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Homework assignment

Chapter 7

turn in complete assignment electronically in pdf format:

send as attachment via email at maxim.sukharev@asu.edu
use subject line "PHY314, HW #7, Your Last Name Your First Name"

Problem #1 (show all your work!)

Show by explicit calculations that the Laplacian in spherical coordinates has the following form (see lecture for Ch.7, part 1, problem #3)

calculations are quite involved thus do only $\frac{\partial^2}{\partial x^2}$ in spherical coordinates

for extra-credit (30 additional points on top of 100 for this HW) do the entire calculations

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$x = r\cos\phi\sin\theta \quad y = r\sin\phi\sin\theta \quad z = r\cos\theta$

$r = \sqrt{x^2+y^2+z^2} \quad \phi = \text{Arctan}(y/x) \quad \theta = \arctan(\sqrt{x^2+y^2}/z)$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{r\cos\phi\sin\theta}{\sqrt{r^2\cos^2\phi\sin^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\theta}}$$

$$= \frac{r\cos\phi\sin\theta}{\sqrt{r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) + r^2\cos^2\theta}} = \frac{r\cos\phi\sin\theta}{r} = \cos\phi\sin\theta$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{1+(y/x)^2}(-yx^{-2}) = \frac{-r\sin\phi\sin\theta}{r^2\cos^2\phi\sin^2\theta\left(1+\left(\frac{r^2\sin^2\phi\sin^2\theta}{r^2\cos^2\phi\sin^2\theta}\right)\right)} = \frac{-\sin\phi}{r\cos^2\phi\sin\theta + r\sin^2\phi\sin\theta}$$

$$= \frac{-\sin\phi}{r\sin\theta}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\left(\frac{1}{z}\frac{1}{2}(x^2+y^2)^{-1/2}(2x)\right) = \frac{2x}{2z\sqrt{x^2+y^2}\left(1+\frac{x^2+y^2}{z^2}\right)}$$

$$= \frac{r\cos\phi\sin\theta}{\cos\theta\sqrt{r^2\cos^2\phi\sin^2\theta+r^2\sin^2\phi\sin^2\theta}\left(1+\frac{r^2\cos^2\phi\sin^2\theta+r^2\sin^2\theta\sin^2\phi}{r^2\cos^2\theta}\right)} = \frac{\cos\phi\sin\theta}{\cos\theta\, r\sin\theta\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)}$$

$$= \frac{\cos\phi}{r\left(\cos\theta+\frac{\sin^2\theta}{\cos\theta}\right)}$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial\theta}{\partial x} = \frac{+1}{\sqrt{1-\left(\frac{z^2}{x^2+y^2+z^2}\right)}}\left(\frac{1}{2}z(x^2+y^2+z^2)^{-3/2}(2x)\right) = \frac{+xz}{\sqrt{(x^2+y^2+z^2)^3}\sqrt{1-\left(\frac{z^2}{x^2+y^2+z^2}\right)}} = \frac{+xz}{\sqrt{(x^2+y^2+z^2)^3-z^2(x^2+y^2+z^2)^2}}$$

$$= \frac{+xz}{(x^2+y^2+z^2)\sqrt{x^2+y^2+\cancel{z^2}-\cancel{z^2}}} = \frac{+r\cos\phi\sin\theta\, r\cos\theta}{(r^2\cos^2\phi\sin^2\theta+r^2\sin^2\phi\sin^2\theta+r^2\cos^2\theta)\sqrt{r^2\cos^2\phi\sin^2\theta+r^2\sin^2\phi\sin^2\theta}}$$

$$= \frac{+r\cos\phi\,\cancel{\sin\theta}\cos\theta}{\cancel{r\sin\theta}\, r^2(1)} = \frac{+\cos\phi\cos\theta}{r}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \cos\phi\sin\theta\frac{\partial}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial x^2} = \cos\phi\sin\theta\frac{\partial}{\partial r}\left[\cos\phi\sin\theta\frac{\partial}{\partial r}\right] + \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{-\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right] + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}\left[\frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}\right] + \cos\phi\sin\theta\frac{\partial}{\partial r}\left[\frac{-\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right] - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\cos\phi\sin\theta\frac{\partial}{\partial r}\right]$$

