Minimal mutation-infinite quivers

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Workshop on Cluster Algebras
and finite dimensional algebras

Introduction

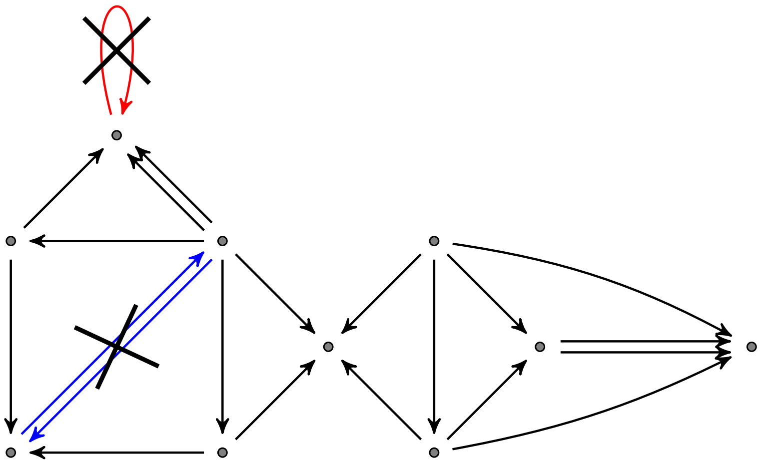
Mutations on quivers studied following the introduction of cluster algebras by Fomin and Zelevinsky in 2002.

This work follows:

- ⋆ Classification of minimal infinite-type diagrams by Seven published in 2007
- ⋆ Classification of mutation-finite quivers by Felikson, Shapiro and Tumarkin published in 2012

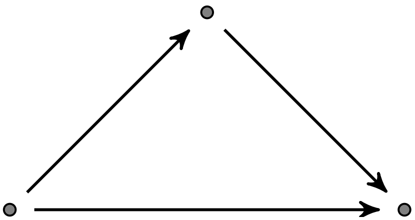
Quivers

directed (multi-)graphs with no loops or 2-cycles



Adjacency matrix

$A = (a_{i,j})$ where $a_{i,j} = \#(i \to j) - \#(j \to i)$

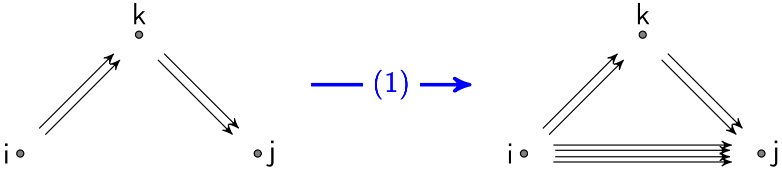


$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Mutations

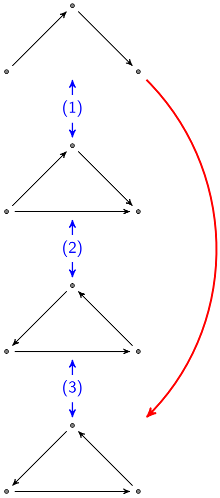
Mutation is a function on the quiver which acts at a vertex k through 3 steps:

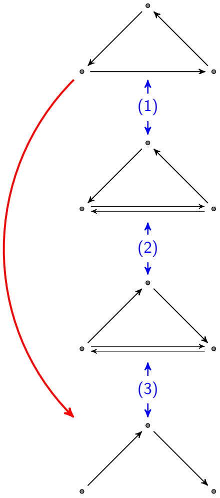
1. For each pair of arrows $i \to k \to j$ add an arrow $i \to j$.
2. Reverse direction of arrows adjacent to k.
3. Remove any 2-cycles created in step (1).



Mutation examples

mutate at top vertex





Mutations

Mutations are involutions.

