Lattice QCD


Jong-Wan Lee

(IBS, CTPU)

1st Nuclear Physics Turtles lecture Series
Jan. 19-24, 2025

Goals

- Able to read papers in hep-lat (crossed by nucl-th, nucl-ex, hep-ph, hep-th)
- “Hands-on” Monte Carlo calculations and data analysis
- Ready to do any projects in the field of lattice QCD

References

- An introduction to Quantum Field Theory, Michael E. Peskin, Daniel V. Schroeder, 1990
- Lattice Gauge Theories: An introduction, Heinz J. Rothe, 1992
- Quarks, gluons and lattices, Michael Creutz, 1983
- Quantum fields on a lattice, Istvan Montvay, Gernot Munster, 2003
- Lattice QCD for Nuclear Physics, Huey-Wen Lin, Harvey B. Meyer, 2014
- Five lectures on effective field theory, David B. Kaplan, 2005

Sample codes

- https://github.com/jwlee823/LQCD-tutorials

- https://github.com/claudiopica/HiRep

Higher representations on the lattice: Numerical simulations. SU(2) with adjoint fermions #1

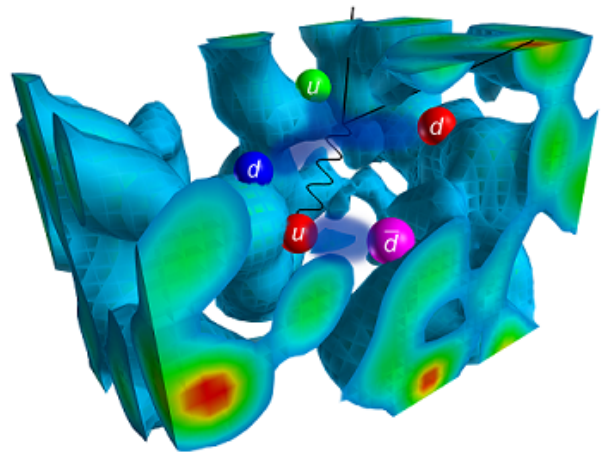
Luigi Del Debbio (Edinburgh U. and Newton Inst. Math. Sci., Cambridge), Agostino Patella (Swansea U.), Claudio Pica (Brookhaven) (May, 2008)

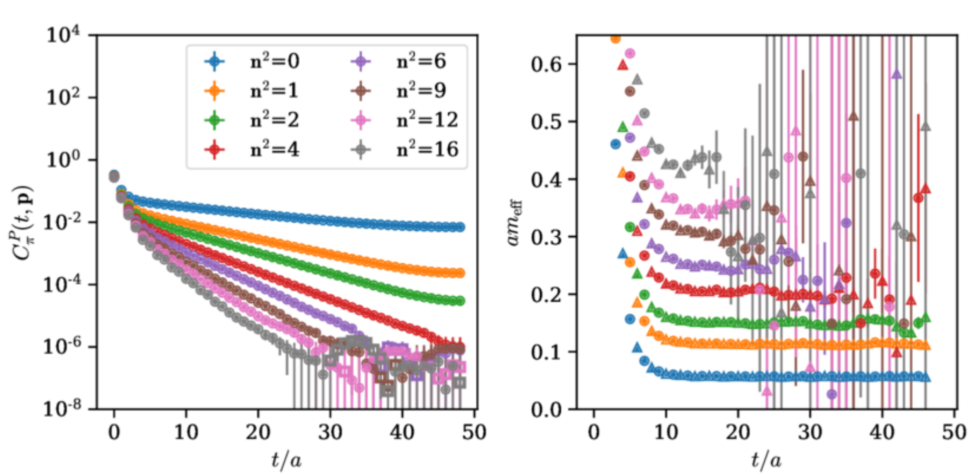
Published in: *Phys.Rev.D* 81 (2010) 094503 • e-Print: 0805.2058 [hep-lat]

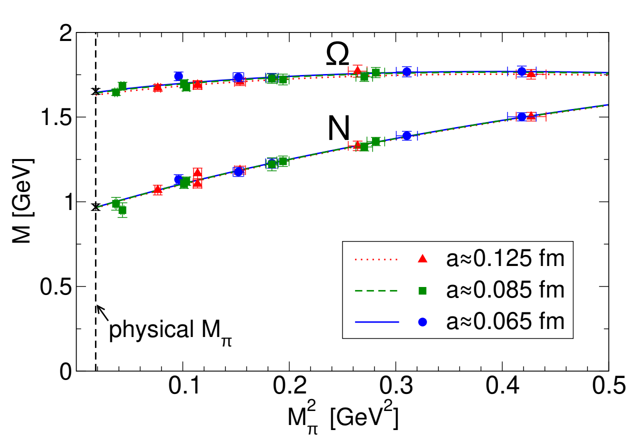
pdf DOI cite claim reference search 257 citations

Lattice QCD

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

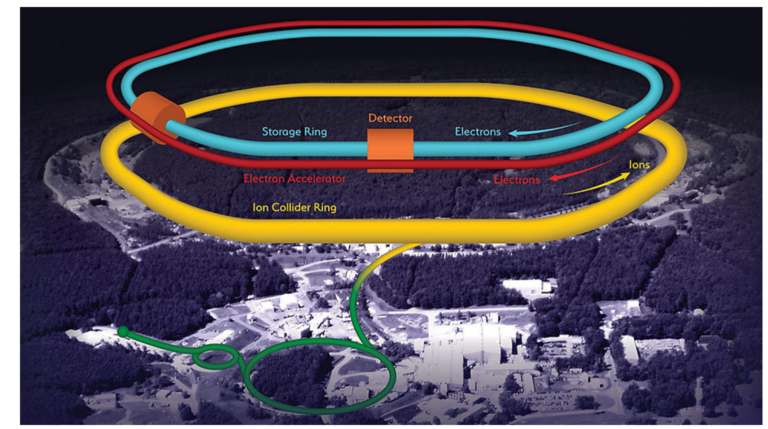


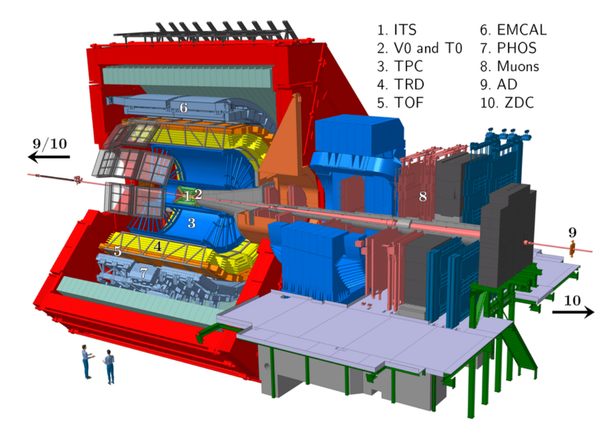


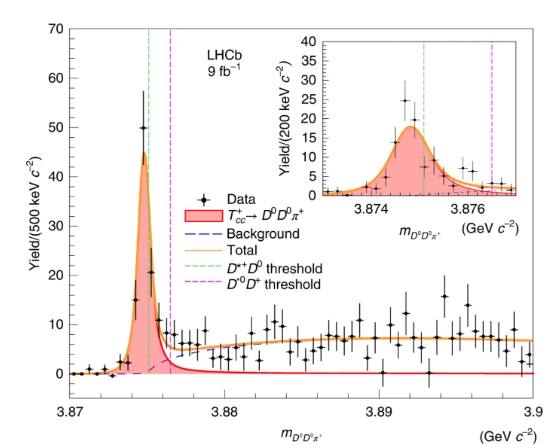


Collider experiments

Nature







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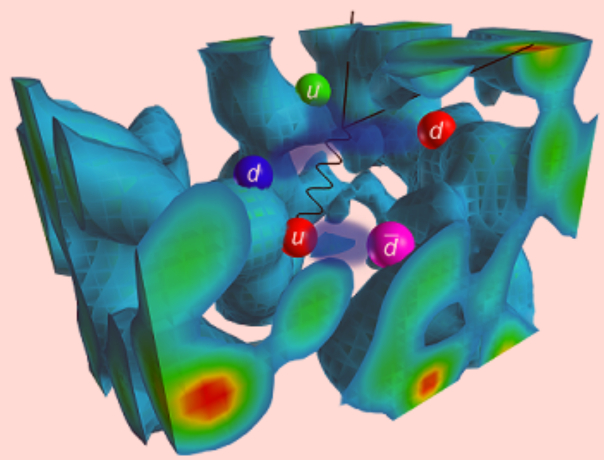
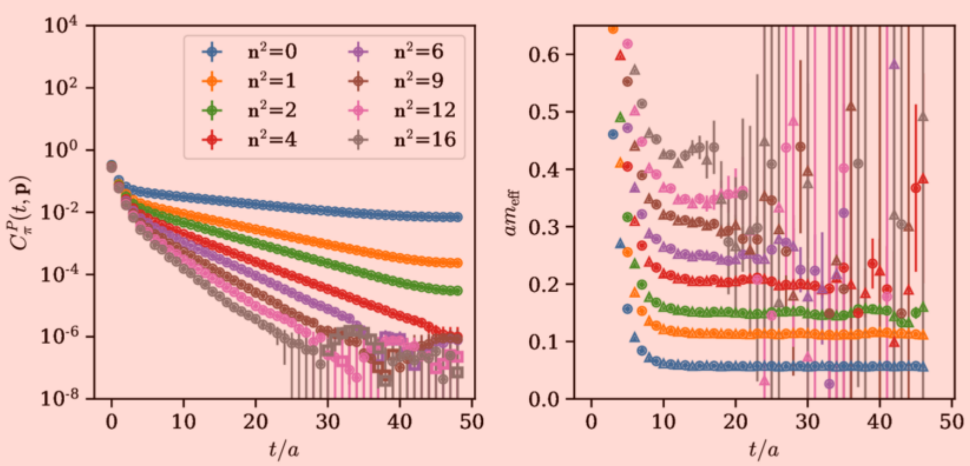
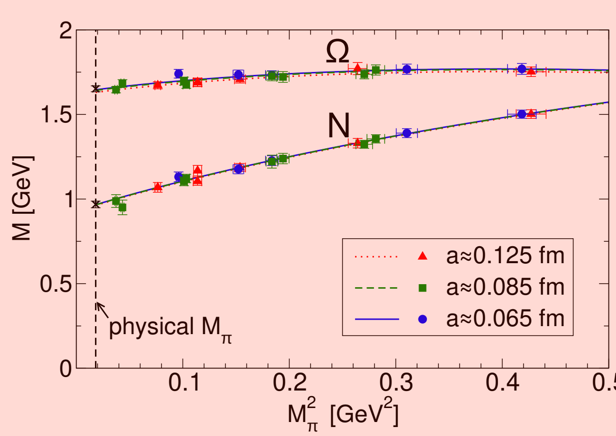




Today!



