# Minimal mutation-infinite quivers

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Workshop on Cluster Algebras
and finite dimensional algebras

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Thank organisers for the conference.

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## Introduction

Mutations on quivers studied following the introduction of cluster algebras by Fomin and Zelevinsky in 2002.

This work follows:

- ⋆ Classification of minimal infinite-type diagrams by Seven published in 2007
- ⋆ Classification of mutation-finite quivers by Felikson, Shapiro and Tumarkin published in 2012

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Quivers have been studied for a number of years in representation theory.

Mutations on quivers were introduced by Fomin and Zelevinsky in 2002 in the study of Cluster Algebras. These mutations provide a number of interesting properties to study.

My work on this is in the same vein as Seven's classification of minimal infinite-type diagrams and closely follows work done by Felikson, Shapiro and Tumarkin classifying all mutation-finite quivers.

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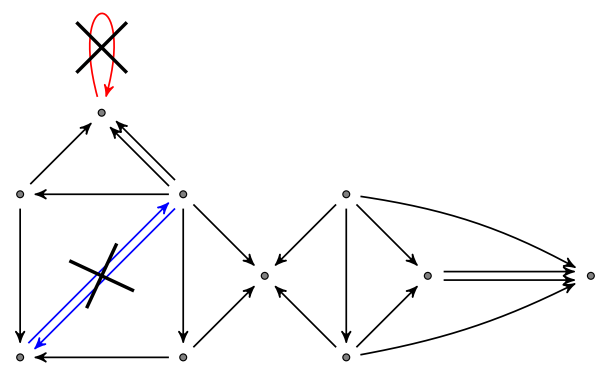
## Quivers
directed (multi-)graphs with no loops or 2-cycles



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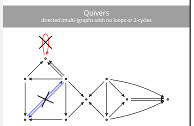
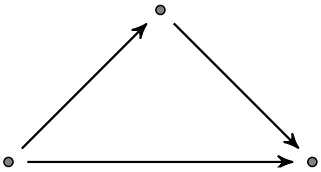


We do not allow quivers to have loops, like the red arrow in the picture, or 2-cycles, like the blue pair.

By restricting in this way the quiver can be represented by a unique (up to relabelling the vertices) skew-symmetric matrix.

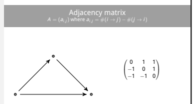
## Adjacency matrix

$A = (a_{i,j})$ where $a_{i,j} = \#(i \to j) - \#(j \to i)$



$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$



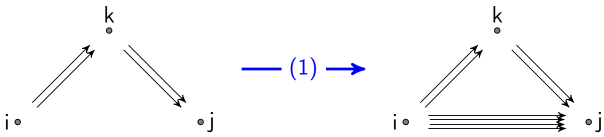


The adjacency matrix of a quiver is formed by setting the i,j-th entry to be (number of arrows i -> j) - (number of arrows j -> i)

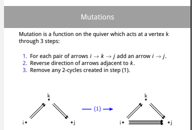
## Mutations

Mutation is a function on the quiver which acts at a vertex k through 3 steps:

1. For each pair of arrows $i \to k \to j$ add an arrow $i \to j$.
2. Reverse direction of arrows adjacent to $k$.
3. Remove any 2-cycles created in step (1).





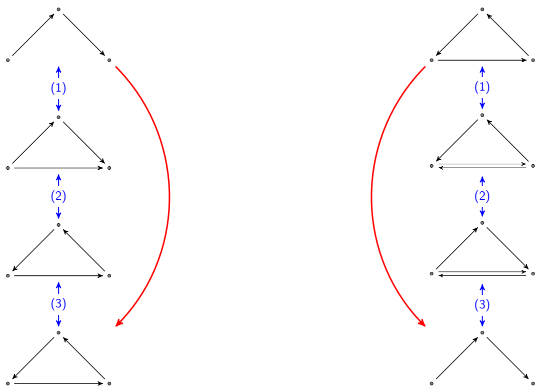



Mutation is defined in three steps.

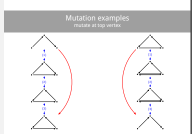
The first step adds an arrow for each possible path through the vertex k. For example in the picture shown here there are 4 possible paths from i to j, so add 4 arrows from i to j.

