On the Mean Curvature Flow on Graphs with Applications in Image and Manifold Processing

Jia-Wei Liao, Kuok-Tong Ng

Department of Applied Mathematics
National Chiao Tung University

January 5, 2021

Outline

Level set method

Given a parametrized curve $\Gamma : [0, 1] \to \Omega$, evolving on a domain $\Omega \subset \mathbb{R}^d$ due to effect of a scalar field $\mathcal{F} : \Omega \to \mathbb{R}$. The level set method aims to find a function $f(x, y)$ such that at each times t the evolving curve Γ_t can be provided by the 0-level set of $f(x, t)$. In the other words,

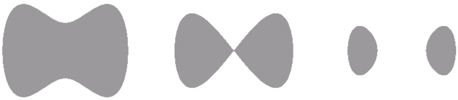
$$\Gamma_t = \{x | f(x, t) = 0\}$$

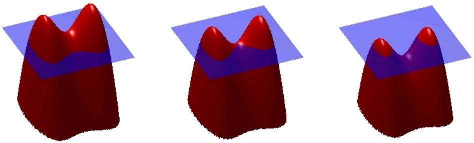
and the curve evolution can be done solving

$$\begin{cases} \frac{\partial f}{\partial t} = \mathcal{F} \cdot |\nabla f(x, t)| \\ f(x, 0) = f_0(x) \end{cases}$$

where f_0 corresponds to the given noisy image or to an implicit representation of a front (surface).

Level set method





$$\Gamma_t = \{x | f(x, t) = 0\}$$

Mean curvature flow

Mean Curvature

$\mathcal{K} = \operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right)$, where $\left(\frac{\nabla f}{|\nabla f|}\right)$ is a unit normal vector of f

When the normal velocity $\mathcal{F}$ also depends on the spatial derivative of the normal vector, we obtain the following mean curvature level set equation:

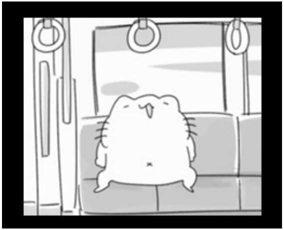
$$\left\{ \frac{\partial f}{\partial t} = \mathcal{K} \cdot |\nabla f(x,t)| f(x,0) = f_0(x) \right.$$

Image denoising

As usual, in image denoising, $f : V \to \mathbb{R}$,

$$\frac{\partial f}{\partial t} = \mathcal{K} \cdot |\nabla f(x,t)|,$$

where $V \subseteq \mathbb{Z}^2$ contains all pixels.



image

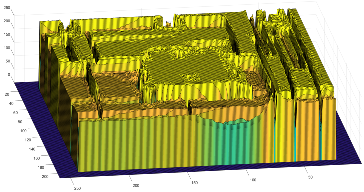
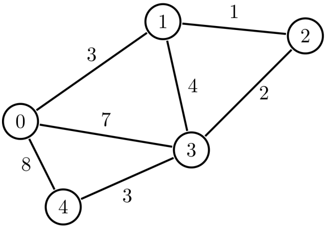
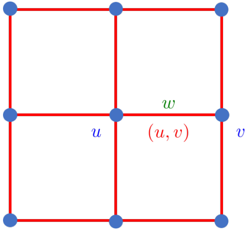


image manifold

Graph theory

- V: set of vertice
- E: set of edges
- w: weight function
- $G = (V, E, w)$: weighted graph
- (u, v): the edge that connects vertices u and v
- $\mu(u)$: the degree of a vertex u



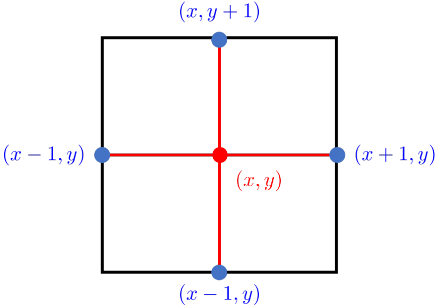


Graph structure

4-Adjacency grid graph

Let $G = (V, E, w)$. For any vertex $v = (x, y) \in V$, there are 4 vertices (possibly 2 or 3 only) adjacent to V say

