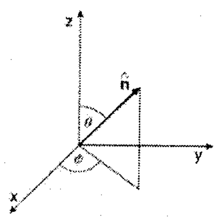
Jacob Knaup

Problem #1 – Ch2, part 1 lecture, spin ½ in general direction (show all your work!)



$$0 \le \theta < \pi \text{ and } 0 \le \phi < 2\pi$$

$$\hat{\mathbf{n}} = \hat{\mathbf{i}}\sin\theta\cos\phi + \hat{\mathbf{j}}\sin\theta\sin\phi + \hat{\mathbf{k}}\cos\theta$$

$$\begin{aligned} S_n &= \mathbf{S}\cdot\hat{\mathbf{n}} \\ &= S_x\sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta \end{aligned}$$

Prove that these expressions are correct

$$\boxed{S_n \doteq \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta\end{pmatrix}}$$

$$\boxed{\begin{aligned} |+\rangle_n &= \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle \\ |-\rangle_n &= \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}e^{i\phi}|-\rangle \end{aligned}}$$

$$S_n = S_x \sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta$$

$$= \sin\theta\cos\phi\,\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \sin\theta\sin\phi\,\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \frac{\hbar}{2}\cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix}0 & \sin\theta\cos\phi\\ \sin\theta\cos\phi & 0\end{pmatrix} + \begin{pmatrix}0 & -i\sin\theta\sin\phi\\ i\sin\theta\sin\phi & 0\end{pmatrix} + \begin{pmatrix}\cos\theta & 0\\ 0 & -\cos\theta\end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta(\cos\phi - i\sin\phi)\\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$$

1) $|+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$

$$\left|\frac{\hbar}{2}\begin{pmatrix}\cos\theta-\lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -(\cos\theta-\lambda)\end{pmatrix}\right| = 0$$

$$-\frac{\hbar^2}{4}\cos^2\theta + \lambda^2 - \frac{\hbar^2}{4}\sin^2\theta = 0$$

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\lambda = \frac{\hbar}{2}$$

$$S_n|+\rangle_n = \lambda|+\rangle_n$$

$$\begin{pmatrix}\frac{\hbar}{2}\cos\theta-\lambda & \frac{\hbar}{2}\sin\theta e^{-i\phi} \\ \frac{\hbar}{2}\sin\theta e^{i\phi} & \frac{\hbar}{2}\cos\theta-\lambda\end{pmatrix}\begin{pmatrix}a \\ b\end{pmatrix} = 0$$

$$\begin{pmatrix}a\left(\frac{\hbar}{2}\cos\theta-\frac{\hbar}{2}\right)+b\frac{\hbar}{2}\sin\theta e^{-i\phi} \\ a\left(\frac{\hbar}{2}\sin\theta e^{i\phi}\right)+b\left(\frac{\hbar}{2}\cos\theta-\frac{\hbar}{2}\right)\end{pmatrix} = 0$$

$$a(\cos\theta-1)+b\sin\theta e^{-i\phi} = 0$$

$$\sin\theta\left(-a\frac{1-\cos\theta}{\sin\theta}+be^{-i\phi}\right) = 0$$

$$\sin\theta\left(\cos\frac{\theta}{2}\tan\frac{\theta}{2}+\sin\frac{\theta}{2}\right) = 0$$

$$\sin\theta\left(\cancel{-\sin\frac{\theta}{2}+\sin\frac{\theta}{2}}\right)^0 = 0$$

