\right|^2 = \left(\frac{1}{\sqrt3} - \frac{1}{\sqrt6}\right)^2 \approx .029$$

$$P_{-y} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1\\-i\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt3\\ i\sqrt2/\sqrt3\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt3} - i\frac{1}{\sqrt2}i\frac{\sqrt2}{\sqrt3}\right|^2 = \left|\frac{1}{\sqrt6} + \frac{\sqrt2}{\sqrt6}\right|^2 = \left(\frac{1}{\sqrt3} + \frac{1}{\sqrt6}\right)^2 \approx .971$$

$|\psi_2\rangle = \frac{1}{\sqrt5}|+\rangle - \frac{2}{\sqrt5}|-\rangle = \begin{pmatrix}\frac{1}{\sqrt5}\\ \frac{-2}{\sqrt5}\end{pmatrix}$

$$P_{+z} = \left|\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt5\\ -2/\sqrt5\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt5}\right|^2 = \frac15$$

$$P_{-z} = \left|\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1/\sqrt5\\ -2/\sqrt5\end{pmatrix}\right|^2 = \left|-2/\sqrt5\right|^2 = 4/5$$

$$P_{+x} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1/\sqrt5\\ -2/\sqrt5\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt5} - \frac{1}{\sqrt2}\frac{2}{\sqrt5}\right|^2 = \left|\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}}\right|^2 = \frac{1}{10}$$

$$P_{-x} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1/\sqrt5\\ -2/\sqrt5\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt5} + \frac{1}{\sqrt2}\frac{2}{\sqrt5}\right|^2 = \left|\frac{1}{\sqrt{10}} + \frac{2}{\sqrt{10}}\right|^2 = \frac{9}{10}$$

$$P_{+y} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1\\i\end{pmatrix}\begin{pmatrix}1/\sqrt5\\ -2/\sqrt5\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt5} - \frac{1}{\sqrt2}i\frac{2}{\sqrt5}\right|^2 = \left|\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}}i\right|^2 = \left(\frac{1}{\sqrt{10}}\right)^2 + \left(\frac{-2}{\sqrt{10}}\right)^2 = \frac12$$

$$P_{-y} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1\\-i\end{pmatrix}\begin{pmatrix}1/\sqrt5\\ -2/\sqrt5\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt5} + \frac{1}{\sqrt2}i\frac{2}{\sqrt5}\right|^2 = \left|\frac{1}{\sqrt{10}} + i\frac{2}{\sqrt{10}}\right|^2 = \left(\frac{1}{\sqrt{10}}\right)^2 + \left(\frac{2}{\sqrt{10}}\right)^2 = \frac12$$

$$|\psi_3\rangle = \frac{1}{\sqrt2}|+\rangle + e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}|-\rangle = \begin{pmatrix}1/\sqrt2\\ e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}\end{pmatrix}$$

$$P_{+z} = \left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1/\sqrt2\\ e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\right|^2 = \frac{1}{2}$$

$$P_{-z} = \left|\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1/\sqrt2\\ e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}\end{pmatrix}\right|^2 = \left|e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}\right|^2 = \frac{1}{\sqrt2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt2}\frac{1}{\sqrt2}\right)^2 + \left(\frac{1}{\sqrt2}\frac{1}{\sqrt2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P_{+x} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1/\sqrt2\\ e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt2} + \frac{1}{\sqrt2}\frac{1}{\sqrt2}e^{i\frac{\pi}{4}}\right|^2 = \left|\frac{1}{\sqrt4} + \frac{1}{\sqrt4}\cos\frac{\pi}{4} + \frac{1}{\sqrt4}i\sin\frac{\pi}{4}\right|^2 = \left(\frac{1}{\sqrt4}\frac{\sqrt2}{\sqrt2} + \frac{1}{\sqrt4}\frac{1}{\sqrt2}\right)^2 + \left(\frac{1}{\sqrt4}\frac{1}{\sqrt2}\right)^2$$
$$= \frac{(1+\sqrt2)^2}{8} + \frac{1}{8} = \frac{2+\sqrt2}{4}$$