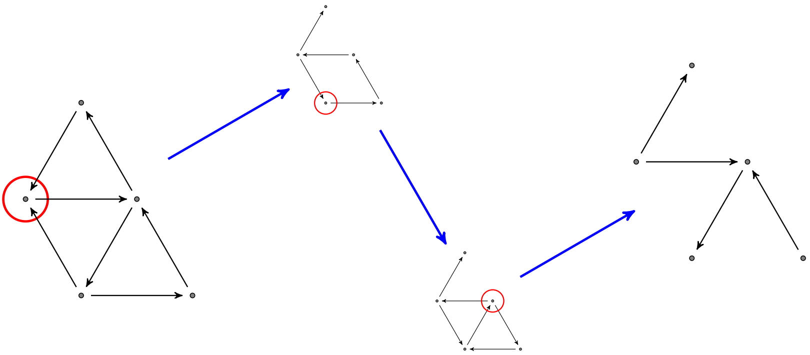
Matrix mutations

Mutation at vertex k takes an adjacency matrix $B = (b_{i,j})$ to $B' = (b'_{i,j})$ where

$$b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k \\ b_{i,j} + \frac{|b_{i,k}| b_{k,j} + b_{i,k} |b_{k,j}|}{2} & \text{otherwise} \end{cases}$$