✓

$$|-\rangle_n = \overbrace{\sin\tfrac{\theta}{2}}^{a}|+\rangle - \overbrace{\cos\tfrac{\theta}{2}e^{i\phi}}^{b}|-\rangle$$

$$\lambda = -\frac{\hbar}{2}$$

$$\begin{pmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin\theta e^{-i\phi} \\ \frac{\hbar}{2}\sin\theta e^{i\phi} & \frac{\hbar}{2}\cos\theta - \lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\begin{pmatrix} a\left(\frac{\hbar}{2}\cos\theta + \frac{\hbar}{2}\right) + b\,\frac{\hbar}{2}\sin\theta e^{-i\phi} \\ a\left(\frac{\hbar}{2}\sin\theta e^{i\phi}\right) + b\left(\frac{\hbar}{2}\cos\theta + \frac{\hbar}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a(\cos\theta + 1) + b\sin\theta e^{-i\phi} = 0$$

$$a\sin\theta e^{i\phi} + b(\cos\theta + 1) = 0$$

$$(\cos\theta + 1)\left(a + b\,\frac{\sin\theta e^{-i\phi}}{1+\cos\theta}\right) = 0$$

$$(\cos\theta + 1)\left(\sin\tfrac{\theta}{2} - \cos\tfrac{\theta}{2}\cancel{e^{i\phi}}\tan\tfrac{\theta}{2}\cancel{e^{-i\phi}}\right) = 0$$

$$(\cos\theta + 1)\cancel{\left(\sin\tfrac{\theta}{2} - \sin\tfrac{\theta}{2}\right)}^{0} = 0$$

$$\cancel{(\cos\theta + 1)}(0)^{0} = 0$$

✓

Problem #2 (show all your work!) spin ½

$\psi = |+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

For the state $|+\rangle_y$, calculate the expectation values and uncertainties for measurements of S_x, S_y, and S_z

$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\langle S_x \rangle = {}_y\langle + | S_x | + \rangle_y = \frac{1}{\sqrt{2}}(1 \ -i)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\frac{\hbar}{2\sqrt{2}}(-i \ \ 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = 0$

$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} \qquad \langle S_x^2 \rangle = {}_y\langle + | S_x S_x | + \rangle_y$

$= \frac{1}{\sqrt{2}}(1 \ -i)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{\hbar}{2\sqrt{2}}(-i \ 1)\frac{\hbar}{2\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{\hbar^2}{8}(1 + 1) = \frac{\hbar^2}{4}$

$\Delta S_x = \sqrt{\left(\frac{\hbar^2}{4}\right) - 0} = \frac{\hbar}{2}$

2) $|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

$\langle S_y\rangle = {}_y\langle +|S_y|+\rangle_y = \frac{1}{\sqrt{2}}(1\ -i)\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{2\sqrt{2}}(1\ -i)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{4}(2) = \frac{\hbar}{2}$

$\Delta S_y = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2}$

$\langle S_y^2\rangle = {}_y\langle +|S_y S_y|+\rangle_y = \frac{1}{\sqrt{2}}(1\ -i)\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$= \frac{\hbar}{2\sqrt{2}}(1\ -i)\frac{\hbar}{2\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar^2}{8}(1+1) = \frac{\hbar^2}{4}$

$\Delta S_y = \sqrt{\frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2} = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{4}} = 0$

$|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$\langle S_z\rangle = {}_y\langle +|S_z|+\rangle_y = \frac{1}{\sqrt{2}}(1\ -i)\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{2\sqrt{2}}(1\ i)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{4}(1 + -1) = 0$

$\langle S_z^2\rangle = {}_y\langle +|S_z S_z|+\rangle_y = \frac{\hbar}{2\sqrt{2}}(1\ i)\frac{\hbar}{2\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{\hbar^2}{8}(1+1) = \frac{\hbar^2}{4}$

$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\frac{\hbar^2}{4} - 0^2} = \frac{\hbar}{2}$

Problem #3 (show all your work!)