Collider experiments

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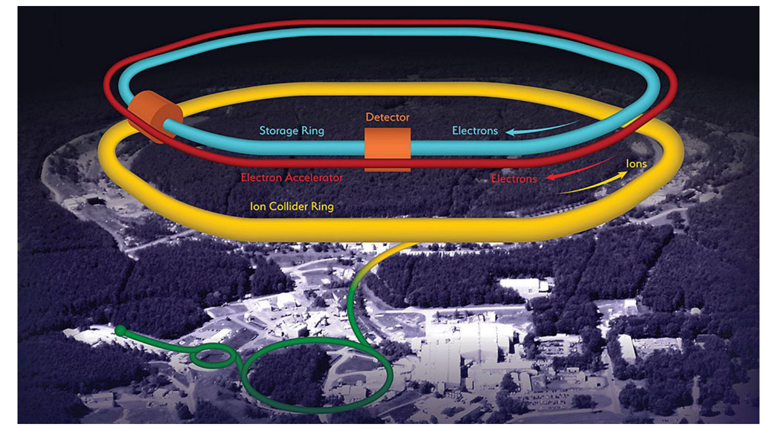
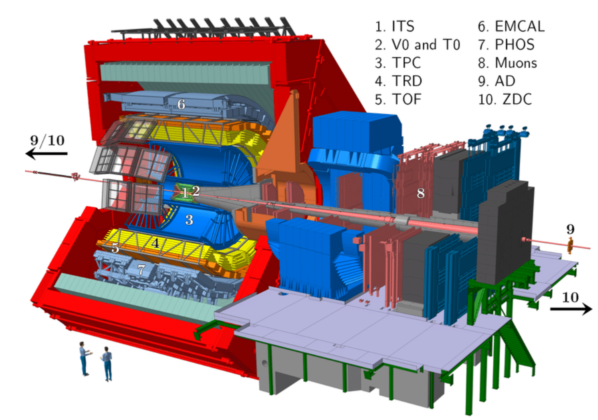
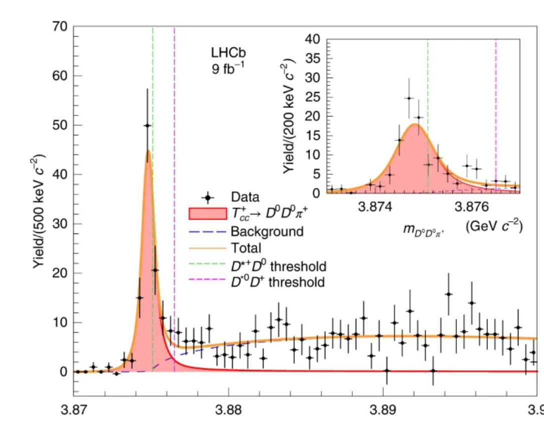
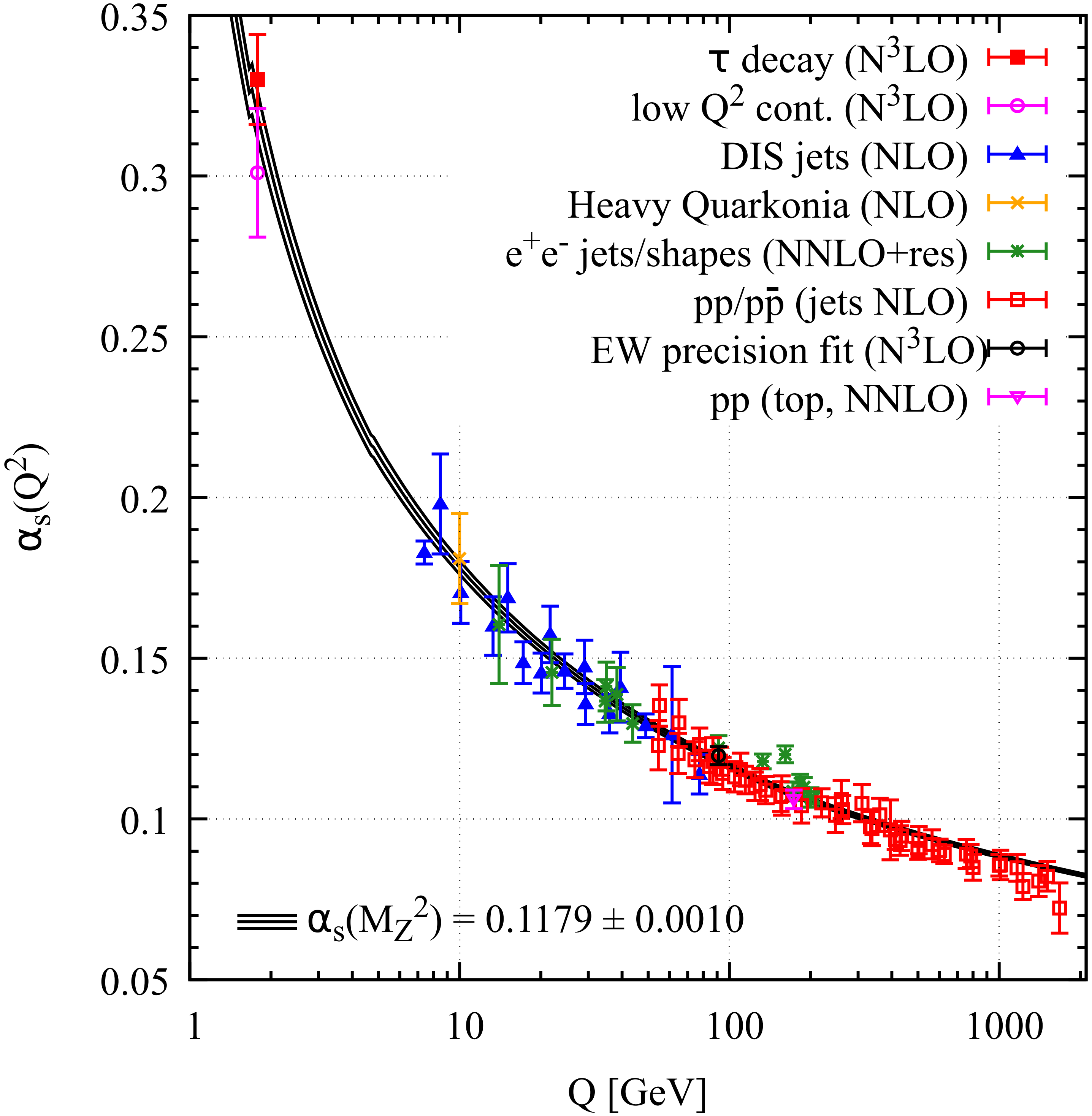
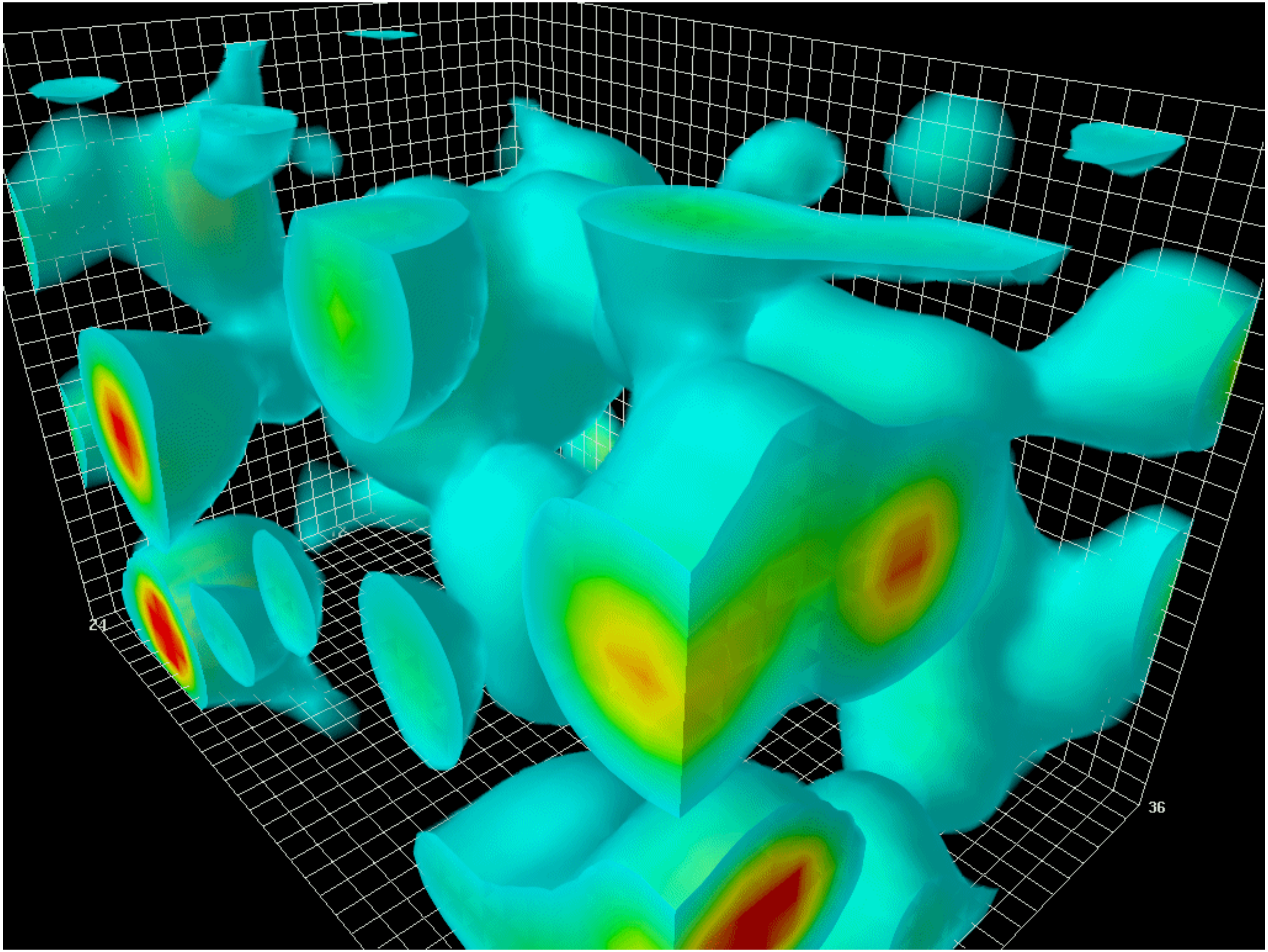






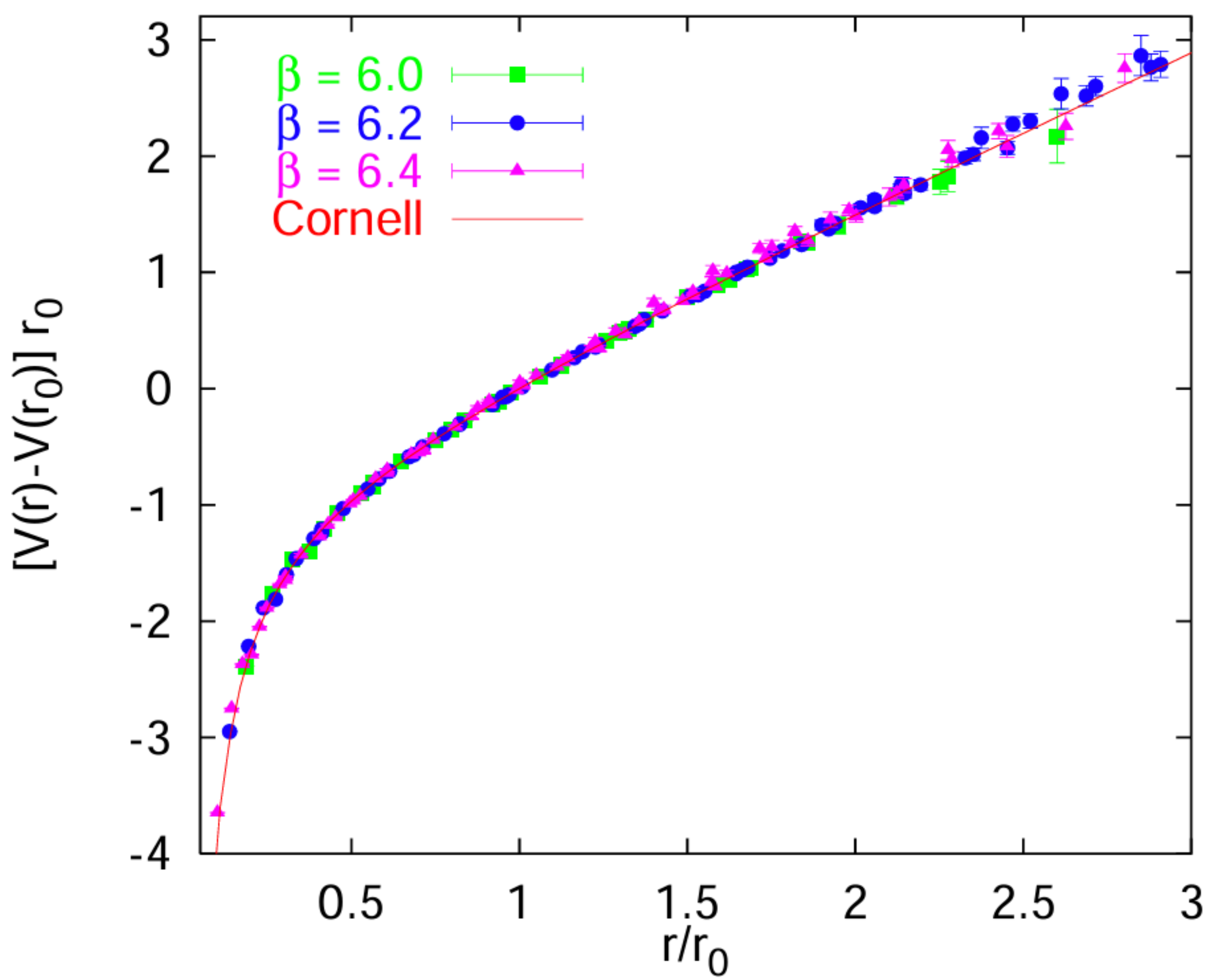
Asymptotic freedom





Credit: Derek B. Leinweber, Univ. Of Adelaide

Confinement



Gunnar Bali, Phys. Rept. 343 (2001)

