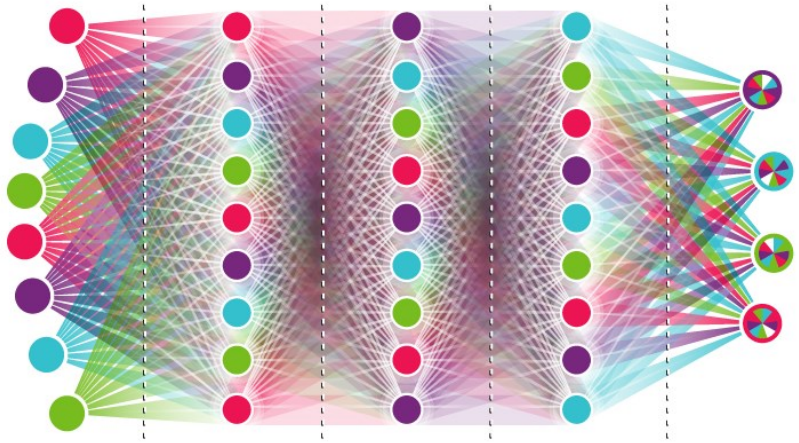
# Multi-Layer Perceptron

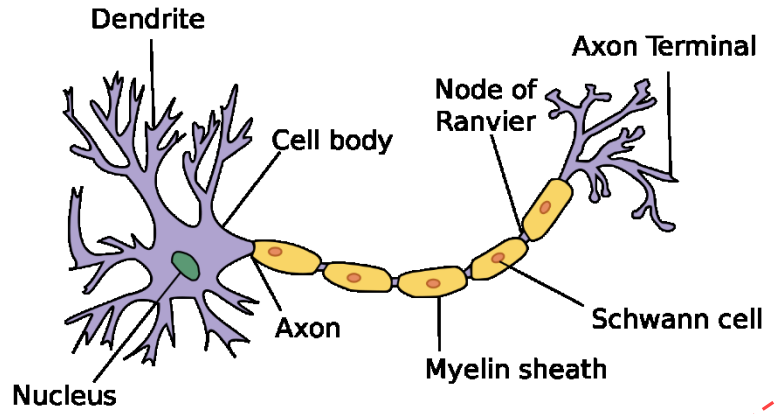
## The Road to Deep Learning

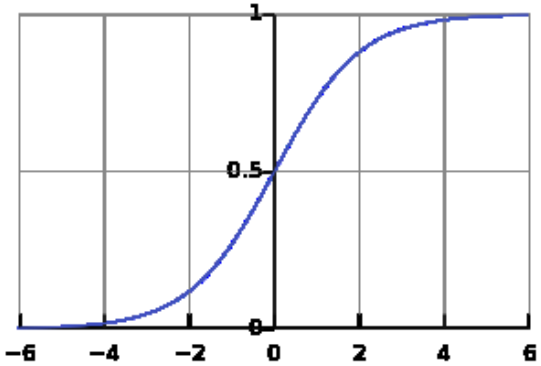


Fast Campus

Start Deep Learning with Tensorflow

# Recap - Perceptron



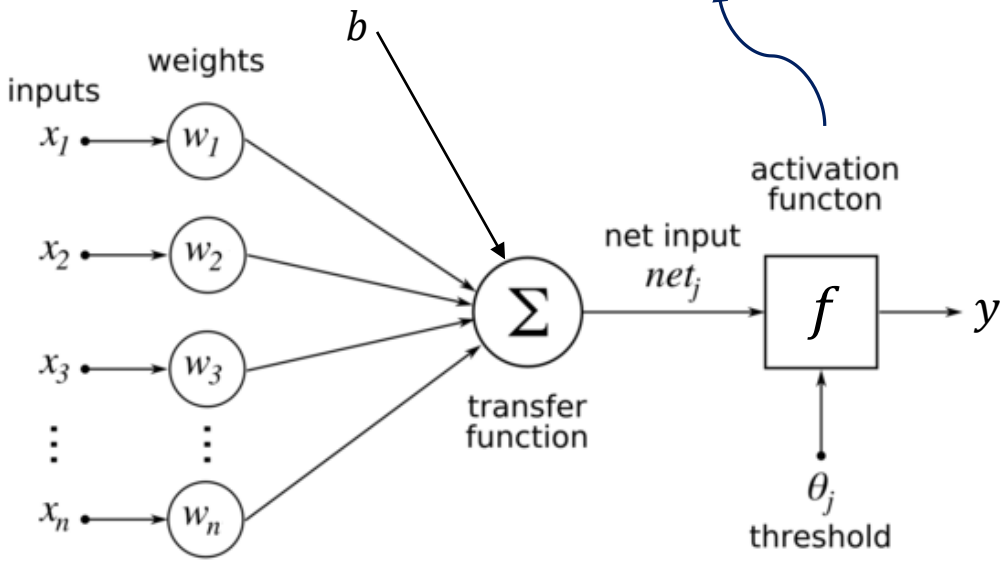

sigmoid activation function

$$f(x) = \frac{1}{1 + e^{-x}}$$

$$y = f(\mathbf{wx} + \mathrm{b})$$

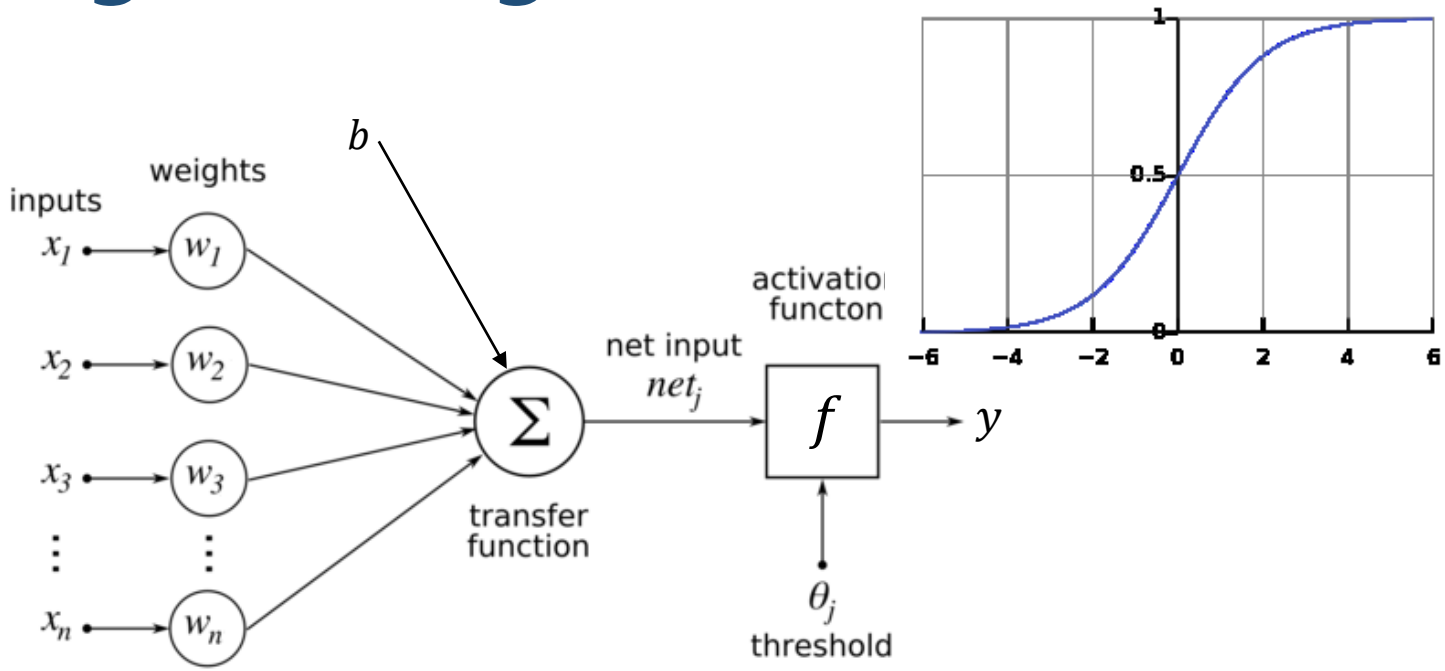
$$\boldsymbol{w} = [w_1\ w_2\ w_3\ \ldots\ w_n]$$

$$\boldsymbol{x} = [x_1\ x_2\ x_3\ \ldots\ x_n]^T$$




<Perceptron>

# Recap – Logistic Regression




<Perceptron>

$$\text{output} = \begin{cases} 0 & \text{if } \sum_j w_j x_j \leq \text{ threshold} \\ 1 & \text{if } \sum_j w_j x_j > \text{ threshold} \end{cases} \xrightarrow{\text{sigmoid activation}} \text{output} = \begin{cases} 0 & \text{if } w \cdot x + b \leq 0 \\ 1 & \text{if } w \cdot x + b > 0 \end{cases}$$

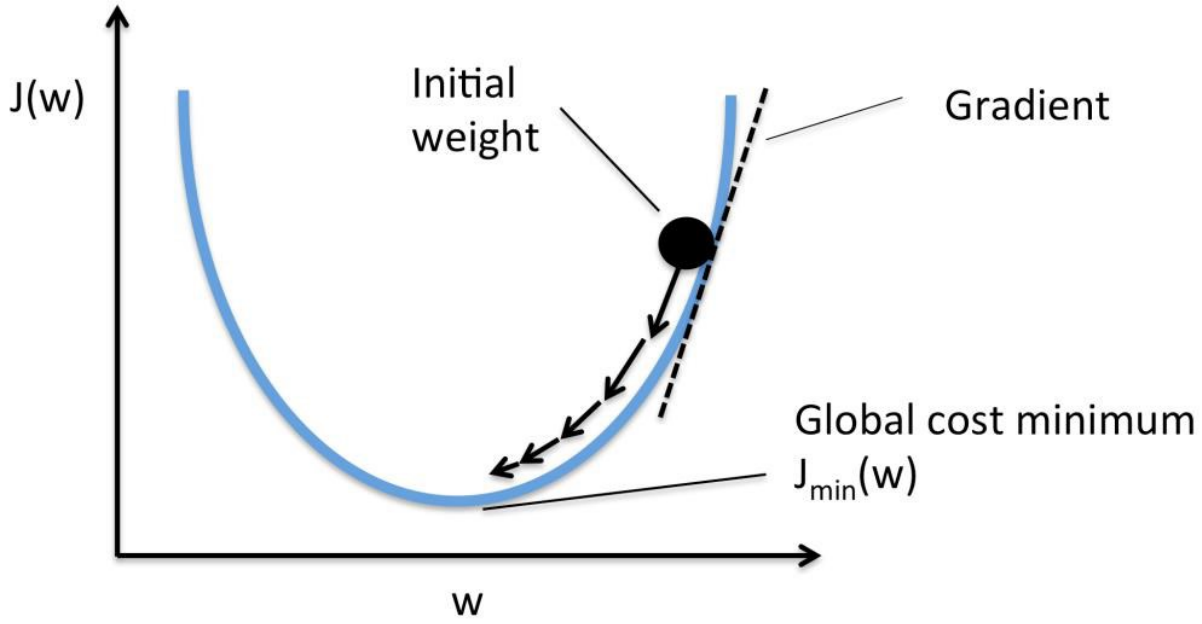
Sigmoid activation : small changes in their weights and bias cause only a small change in their output