$$(x-1, y), (x+1, y), (x, y-1), (x, y+1).$$

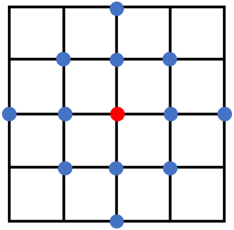


Graph structure

We use L_1-norm as our distance. That is, if $P = (x_1, y_1), Q = (x_2, y_2)$, then

$$d(P, Q) = |x_1 - x_2| + |y_1 - y_2|.$$

For example, if we say $u \sim v \Leftrightarrow d(u, v) \leq 2$, then we have

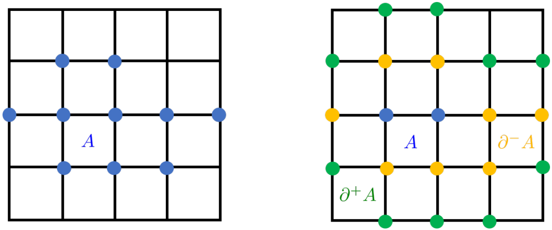


In general, $u \sim v \Leftrightarrow d(u, v) \leq n$, then u has $2n(n+1)$ neighborhood.

Graph theory

Let A be a set of connected vertices with $A \subset V$.

- Outer vertex boundary: $\partial^+ A = \{u \in A^c \mid \exists v \in A, v \sim u\}$
- Inner vertex boundary: $\partial^- A = \{u \in A \mid \exists v \in A^c, v \sim u\}$
- Vertex boundary: $\partial A = \partial^+ A \cup \partial^- A$



Operator norm

Let $f : V \to \mathbb{R}$. The integral of f is defined as:

$$\int_V f = \sum_{u \in V} f(u)$$

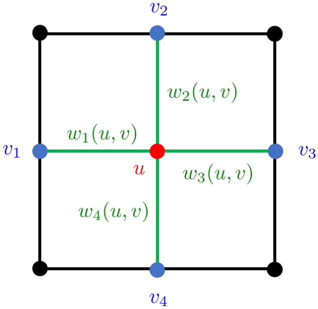
and L^p norm is defined by

- $1 \leq p < \infty$: $\|f\|_p = \left(\sum_{u \in V} |f(u)|^p \right)^{\frac{1}{p}}$
- $p = \infty$: $\|f\|_\infty = \max_{u \in V} |f(u)|$

Weighted operators

Weight directional derivative

- General: $(\partial_v f)(u) = w(u,v)(f(v) - f(u))$
- External: $(\partial_v^+ f)(u) = w(u,v)(f(v) - f(u))^+$ [1]
- Internal: $(\partial_v^- f)(u) = w(u,v)(f(v) - f(u))^-$ [2]



[1] $f^+ = \max(f, 0)$
[2] $f^- = \max(-f, 0)$

Weighted operators

Weight directional derivative

- General: $(\partial_v f)(u) = w(u,v)(f(v) - f(u))$
- External: $(\partial_v^+ f)(u) = w(u,v)(f(v) - f(u))^+$
- Internal: $(\partial_v^- f)(u) = w(u,v)(f(v) - f(u))^-$

Weight gradient

- General: $(\nabla_w f)(u) = (\partial_v f(u))_{v\in V}^\top$
- External: $(\nabla_w^+ f)(u) = (\partial_v^+ f(u))_{v\in V}^\top$
- Internal: $(\nabla_w^- f)(u) = (\partial_v^- f(u))_{v\in V}^\top$

Gradient norm

Weight gradient

$$(\nabla_w f)(u) = [w(u,v)(f(v) - f(u))]_{v \in V}^{\top}$$

Gradient norm

- $1 \leq p < \infty$: $\|(\nabla_w^{\pm} f)(u)\|_p = \left(\sum_{v \sim u} \left[w(u,v)(f(v) - f(u)^{\pm}) \right]^p \right)^{\frac{1}{p}}$
- $p = \infty$: $\|(\nabla_w^{\pm} f)(u)\|_\infty = \max_{v \sim u} \left[w(u,v)(f(u) - f(v))^{\pm} \right]$