Find the matrix representation of the $\mathbf{S}^2$ operator for a spin-1 system. Do this once by explicit matrix calculation

$$S^2 = S_x^2 + S_y^2 + S_z^2 \qquad S_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix} \qquad S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_z^2 = S_z S_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix} \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix} = \begin{pmatrix} \hbar^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar^2 \end{pmatrix} = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_x^2 = S_x S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$S_y^2 = S_y S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{\hbar^2}{2} \begin{pmatrix} -i^2 & 0 & i^2 \\ 0 & -2i^2 & 0 \\ i^2 & 0 & -i^2 \end{pmatrix}$$

$$S^2 = \frac{\hbar^2}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$S^2 = \frac{\hbar^2}{2} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem #4 (show all your work!)

A beam of spin-1 particles is prepared in the state

$$|\psi\rangle = \frac{2}{\sqrt{29}}|1\rangle_y + i\frac{3}{\sqrt{29}}|0\rangle_y - \frac{4}{\sqrt{29}}|-1\rangle_y.$$

a) What are the possible results of a measurement of the spin component S_z, and with what probabilities would they occur?

$|-1\rangle$, $|0\rangle$, or $|1\rangle$

b) What are the possible results of a measurement of the spin component S_y, and with what probabilities would they occur?

$|+1\rangle_y$, $|0\rangle_y$, $|-1\rangle_y$

See probabilities on next page

4) a) $|\psi\rangle = \frac{2}{\sqrt{29}}|1\rangle_y + i\frac{3}{\sqrt{29}}|0\rangle_y - \frac{4}{\sqrt{29}}|-1\rangle_y$

$= \frac{2}{\sqrt{29}}\left(\frac{1}{2}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle\right) + \frac{3i}{\sqrt{29}}\left(\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|-1\rangle\right) - \frac{4}{\sqrt{29}}\left(\frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle\right)$

$P_{+1,z} = |\langle +1|\psi\rangle|^2 = \left|\frac{\cancel{2}}{\sqrt{29}}\frac{1}{\cancel{2}} + \frac{3i}{\sqrt{29}}\frac{1}{\sqrt{2}} + \frac{-\cancel{4}}{\sqrt{29}}\frac{1}{\cancel{2}}\right|^2 = \left|\frac{-1}{\sqrt{29}} + \frac{3i}{\sqrt{58}}\right|^2 = \left(\frac{-1}{\sqrt{29}}\right)^2 + \left(\frac{3}{\sqrt{58}}\right)^2$

$= \frac{11}{58}$

$P_{0,z} = |\langle 0|\psi\rangle|^2 = \left|\frac{2}{\sqrt{29}}\frac{i}{\sqrt{2}} + \frac{4}{\sqrt{29}}\frac{i}{\sqrt{2}}\right|^2 = \frac{36}{58}$

$P_{-1,z} = |\langle -1|\psi\rangle|^2 = \left|\frac{\cancel{2}}{\sqrt{29}}\frac{-1}{\cancel{2}} + \frac{3i}{\sqrt{29}}\frac{1}{\sqrt{2}} + \frac{\overset{2}{\cancel{4}}}{\sqrt{29}}\frac{1}{\cancel{2}}\right|^2 = \left|\frac{-1}{\sqrt{29}} + \frac{2}{\sqrt{29}} + \frac{3i}{\sqrt{58}}\right|$

$= \left(\frac{1}{\sqrt{29}}\right)^2 + \left(\frac{3}{\sqrt{58}}\right)^2 = \frac{11}{58}$

b) $|\psi\rangle = \frac{2}{\sqrt{29}}|1\rangle_y + i\frac{3}{\sqrt{29}}|0\rangle_y - \frac{4}{\sqrt{29}}|-1\rangle_y$

$P_{+1,y} = |{}_y\langle +1|\psi\rangle|^2 = \left|\frac{2}{\sqrt{29}}\right|^2 = \frac{4}{29}$

$P_{0,y} = |\langle 0|\psi\rangle|^2 = \left|\frac{i3}{\sqrt{29}}\right|^2 = \frac{9}{29}$

$P_{-1,y} = |\langle -1|\psi\rangle|^2 = \left|-\frac{4}{\sqrt{29}}\right|^2 = \frac{16}{29}$