Bias(b) : The bias is a measure of how easy it is to get the perceptron to fire

# Recap – Training(Gradient Descent)

$$w_{new} = w - \alpha \frac{\delta L}{\delta w}$$

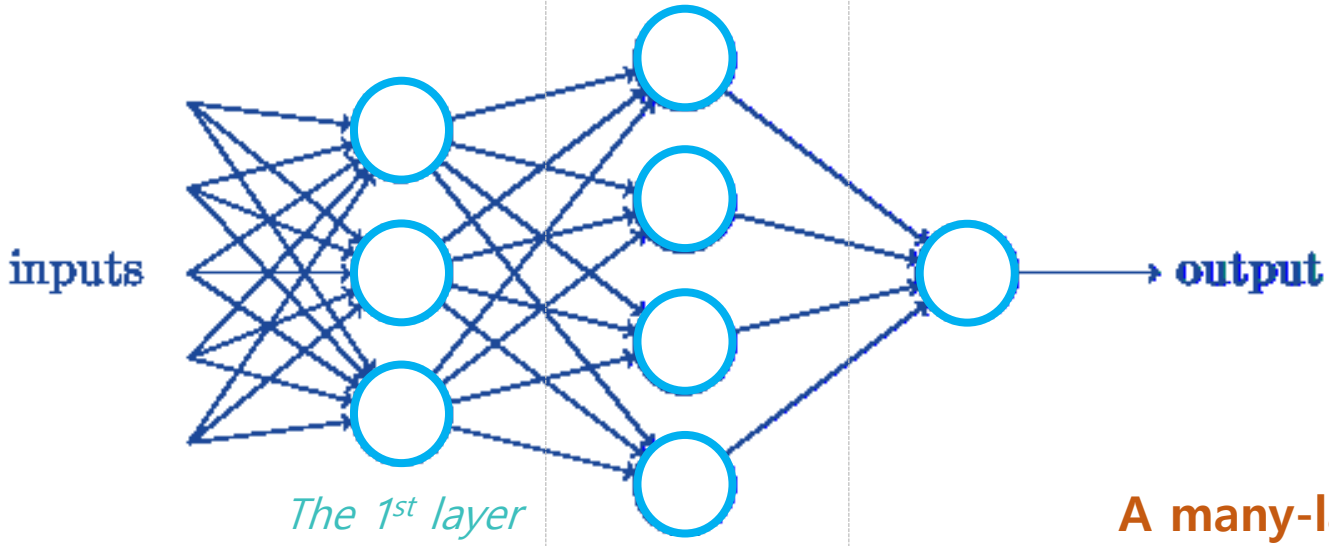
- 방향 : 그 지점에서의 – gradient
- 속력(보폭) : learning rate(α)



# Linear Regression vs Logistic Regression

| Type | Pros | Cons |
| --- | --- | --- |
| Linear Regression | • Simple to implement | • Not guaranteed to work<br>• Only supports binary labels |
| Logistic Regression | • Highly accurate<br>• Model responses are measures of probability | • Only supports binary labels |

# How about This? Multi-Layer Perceptron




*The 1st layer*

simple decisions are made

*The 2nd layer*

more complex and more complex level decisions are made

*The 3rd layer*

Most complex decision is made

**A many-layer network of perceptrons can engage in sophisticated decision making.**

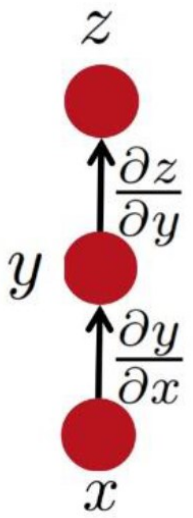
Network을 deep하게 쌓고, class도 여러 개일 때는
어떻게 학습할 수 있을까?

# 먼저 Multi-Layer부터 생각해봅시다

# Back Propagation!

- $w_{new} = w - \alpha \frac{\delta L}{\delta w}$, 결국 $\frac{\delta L}{\delta w}$을 구하고 싶은데, input에 가까운 w로 L을 어떻게 미분할까?
- Chain Rule을 이용!

**Simple Chain Rule**

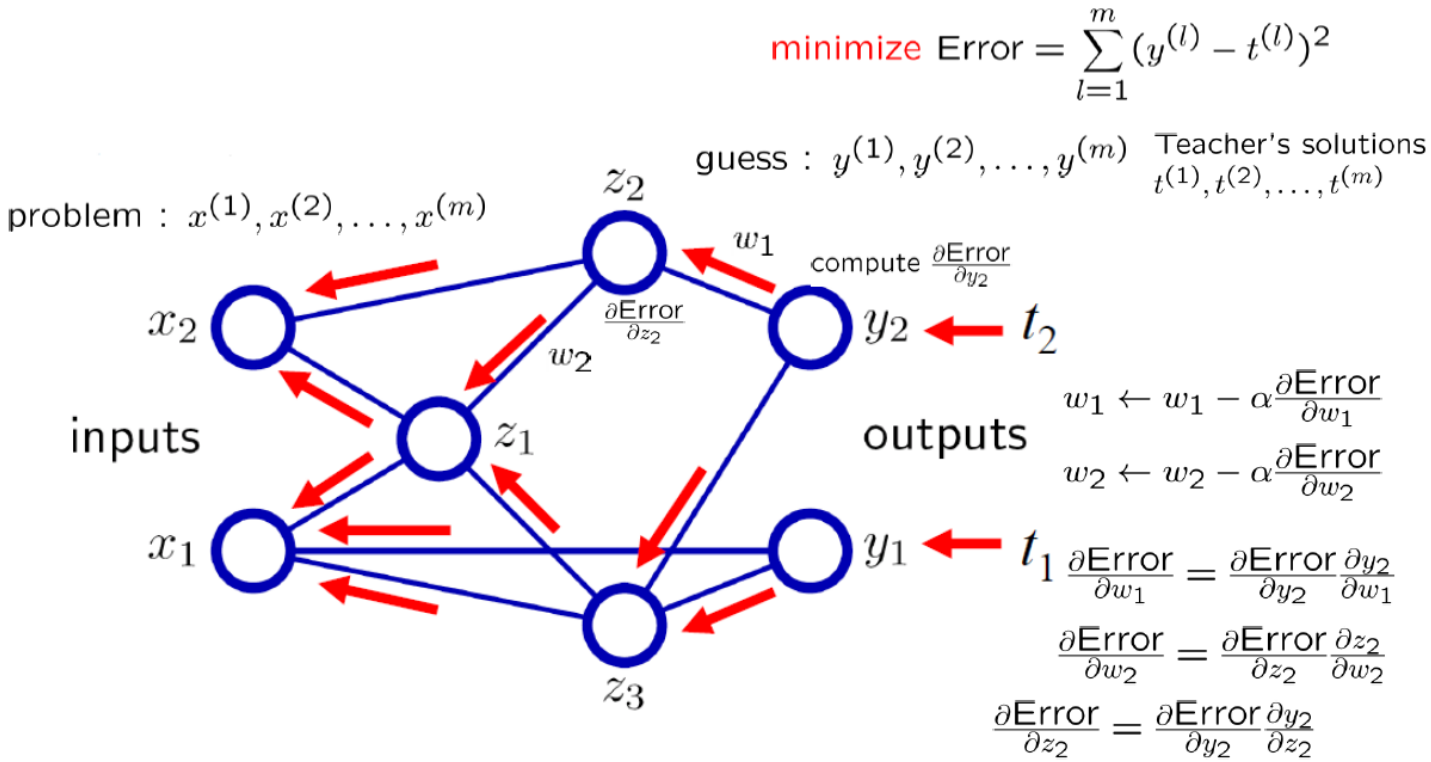



$$\Delta z = \frac{\partial z}{\partial y}\Delta y$$

$$\Delta y = \frac{\partial y}{\partial x}\Delta x$$

$$\Delta z = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}\Delta x$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}$$

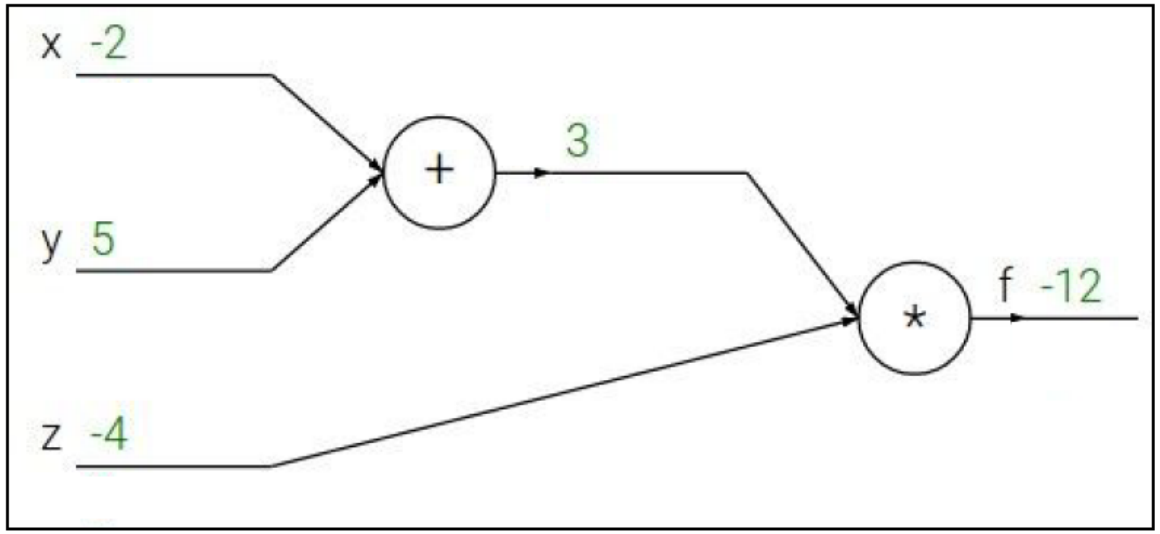


# Back Propagation

Backpropagation: a simple example

$$f(x, y, z) = (x + y)z$$

e.g. x = -2, y = 5, z = -4



# Back Propagation

Backpropagation: a simple example

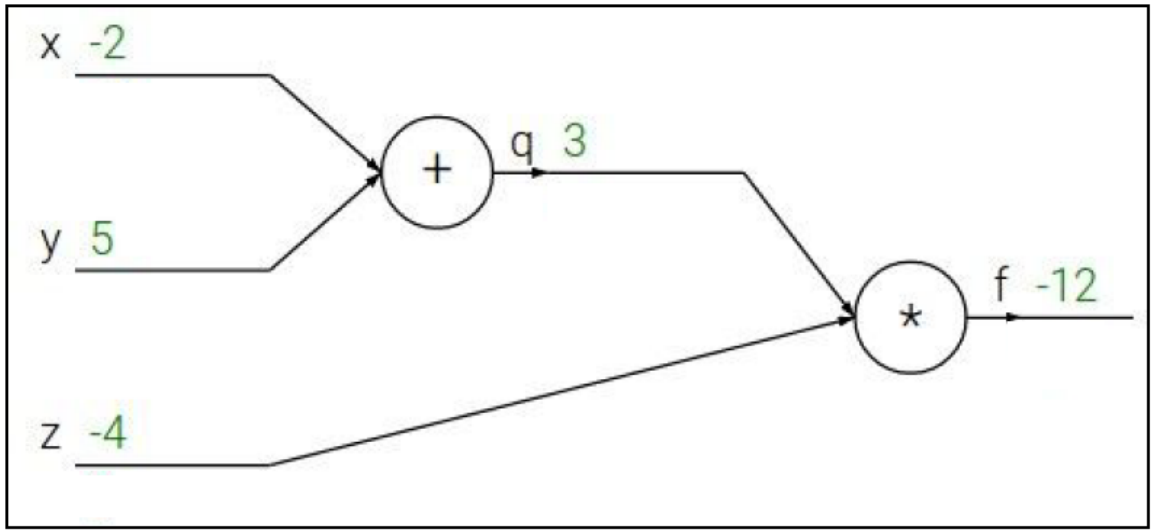
$$f(x, y, z) = (x + y)z$$

e.g. x = -2, y = 5, z = -4

$$q = x + y \qquad \frac{\partial q}{\partial x} = 1, \frac{\partial q}{\partial y} = 1$$

$$f = qz \qquad \frac{\partial f}{\partial q} = z, \frac{\partial f}{\partial z} = q$$