$$+ \cos\phi\sin\theta\frac{\partial}{\partial r}\left[\frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}\right] + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}\left[\cos\phi\sin\theta\frac{\partial}{\partial r}\right] - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}\right] + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta}\left[\frac{-\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$= \cos\phi\sin\theta\left(0+\cos\phi\sin\theta\frac{\partial^2}{\partial r^2}\right) - \frac{\sin\phi}{r\sin\theta}\left(\frac{-\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{-\sin\phi}{r\sin\theta}\frac{\partial^2}{\partial\phi^2}\right) + \frac{\cos\phi\cos\theta}{r}\left(\frac{-\cos\phi\sin\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi\cos\theta}{r}\frac{\partial^2}{\partial\theta^2}\right)$$

$$+ \cos\phi\sin\theta\left(\frac{-\sin\phi}{-r^2\sin\theta}\frac{\partial}{\partial\phi} + \frac{-\sin\phi}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}\right) - \frac{\sin\phi}{r\sin\theta}\left(-\sin\phi\sin\theta\frac{\partial}{\partial r} + \cos\phi\sin\theta\frac{\partial^2}{\partial\phi\partial r}\right)$$

$$+ \cos\phi\sin\theta\left(\frac{\cos\phi\cos\theta}{-r^2}\frac{\partial}{\partial\theta} + \frac{\cos\phi\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}\right) + \frac{\cos\phi\cos\theta}{r}\left(\cos\phi\cos\theta\frac{\partial}{\partial r} + \cos\phi\sin\theta\frac{\partial^2}{\partial\theta\partial r}\right)$$

$$- \frac{\sin\phi}{r\sin\theta}\left(-\frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi\cos\theta}{r}\frac{\partial^2}{\partial\phi\partial\theta}\right) + \frac{\cos\phi\cos\theta}{r}\left(\frac{-\sin\phi}{-r\sin^2\theta\cos\theta}\frac{\partial}{\partial\phi} - \frac{\sin\phi}{r\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\right)$$

$$= \cos^2\phi \sin^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin\phi\cos\phi}{r^2\sin\theta^2}\frac{\partial}{\partial\phi} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{\cos^2\phi\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\phi\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$+ \frac{\cos\phi\sin\phi}{r^2}\frac{\partial}{\partial\phi} - \frac{\cos\phi\sin\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r} - \frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial\phi\partial r} - \frac{\cos^2\phi\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\phi\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}$$

$$+ \frac{\cos^2\phi\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\phi\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\sin^2\phi\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} - \frac{\sin\phi\cos\phi\cos\theta}{r^2\sin\theta}\frac{\partial^2}{\partial\phi\partial\theta}$$

$$+ \frac{\cos\phi\sin\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi} - \frac{\cos\phi\sin\phi\cos\theta}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}$$

Problem #2 (show all your work!)

Show by explicit calculations that (see hint in Lecture, Ch. 7, part 2, problem #4)

$L_z \doteq -i\hbar\frac{\partial}{\partial\phi}$

$L_z = xp_y - yp_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

$= -i\hbar\Big[r\cos\phi\sin\theta\left(\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\sin\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right) - r\sin\phi\sin\theta\left(\cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\Big]$

$= -i\hbar\Big[\underline{r\cos\phi\sin\phi\sin^2\theta\frac{\partial}{\partial r}} + \cancel{\cos\phi\sin\phi\sin\theta\cos\theta\frac{\partial}{\partial\theta}} + \cos^2\phi\frac{\partial}{\partial\phi} - \underline{r\sin\phi\cos\phi\sin^2\theta\frac{\partial}{\partial r}} - \cancel{\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial}{\partial\theta}} + \sin^2\phi\frac{\partial}{\partial\phi}\Big]$

$= -i\hbar\left[(\cos^2\phi + \sin^2\phi)\frac{\partial}{\partial\phi}\right] = -i\hbar\frac{\partial}{\partial\phi}$

Problem #3 (show all your work!)

Using Rodrigues' formula (7.147, p. 231), by hand, to generate first **four** Legendre polynomials.

Also show by direct integration that:
(a) $P_2(\cos\theta)$ is orthogonal to $P_4(\cos\theta)$;
(b) $P_2(\cos\theta)$ is normalized according to $\int_{-1}^{1} P_k^*(z)P_\ell(z)\,dz = \frac{2}{2\ell+1}\delta_{k\ell}$.