## Mutation examples

mutate at top vertex





This slide shows two examples of mutations.

Mutation at the top vertex is shown by the large red arrow, and is made up of the three steps outlines above.

## Mutations

Mutations are involutions.



Mutations
Mutations are involutions.

The previous examples illustrate the important property that mutations are involutions (or self-inverting). This fact is not immediately obvious from the definition, but is fairly easy to check.

## Matrix mutations

Mutation at vertex $k$ takes an adjacency matrix $B = (b_{i,j})$ to $B' = (b'_{i,j})$ where

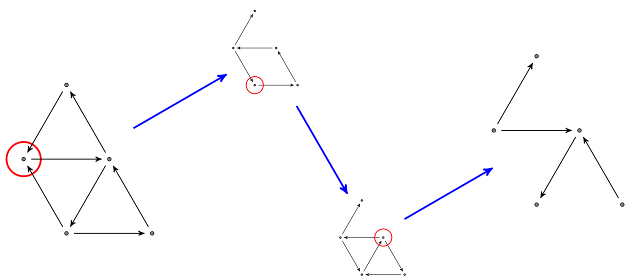
$$b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k \\ b_{i,j} + \frac{|b_{i,k}|b_{k,j} + b_{i,k}|b_{k,j}|}{2} & \text{otherwise} \end{cases}$$



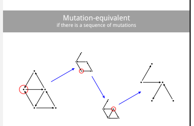
Matrix mutations

One can see that this formula acts on a matrix in the same way that mutation acted on a quiver. The first case corresponds to reversing the arrows adjacent to $k$, while the second case adds the appropriate arrows.

## Mutation-equivalent
if there is a sequence of mutations







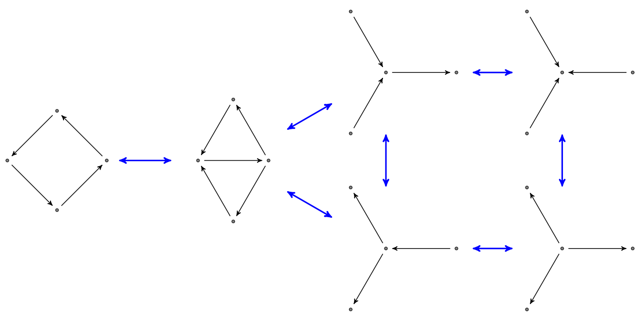
Quivers $P$ and $Q$ are **mutation-equivalent** if there is a sequence of mutations $\mu_{i_1}, \ldots, \mu_{i_n}$ such that

$$P = \mu_{i_1} \circ \cdots \circ \mu_{i_n}(Q)$$

This is an equivalence relation, and the equivalence classes under this relation are called **mutation-classes**.

## Mutation-finite

or conversely mutation-infinite



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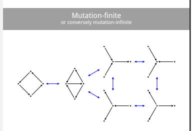
Minimal mutation-infinite quivers
└Quivers and mutations
  └Mutation-finite

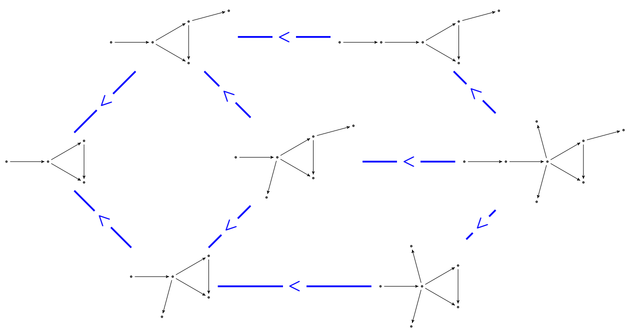


A quiver is **mutation-finite** if its mutation class is of finite size.

Conversely, a quiver which is not mutation-finite is **mutation-infinite**.