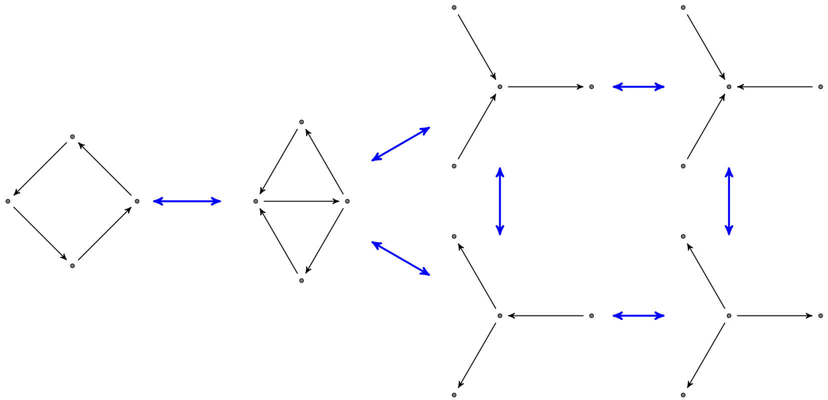
Mutation-equivalent

if there is a sequence of mutations



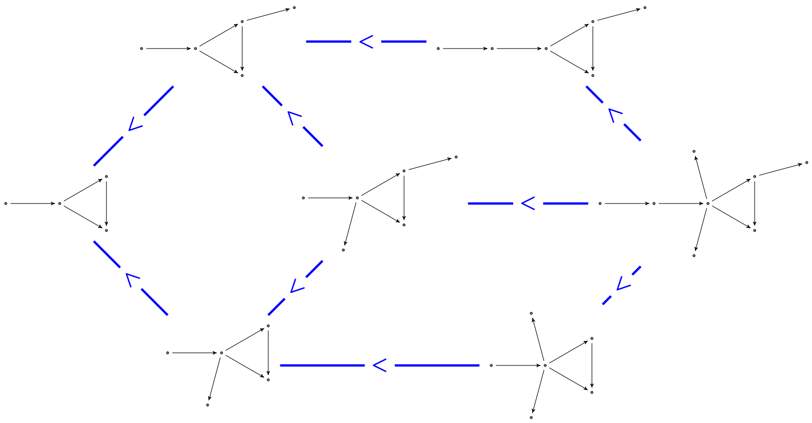
Mutation-finite

or conversely mutation-infinite

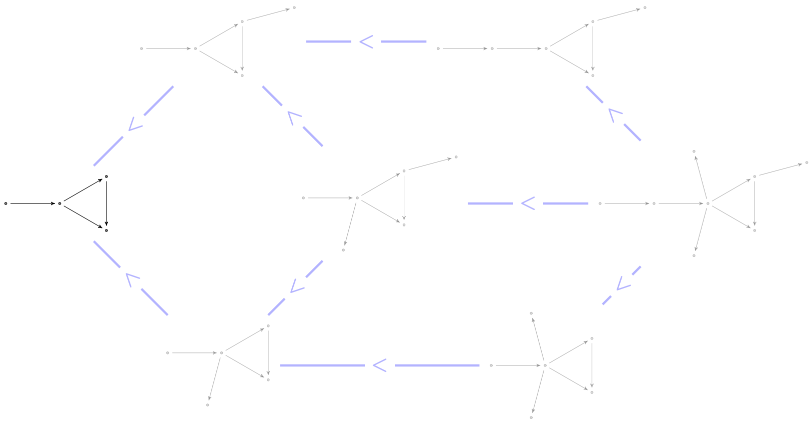


Partial ordering

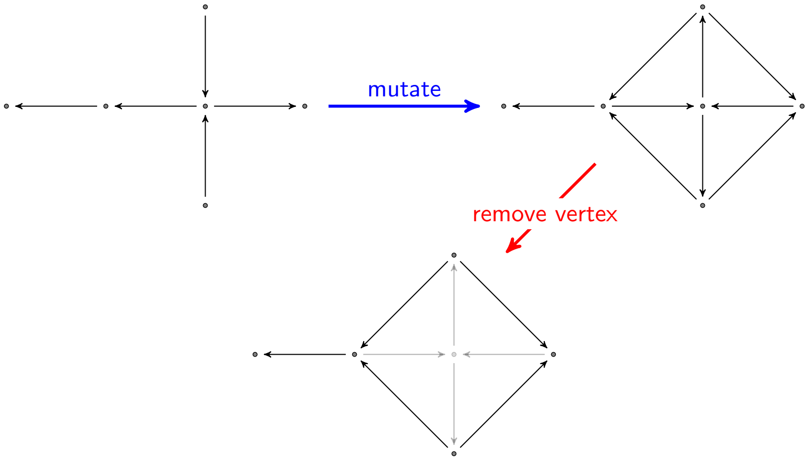
on mutation-infinite quivers given by inclusion



Minimal mutation-infinite quivers



Mutations do not preserve
minimal mutation-infinite property



Why
are minimal mutation-infinite quivers interesting?

Any quiver containing a minimal mutation-infinite subquiver is necessarily mutation-infinite.

A complete classification would give a systematic approach to check whether any given quiver is mutation-infinite or not.

Ahmet Seven’s classification
of minimal infinite-type diagrams

Seven classified all the infinite-type diagrams such that removing a vertex yielded a finite-type diagram.

Felikson, Shapiro and Tumarkin

started studying minimal mutation-infinite quivers

In their paper on classifying mutation-finite quivers, FST proved that there were no minimal mutation-infinite quivers with more than 10 vertices.

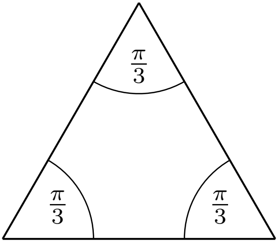
Minimal mutation-infinite quivers

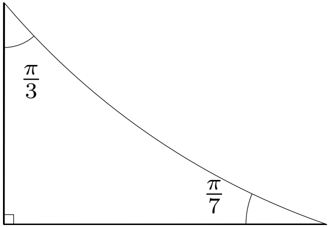
Distinguished family of minimal mutation-infinite quivers which are orientations of simply-laced Coxeter diagrams of hyperbolic Coxeter simplices.

Coxeter simplex
convex hull of $n+1$ points

Considered inside spherical, Euclidean or hyperbolic space.

$n+1$ hyper-planes H_i with dihedral angles $\frac{\pi}{k_{ij}}$ (or possibly 0) between H_i and H_j.

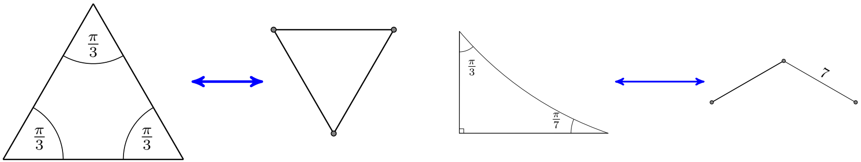




Coxeter diagram

from simplex bounded by H_i with angles $\frac{\pi}{k_{ij}}$

- $\star$ vertex i for each H_i
- $\star$ edge $i - j$ with no weight when $k_{ij} = 3$
- $\star$ edge $i - j$ with weight k_{ij} when $k_{ij} > 3$



Coxeter group

from a Coxeter simplex or diagram

A Coxeter group can be constructed from a Coxeter diagram through the following presentation

$$\left\langle s_i \;\middle|\; s_i^2 = 1 = (s_i s_j)^{k_{ij}} \right\rangle .$$

Simply-laced Coxeter diagram

only have $k_{ij} = 2$ or 3

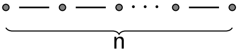
Coxeter diagram with no weighted edges.

Choosing an orientation of the edges gives a quiver.