$$P_{-x} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1/\sqrt2\\ e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt2}\frac{1}{\sqrt2} - \frac{1}{\sqrt4}e^{i\frac{\pi}{4}}\right|^2 = \left|\frac{1}{\sqrt4} - \frac{1}{\sqrt4}\cos\frac{\pi}{4} - \frac{1}{\sqrt4}i\sin\frac{\pi}{4}\right|^2 = \left(\frac{1}{\sqrt4}\frac{\sqrt2}{\sqrt2} - \frac{1}{\sqrt4}\frac{1}{\sqrt2}\right)^2 + \left(\frac{1}{\sqrt4}\frac{1}{\sqrt2}\right)^2 = \frac{(\sqrt2-1)^2}{8} + \frac{1}{4}$$
$$= \frac{2}{8} - \frac{2\sqrt2}{8} + \frac{1}{8} + \frac{1}{8} = \frac{2-\sqrt2}{4}$$

$$P_{+y} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}1/\sqrt2\\ e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt4} + i\frac{1}{\sqrt4}e^{i\frac{\pi}{4}}\right|^2 = \frac{1}{4}\left(\frac{1}{2} + \left(1 - \frac{1}{\sqrt2}\right)^2\right) \approx 0.146$$

$$P_{-y} = \left|\frac{1}{\sqrt2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1/\sqrt2\\ e^{i\frac{\pi}{4}}\frac{1}{\sqrt2}\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt4} - i\frac{1}{\sqrt4}e^{i\frac{\pi}{4}}\right|^2 = \left|\frac{1}{\sqrt4} - i\frac{1}{\sqrt4}\cos\frac{\pi}{4} - \frac{1}{\sqrt4}i\,i\sin\frac{\pi}{4}\right|^2$$
$$= \left(\frac{1}{\sqrt4} + \frac{1}{\sqrt4}\frac{1}{\sqrt2}\right)^2 + \left(\frac{1}{\sqrt4}\frac{1}{\sqrt2}\right)^2 = \left(\frac{\sqrt2+1}{\sqrt8}\right)^2 + \left(\frac{1}{\sqrt8}\right)^2 = \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + \frac{2\sqrt2}{8} = \frac{2+\sqrt2}{4}$$

4) $|\psi_1\rangle = \frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle$

$P_{+z} = |\langle + |(\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle)|^2 = |\frac{4}{5}|^2 = \frac{16}{25}$

$P_{-z} = |\langle + |\psi_1\rangle = |i\frac{3}{5}|^2 = \frac{9}{25}$

$P_{+x} = |(\frac{1}{\sqrt{2}}(\langle +| + \langle -|)(\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{4}{5} + \frac{1}{\sqrt{2}}i\frac{3}{5}|^2 = |\frac{4}{\sqrt{50}} + i\frac{3}{\sqrt{50}}|^2 = \frac{16}{50} + \frac{9}{50} = \frac{1}{2}$

$P_{-x} = |\frac{1}{\sqrt{2}}(\langle +| - \langle -|)(\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{4}{5} - \frac{1}{\sqrt{2}}\frac{3}{5}i|^2 = |\frac{4}{\sqrt{50}} - \frac{3}{\sqrt{50}}|^2 = \frac{16}{50} + \frac{9}{50} = \frac{1}{2}$

$P_{+y} = |(\frac{1}{\sqrt{2}}(\langle +| + i\langle -|)(\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{4}{5} + \frac{1}{\sqrt{2}}i\frac{3}{5}i|^2 = |\frac{4}{\sqrt{50}} - \frac{3}{\sqrt{50}}|^2 = \frac{1}{50}$

$P_{-y} = |\frac{1}{\sqrt{2}}(\langle +| + i\langle -|)(\frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{4}{5} - \frac{1}{\sqrt{2}}i\frac{3}{5}i|^2 = |\frac{4}{\sqrt{50}} + \frac{3}{\sqrt{50}}|^2 = \frac{49}{50}$