Chiral symmetry breaking

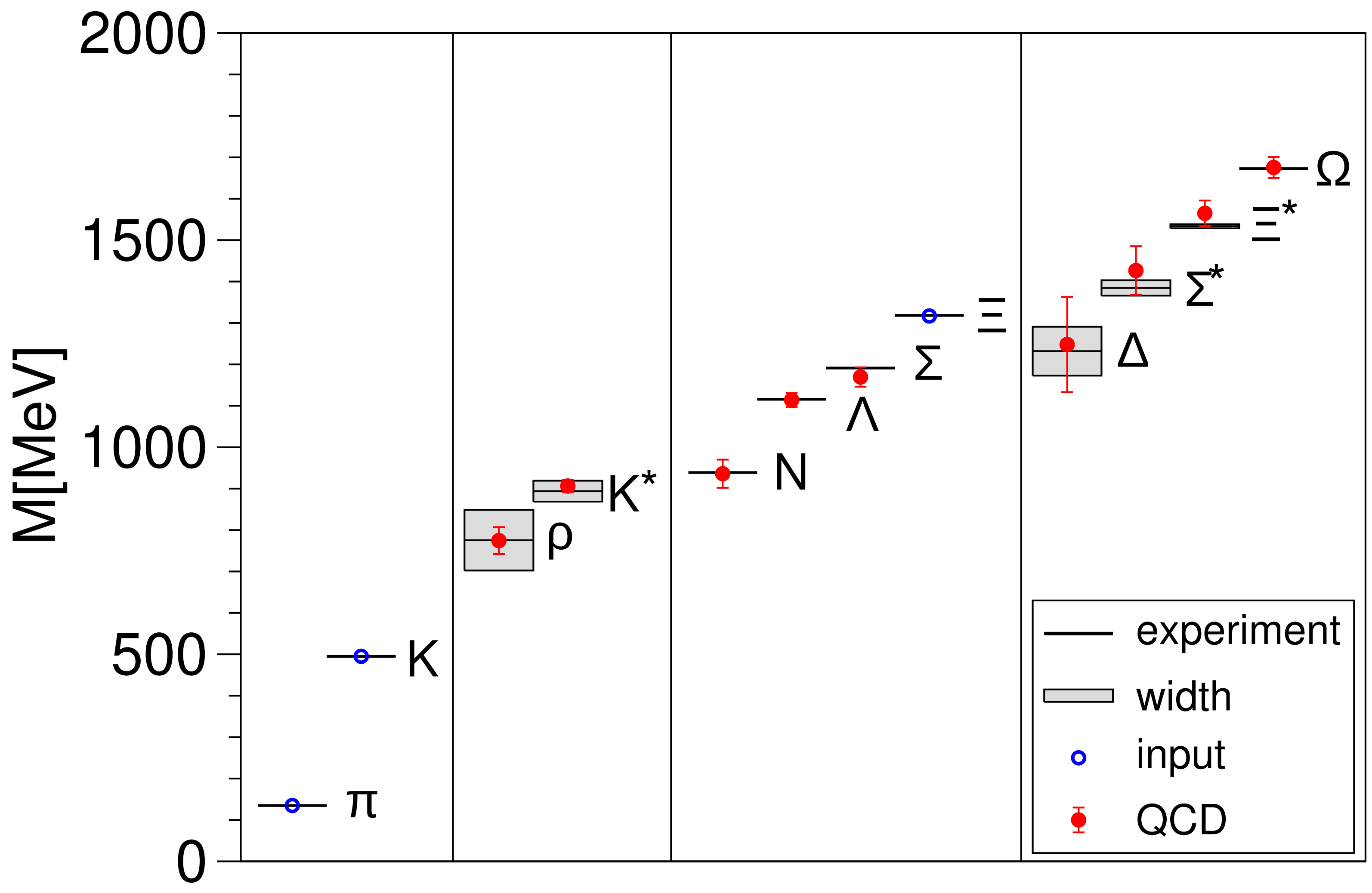


Figure 3: The light hadron spectrum of QCD. Horizontal lines and bands are the experimental values with their decay widths. Our results are shown by solid circles. Vertical error bars represent our combined statistical (SEM) and systematic error estimates. π, K and Ξ have no error bars, because they are used to set the light quark mass, the strange quark mass and the overall scale, respectively.

BMW collaboration, Science (2008)

Chiral fermions on the lattice

Fermions on a lattice

- The naive discretization of the fermion action suffers from the *fermion doubling* problem. To see this, we first consider a free fermion

$$S_f = \int d^4x\, \bar{\psi}(x)\Big(\gamma_\mu \partial_\mu + m\Big)\psi(x) \xrightarrow{\text{discretized}} S_f = \frac{1}{2}\sum_n \bar{\psi}_n \left(\sum_\mu \gamma_\mu(\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}) + m\psi_n\right)$$

Continuum

On the lattice

- The 2-point correlation function becomes

$$\langle \psi(x)\bar{\psi}(y)\rangle = \lim_{a\to 0}\int_{-a/\pi}^{a/\pi} \frac{d^4p}{(2\pi)^2}\frac{-i\sum_\mu \gamma_\mu \tilde{p}_\mu + m}{\sum_\mu \tilde{p}_\mu^2 + m^2} e^{ip(x-y)} \quad \text{with} \quad \tilde{p}_\mu = \frac{\sin ap_\mu}{a}, \qquad -\frac{\pi}{a} \le p_\mu \le \frac{\pi}{a}$$

Brillouin zone

- It should be reduced to the continuum 2-point function as $a \to 0$

$$\tilde{p}_\mu \to p_\mu$$

Dispersion relation for the massless fermion: $\sum_\mu \tilde{p}_\mu^2 = 0$

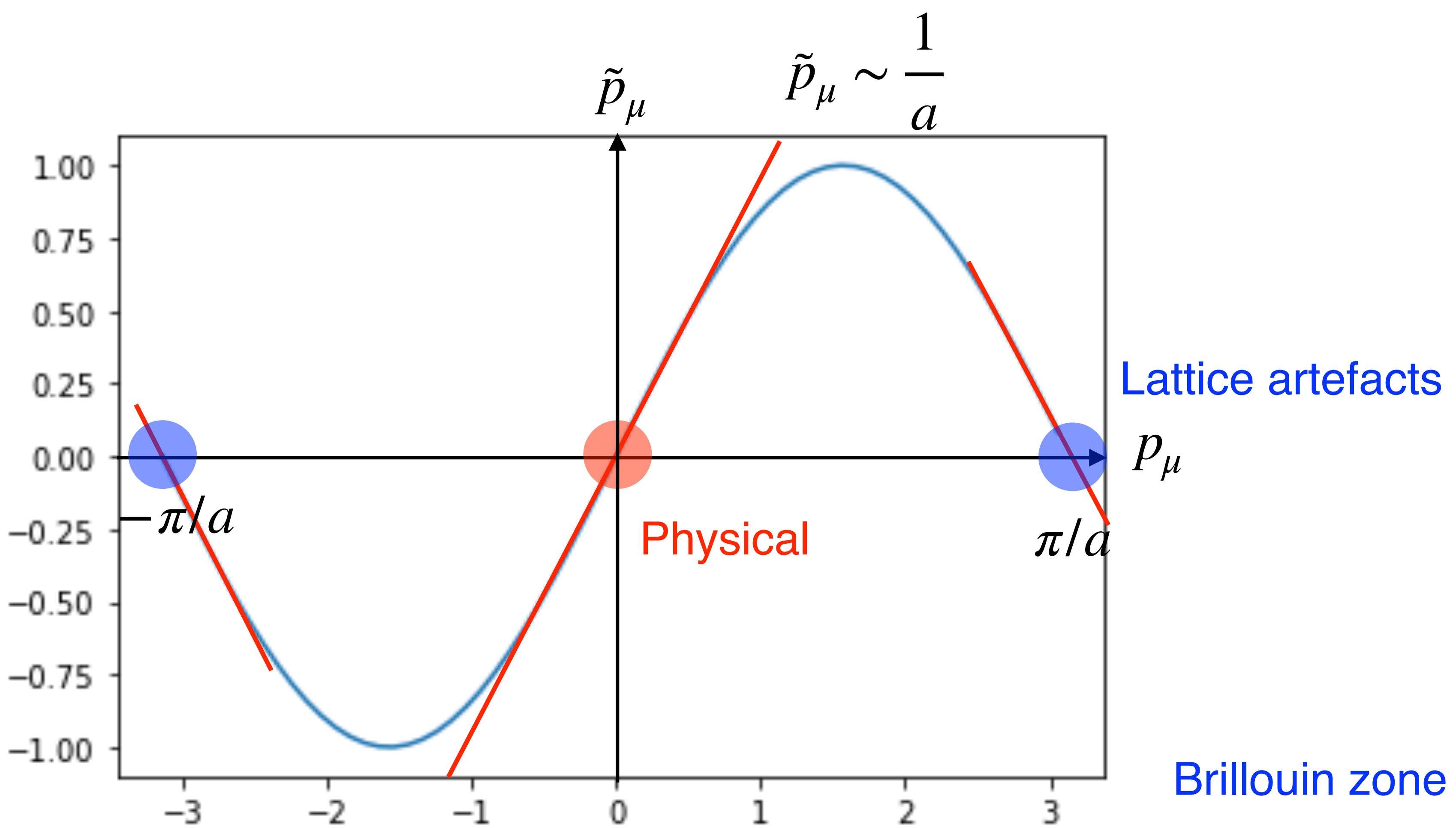
- But, there are *fifteen additional fermion-like excitations* at the edge of the Brillouin zone which have no continuum analogues.

$$\tilde{p}_\mu = \frac{\sin(-\pi + ak_\mu)}{a} \to -k_\mu$$

also satisfies $\sum_\mu \tilde{p}_\mu^2 = 0$

Fermions on a lattice

- The naive discretization of the fermion action suffers from the *fermion doubling* problem in the massless limit.



$$\sum_\mu \tilde{p}_\mu^2 = \sum_\mu \frac{\sin^2 ap_\mu}{a^2} = 0$$

- The doublers have to be removed to recover the correct continuum theory in the limit $a \to 0$!

Wilson's approach

K. Wilson (1974)

- Lift the edge mode by adding a lattice Laplacean to the fermion action

$$S_f = \int d^4x\, \bar{\psi}(x)\left(\gamma_\mu \partial_\mu + m - \frac{r}{2}\partial^2\right)\psi(x)$$

- After setting $r = 1$, the lattice dispersion relation becomes

$$\sum_\mu \sin^2(ap_\mu) + \left(4 - \sum_\mu \cos(ap_\mu)\right)^2 = 0$$

- The doublers except the one at $p_\mu = 0$ are now lifted to the cut-off scale $\Lambda_{\text{cut-off}} \sim \dfrac{1}{a}$

- However, it breaks the chiral symmetry explicitly in the same way of the mass term!

$$\psi_L = \frac{1-\gamma_5}{2}\psi,\ \psi_R = \frac{1+\gamma_5}{2}\psi \qquad \bar{\psi}_{R(L)}(x)\left(m - \frac{r}{2}\partial^2\right)\psi_{L(R)}(x)$$

Nielsen-Ninomiya no-go theorem

"Absence of Neutrino on a lattice"
H. B. Nielsen, M. Ninomiya (1981)

- The fermion doubling problem is unavoidable in a lattice regularization which respects the usual hermiticity, locality, and translation invariance.

- The Dirac operator for a fermion action in $2k$ Euclidean spacetime dimension cannot satisfy the following four conditions simultaneously.

$$S = \int_{\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^4}\, \bar{\psi}_{-\mathbf{p}} \tilde{D}(\mathbf{p}) \psi(\mathbf{p})$$

1. $\tilde{D}(\mathbf{p})$ is a periodic, analytic function of p_μ; — Locality, discrete trans. inv.
2. $\tilde{D}(\mathbf{p}) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$; — Continuum limit
3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_\mu = 0$; — Doublings
4. $\{\gamma_\chi, \tilde{D}(\mathbf{p})\} = 0$. — Chirality

Satisfy at most 3 conditions!