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



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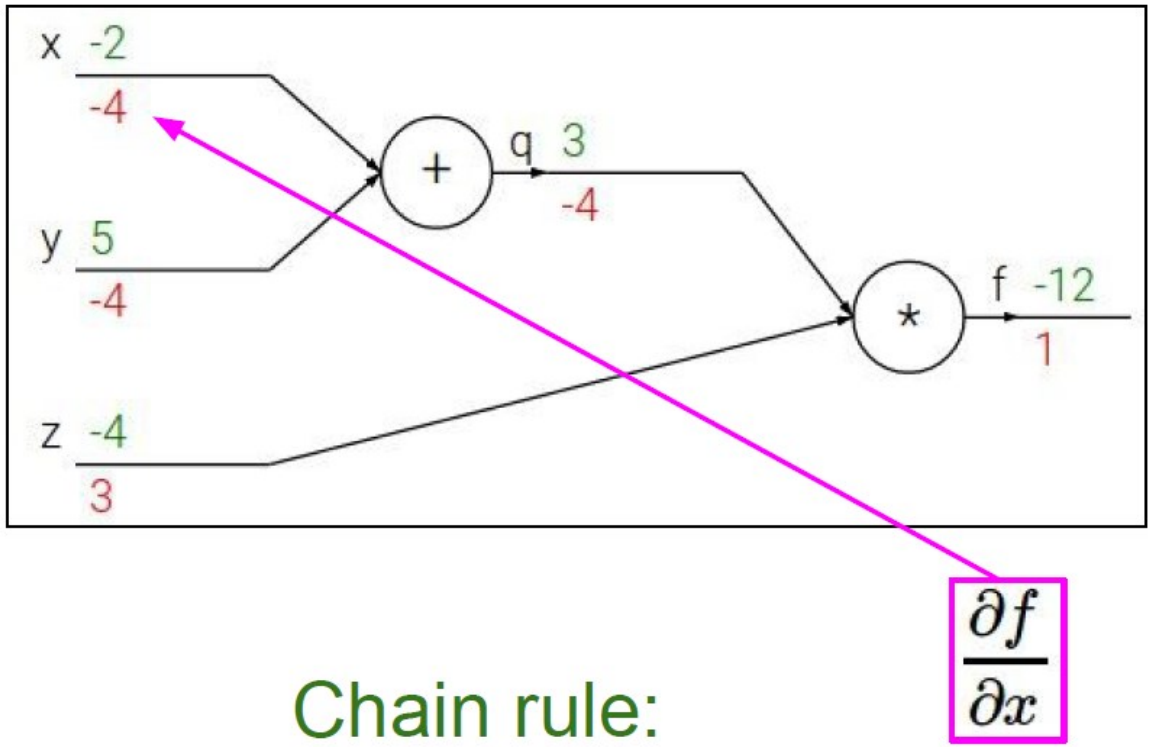
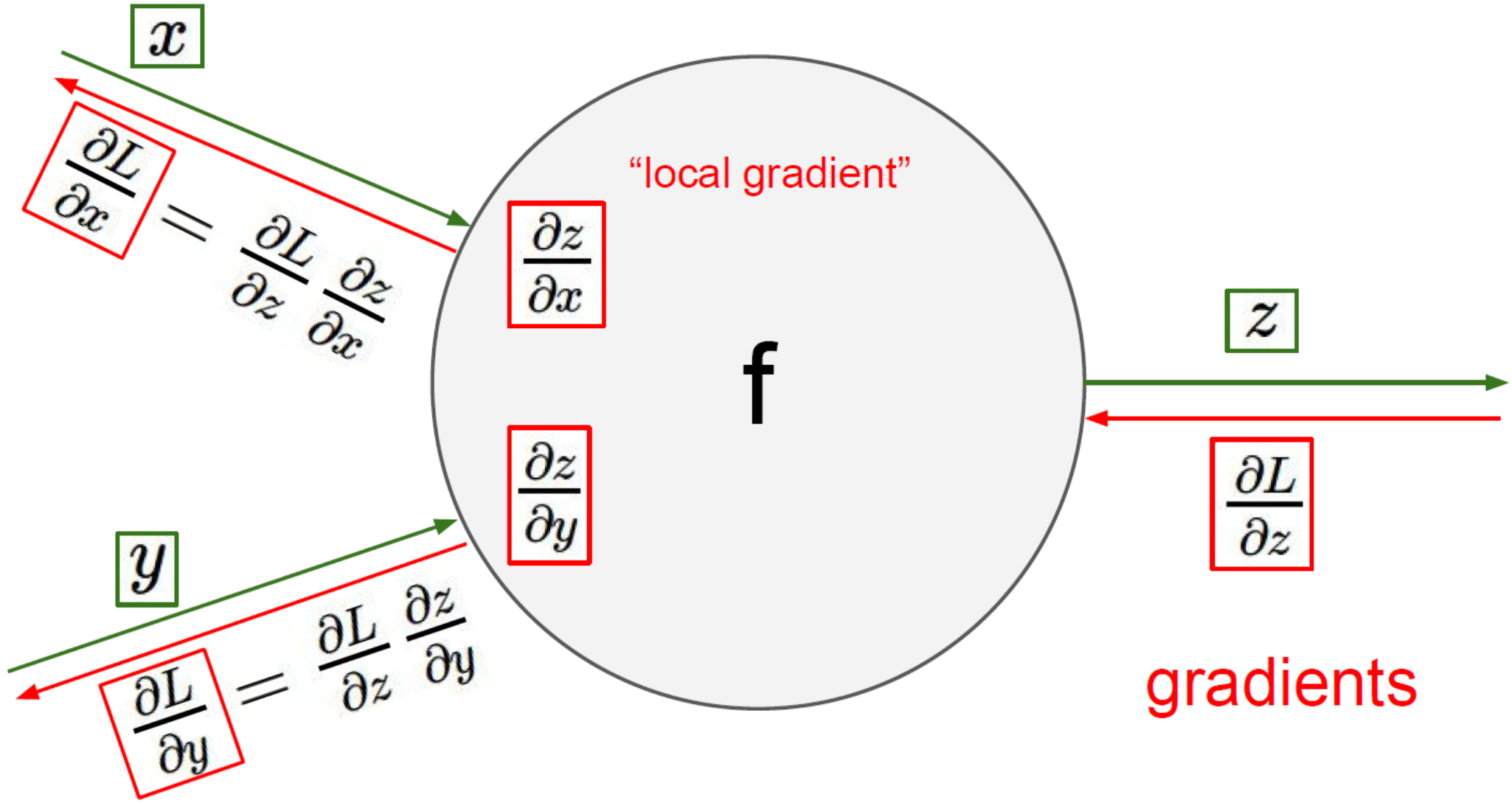



Chain rule:

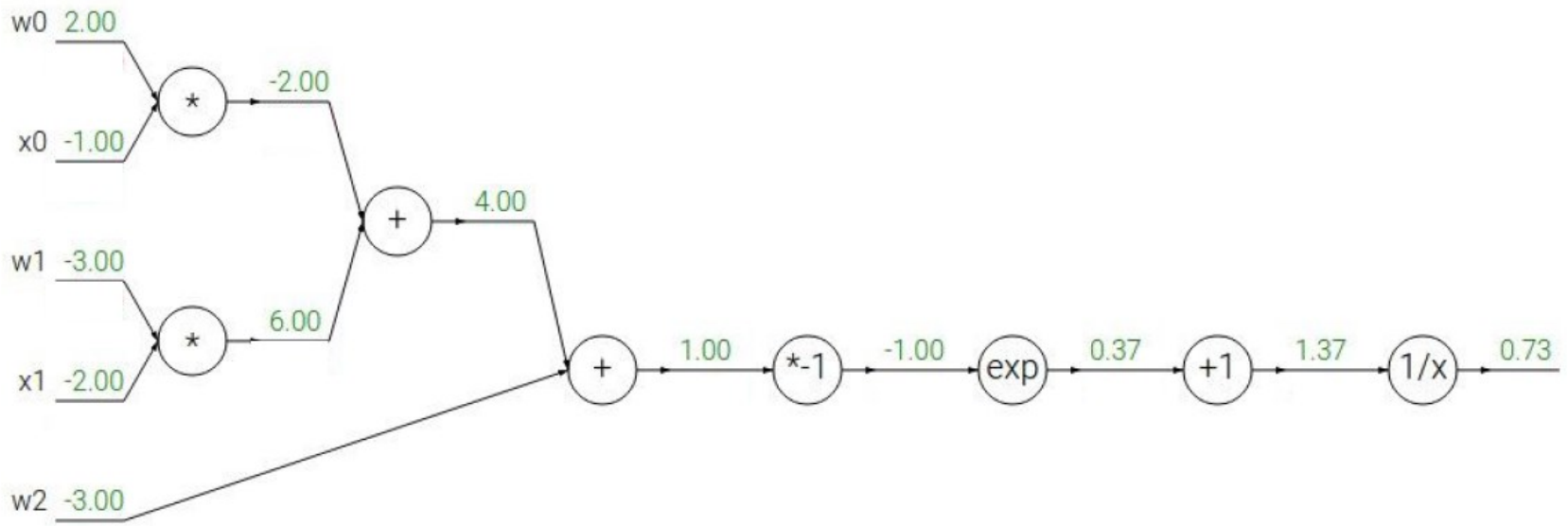
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial x}$$