## Partial ordering

on mutation-infinite quivers given by inclusion



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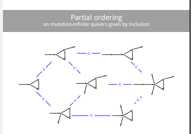
Minimal mutation-infinite quivers
└Quivers and mutations
  └Partial ordering

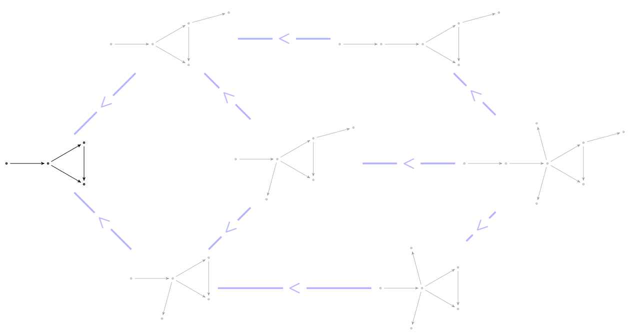


If a quiver contains a mutation-infinite subquiver, then it must be mutation-infinite itself.

This leads to a natural partial ordering on the collection of all mutation-infinite quivers given by inclusion.

Partial ordering is given on all quivers. Then get maximal mutation-finite quivers - those that adding a vertex yields a mutation-infinite quiver - and minimal mutation-infinite quivers - that removing a vertex gives a mutation-finite quiver.

## Minimal mutation-infinite quivers



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Minimal mutation-infinite quivers
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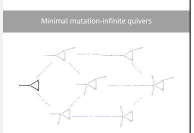
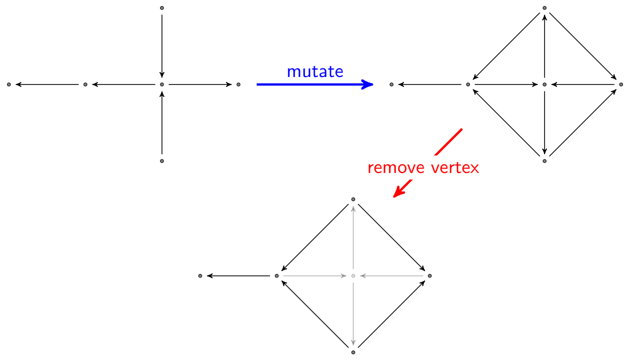


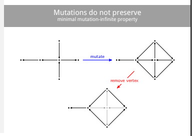
Minimal mutation-infinite quivers are minimal with respect to the partial ordering above.

These are quivers which are mutation-infinite, but any subquiver is mutation-finite.

## Mutations do not preserve
minimal mutation-infinite property





Mutations do not in general preserve minimal mutation-infinite-ness of a quiver.

On this slide is an example of such a mutation. The initial quiver is an orientation of a hyperbolic Coxeter diagram, and so is MMI. Mutation at the central vertex yields the quiver on the right. This quiver is not MMI, as removing the central vertex gives a mutation-infinite quiver, as shown at the bottom. In fact, this quiver with the removed vertex is itself MMI.

## Why
are minimal mutation-infinite quivers interesting?

Any quiver containing a minimal mutation-infinite subquiver is necessarily mutation-infinite.

A complete classification would give a systematic approach to check whether any given quiver is mutation-infinite or not.



MMI quivers provide an computation free way of checking whether a given quiver is mutation-finite or mutation-infinite.

Check whether any minimal mutation-infinite quiver can be embedded as a subquiver in the initial quiver. If it can, then the quiver is mutation-infinite. If the quiver contains no MMI subquiver, then the quiver is mutation-finite.

## Ahmet Seven's classification
of minimal infinite-type diagrams

Seven classified all the infinite-type diagrams such that removing a vertex yielded a finite-type diagram.



This is very similar to the work I am attempting on the minimal mutation-infinite quivers.

## Felikson, Shapiro and Tumarkin

started studying minimal mutation-infinite quivers

In their paper on classifying mutation-finite quivers, FST proved that there were no minimal mutation-infinite quivers with more than 10 vertices.



Felikson, Shapiro and Tumarkin
started studying minimal mutation-infinite quivers

In their paper on classifying mutation-finite quivers, FST proved that there were no minimal mutation-infinite quivers with more than 10 vertices.

Linked to this size restirction - it is known that hyperbolic coxeter simplices only exist up to dimension 10. As will be seen, these simplices admit a coxeter diagram, which in some cases can represent a quiver.

The quivers from simply-laced hyperbolic Coxeter diagrams are minimal mutation-infinite.

## Minimal mutation-infinite quivers

Distinguished family of minimal mutation-infinite quivers which are orientations of simply-laced Coxeter diagrams of hyperbolic Coxeter simplices.