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Staggered fermions ← (2)
Kogut & Susskind (1975)

Wilson fermions ← (4)
K. Wilson (1974)

Chiral anomaly 1+1 dimension

- The occurrence of the fermion doubling problem is actually a consequence of the *chiral (axial) anomaly* in the continuum theory. To see this, let's consider massless Dirac fermion in 1+1 dimensions

$$\mathscr{L} = \bar{\psi}(i\partial_\mu \gamma^\mu)\psi \qquad \psi \text{ is a 2-component spinor}$$

- The gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ , \qquad \eta^{\mu\nu} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\gamma^0 = \sigma_1 \ , \qquad \gamma^1 = -i\sigma_2 \ , \qquad \gamma_\chi = \gamma^0\gamma^1 = \sigma_3$$

- The Lagrangian can be rewritten in the chiral basis, where the vector (V) and axial (A) symmetries are manifest.

$$\mathscr{L} = \bar{\psi}_L(i\partial_\mu \gamma^\mu)\psi_L + \bar{\psi}_R(i\partial_\mu \gamma^\mu)\psi_R$$

$$j_V^\mu = \bar{\psi}\gamma^\mu\psi \ , \qquad j_A^\mu = \bar{\psi}\gamma^\mu\gamma_\chi\psi$$

$$U(1)_V : \qquad \psi \to e^{i\alpha}\psi \ , \qquad \bar{\psi} \to \bar{\psi}e^{-i\alpha}$$

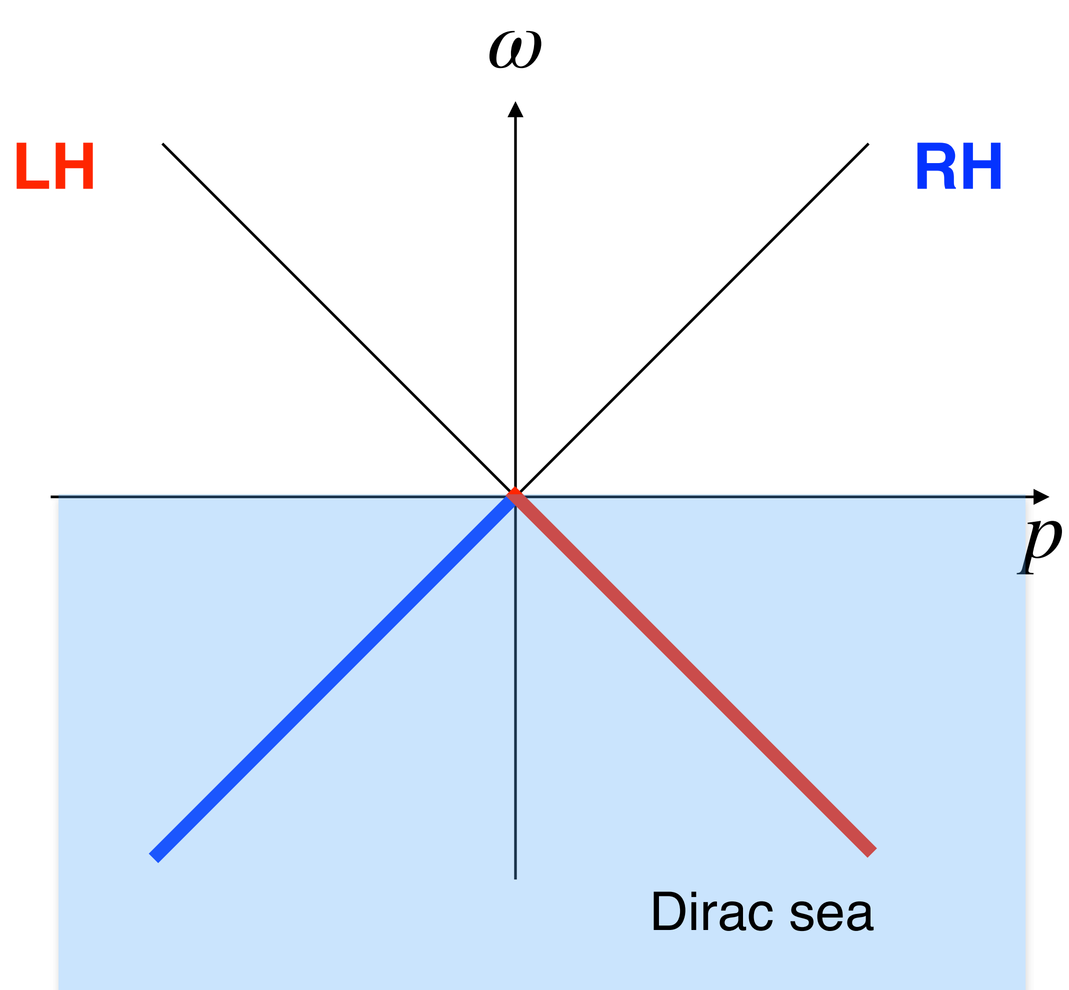
$$U(1)_A : \qquad \psi \to e^{i\gamma_\chi\beta}\psi \ , \qquad \bar{\psi} \to \bar{\psi}e^{i\beta\gamma_\chi}$$

$$\partial_\mu j_V^\mu = 0, \quad \partial_\mu j_A^\mu = 0$$

Chiral anomaly 1+1 dimension

- Now, we gauge the $U(1)_V$ symmetry

$$\mathscr{L} = i\bar{\psi}(\partial_\mu \gamma^\mu - ieA_\mu \gamma^\mu)\psi$$



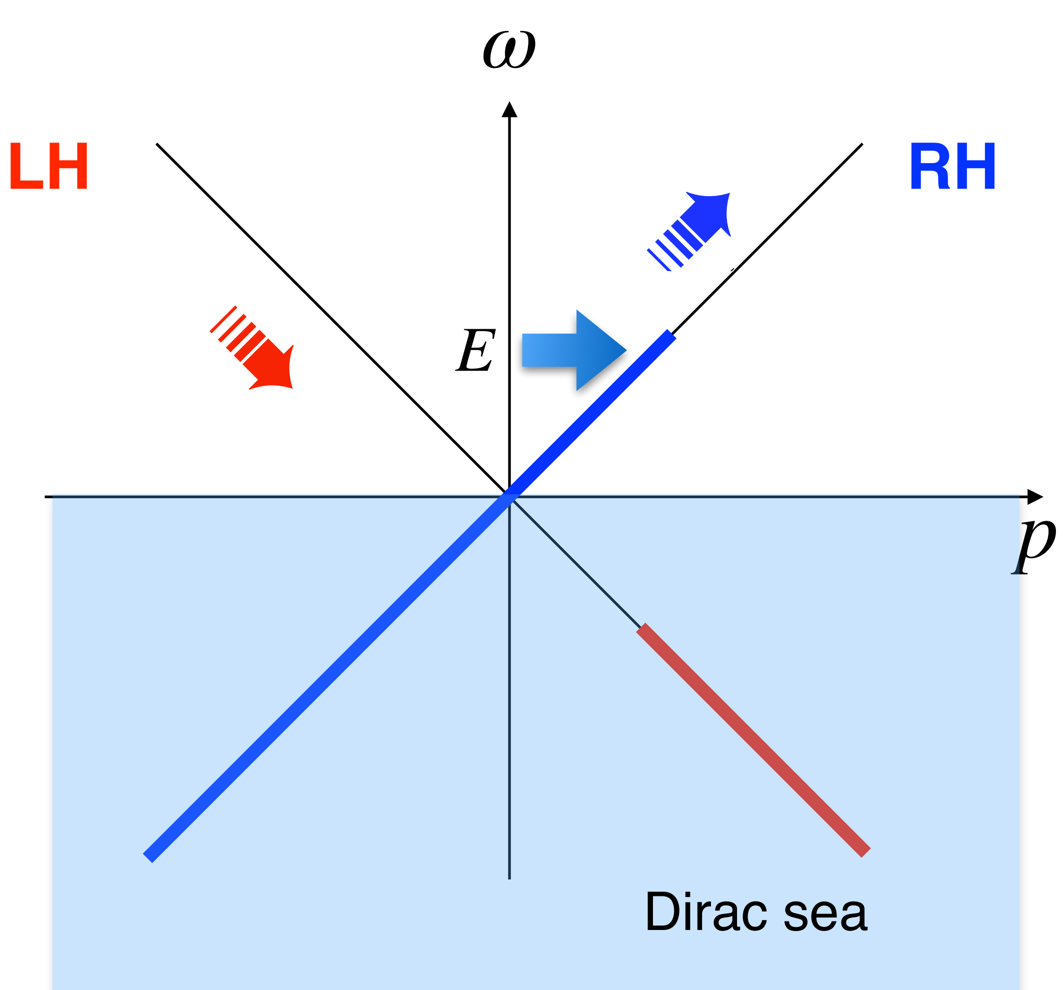
Infinite source & sink for fermions

Chiral anomaly 1+1 dimension

- Now, we gauge the $U(1)_V$ symmetry

$$\mathscr{L} = i\bar{\psi}(\partial_\mu \gamma^\mu - ieA_\mu \gamma^\mu)\psi$$

- Then, apply a constant positive electric current E



$$dp = q\,E\,dt$$

$$dn_R = +\frac{dp}{2\pi} \qquad dn_L = -\frac{dp}{2\pi}$$

$$dn = dn_R + dn_L = 0$$

$$\Rightarrow \quad \partial_\mu j_V^\mu = 0$$

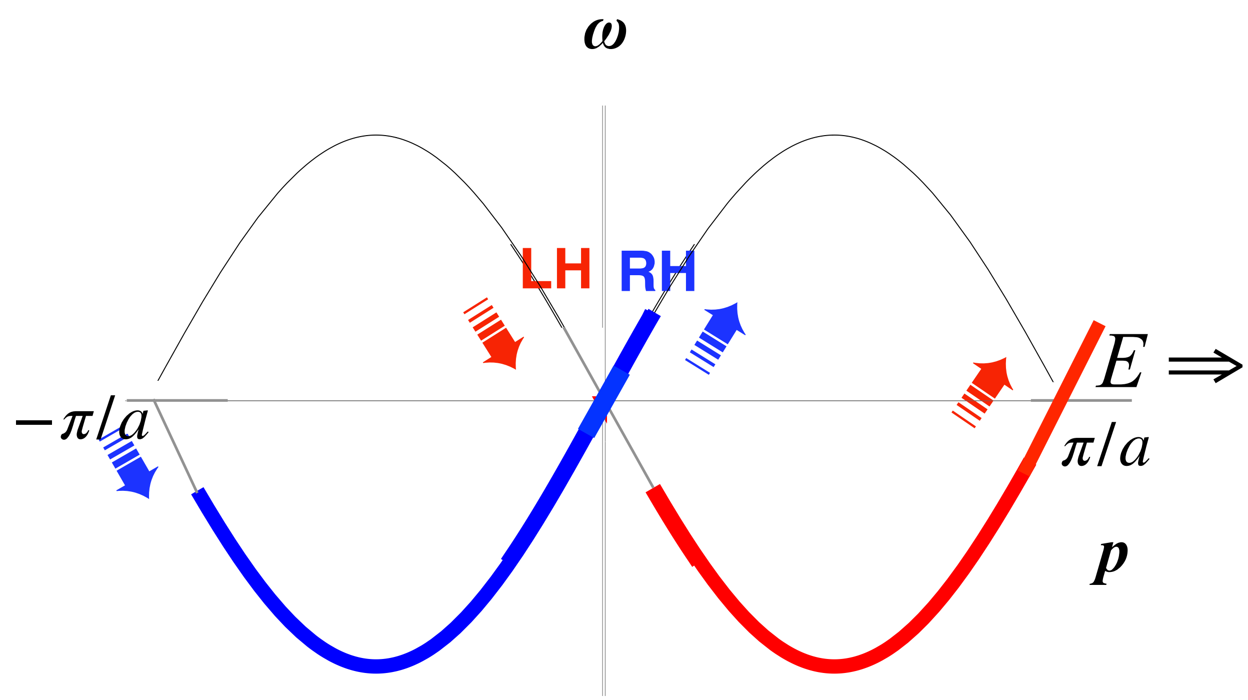
$$dn_A = dn_R - dn_L = \frac{q\,E}{\pi}dt$$

Axial anomaly $\Rightarrow$ $\partial_\mu j_A^\mu = \dfrac{qE}{\pi} = \dfrac{q}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}$

Infinite source & sink for fermions

Absence of chiral anomaly with a naive discretization

- What happens if we formulate it on the discrete space-time?



$\partial_\mu j_A^\mu = 0$

No anomaly in a system with finite number of degrees of freedom, and only reproduces continuum physics for long wavelength modes

Doublers at the edge of BZ, that have no analog in the continuum, cancel the anomaly of the continuum theory arising from momentum excitations around $p_\mu = 0$.

Restoration of the chiral anomaly

- In other words, anomalous symmetry in the continuum must be explicitly broken symmetry on the lattice

Karsten & Smit (1980)

This is exactly what Wilson fermions do!

$$\mathcal{L} = \bar{\psi}\left(\not{D} + m + aD^2\right)\psi$$

violates

chiral symmetry

- Price to pay: additive mass renormalization which requires to fine-tune the bare fermion mass, mixing of operators, ...

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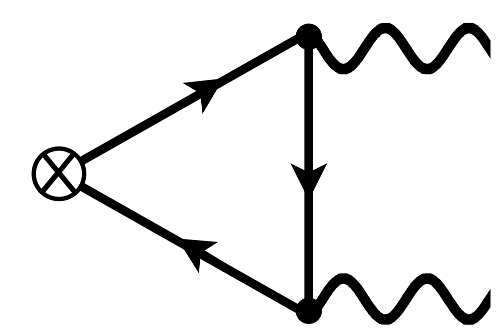
- Price to pay: additive mass renormalization which requires to fine-tune the bare fermion mass, mixing of operators, ...

- Any better way to define lattice fermions with a minimal violation of the chiral symmetry? Yes!

$$\{\gamma_\chi, \hat{D}\} = a\hat{D}\gamma_\chi\hat{D}$$

Ginsparg-Wilson (GW) relation (1982)

$\partial_\mu J^\mu_A$



GW fermion action

- GW fermion action is written

$$S_{GW} = \sum_{x,y} \bar{\psi}(x)(\hat{D}(x,y) + m\delta_{xy})\psi(y)$$

- Massless GW fermions respect *a lattice version of chiral symmetry*!

$$\hat{\gamma}_\chi = \gamma_\chi(1 - \frac{a}{2}\hat{D}) \quad \Rightarrow \quad \psi \to e^{i\alpha\gamma_\chi(1-\frac{a}{2}\hat{D})}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_\chi(1-\frac{a}{2}\hat{D})}$$

- An infinitesimal variation of the GW action δS_{GW} yields the GW relation and the *anomaly is correctly reproduced*, while protecting fermion mass from additive renormalization.