# Chain Rule(Local Gradient)



# Back Propagation(Example)

Another example: $$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$



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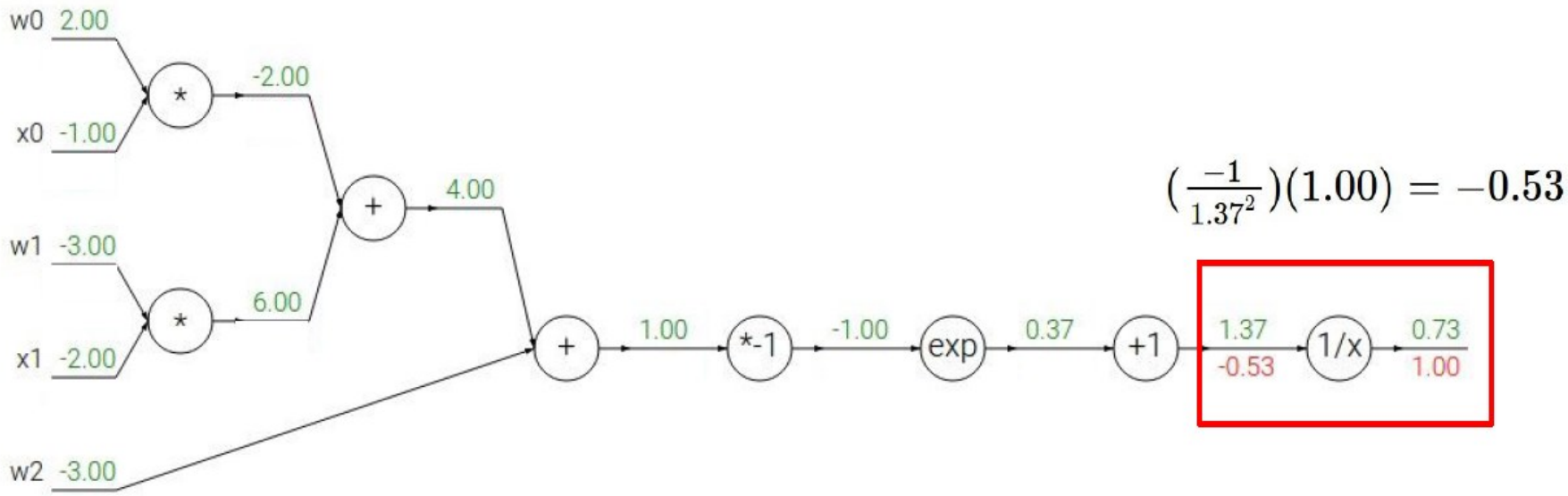


$$(\frac{-1}{1.37^2})(1.00) = -0.53$$

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x$$

$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a$$

$$f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

$$f_c(x) = c + x \quad \rightarrow \quad \frac{df}{dx} = 1$$

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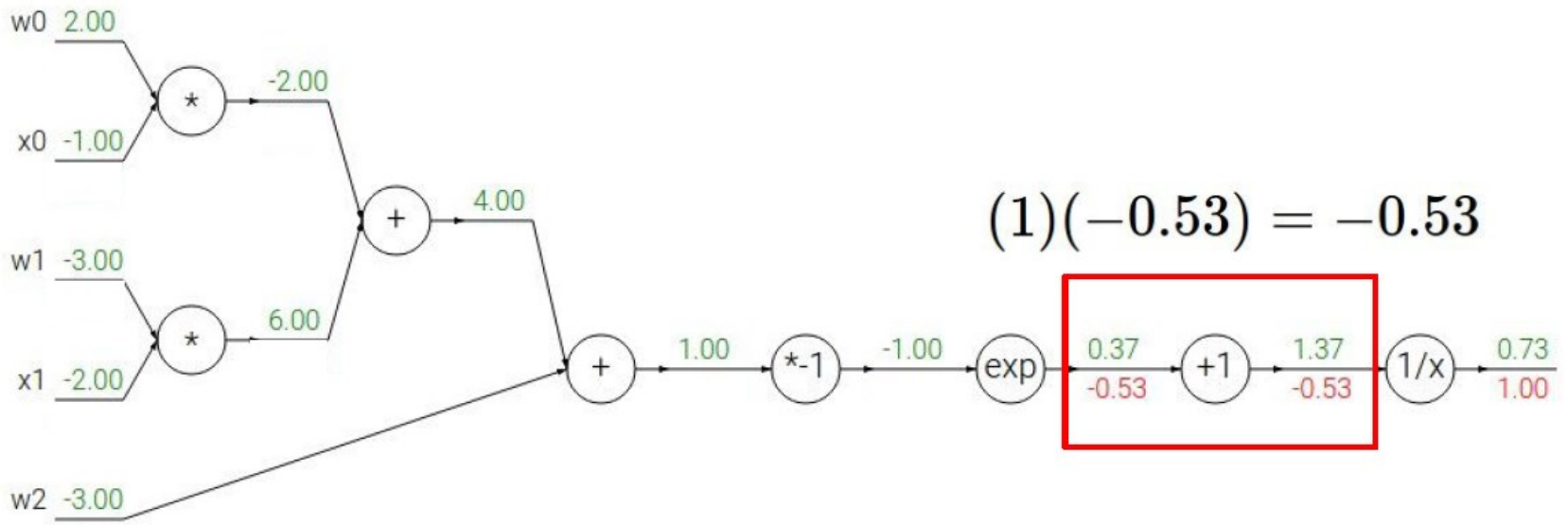


$$(1)(-0.53) = -0.53$$

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x$$

$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a$$

$$f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

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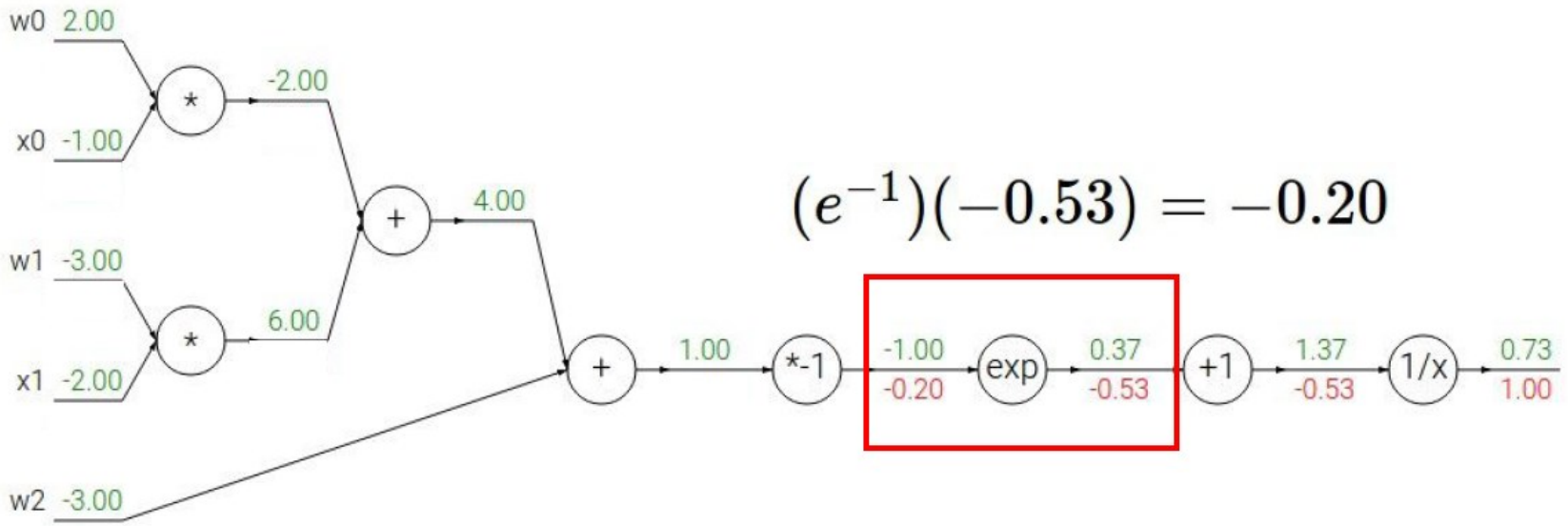
# Back Propagation(Example)

Another example: $$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$




$$(e^{-1})(-0.53) = -0.20$$

| | | |
|---|---|---|
| $f(x) = e^x$ | $\rightarrow$ | $\frac{df}{dx} = e^x$ |
| $f_a(x) = ax$ | $\rightarrow$ | $\frac{df}{dx} = a$ |

| | | |
|---|---|---|
| $f(x) = \frac{1}{x}$ | $\rightarrow$ | $\frac{df}{dx} = -1/x^2$ |
| $f_c(x) = c + x$ | $\rightarrow$ | $\frac{df}{dx} = 1$ |

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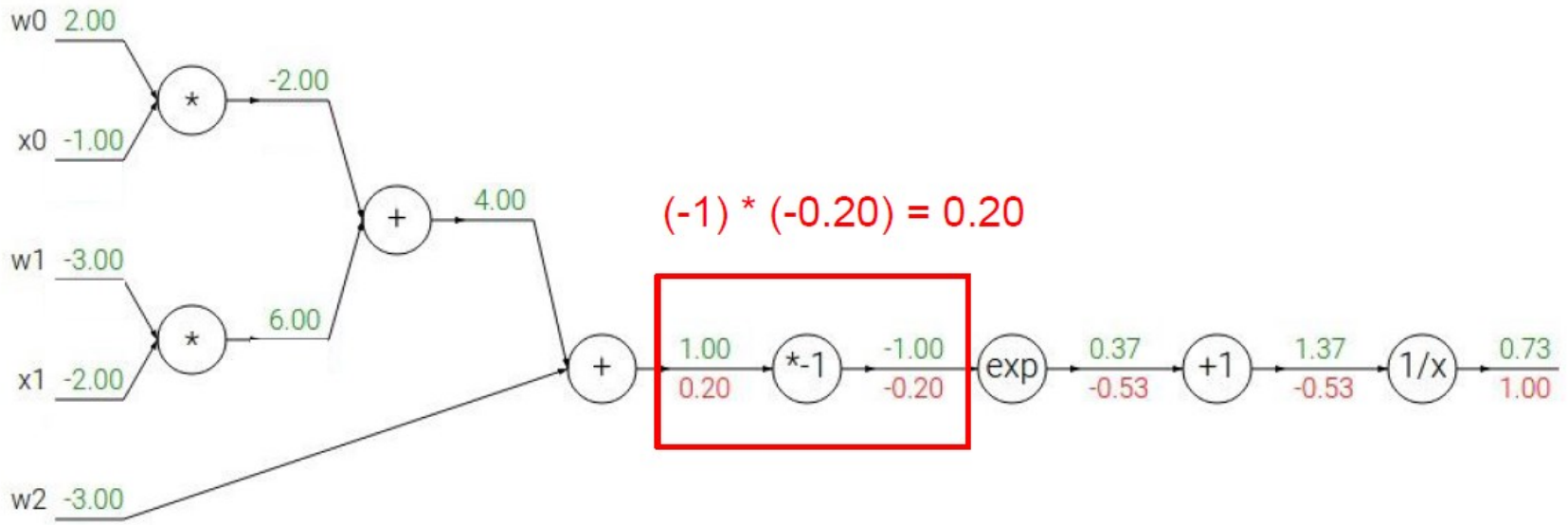


$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x$$

$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a$$

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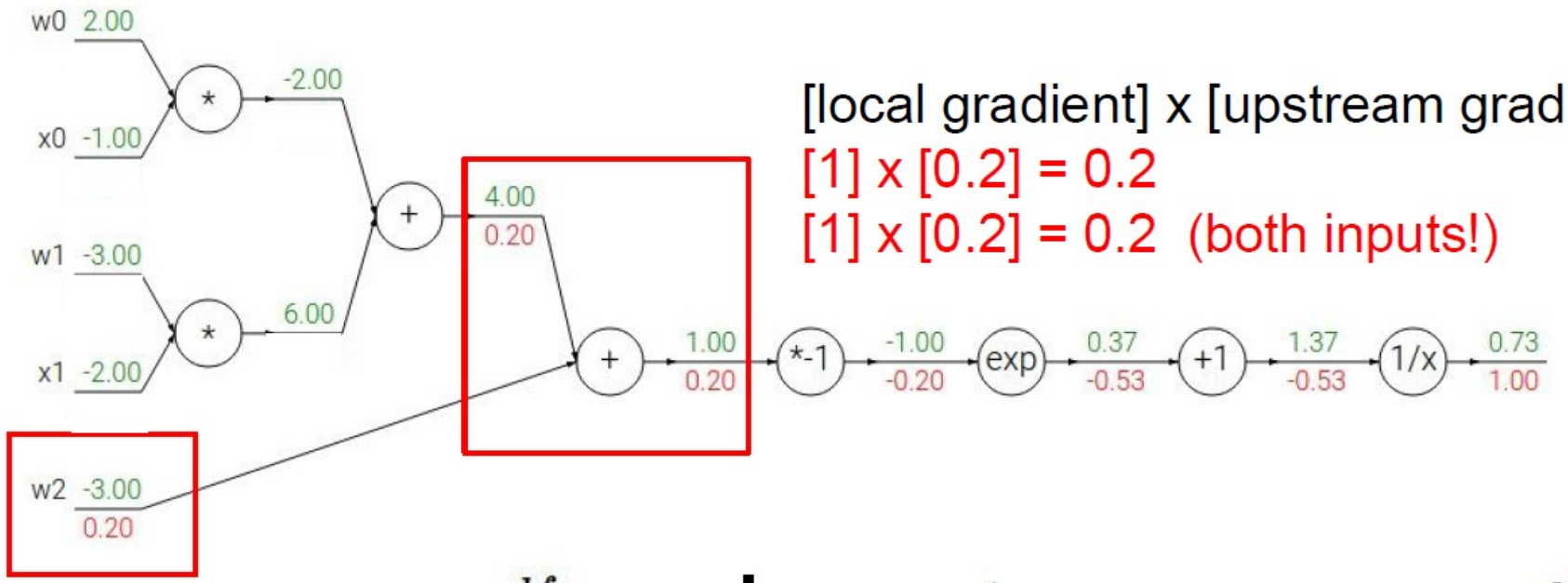