Minimal mutation-infinite quivers

Distinguished family of minimal mutation-infinite quivers which are orientations of simply-laced Coxeter diagrams of hyperbolic Coxeter simplices.

It is a result of Felikson, Shapiro and Tumarkin that the maximum number of vertices in a minimal mutation-infinite quiver is 10.

This coincides with the sizes of hyperbolic Coxeter diagrams, as they exist up to size 10.

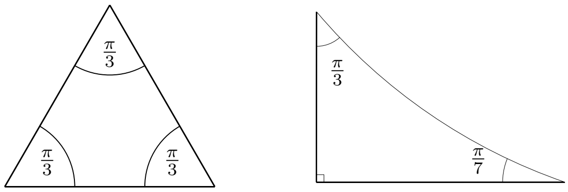
This motivated the study of MMI quivers coming from these hyperbolic Coxeter diagrams.

## Coxeter simplex

convex hull of $n+1$ points

Considered inside spherical, Euclidean or hyperbolic space.

$n+1$ hyper-planes $H_i$ with dihedral angles $\frac{\pi}{k_{ij}}$ (or possibly 0) between $H_i$ and $H_j$.






Coxeter simplex
convex hull of $n+1$ points

Considered inside spherical, Euclidean or hyperbolic space.

$n+1$ hyper-planes $H_i$ with dihedral angles $\frac{\pi}{k_{ij}}$ (or possibly 0) between $H_i$ and $H_j$.

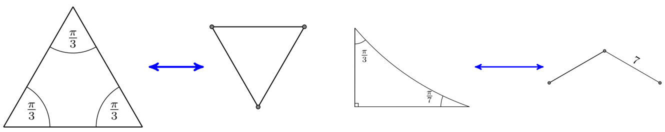
The faces of the simplex can be extended to hyperplanes. As it is a simplex, all such hyperplanes will intersect any other, and will do so at a dihedral angle.

The requirement that the angles are submultiples of $\pi$ is precisely what makes them Coxeter simplices.

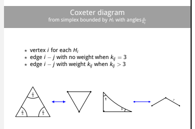
## Coxeter diagram

from simplex bounded by $H_i$ with angles $\frac{\pi}{k_{ij}}$

- ⋆ vertex $i$ for each $H_i$
- ⋆ edge $i - j$ with no weight when $k_{ij} = 3$
- ⋆ edge $i - j$ with weight $k_{ij}$ when $k_{ij} > 3$





When the angle is $\frac{\pi}{2}$ there is no edge between the nodes, and when the angle is 0 (i.e. the hyperplanes meet on the boundary of hyperbolic space) the weight is $\infty$.

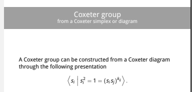
## Coxeter group

from a Coxeter simplex or diagram

A Coxeter group can be constructed from a Coxeter diagram through the following presentation

$$\left\langle s_i \;\middle|\; s_i^2 = 1 = (s_i s_j)^{k_{ij}} \right\rangle .$$





Anna Felikson will *probably* talk about how mutations to the Coxeter diagrams affects the Coxeter groups obtained from them.

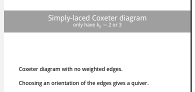
## Simply-laced Coxeter diagram

only have $k_{ij} = 2$ or $3$

Coxeter diagram with no weighted edges.

Choosing an orientation of the edges gives a quiver.

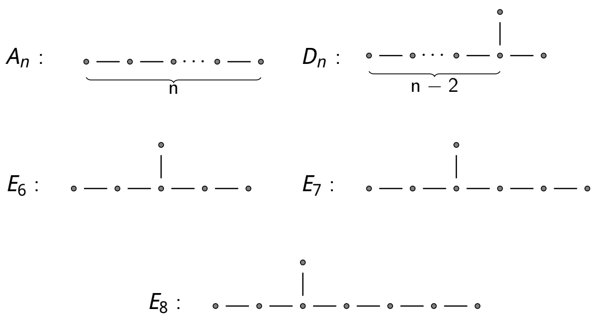




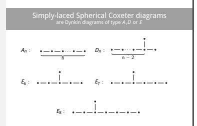
As I have been studying quivers, I only consider those Coxeter diagrams which would give a quiver when an orientation is chosen for each edge.