M. Luscher (1998)

$$a\mathrm{Tr}\ \gamma_\chi\hat{\mathrm{D}} = 2\mathrm{N_f}\,\nu \ \text{ with } \nu = (\mathrm{n}_+ - \mathrm{n}_-)$$

- GW operators (lattice chiral fermions): Overlap Dirac operators

Neuberger (1998)

$$\hat{D} = 1 + \gamma_5\epsilon(H(-m_W))$$
$$= 1 + \frac{D_W - m_W}{\sqrt{(D_W - m_W)^\dagger(D_W - m_W)}}$$

Domain wall fermions

D. B. Kaplan (1992)

Domain wall fermion

D. B. Kaplan (1992)

- A 4-dimensional chiral fermion arises as an edge (boundary) state of the 5-dimensional Dirac fermion. Instead of having a boundary, we consider a compact extra dimension with a sign-flip mass.

$$S_5 = \int d^4x \oint ds\ \bar{\Psi}\left[\not{D}_4 + \gamma_5 \partial_s + \ m\left(s\right)\right]\Psi$$

$$-L < s < L,\ \Psi(-L) = \Psi(L)$$

$$m(s) = +m \text{ for } s > 0,\ -m \text{ for } s < 0$$

- Expand Ψ as a product of 4-d spinor ψ and a function of s

$$\Psi(x,s) = \sum_n [b_n(s)P_+ + f_n(s)P_-]\psi_n(x),\quad P_\pm = \frac{1 \pm \gamma_5}{2}$$

- Satisfying the eigenvalue equations

$$[\partial_s + m(s)]b_n(s) = \mu_n f_n(s)$$

$$[-\partial_s + m(s)]f_n(s) = \mu_n b_n(s)$$

Zero mode solutions

$b_0 \propto e^{-\int^s ds' m(s')}$ Localized at s=0

$f_0 \propto e^{+\int^s ds' m(s')}$ Localized at s=$\pm L$

Domain wall fermion

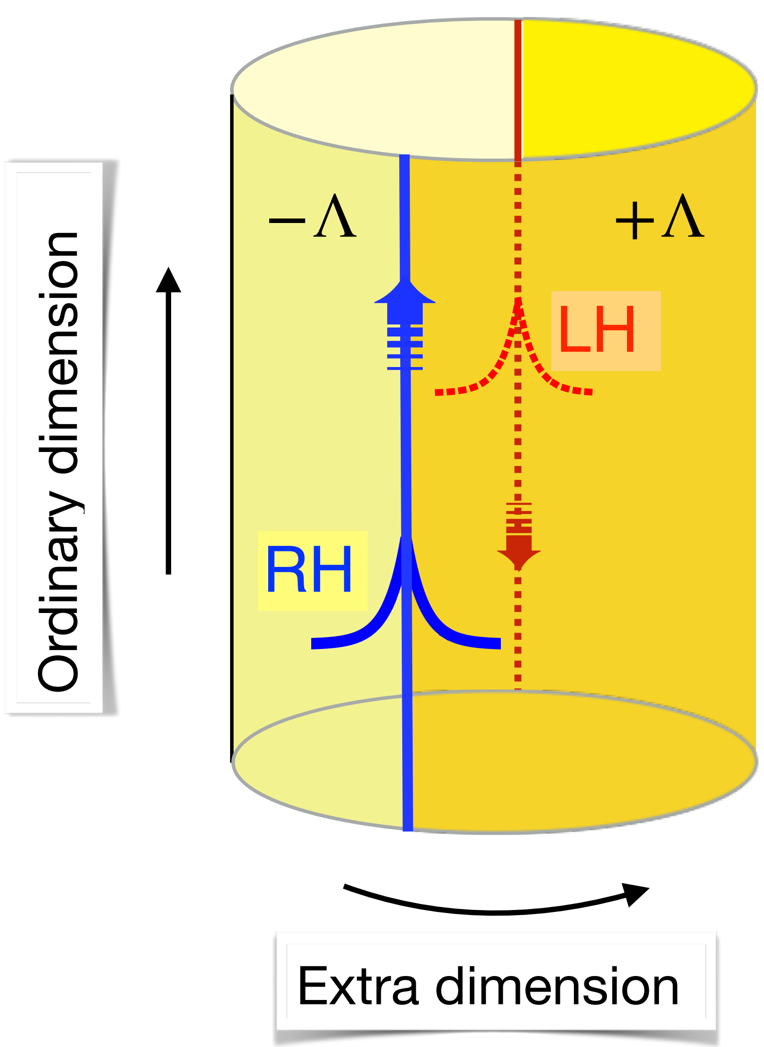
D. B. Kaplan (1992)

- The 5-d action can be written as

$$S_f = \int d^4x \int ds \bar{\Psi}[\gamma^\mu D_{4,\mu} + \gamma_5 \partial_s + m(s)]\Psi \; = \int d^4x \left[\boxed{\bar{\psi}_0 \gamma^\mu D_{4,\mu} P_+ \psi_0} + \sum_{k\neq 0} \bar{\psi}_k (\gamma^\mu D_{4,\mu} + \mu_k)\psi_k \right]$$

- One massless fermion + an infinite tower of heavy states with $\mu_n \gtrsim m$ for $n \neq 0$

Topological insulator!



Mass is protected from radiative corrections in the limit $L \to \infty$ multiplicative renormalisation

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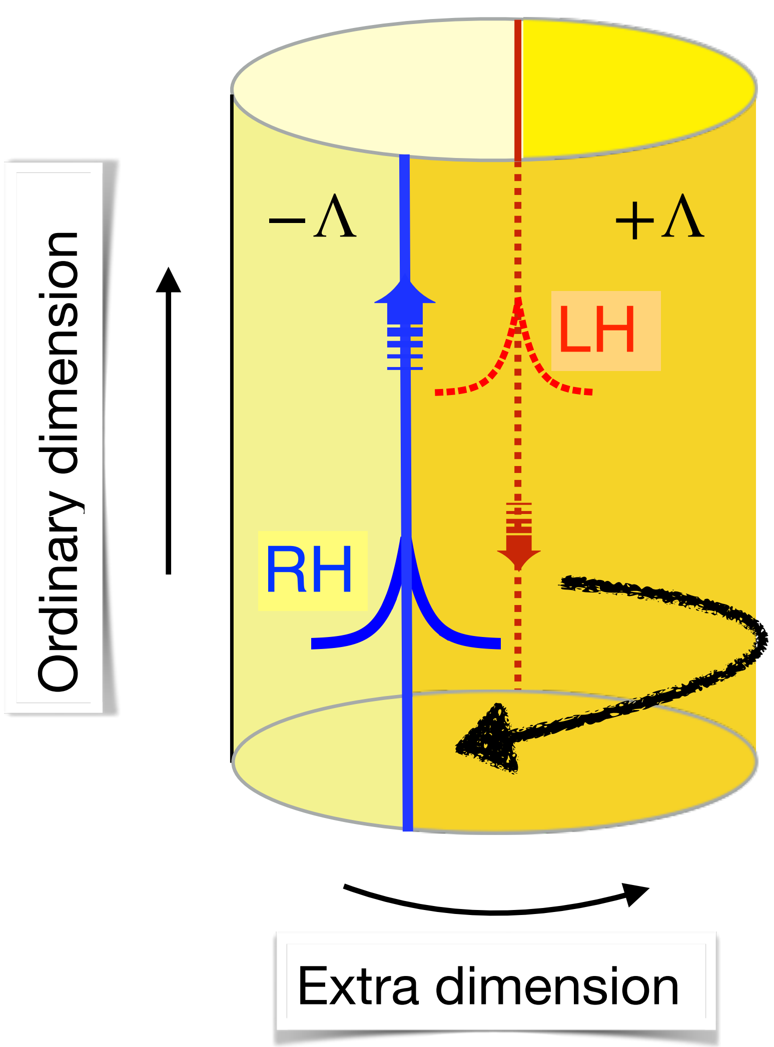


Mass is protected from radiative corrections in the limit $L \to \infty$ multiplicative renormalisation

No $U(1)_A$ anomaly in 5d theory, but *Chern-Simons (CS) current* *(Quantum spin Hall effects)* is induced by integrating out heavy fermion modes

Callan & Harvey (1984)

$$J_5 \propto \frac{m(s)}{|m(s)|} F\tilde{F} \implies \partial_5 J_5 \propto [\delta(s) - \delta(s-L)]\, F\tilde{F}$$

Explains anomalous disappearance of charge on one defect and reappearance on the other!

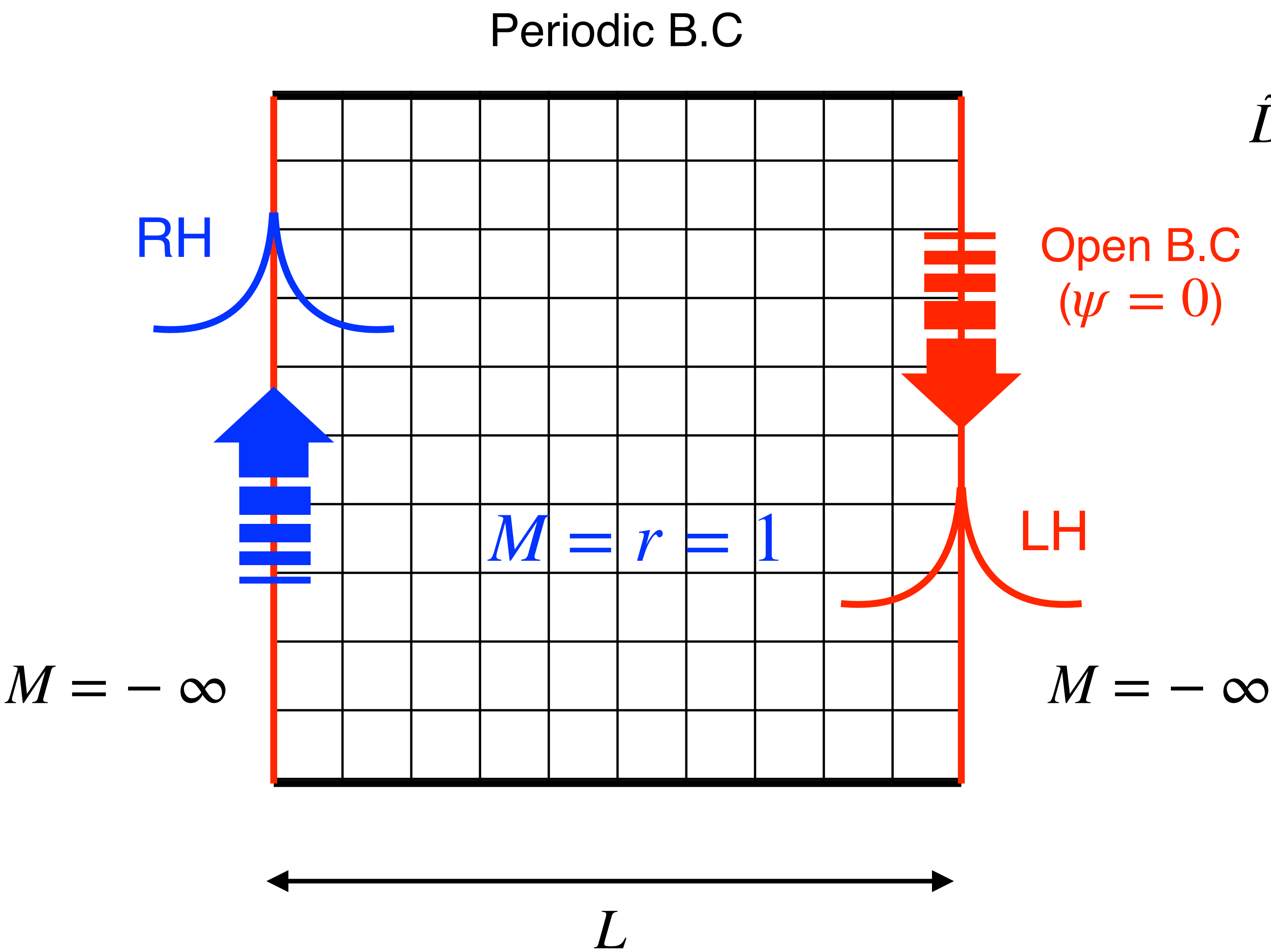
Domain wall fermion on the lattice

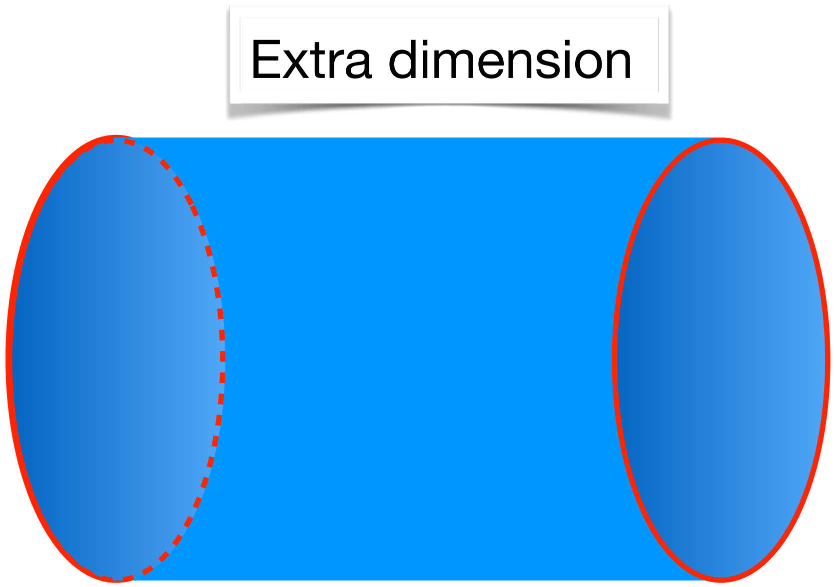
Y. Shamir (1993)