[local gradient] x [upstream gradient]

[1] x [0.2] = 0.2

[1] x [0.2] = 0.2 (both inputs!)

$$f(x) = e^x \rightarrow \frac{df}{dx} = e^x$$

$$f_a(x) = ax \rightarrow \frac{df}{dx} = a$$

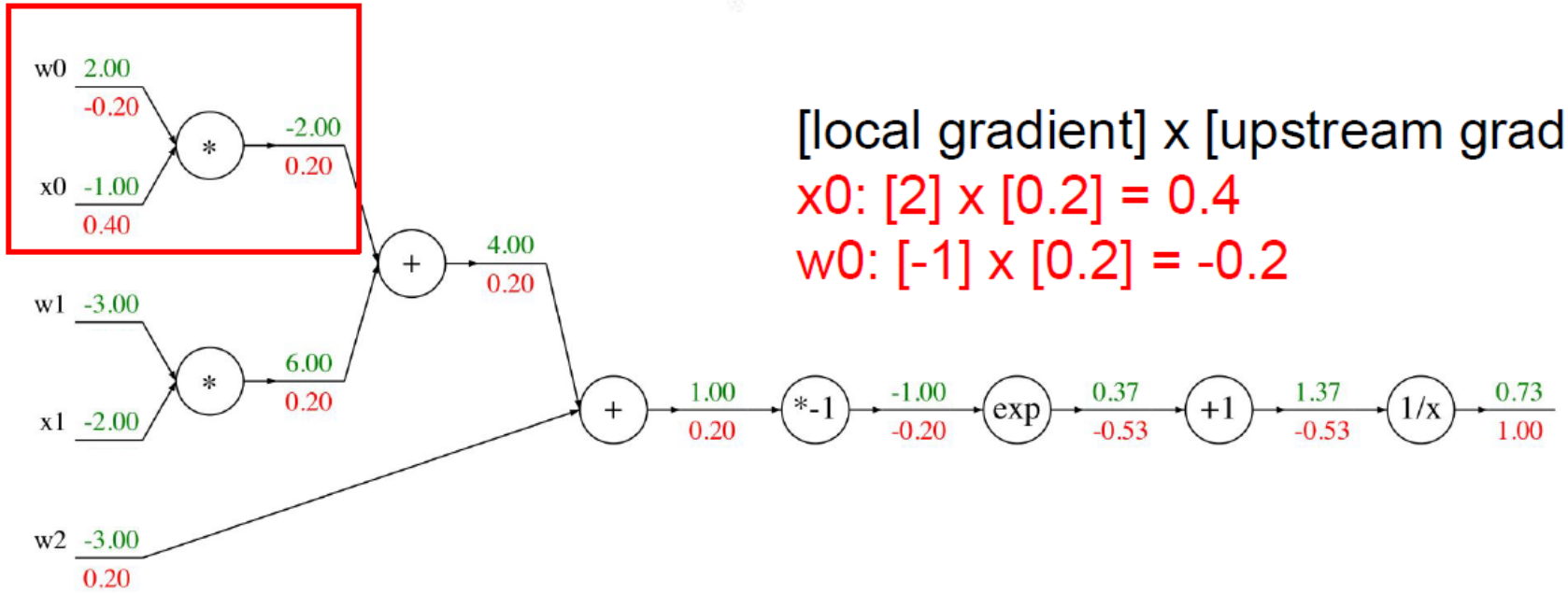
$$f(x) = \frac{1}{x} \rightarrow \frac{df}{dx} = -1/x^2$$

$$f_c(x) = c + x \rightarrow \frac{df}{dx} = 1$$

# Back Propagation(Example)

Another example:

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$




[local gradient] x [upstream gradient]

x0: [2] x [0.2] = 0.4

w0: [-1] x [0.2] = -0.2

$$f(x) = e^x \rightarrow \frac{df}{dx} = e^x$$

$$f_a(x) = ax \rightarrow \frac{df}{dx} = a$$

$$f(x) = \frac{1}{x} \rightarrow \frac{df}{dx} = -1/x^2$$

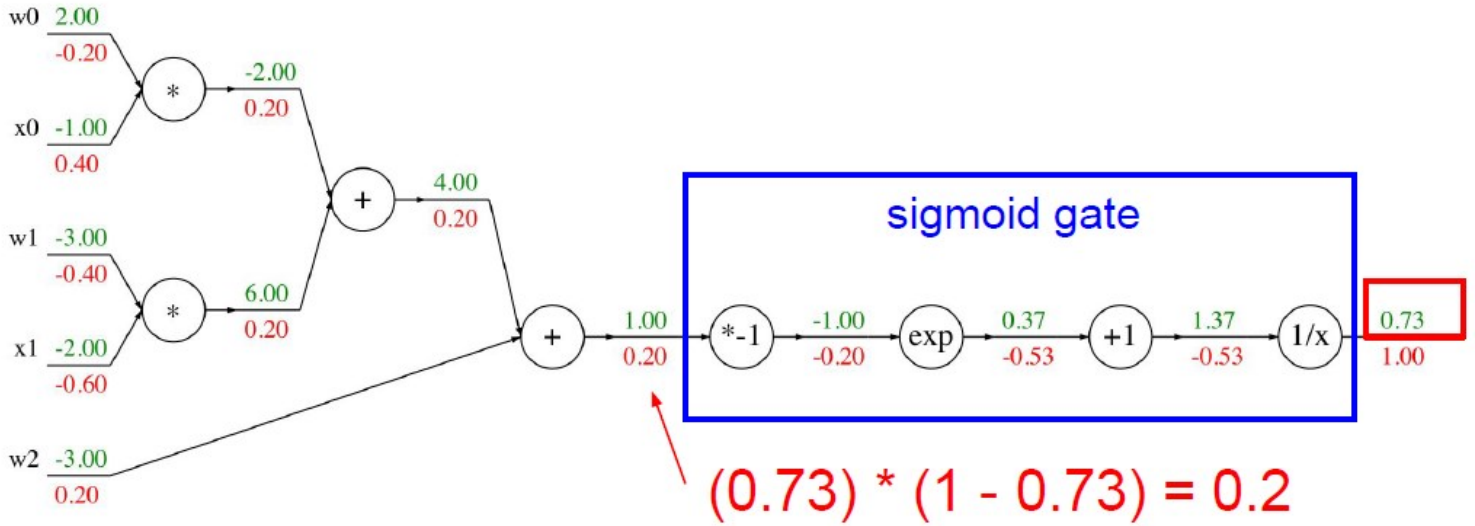
$$f_c(x) = c + x \rightarrow \frac{df}{dx} = 1$$

# Back Propagation(Example)

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$\boxed{\sigma(x) = \frac{1}{1+e^{-x}}}$$ sigmoid function

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right) = (1-\sigma(x))\,\sigma(x)$$



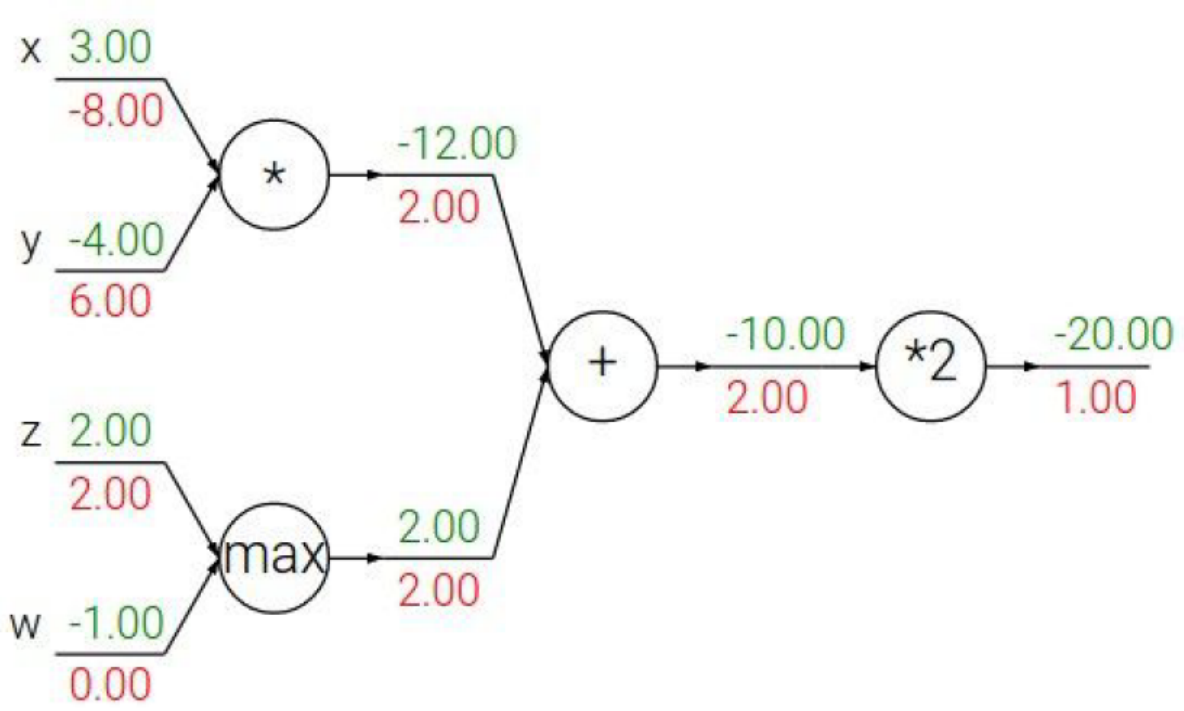
# Back Propagation(Example)

## Patterns in backward flow

**add** gate: gradient distributor

**max** gate: gradient router

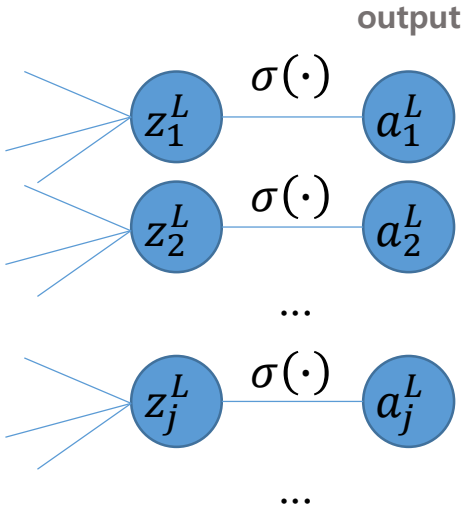
**mul** gate: gradient switcher



# 이제 Multi-Layer인 Network도 학습하는 방법은 알았는데, 그럼 Multi-Class는?

# Softmax!

## Original Output Layer



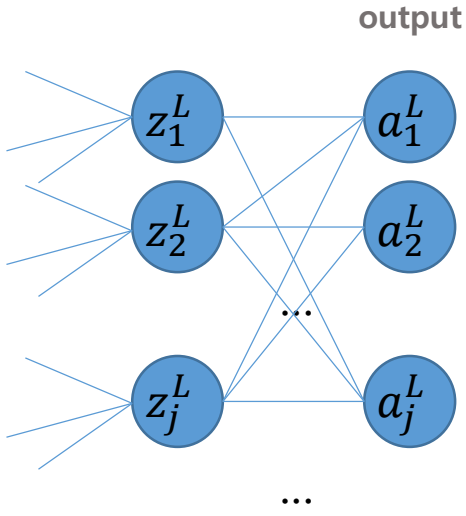

$$a_j^L = \sigma(z_j^L) = \frac{1}{1+e^{-z_j^L}} \in [0,1]$$

$$0 \leq \sum_j a_j^L \leq N^{(L)}$$

In classification problem, a desired output vector contains zero elements except only one '1' element .