These diagrams are precisely those given by simply-laced diagrams. This condition ensures that there are no weighted edges in the diagram, so the only possible anglees in the simplex are $\frac{\pi}{2}$ and $\frac{\pi}{3}$.

## Simply-laced Spherical Coxeter diagrams
are Dynkin diagrams of type $A$,$D$ or $E$



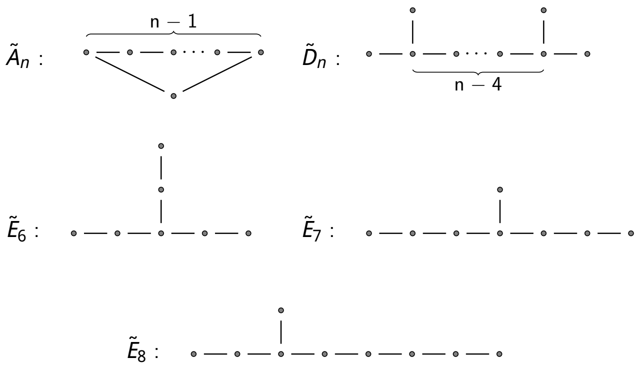


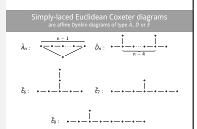
Simply-laced spherical Coxeter simplices give a Coxeter diagram which is one of the Dynkin diagrams of type A,D or E.

Orientations of such diagrams give mutation-finite quivers. In fact such quivers are of finite-type and give a finite-type cluster algebra.

## Simply-laced Euclidean Coxeter diagrams
are affine Dynkin diagrams of type $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$





Orientations of affine Dynkin diagrams are mutation-finite.

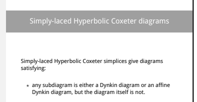
These quivers are of affine-type.

## Simply-laced Hyperbolic Coxeter diagrams

Simply-laced Hyperbolic Coxeter simplices give diagrams satisfying:

- ⋆ any subdiagram is either a Dynkin diagram or an affine Dynkin diagram, but the diagram itself is not.





The quivers attained from those hyperbolic Coxeter simplices which admit quivers necessary have the property of being minimal mutation-infinite.

Quivers from both affine-type and finite-type are mutation-finite.

## Orientations of simply-laced Hyperbolic
### Coxeter diagrams are mutation-infinite quivers

Felikson, Shapiro and Tumarkin classified all mutation-finite quivers - (almost all) orientations of simply-laced Hyperbolic Coxeter diagrams do not lie in this classification.

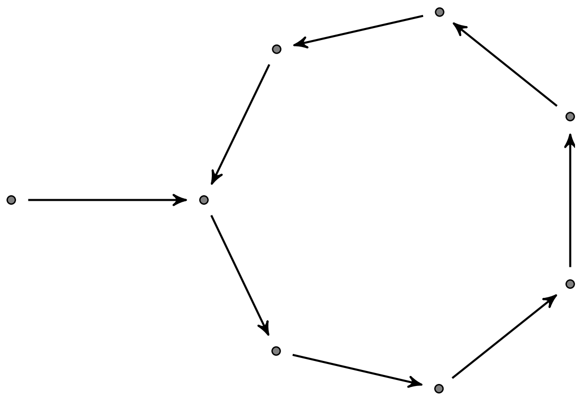


Orientations of simply-laced Hyperbolic
Coxeter diagrams are mutation-infinite quivers
Felikson, Shapiro and Tumarkin classified all mutation-finite quivers - (almost all) orientations of simply-laced Hyperbolic Coxeter diagrams do not lie in this classification.

A few orientations of simply-laced hyperbolic Coxeter diagrams are in fact mutation-finite, but the majority are mutation-infinite.

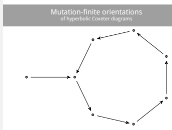
This is known as Felikson, Shapiro and Tumarkin gave a classification of all mutation-finite quivers, and orientations of hyperbolic Coxeter diagrams do no lie in this classification.

## Mutation-finite orientations
### of hyperbolic Coxeter diagrams





Mutation-finite orientations
of hyperbolic Coxeter diagrams



This is an example of one of the mutation-finite orientations of a hyperbolic Coxeter diagram.

In this case the quiver is mutation-equivalent to the $D_8$ dynkin diagram.