- Chiral fermions (RH & LH Weyl fermions) with residual mass ~ e^{-2ML} arise in the 4-d boundary of 5-d model with Wilson fermions & an open b.c.

$$D_{DW} = \gamma_\mu \partial^\mu + M + \frac{r}{2}\Delta$$

$$\tilde{D}_{DW}(p) = M + \sum_\mu \left(i \sin p_\mu + \frac{r}{2}(1 - \cos p_\mu)\right)$$





Topology of Shamir's cylinder

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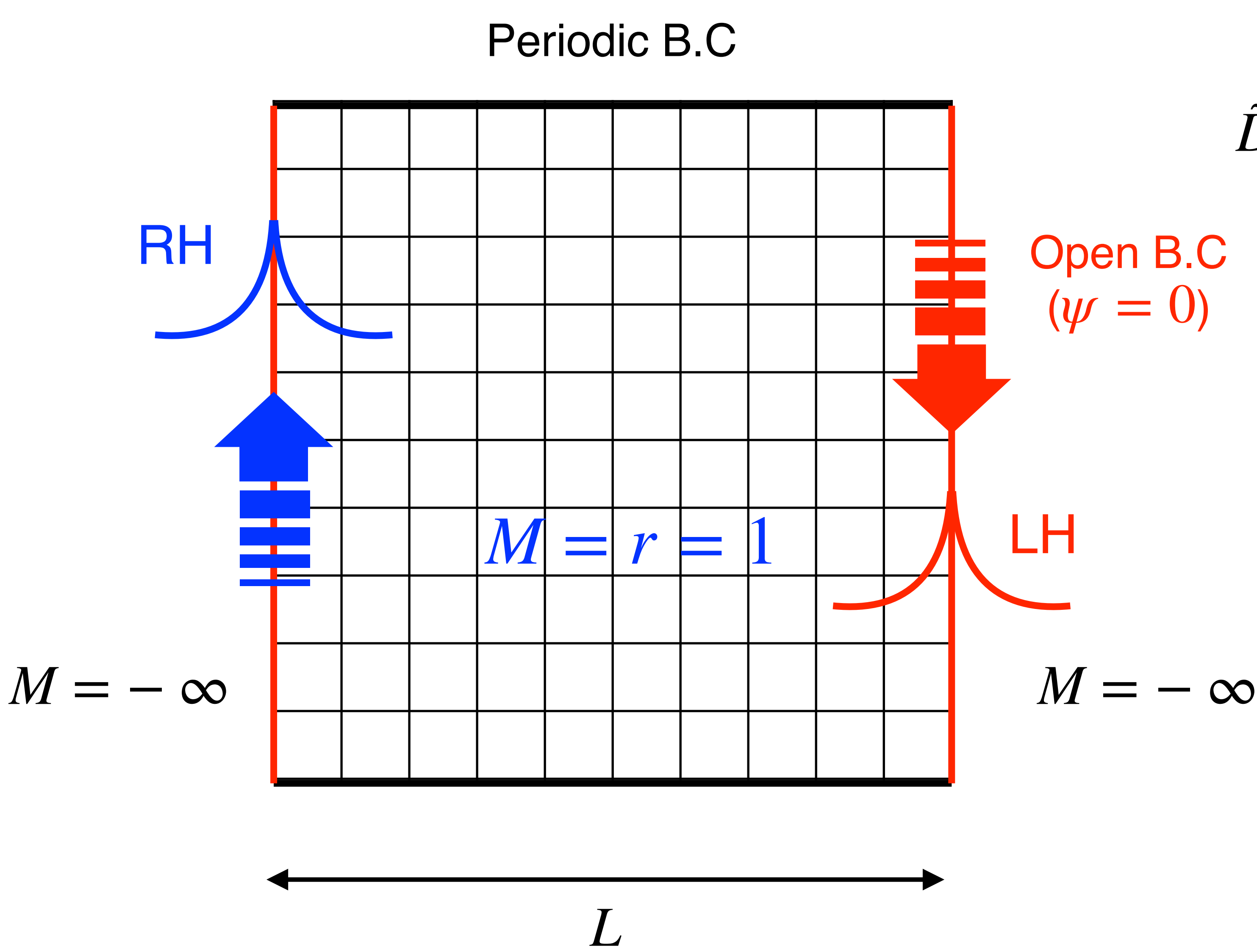

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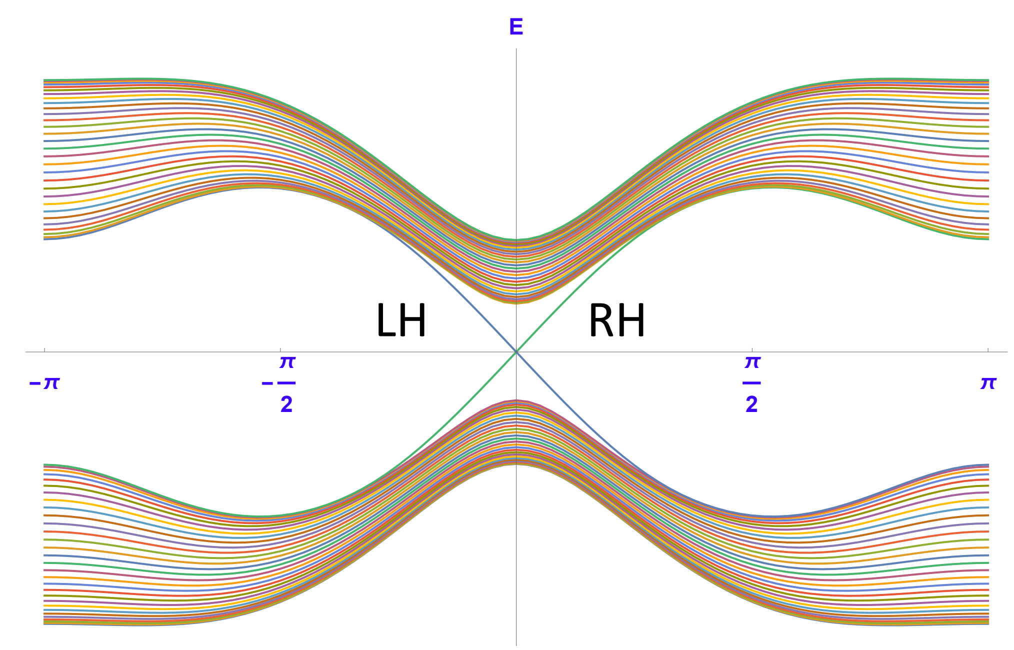
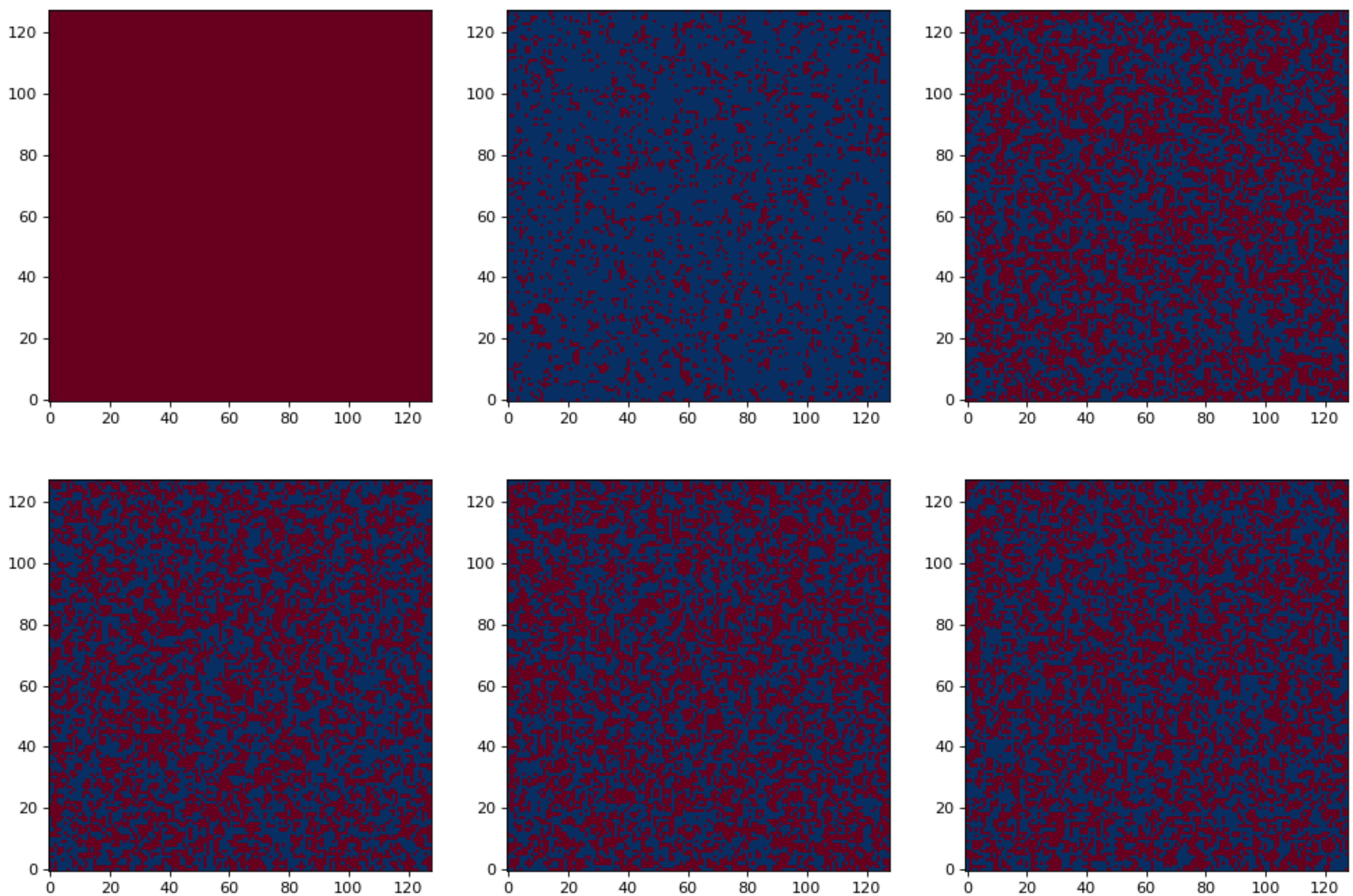
Dispersion relation of edge modes