$z = \cos\theta$ $\quad dz = -\sin\theta\, d\theta$

$$P_\ell(z) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dz^\ell}(z^2-1)^\ell$$

$$P_0(z) = \frac{1}{2^0 0!}\frac{d^0}{dz^0}(z^2-1)^0 = 1 \qquad P_0(z) = 1$$

$$P_1(z) = \frac{1}{2^1 1}\frac{d}{dz}(z^2-1) = \frac{1}{2}(2z) = z \qquad P_1(z) = z$$

$$P_2(z) = \frac{1}{2^2 2\cdot 1}\frac{d^2}{dz^2}(z^2-1)^2 = \frac{1}{8}\frac{d}{dz}\left[2(z^2-1)(2z)\right] = \frac{1}{8}\frac{d}{dz}\left[4z^3-4z\right] = \frac{1}{8}(12z^2-4)$$

$$P_2(z) = \frac{1}{2}(3z^2-1)$$

$$P_3(z) = \frac{1}{2^3 3\cdot 2\cdot 1}\frac{d^3}{dz^3}(z^2-1)^3 = \frac{1}{48}\frac{d^2}{dz^2}\left[3(z^2-1)^2(2z)\right]$$

$$= \frac{6}{48}\frac{d^2}{dz^2}\left[z(z^2-1)^2\right] = \frac{1}{8}\frac{d}{dz}\left[(z^2-1)^2 + z2(z^2-1)(2z)\right] = \frac{1}{8}\frac{d}{dz}\left[(z^2-1)^2 + 4z^2(z^2-1)\right]$$

$$= \frac{1}{8}\frac{d}{dz}\left[z^4 - 2z^2 + 1 + 4z^4 - 4z^2\right] = \frac{1}{8}\frac{d}{dz}\left[5z^4 - 6z^2 + 1\right] = \frac{1}{8}(20z^3 - 12z)$$

$$P_3(z) = \frac{1}{2}(5z^3 - 3z)$$

a) $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ $\quad P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)$

$\frac{1}{2}\int P_2(\cos\theta) P_4\cos(\theta) \sin\theta \, d\theta = 0$

$\frac{1}{2}\frac{1}{8}\int \sin\theta (3\cos^2\theta - 1)(35\cos^4\theta - 30\cos^2\theta + 3) = \frac{1}{16}\int \sin\theta (105\cos^6\theta - 90\cos^4\theta + 9\cos^2\theta$

$- 35\cos^4\theta + 30\cos^2\theta - 3)\,d\theta = \frac{1}{16}\int \sin\theta (105\cos^6\theta - 125\cos^4\theta + 39\cos^2\theta - 3)\,d\theta$

$u = \cos\theta \quad du = -\sin\theta \, d\theta$

$\frac{1}{16}\int 105u^6 - 125u^4 + 39u^2 - 3 \; du = \frac{1}{16}(15u^7 - 25u^5 + 13u^3 - 3u)\Big|_0^{2\pi}$

$\cos(2\pi) = 1 \quad \cos(0) = 1$

$\frac{1}{16}(15 - 25 + 13 - 3) = 0$

b) $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ $\quad \int_{-1}^{1} P^*_{(z)} P(z)\,dz = \frac{2}{4+1}$ ✓ $\quad z = \cos\theta$, $dz = -\sin\theta \, d\theta$

$\int_\pi^0 \frac{1}{2}(3\cos^2\theta - 1)\frac{1}{2}(3\cos^2\theta - 1)(-\sin\theta\,d\theta) = \frac{1}{4}\int_\pi^0 (9\cos^4\theta - 6\cos^2\theta + 1)(-\sin\theta\,d\theta)$

$u = \cos\theta \quad du = -\sin\theta\,d\theta$

$\frac{1}{4}\int (9u^4 - 6u^2 + 1)\,du = \frac{1}{4}\left(\frac{9}{5}u^5 - 2u^3 + u\right) = \frac{1}{4}\left(\frac{9}{5}\cos^5\theta - 2\cos^3\theta + \cos\theta\right)\Big|_0^\pi$

$\cos\pi = -1 \quad \cos 0 = 1$

$\frac{1}{4}\left[\left(-\frac{9}{5} - 2 + 1\right) - \left(-\frac{9}{5} + 2 - 1\right)\right] = \frac{2}{5} = \frac{2}{5}$ ✓