## Orientations of simply-laced Hyperbolic
### Coxeter diagrams are **minimal** mutation-infinite quivers

Orientations of Dynkin diagrams and affine Dynkin diagrams are mutation-finite.

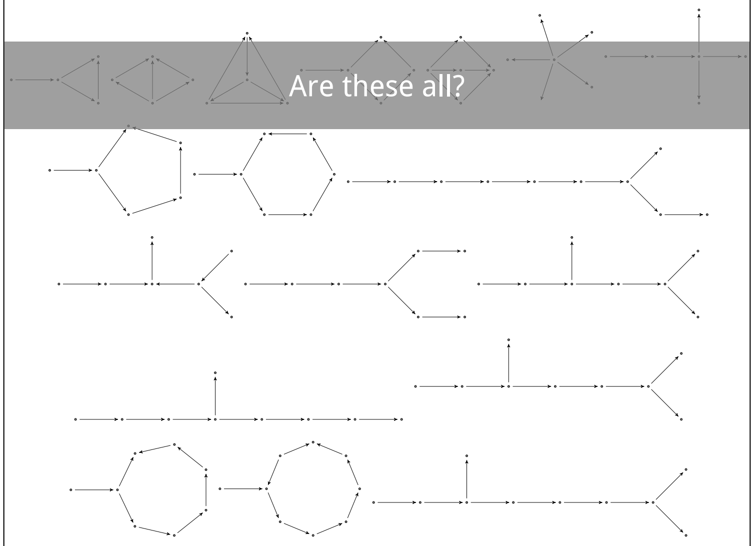
Orientations of simply-laced Hyperbolic Coxeter diagrams are mutation-infinite.



Orientations of simply-laced Hyperbolic
Coxeter diagrams are **minimal** mutation-infinite quivers
Orientations of Dynkin diagrams and affine Dynkin diagrams are mutation-finite.
Orientations of simply-laced Hyperbolic Coxeter diagrams are mutation-infinite.

Put these two facts together to show that these quivers are minimal mutation-infinite.

## Are these all?



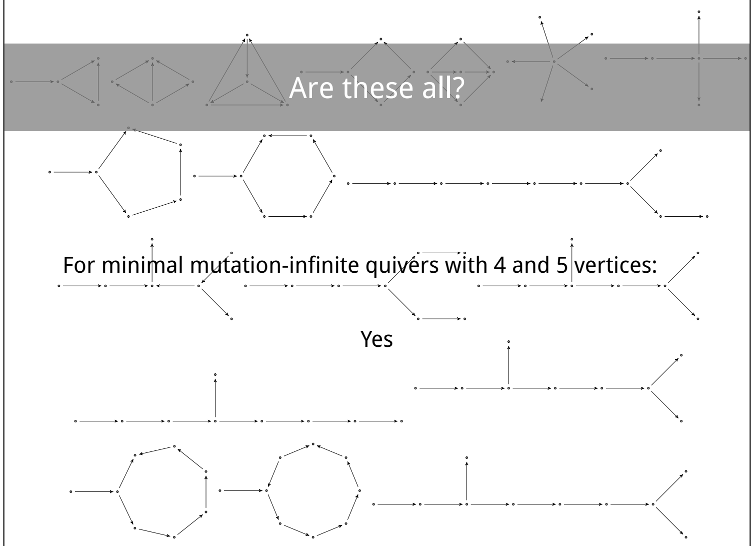






Are there any quivers which are minimal mutation-infinite but do not come from hyperbolic Coxeter simplices?

## Are these all?



For minimal mutation-infinite quivers with 4 and 5 vertices:

Yes

Are these all?