2-d ising model

T>Tc



2-d ising model

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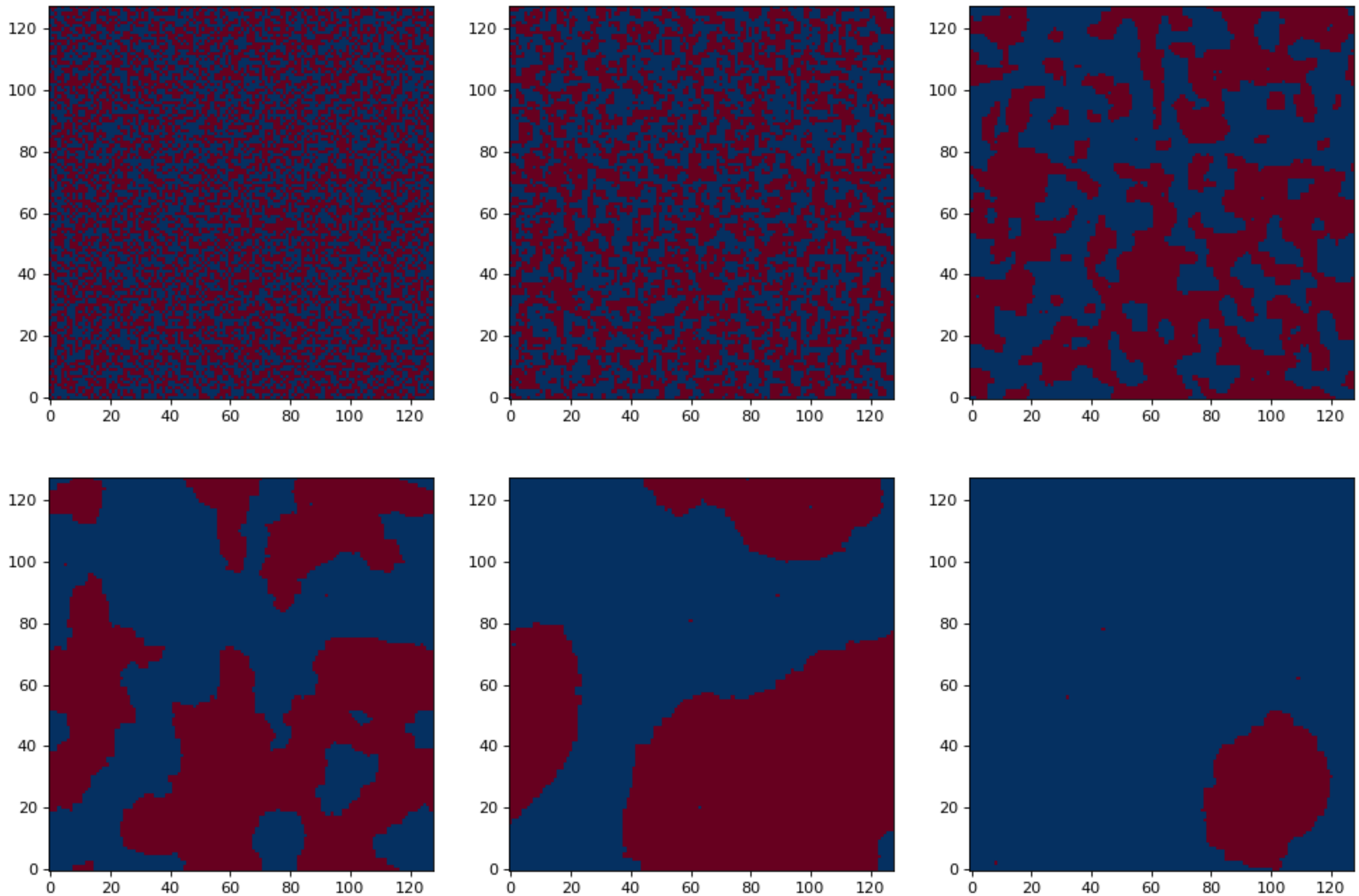

2-d ising model

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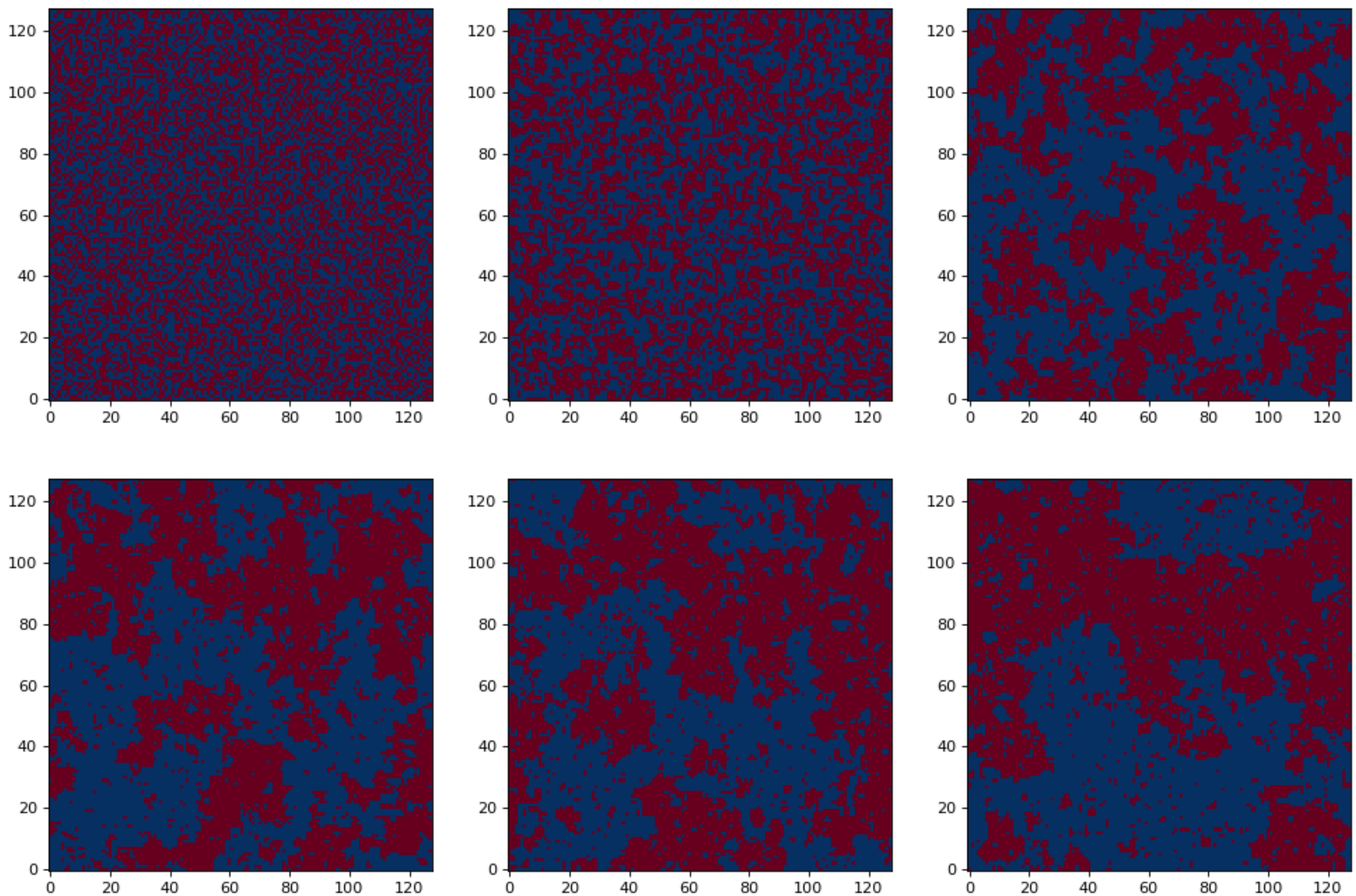
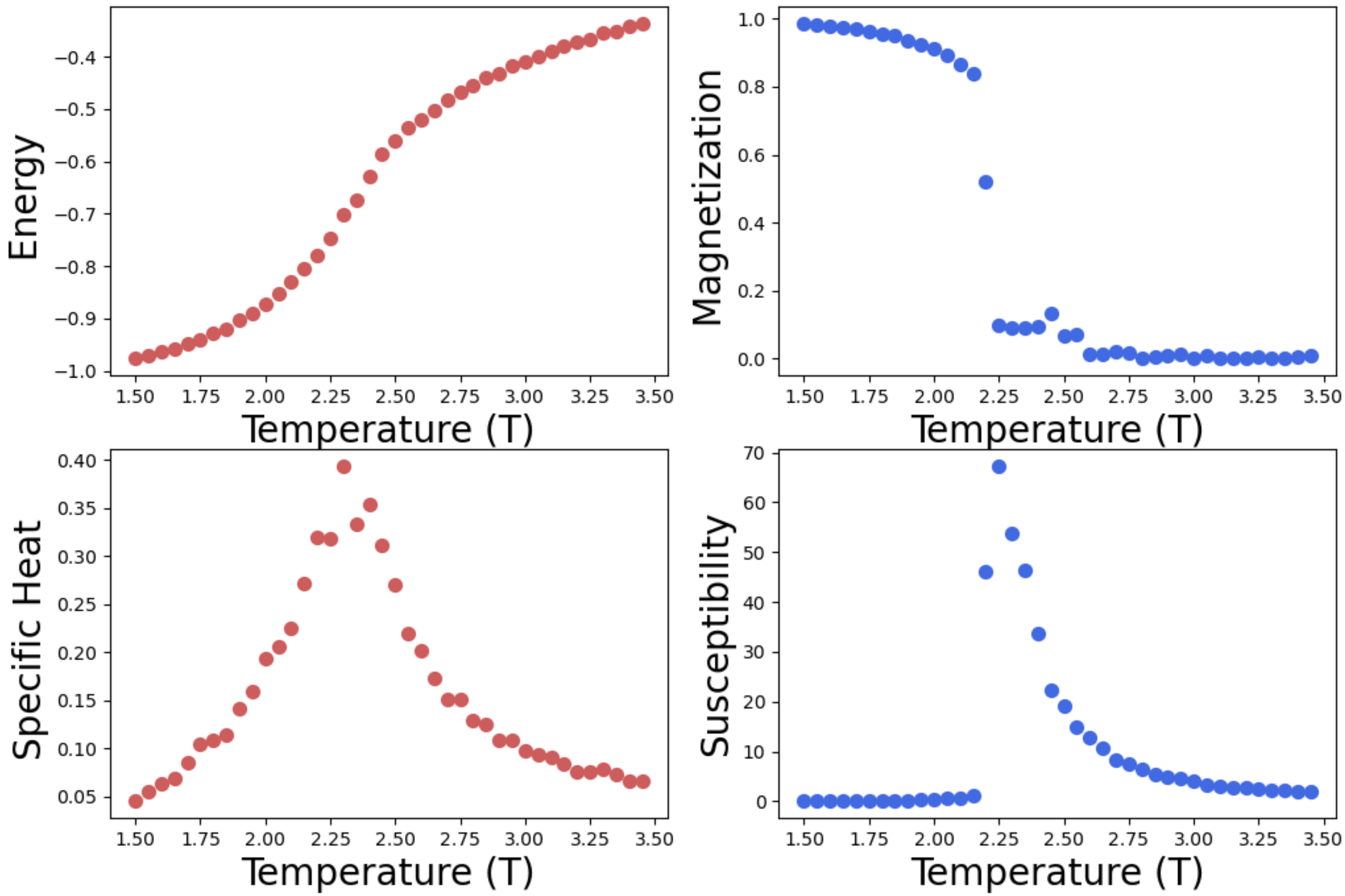