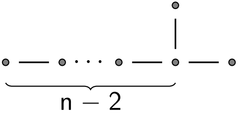
Simply-laced Spherical Coxeter diagrams

are Dynkin diagrams of type A, D or E

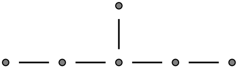
A_n :



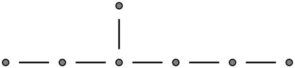
D_n :



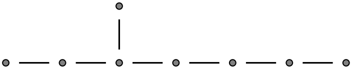
E_6 :



E_7 :



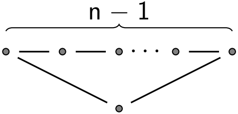
E_8 :



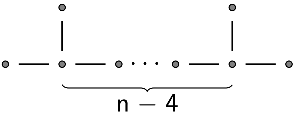
Simply-laced Euclidean Coxeter diagrams

are affine Dynkin diagrams of type $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$

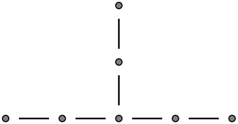
$\tilde{A}_n$:



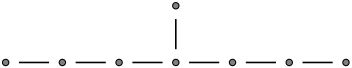
$\tilde{D}_n$:



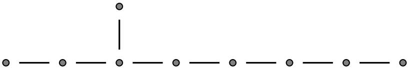
$\tilde{E}_6$:



$\tilde{E}_7$:



$\tilde{E}_8$:



Simply-laced Hyperbolic Coxeter diagrams

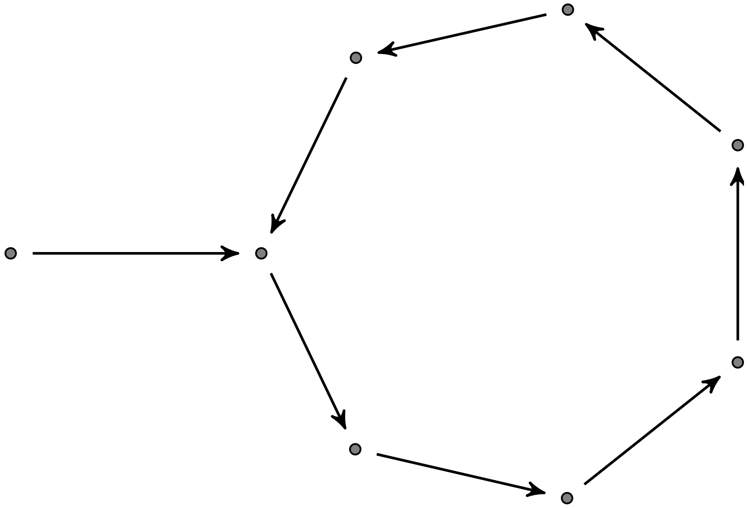
Simply-laced Hyperbolic Coxeter simplices give diagrams satisfying:

- ⋆ any subdiagram is either a Dynkin diagram or an affine Dynkin diagram, but the diagram itself is not.

Orientations of simply-laced Hyperbolic
Coxeter diagrams are mutation-infinite quivers

Felikson, Shapiro and Tumarkin classified all mutation-finite quivers - (almost all) orientations of simply-laced Hyperbolic Coxeter diagrams do not lie in this classification.