$|\psi_2\rangle = \frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle$

$P_{+z} = |(\langle +|)(\frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle)|^2 = |\frac{4}{5}|^2 = \frac{16}{25}$

$P_{-z} = |(\langle -|)(\frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle)|^2 = |-i\frac{3}{5}|^2 = \frac{9}{25}$

$P_{+x} = |(\frac{1}{\sqrt{2}}(\langle +| + \langle -|)(\frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{4}{5} + \frac{1}{\sqrt{2}}(-i\frac{3}{5})|^2 = |\frac{4}{\sqrt{50}} - \frac{3}{\sqrt{50}}i|^2 = (\frac{4}{\sqrt{50}})^2 + (\frac{-3}{\sqrt{50}})^2$

$= \frac{16}{50} + \frac{9}{50} = \frac{1}{2}$

$P_{-x} = |\frac{1}{\sqrt{2}}(\langle +| - \langle -|)(\frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{4}{5} + \frac{1}{\sqrt{2}}\frac{3}{5}i|^2 = (\frac{4}{\sqrt{50}})^2 + (\frac{3}{\sqrt{50}})^2 = \frac{16}{50} + \frac{9}{50} = \frac{1}{2}$

$P_{+y} = |\frac{1}{\sqrt{2}}(\langle +| + i\langle -|)(\frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{4}{5} - \frac{1}{\sqrt{2}}\frac{3}{5}i^2|^2 = |\frac{4}{\sqrt{50}} + \frac{3}{\sqrt{50}}|^2 = |\frac{7}{\sqrt{50}}|^2 = \frac{49}{50}$

$P_{-y} = |\frac{1}{\sqrt{2}}(\langle +| - i\langle -|)(\frac{4}{5}|+\rangle - i\frac{3}{5}|-\rangle)|^2 = |\frac{1}{\sqrt{2}}\frac{4}{5} + \frac{1}{\sqrt{2}}\frac{3}{5}i^2|^2 = |\frac{4}{\sqrt{50}} - \frac{3}{\sqrt{50}}|^2 = \frac{1}{50}$

$|\psi_3\rangle = -\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle$

$P_{+z} = \left|\langle +|\left(-\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle\right)\right|^2 = \left|-\frac{4}{5}\right|^2 = \frac{16}{25}$

$P_{-z} = \left|\langle -|\left(-\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle\right)\right|^2 = \left|i\frac{3}{5}\right|^2 = \frac{9}{25}$

$P_{+x} = \left|\frac{1}{\sqrt{2}}(\langle +| + \langle -|)\left(-\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{2}}\frac{-4}{5} + \frac{1}{\sqrt{2}}\frac{3}{5}i\right|^2 = \left(\frac{-4}{\sqrt{50}}\right)^2 + \left(\frac{3}{\sqrt{50}}\right)^2 = \frac{1}{2}$

$P_{-x} = \left|\frac{1}{\sqrt{2}}(\langle +| - \langle -|)\left(-\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{2}}\frac{-4}{5} - \frac{1}{\sqrt{2}}\frac{3}{5}i\right|^2 = \left(\frac{-4}{\sqrt{50}}\right)^2 + \left(-\frac{3}{\sqrt{50}}\right)^2 = \frac{16}{50} + \frac{9}{50} = \frac{1}{2}$

$P_{+y} = \left|\frac{1}{\sqrt{2}}(\langle +| + i\langle -|)\left(-\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{2}}\frac{-4}{5} + \frac{1}{\sqrt{2}}i\frac{3}{5}i\right|^2 = \left|\frac{-4}{\sqrt{50}} - \frac{3}{\sqrt{50}}\right|^2 = \frac{49}{50}$

$P_{-y} = \left|\frac{1}{\sqrt{2}}(\langle +| - i\langle -|)\left(-\frac{4}{5}|+\rangle + i\frac{3}{5}|-\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{2}}\frac{-4}{5} - \frac{1}{\sqrt{2}}i\frac{3}{5}i\right|^2 = \left|\frac{-4}{\sqrt{50}} + \frac{3}{\sqrt{50}}\right|^2 = \frac{1}{50}$