→ This can be view as sum of all elements is 1

## Output Layer of Softmax




$$a_j^L = \frac{e^{z_j^L}}{\sum_k e^{z_k^L}} \in [0,1]$$

***Non-locality***

$$\sum_j a_j^L = 1$$

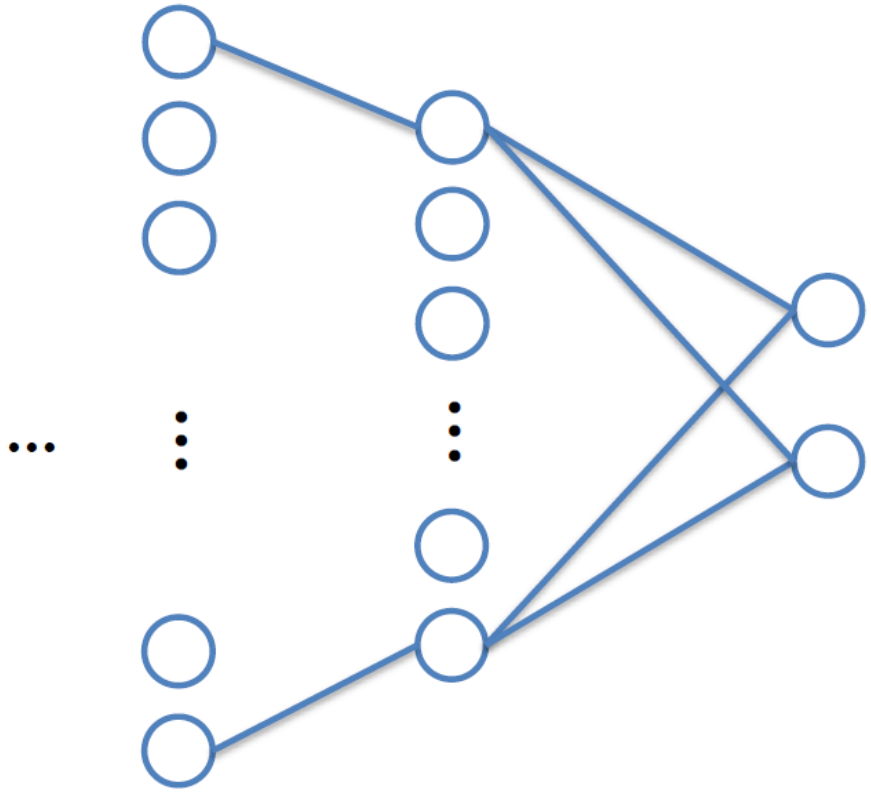
A softmax layer outputs a probability distribution!!

***Monotonicity*** $\frac{\partial a_j^L}{\partial z_k^L}$ = *positive* if j=k, *negative* otherwise

# Softmax

- Output을 확률처럼 나타내보자

$$P(y = j \mid \mathbf{x}) = \frac{e^{\mathbf{x}^\top \mathbf{w}_j}}{\sum_{k=1}^{K} e^{\mathbf{x}^\top \mathbf{w}_k}}$$



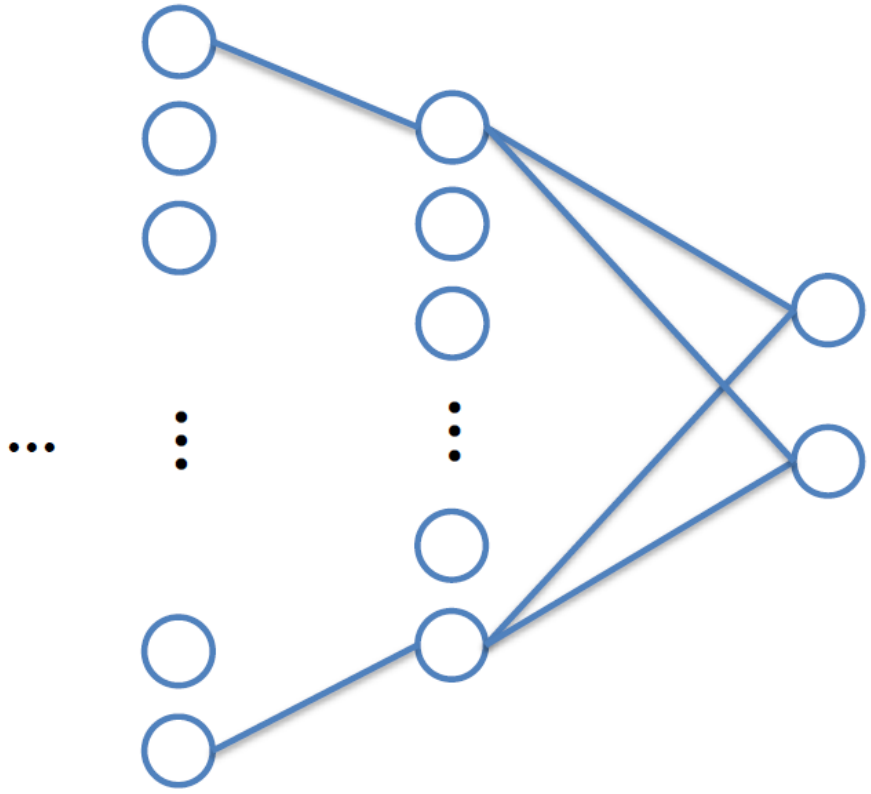
| $\mathbf{x}^\top \mathbf{w}_k$ | $e^{\mathbf{x}^\top \mathbf{w}_k}$ | $\frac{e^{\mathbf{x}^\top \mathbf{w}_j}}{\sum_{k=1}^{K} e^{\mathbf{x}^\top \mathbf{w}_k}}$ | one-hot vector |
|---|---|---|---|
| 강아지 0.37 | 1.4477 | 0.2296 | 0 |
| 고양이 1.58 | 4.8549 | 0.7704 | 1 |

수치 → exp → 확률 ↔ 확률

softmax!

# Loss Function of Softmax

- Loss function은 어떻게 정의할까?
  - L1 loss or L2 loss(MSE)?



| $\mathbf{x}^\top \mathbf{w}_k$ | $e^{\mathbf{x}^\top \mathbf{w}_k}$ | $\frac{e^{\mathbf{x}^\top \mathbf{w}_j}}{\sum_{k=1}^{K} e^{\mathbf{x}^\top \mathbf{w}_k}}$ | one-hot vector |
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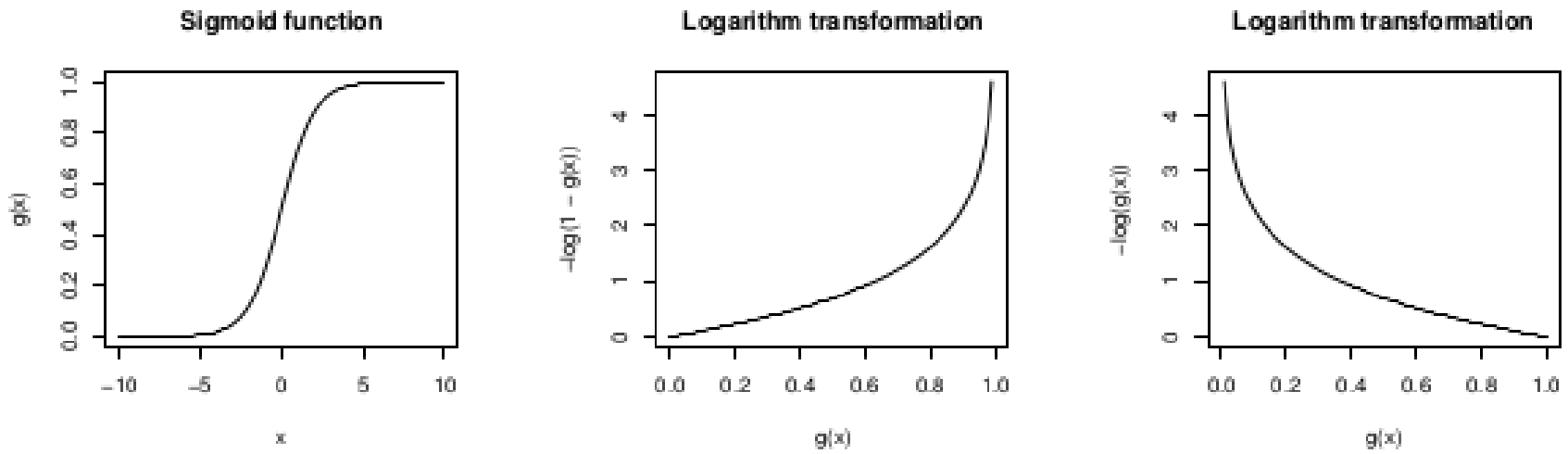
수치 → exp → 확률 ↔ 확률

softmax!

# Loss Function of Logistic Regression

- Cross Entropy Loss!