Mutation-finite orientations
of hyperbolic Coxeter diagrams

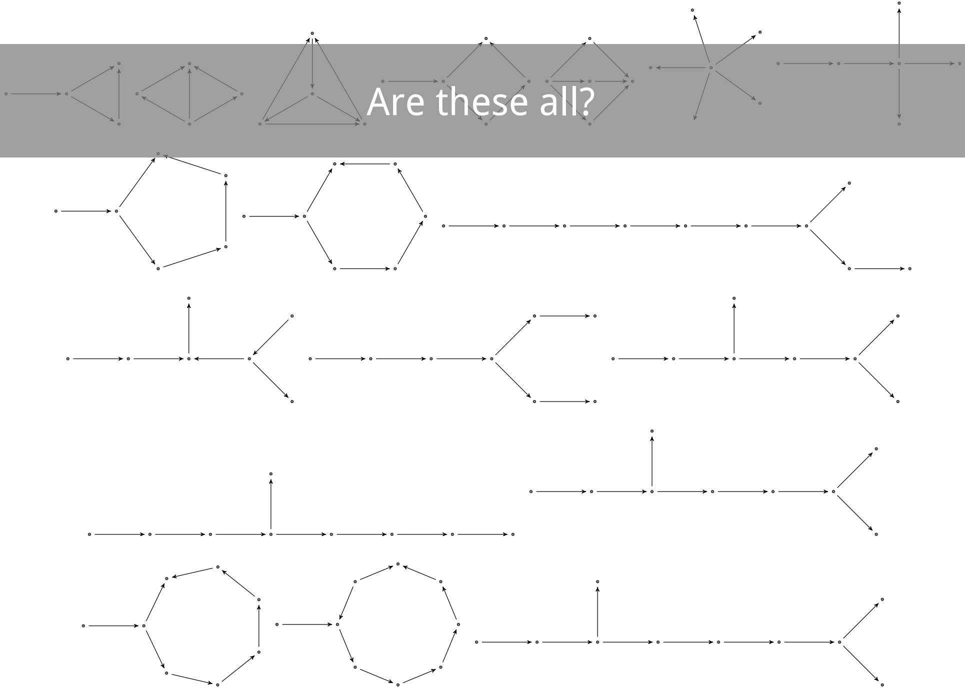


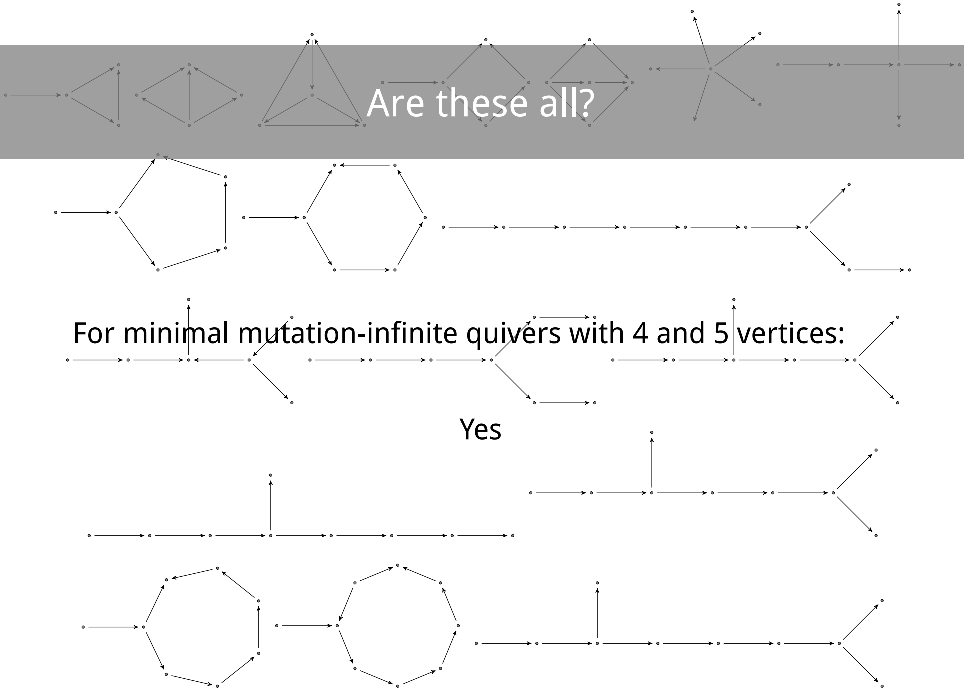
Orientations of simply-laced Hyperbolic Coxeter diagrams are **minimal** mutation-infinite quivers

Orientations of Dynkin diagrams and affine Dynkin diagrams are mutation-finite.

Orientations of simply-laced Hyperbolic Coxeter diagrams are mutation-infinite.

Are these all?