For minimal mutation-infinite quivers with 4 and 5 vertices:

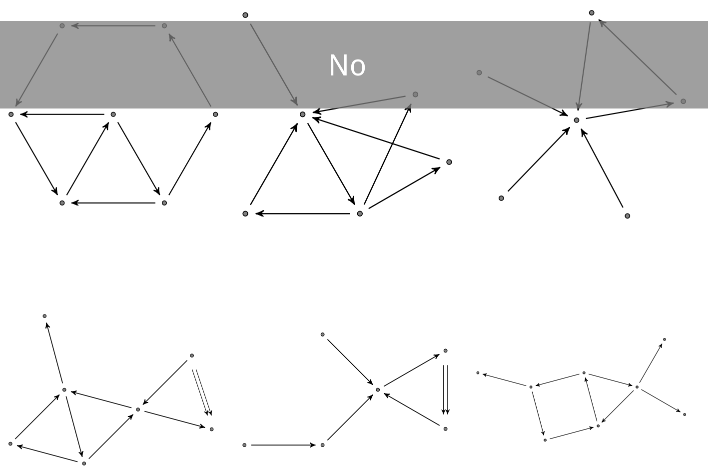
Yes





MMI Quivers exist with between 4 and 10 vertices. Those with 4 or 5 vertices are in fact all hyperbolic Coxeter diagrams.

## No



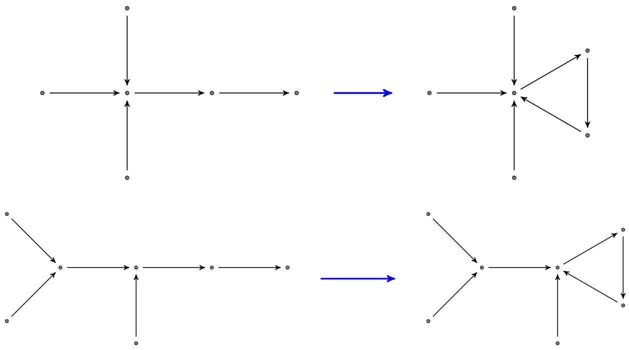
No





In general examples of minimal mutation-infinite quivers of size 6 and up were found which were not orientations of Coxeter simplices.

## Patterns among the quivers



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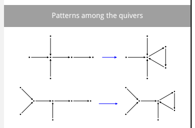
Minimal mutation-infinite quivers
└─Coxeter simplices, groups and diagrams
└─Patterns among the quivers



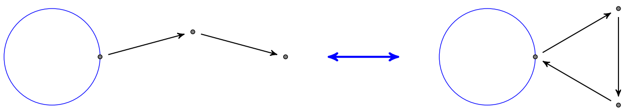
Many of the quivers that we found were only one or two mutations away from hyperbolic Coxeter simplices.

Among the quivers there are a number of patterns in the subquivers which appear again and again.

For example the pattern shown here, when an MMI quiver had two arrows coming out of it (like in the pictures) then replacing those with an oriented triangle would also give an MMI quiver.

## Moves

replace a subquiver while staying minimal mutation-infinite



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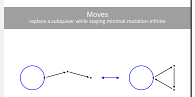
Minimal mutation-infinite quivers
└─Coxeter simplices, groups and diagrams
└─Moves



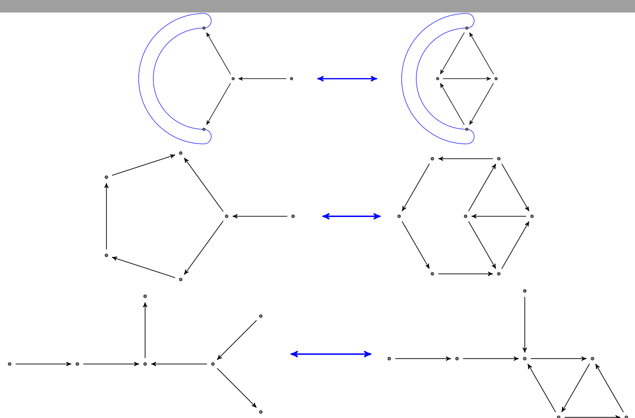
These patterns lead to the idea of 'moves' to classify these quivers.

We found and proved a number of equivalences for generic subquivers of the form:

A quiver containing such a subquiver is MMI if and only if the same quiver with the subquiver replaced according to the move is MMI.

Given an MMI quiver, swapping a subquiver according to some move will give another MMI quiver.

## Another example



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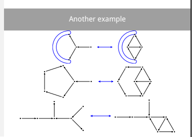
Minimal mutation-infinite quivers
└─Coxeter simplices, groups and diagrams
└─Another example



A quiver taking the form on the right is MMI if and only if the one on the left is MMI.

These two examples show how the move works. In the bottom one the blue subquiver is not actually connected to the other vertex, but the move still works.