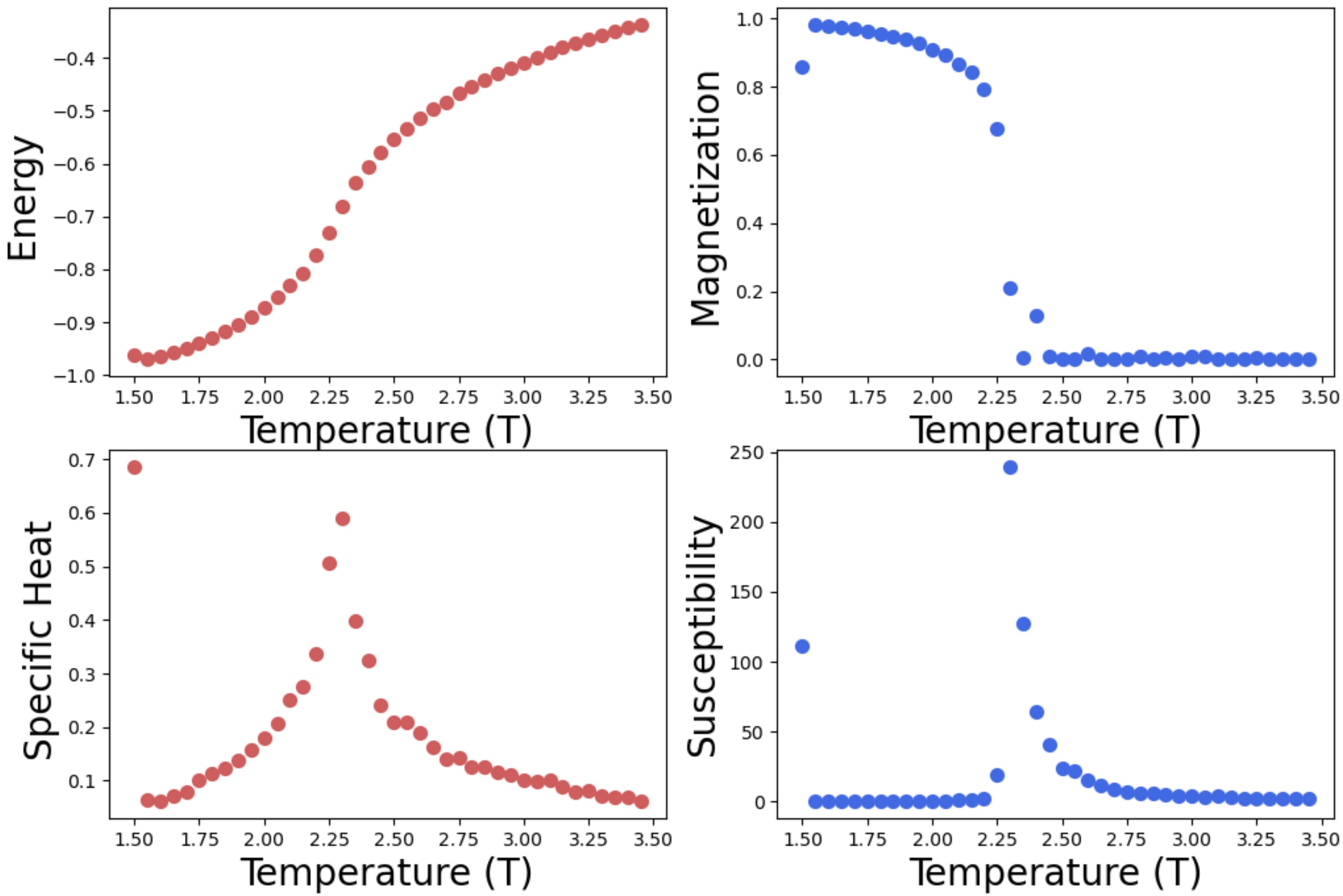
2-d ising model

NxN=16x16



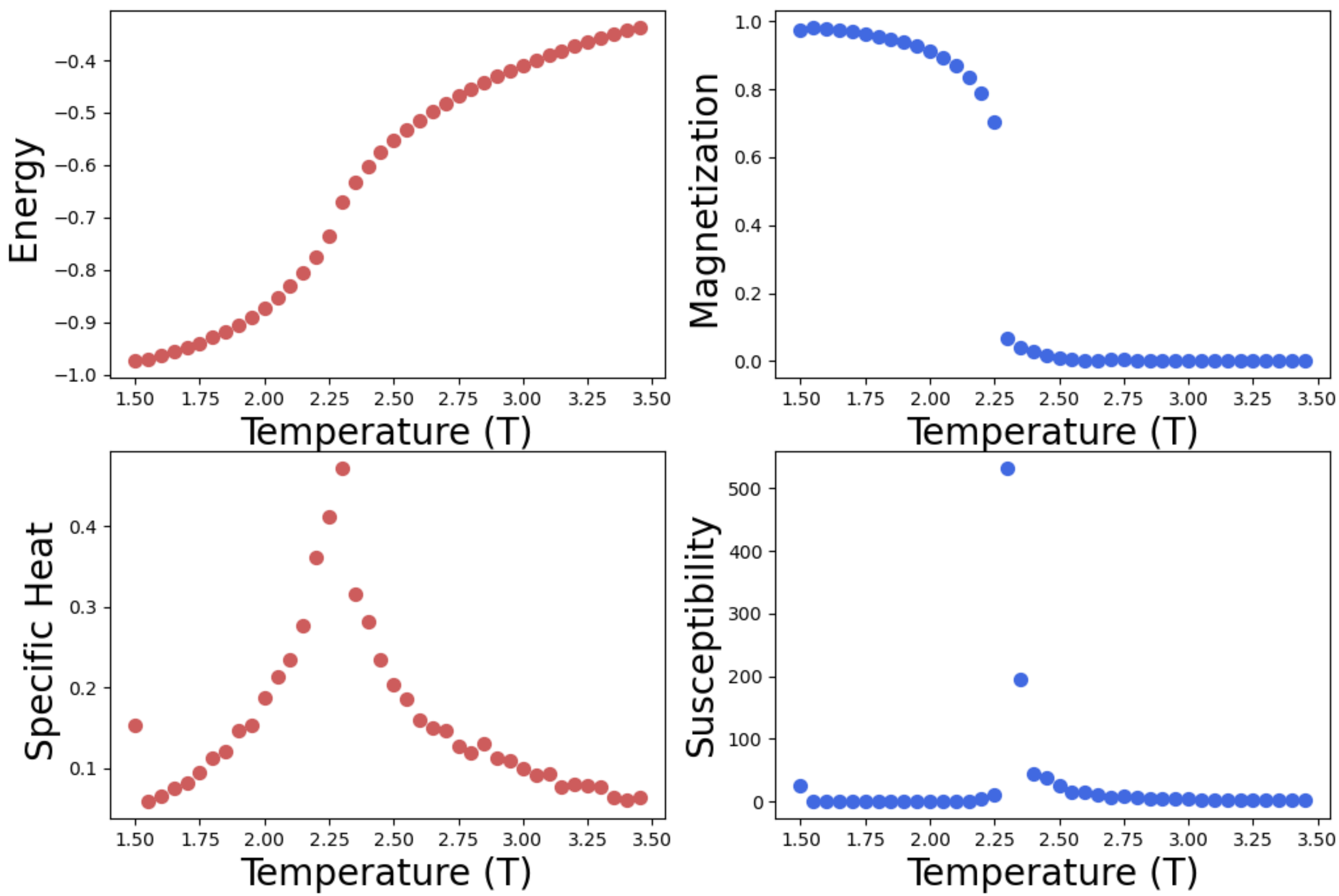
2-d ising model

NxN=48x48

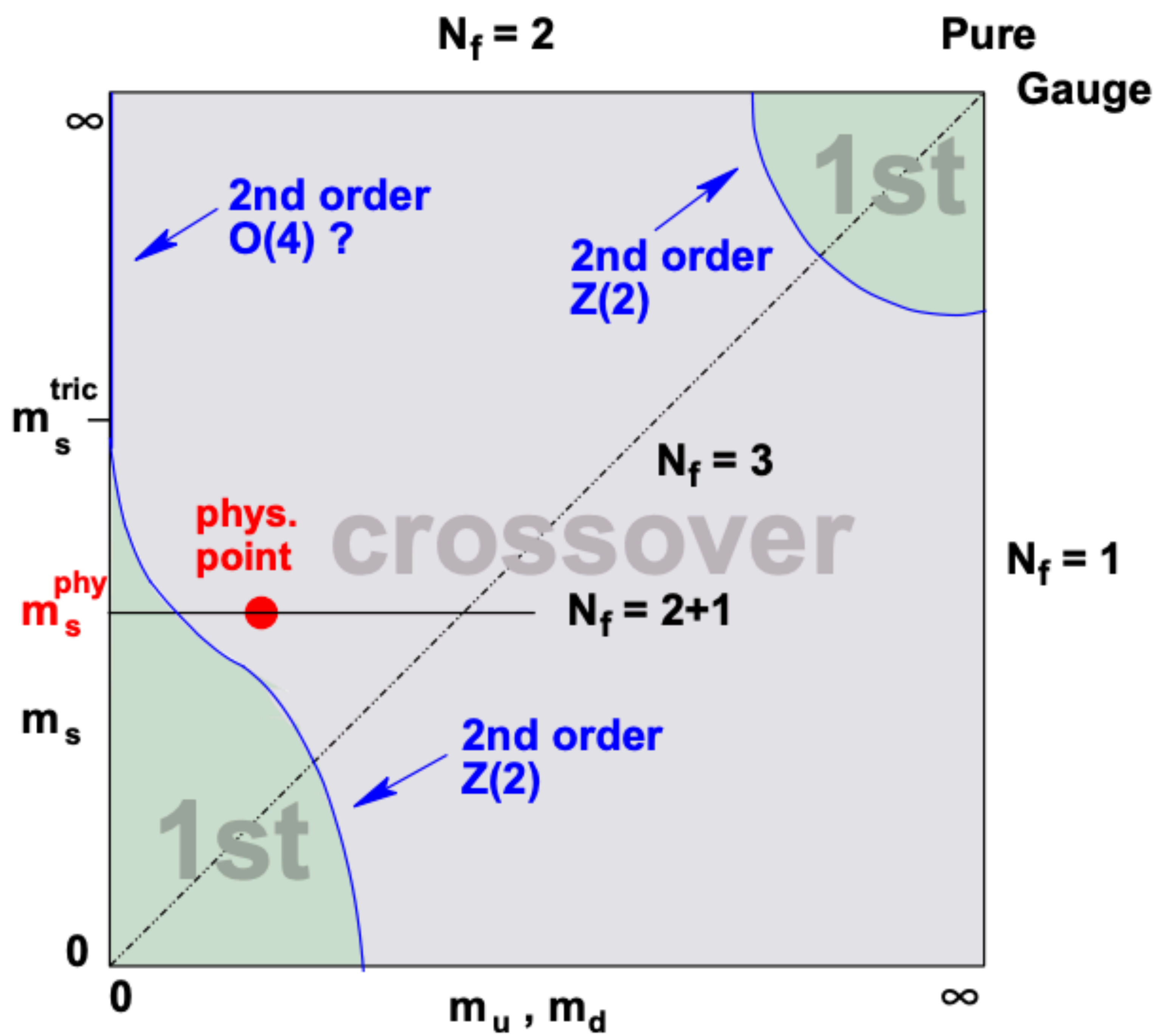


2-d ising model

NxN=128x128



Columbia plot



QCD phase diagram

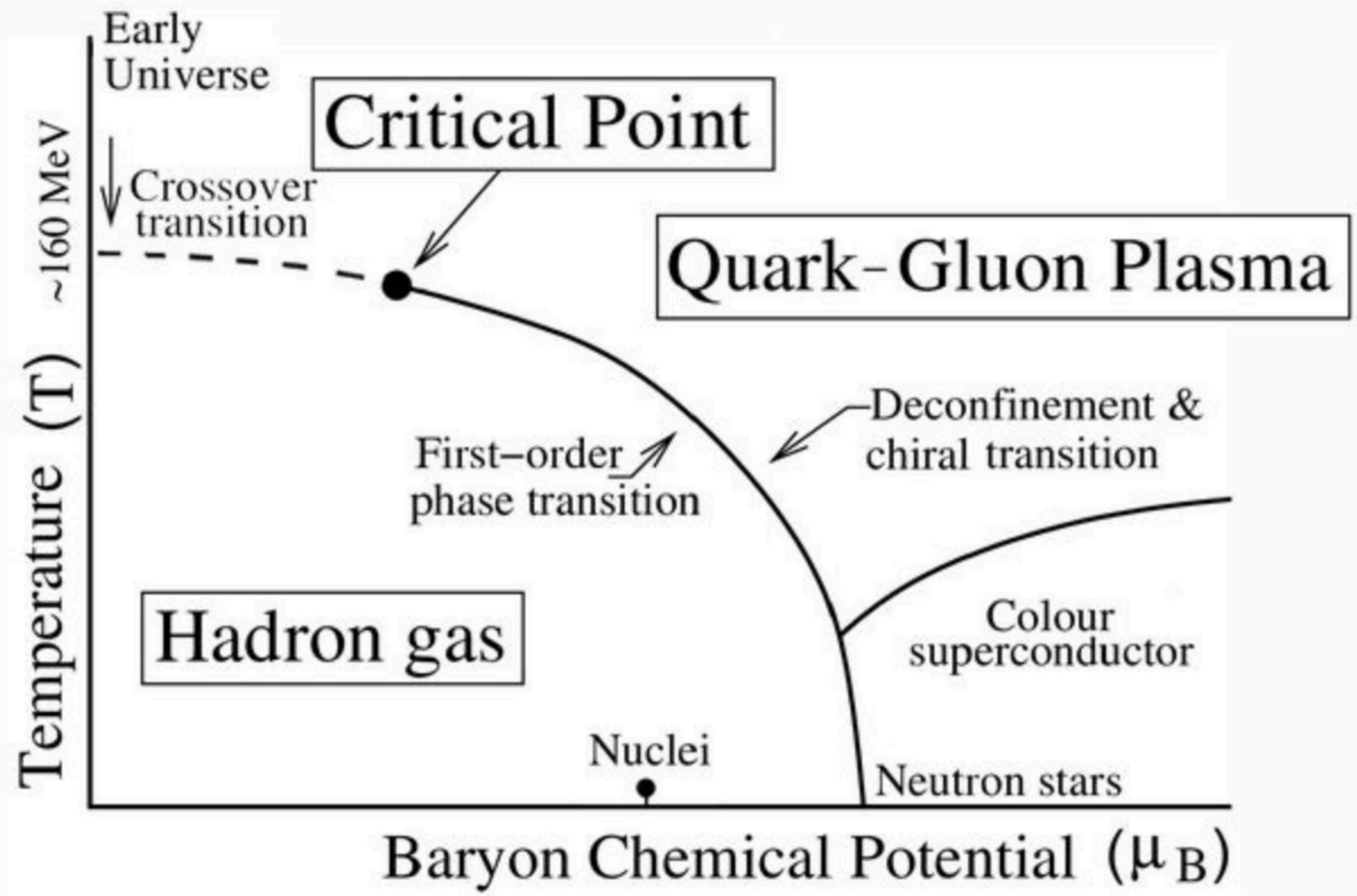


Figure 1. *A sketched view of the phase diagram of strongly interacting matter, in the plane of temperature and (quark) chemical potential.*

Lattice fermions	Advantages	Disadvantages
Wilson	- Simple, intuitive - Cheap - Improved action (clover, exponential, …)	- Explicit breaking of chiral symmetry - Additive renormalization ($m_q = m_0 - m_0^c$) - Discretization errors of $\mathcal{O}(a)$
Staggered	- Very cheap - (Partly) preserve chiral symmetry - Discretization errors of $\mathcal{O}(a^2)$	- Difficult to analyze the data (taste) - Chiral symmetry breaking pattern is different to QCD
Ginsparg-Wilson (domain wall, overlap)	- (continuum) Chiral symmetry is preserved - Discretization errors of $\mathcal{O}(a^2)$	- Expensive - Involve some tunings of parameters (5th dimension)