$$cost(W) = -\frac{1}{m}\sum ylog(H(x)) + (1-y)log(1-H(x))$$



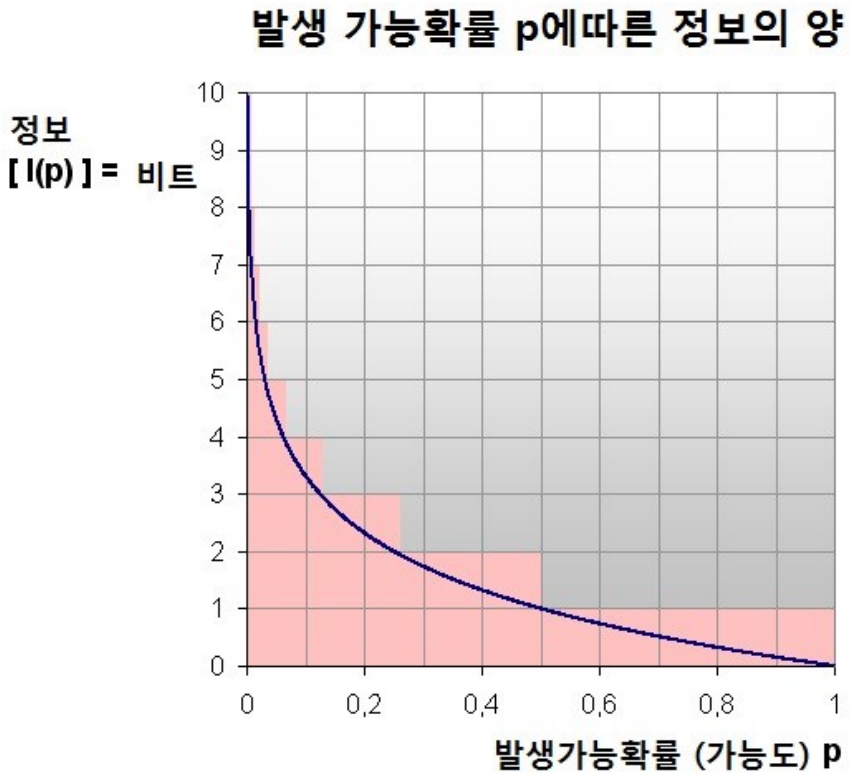

**(a)** *Sigmoid function.* **(b)** *Cost for* $y = 0$. **(c)** *Cost for* $y = 1$.

**Figure B.1:** *Logarithmic transformation of the sigmoid function.*

# entropy

- Information Theory에서 Entropy란 정보량의 기댓값(평균)이다

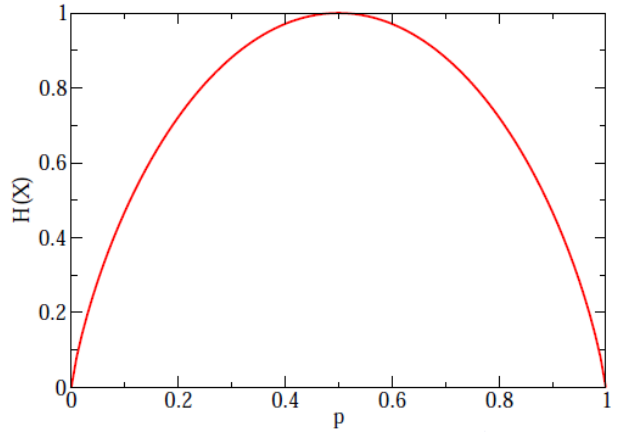
$$\mathrm{H}(X) = \mathrm{E}[\mathrm{I}(X)] = \mathrm{E}[-\ln(\mathrm{P}(X))].$$




- 항상 일어나는 사건이라면 p=1 이고 이것이 가지고 있는 정보량은 0이다.
- 일어날 확률이 적을 수록 많은 정보를 담고 있다.

# Entropy

$$\mathrm{H}(X) = \mathrm{E}[\mathrm{I}(X)] = \mathrm{E}[-\ln(\mathrm{P}(X))].$$
$$= \sum_{i=1}^{n} \mathrm{P}(x_i)\,\mathrm{I}(x_i) = -\sum_{i=1}^{n} \mathrm{P}(x_i)\log_b \mathrm{P}(x_i),$$




Ex) 동전던지기

P(앞) = ½
P(뒤) = ½

$H(X) = -(0.5 * \log_2 0.5 + 0.5 * \log_2 0.5) = 0.5 + 0.5 = 1$

이 확률 분포를 표현하기 위하여 평균 1 bit가 필요하다는 의미!

# Cross Entropy

True distribution p,
Model이 예측한 distribution을 q라고 하면

Cross entropy를 minimize 하는 것은 p와 q의 분포가 가까워지도록 만드는 것과 같다

$$H(p,q) = -\sum_i p_i \log q_i$$

$$= -\sum_i p_i \log q_i \boxed{- \sum_i p_i \log p_i} + \sum_i p_i \log p_i$$

$$= \boxed{H(p)} + \sum_i p_i \log p_i - \sum_i p_i \log q_i$$

$$= H(p) + \sum_i p_i \log \frac{p_i}{q_i}$$

$$= H(p) + D_{KL}(p \parallel q)$$

P 자체의 entropy(불변)

p를 기준으로 q가 얼마나 다른가
(p의 distribution과 q의 distribution의 거리)

# Cross Entropy vs MSE

| | Inference(softmax output) | | | True Label(onehot) | | | Correct? |
|---|---|---|---|---|---|---|---|
| | class1 | class2 | class3 | class1 | class2 | class3 | |
| model1 | 0.3 | 0.3 | 0.4 | 0 | 0 | 1 | Yes |
| | 0.3 | 0.4 | 0.3 | 0 | 1 | 0 | Yes |
| | 0.1 | 0.2 | 0.7 | 1 | 0 | 0 | No |
| model2 | 0.1 | 0.2 | 0.7 | 0 | 0 | 1 | Yes |
| | 0.1 | 0.7 | 0.2 | 0 | 1 | 0 | Yes |
| | 0.3 | 0.4 | 0.3 | 1 | 0 | 0 | No |

- 2가지 model 모두 classification error는 33%이다
- Cross entrop와 squared loss에 따른 loss 값은?

# Cross Entropy vs MSE

| | Inference(softmax output) | | | True Label(onehot) | | | Correct? |
|---|---|---|---|---|---|---|---|
| | class1 | class2 | class3 | class1 | class2 | class3 | |
| model1 | 0.3 | 0.3 | 0.4 | 0 | 0 | 1 | Yes |
| | 0.3 | 0.4 | 0.3 | 0 | 1 | 0 | Yes |
| | 0.1 | 0.2 | 0.7 | 1 | 0 | 0 | No |
| model2 | 0.1 | 0.2 | 0.7 | 0 | 0 | 1 | Yes |
| | 0.1 | 0.7 | 0.2 | 0 | 1 | 0 | Yes |
| | 0.3 | 0.4 | 0.3 | 1 | 0 | 0 | No |

| | Cross Entropy | Squared Loss |
|---|---|---|
| model1 | -1/3 x (log0.4 + log0.4 + log0.1) = 1.38 | 1/3 x ($0.3^2$ + $0.3^2$ + $(1-0.4)^2$ + $0.3^2$ + $(1-0.4)^2$ + $0.3^2$ + $(1-0.1)^2$ + $0.2^2$ + $0.7^2$) = 0.81 |
| model2 | -1/3 x (log0.7 + log0.7 + log0.4) = 0.64 | 1/3 x ($0.1^2$ + $0.2^2$ + $(1-0.7)^2$ + $0.1^2$ + $(1-0.7)^2$ + $0.2^2$ + $(1-0.3)^2$ + $0.4^2$ + $0.3^2$ = 0.34 |

# Cross Entropy vs MSE

| | Inference(softmax output) | | | True Label(onehot) | | | Correct? |
|---|---|---|---|---|---|---|---|
| | class1 | class2 | class3 | class1 | class2 | class3 | |
| model1 | 0.3 | 0.3 | 0.4 | 0 | 0 | 1 | Yes |
| | 0.3 | 0.4 | 0.3 | 0 | 1 | 0 | Yes |
| | 0.1 | 0.2 | 0.7 | 1 | 0 | 0 | No |
| model2 | 0.1 | 0.2 | 0.7 | 0 | 0 | 1 | Yes |
| | 0.1 | 0.7 | 0.2 | 0 | 1 | 0 | Yes |
| | 0.3 | 0.4 | 0.3 | 1 | 0 | 0 | No |

- Cross entropy focusses on correct classification
- MSE focusses on fitting values of all classes

# Fundamental Equation behind Back-Prop

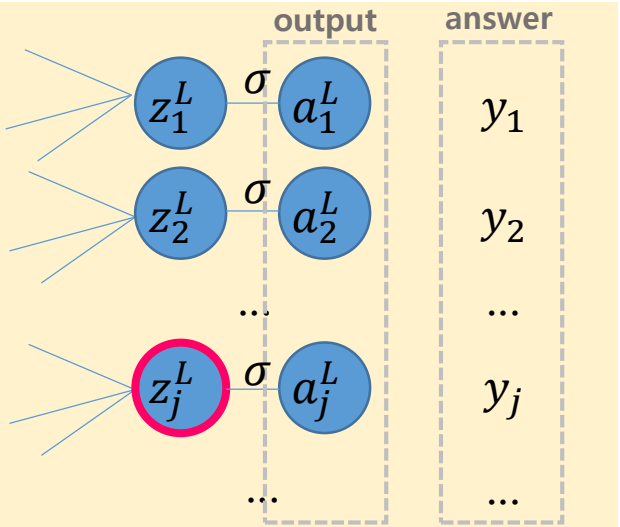
| Cost | Error Signal | Error Gradient |
|---|---|---|
| C | $\delta_j^l \equiv \frac{\partial C}{\partial z_j^l}$ | $\partial C/\partial w_{jk}^l$ and $\partial C/\partial b_j^l$ |

## Equation for the error in the output layer, $\delta^L$

$$\frac{\partial C}{\partial z_j^L} = \frac{\partial C}{\partial a_j^L} \cdot \frac{\partial a_j^L}{\partial z_j^L} = \frac{\partial C}{\partial a_j^L} \cdot \sigma'(z_j^L)$$

Chain rule

$$\delta_j^L = \frac{\partial C}{\partial z_j^L} = \frac{\partial C}{\partial a_j^L} \cdot \sigma'(z_j^L)$$




***Quadratic Cost Function :*** $C = \frac{1}{2}\sum_k (y_k^L - a_k^L)^2$

$$\frac{\partial C}{\partial z_j^L} = \frac{\partial}{\partial z_j^L}\left(\frac{1}{2}\sum_k (y_k - a_k^L)^2\right) = \frac{\partial}{\partial z_j^L}\left(\frac{1}{2}\sum_k \left(y_k - \sigma(z_k^L)\right)^2\right) = \frac{\partial}{\partial z_j^L}\left(\frac{1}{2}\left(y_j - \sigma(z_j^L)\right)^2\right)$$

Only the j[th] term is valid

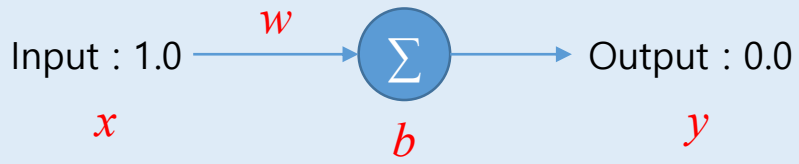
$$\frac{\partial C}{\partial z_j^L} = -\left(y_j - \sigma(z_j^L)\right)\sigma'(z_j^L) = (a_j^L - y_j)\sigma'(z_j^L)$$

Slide credit : Hwalsuk Lee@CLAIR

# Mean Squared Error

**We hope and expect that our neural networks will learn fast from their errors.**
**An example :**

An Object




$$C = \frac{(a-y)^2}{2} = \frac{a^2}{2}$$

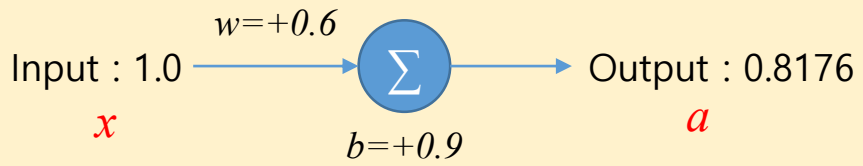
$$a = \sigma(z) = \sigma(wx + b) = \sigma(w + b)$$