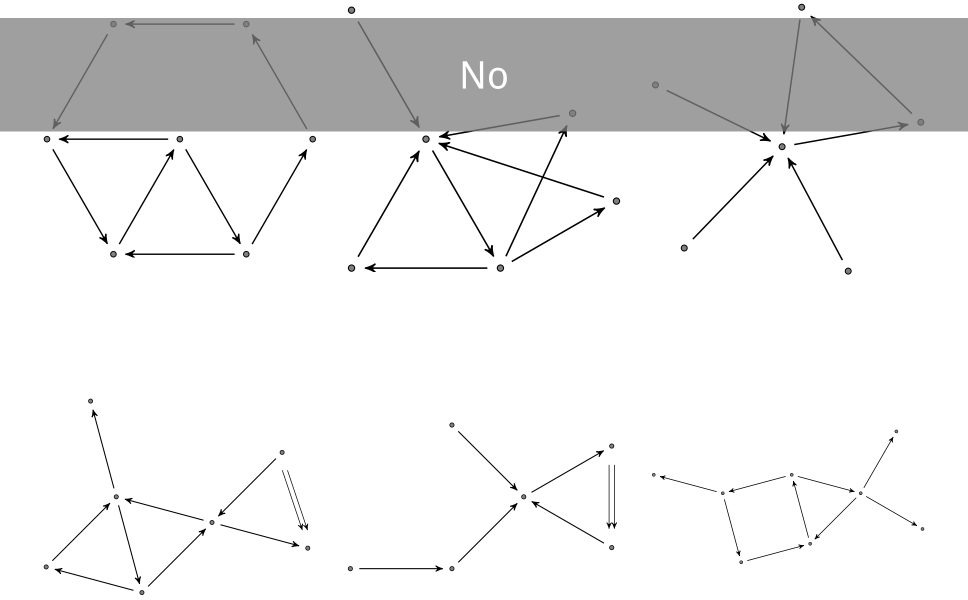




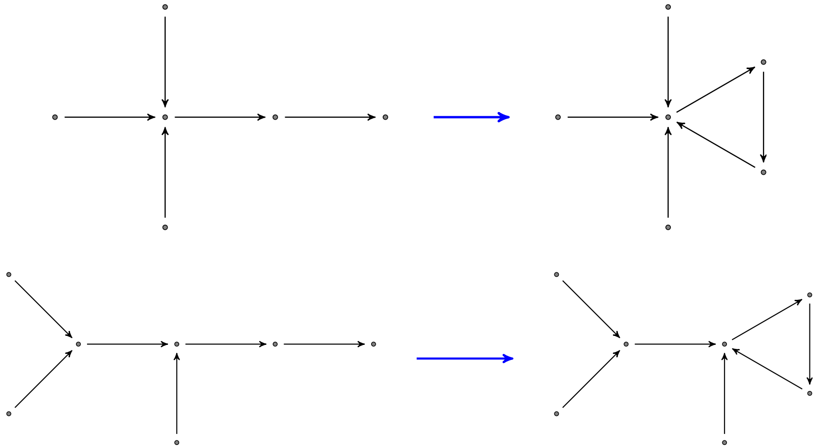
Are these all?

For minimal mutation-infinite quivers with 4 and 5 vertices:

Yes

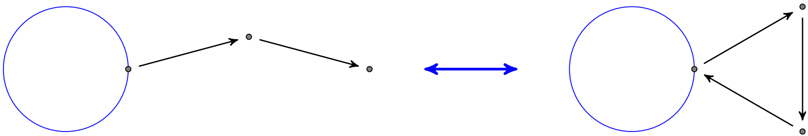

No

Patterns among the quivers

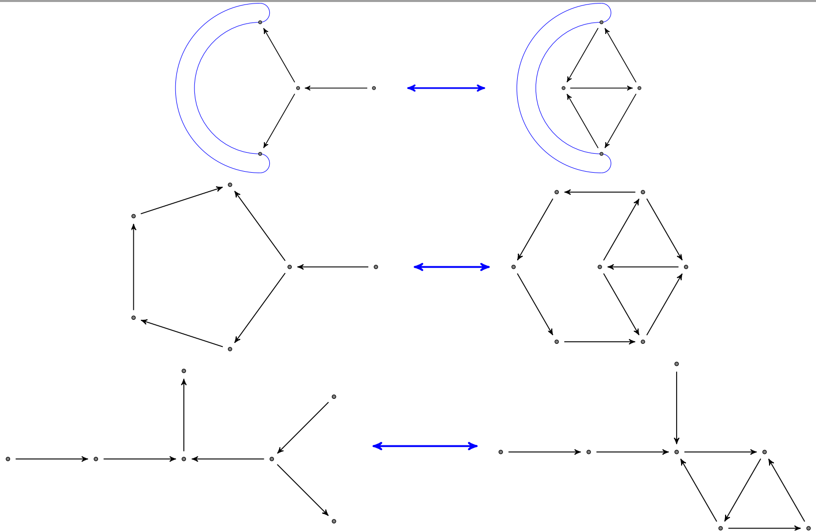


Moves

replace a subquiver while staying minimal mutation-infinite



Another example



and many more

Sink-source mutations preserve
minimal mutation-infinite-ness

A sink-source mutation does not affect the underlying unoriented graph of a quiver and does not change the mutation class of any subquivers.