Nonlocal perimeter on graph

Perimeter

For $0 < p < \infty$ and $A \subset V$, the perimeters of A are defined as

- $Per^{+}_{w,p}(A) = \dfrac{1}{2p} \displaystyle\sum_{u \in V} \left\| \nabla^{+}_{w} \chi_A(u) \right\|_p$
- $Per^{-}_{w,p}(A) = \dfrac{1}{2p} \displaystyle\sum_{u \in V} \left\| \nabla^{-}_{w} \chi_A(u) \right\|_p$
- $Per_{w,p}(A) = \dfrac{1}{p} \left(Per^{+}_{w,p}(A) + Per^{-}_{w,p}(A) \right)$

Remark:

- $Per^{+}_{w,1}(A) = Per^{-}_{w,1}(A)$
- $Per_{w,1}(A) = 2 Per^{+}_{w,1}(A)$

Nonlocal perimeter on graph

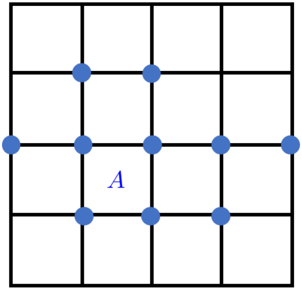
Recall: $\|(\nabla_w^{\pm} f)(u)\|_p = \left(\sum_{v \sim u} \left[w(u,v)(f(v) - f(u)^{\pm}) \right]^p \right)^{\frac{1}{p}}$

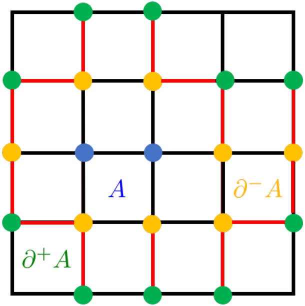
Note that

$$\begin{aligned} Per_{w,1}^{+}(A) &= \frac{1}{2} \sum_{u \in V} \left\| \nabla_w^{+} \chi_A(u) \right\|_1 \\ &= \frac{1}{2} \sum_{v \sim u} w(u,v) \left(\chi_A(v) - \chi_A(u) \right)^{+} \\ &= \frac{1}{2} \sum_{u \in A^c} \sum_{v \in A} w(u,v) \end{aligned}$$

Hence $Per_{w,1}(A) = \sum_{u \in A^c} \sum_{v \in A} w(u,v)$.

Nonlocal perimeter on graph





$$Per_{w,1}(A) = \sum_{u \in A^c} \sum_{v \in A} w(u,v)$$

Nonlocal mean curvature on graph

Mean Curvature

Let $u_0 \in \partial A$. For $u_0 \in \partial^+ A$, the mean curvature of u_0 is defined as

$$\mathcal{K}_w^+(u_0, A) = \frac{Per_{w,1}^+(A \cup \{u_0\}) - Per_{w,1}^+(A)}{\mu(u_0)},$$

and for $u_0 \in \partial^- A$, the mean curvature of u_0 is defined as

$$\mathcal{K}_w^-(u_0, A) = \frac{Per_{w,1}^-(A) - Per_{w,1}^-(A - \{u_0\})}{\mu(u_0)}.$$

Nonlocal mean curvature on graph

Note that

$$Per_{w,1}^{+}\left(A \cup \{u_0\}\right) - Per_{w,1}^{+}(A) = \sum_{v \in A^c} w(u_0, v) - \sum_{v \in A} w(u_0, v)$$

Therefore, we can get

$$\mathcal{K}_w^{+}\left(u_0, A\right) = \frac{\sum_{v \in A^c} w(u_0, v) - \sum_{v \in A} w(u_0, v)}{\mu\left(u_0\right)}$$

Similarly,

$$\mathcal{K}_w^{-}\left(u_0, A\right) = \frac{\sum_{v \in A^c} w(u_0, v) - \sum_{v \in A} w(u_0, v)}{\mu\left(u_0\right)}$$

Nonlocal mean curvature on graph

Note that

$$Per^{+}_{w,1}\left(A\cup\{u_0\}\right)-Per^{+}_{w,1}(A)=\sum_{v\in A^c}w(u_0,v)-\sum_{v\in A}w(u_0,v)$$