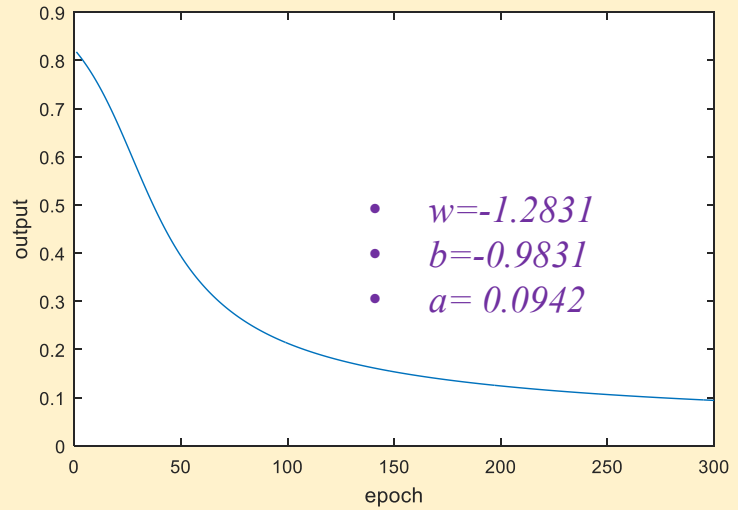
$$\delta = a\sigma'(w + b)$$

$$w = w - \eta\delta$$

$$b = b - \eta\delta$$

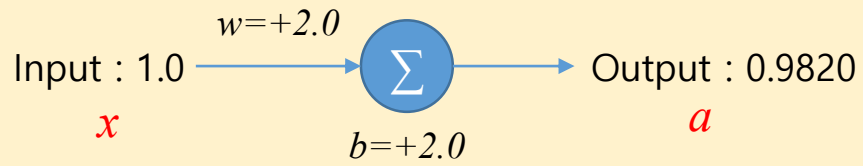
Initialization 1

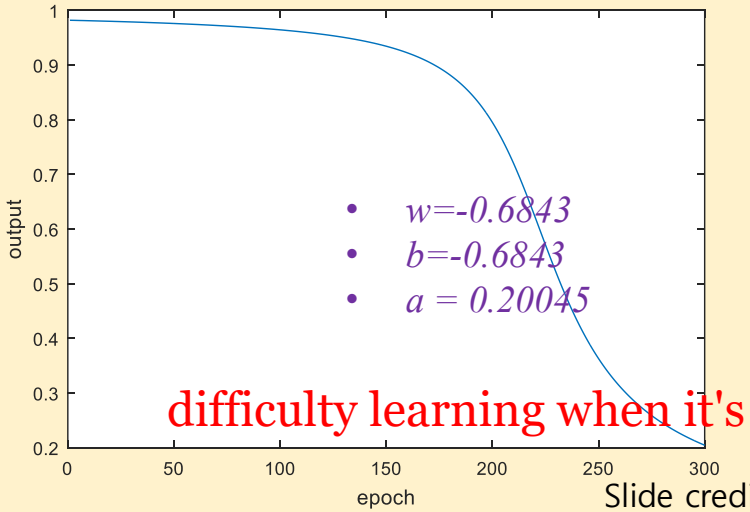



- *w=-1.2831*
- *b=-0.9831*
- *a= 0.0942*

Initialization 2



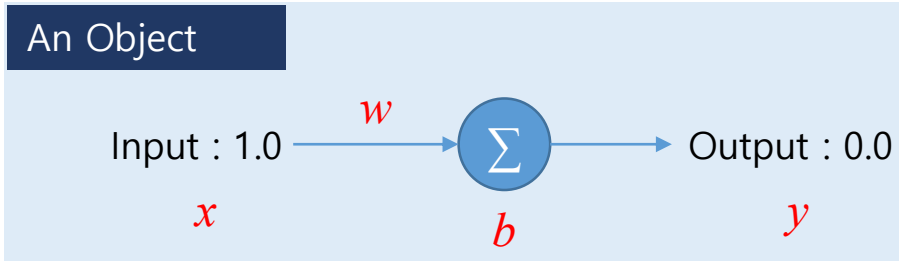

- *w=-0.6843*
- *b=-0.6843*
- *a = 0.20045*

difficulty learning when it's badly wrong

Slide credit : Hwalsuk Lee@CLAIR

# Mean Squared Error

**We hope and expect that our neural networks will learn fast from their errors.**
**An example :**




$$C = \frac{(a-y)^2}{2} = \frac{a^2}{2}$$

$$a = \sigma(z) = \sigma(wx + b) = \sigma(w + b)$$

$$\delta = a\sigma'(w + b)$$

$$w = w - \eta\delta$$

$$b = b - \eta\delta$$

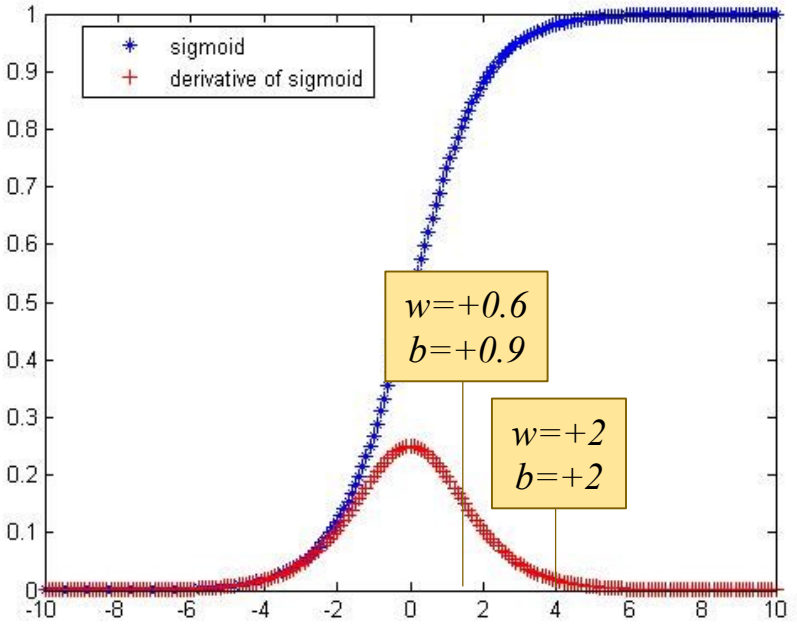
Learning slow means are $\partial C / \partial w$, $\partial C / \partial b$ small

Why they are small???

$$C = \frac{(y-a)^2}{2}$$

$$\frac{\partial C}{\partial w} = (a-y)\sigma'(z)x = a\sigma'(z)$$

$$\frac{\partial C}{\partial b} = (a-y)\sigma'(z) = a\sigma'(z)$$

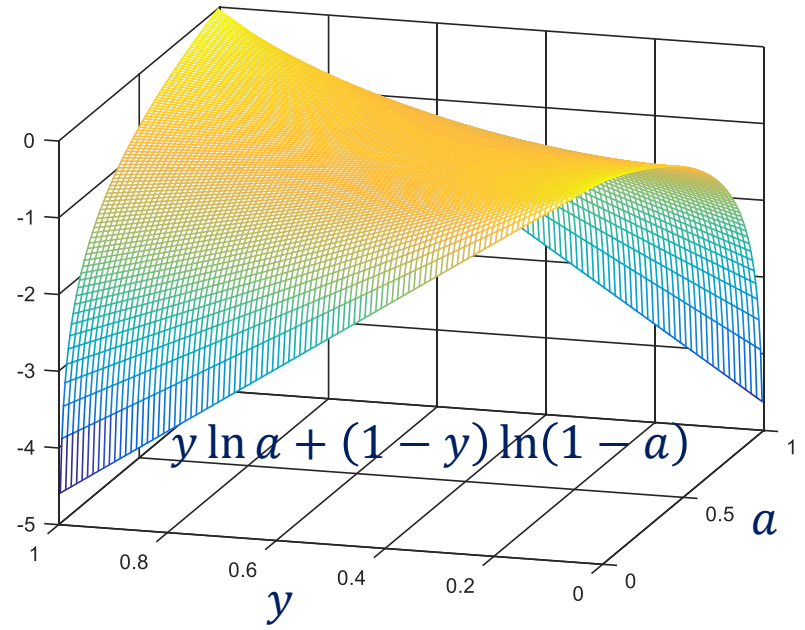



Slide credit : Hwalsuk Lee@CLAIR

# Cross Entropy Cost Function

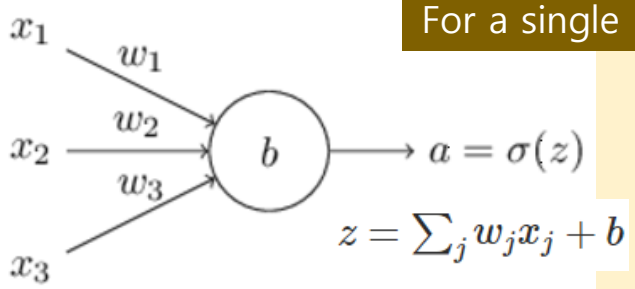
## Introducing the cross-entropy cost function

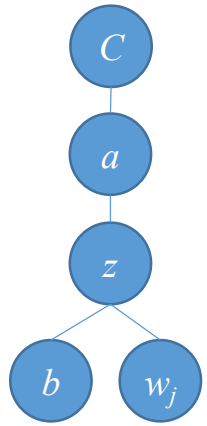
$$C = \frac{-1}{n}\sum_x [y\, ln\, a + (1-y) ln(1-a)]$$




[Two features]
- Non-negative → cost function
- Zero at y=a

For a single neuron




Error gradient

$$\frac{\partial C}{\partial w_j} = \frac{-1}{n}\sum_x \frac{\partial C}{\partial a}\cdot\frac{\partial a}{\partial z}\cdot\frac{\partial z}{\partial w_j}$$

$$\frac{\partial C}{\partial a} = \frac{y}{a} + (1-y)\frac{-1}{1-a} = \frac{y-a}{(1-a)a}$$

$$\frac{\partial a}{\partial z} = \sigma'(z) = \big(1-\sigma(z)\big)\sigma(z) = (1-a)a \qquad \frac{\partial z}{\partial w_j} = x_j$$

$$\frac{\partial C}{\partial w_j} = \frac{-1}{n}\sum_x (y-a)x_j = \frac{1}{n}\sum_x (\sigma(z)-y)x_j$$

$$\frac{\partial C}{\partial b} = \frac{1}{n}\sum_x (\sigma(z)-y)$$

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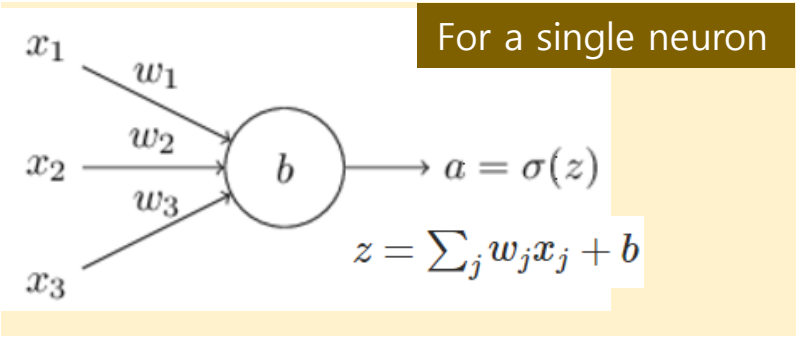
# Cross Entropy vs MSE

## Error gradient comparison




| $Cross - entropy\ cost$ | $Quadratic\ cost$ |
|---|---|
| $C = -\frac{1}{n}\sum_x [y \ln a + (1-y)\ln(1-a)\ ]$ | $C = \frac{1}{n}\sum_x \frac{1}{2}(a-y)^2$ |
| $\frac{\partial C}{\partial w_j} = \frac{1}{n}\sum_x (\sigma(z)-y)x_j$ | $\frac{\partial C}{\partial w_j} = \frac{1}{n}\sum_x (\sigma(z)-y)\cdot \boxed{\sigma'(z)} \cdot x_j$ |
| | Error signal decreaser!!! |
| $\frac{\partial C}{\partial b} = \frac{1}{n}\sum_x (\sigma(z)-y)$ | $\frac{\partial C}{\partial b} = \frac{1}{n}\sum_x (\sigma(z)-y)\cdot \boxed{\sigma'(z)}$ |

Slide credit : Hwalsuk Lee@CLAIR