Result

Any minimal mutation-infinite quiver can be transformed through sink source mutations and at most 10 moves to either

- ⋆ a hyperbolic Coxeter simplex diagram
- ⋆ a double arrow quiver
- ⋆ an exceptional quiver

Result
for quivers up to size 9

Any minimal mutation-infinite quiver can be transformed through sink-source mutations and at most **5** moves to either

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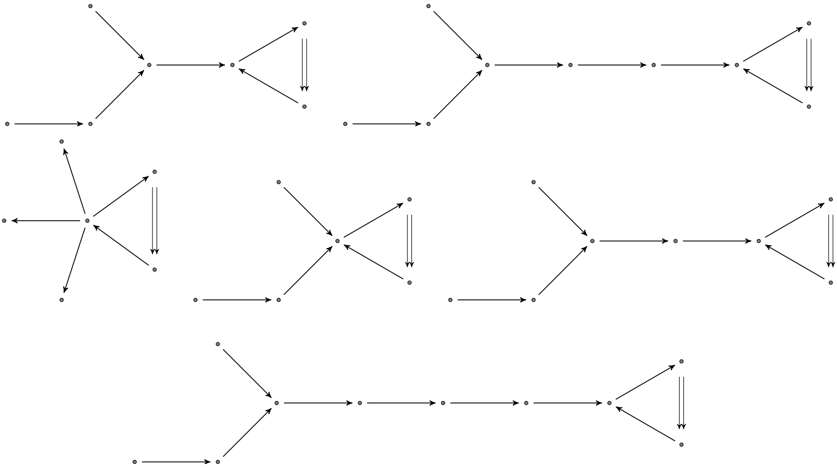
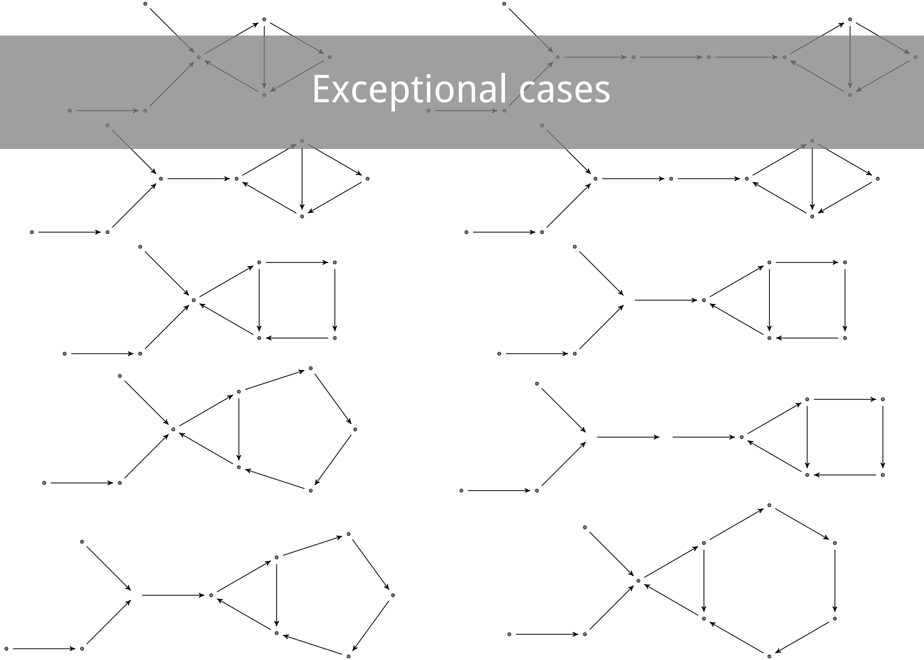
- ⋆ a double arrow quiver
- ⋆ an exceptional quiver

Double arrow quivers



Exceptional cases



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