## and many more

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Minimal mutation-infinite quivers

└Coxeter simplices, groups and diagrams

└and many more

Currently have a list of 35 moves of these kind.

## Sink-source mutations preserve
minimal mutation-infinite-ness

A sink-source mutation does not affect the underlying unoriented graph of a quiver and does not change the mutation class of any subquivers.

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Minimal mutation-infinite quivers

└Coxeter simplices, groups and diagrams

└Sink-source mutations preserve

Using sink-source mutations, the orientations of arrows in any tree-like parts of the quiver are not important.

## Result

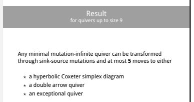
Any minimal mutation-infinite quiver can be transformed through sink source mutations and at most 10 moves to either

- a hyperbolic Coxeter simplex diagram
- a double arrow quiver
- an exceptional quiver

## Result
for quivers up to size 9

Any minimal mutation-infinite quiver can be transformed through sink-source mutations and at most **5** moves to either

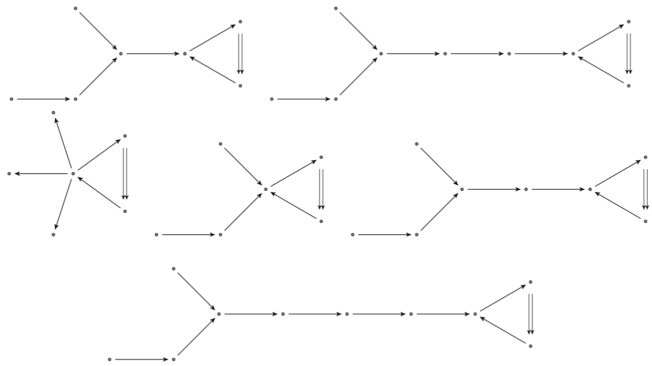
- a hyperbolic Coxeter simplex diagram
- a double arrow quiver
- an exceptional quiver



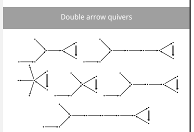

For smaller quivers, the number of moves needed is capped at 5.

Th number of sink-source mutations may seem large, but if you require that the quiver is exactly one of the double arrow quivers, or exceptional quivers given below, then the tree-like sections of the quiver must all have the correct orientations. In the worst case this will require enough sink-source mutations to reverse the direction of all arrows in a tree-like section.

## Double arrow quivers

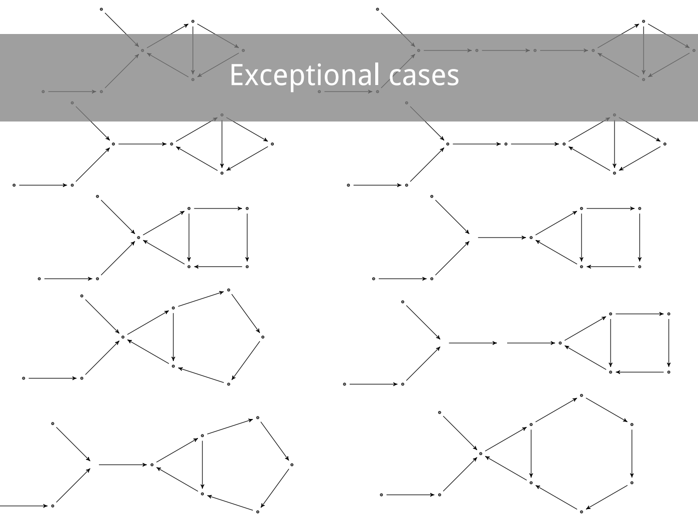




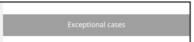



There is one double arrow representative for each size of quiver between 6 and 10. (2 for size 6)

## Exceptional cases








In the exceptional case, the orientation of the cycle at the end is important.

When the cycle is oriented in different ways, then the quiver can be transformed to an orientation of a hyperbolic Coxeter diagram, or a double arrow quiver. However there are some orientations where no such tranformation exists, so these cases must be considered separately.

## Result

Any minimal mutation-infinite quiver can be transformed through sink source mutations and at most 10 moves to either

- ⋆ a hyperbolic Coxeter simplex diagram
- ⋆ a double arrow quiver
- ⋆ an exceptional quiver