Therefore, we can get

$$\mathcal{K}^{+}_{w}\left(u_0,A\right)=\frac{\sum_{v\in A^c}w(u_0,v)-\sum_{v\in A}w(u_0,v)}{\mu\left(u_0\right)}$$

Similarly,

$$\mathcal{K}^{-}_{w}\left(u_0,A\right)=\frac{\sum_{v\in A^c}w(u_0,v)-\sum_{v\in A}w(u_0,v)}{\mu\left(u_0\right)}$$

Then for $u_0\in\partial A$, the mean curvature is defined as:

$$\mathcal{K}_{w}\left(u_0,A\right)=\frac{\sum_{v\in A^c}w(u_0,v)-\sum_{v\in A}w(u_0,v)}{\mu\left(u_0\right)}$$

Nonlocal mean curvature on graph

Let $f : V \to \mathbb{R}$ and $u \in V$. The mean curvature $\mathcal{K}_w$ of f at u on a graph is defined as:

$$\begin{aligned}\mathcal{K}_w(u, f) &= \mathcal{K}_w(u, \{f(v) \geq f(u)\}) \\ &= \frac{\sum_{f(v)-f(u)\geq 0} w(u,v) - \sum_{f(v)-f(u)<0} w(u,v)}{\mu(u)} \\ &= \frac{\sum_{u\in V} w(u,v)\,\mathrm{sign}(f(v)-f(u))}{\mu(u)}\end{aligned}$$

Fractional perimeter and curvature

The s-perimeter of $A \subset \mathbb{R}^n$ is defined as

$$Per(A) = c_n \int_A \int_{A^c} \frac{1}{|x-y|^{n+s}} dxdy$$

where c_n is a normalization constant.

The continuous fractional curvature is defined formally as the first variation of these s-perimeters as follows:

$$\begin{aligned} \mathcal{K}(x, A) &= c_n \int \frac{\chi_A(y) - \chi_{A^c}(y)}{|x-y|^{n+s}} dy \\ &= c_n \int_A \frac{1}{|x-y|^{n+s}} dy - c_n \int_{A^c} \frac{1}{|x-y|^{n+s}} dy \end{aligned}$$

Mean curvature on graph vs fractional mean curvature

Let $G = (V, E, w)$, for $w(x, y) = \begin{cases} \frac{1}{|x-y|^{n+s}}, \text{ with } 0 < s < 1 \\ 0, \text{ otherwise} \end{cases}$

Our mean curvature is defined as:

$$\mathcal{K}_w(u_0, A) = \frac{\sum_{v \in A^c} w(u_0, v) - \sum_{v \in A} w(u_0, v)}{\mu(u_0)}$$

and the fractional curvature is

$$\mathcal{K}(x, A) = c_n \int_A \frac{1}{|x-y|^{n+s}} dy - c_n \int_{A^c} \frac{1}{|x-y|^{n+s}} dy$$

Discretization of mean curvature flow

Based on the definition of our mean curvature and the mean curvature flow can be expressed for the case of L_∞ norm as follows:

$$\begin{cases} \frac{\partial f}{\partial t}(u) = \mathcal{K}_w^+(f(u)) \left\| \left(\nabla_w^+ f\right)(u) \right\|_\infty - \mathcal{K}_w^-(f(u)) \left\| \left(\nabla_w^- f\right)(u) \right\|_\infty \\ f(u,0) = f_0(u) \end{cases}$$

where $\mathcal{K}_w(f(u)) = \dfrac{\sum_{v \in V} w(u,v)\, \mathrm{sign}(f(v) - f(u))}{\mu(u)}$.

In particular,

- If $\mathcal{K}_w > 0$, then $\frac{\partial f}{\partial t}(u) = \mathcal{K}_w^+(f(u)) \left\| \left(\nabla_w^+ f\right)(u) \right\|_p$
- If $\mathcal{K}_w < 0$, then $\frac{\partial f}{\partial t}(u) = \mathcal{K}_w^-(f(u)) \left\| \left(\nabla_w^- f\right)(u) \right\|_p$

Discretization of mean curvature flow

The time variable can be discretized using Explicit Euler method as

$$\frac{\partial f}{\partial t}(u) = \frac{f^{n+1}(u) - f^{n}(u)}{\Delta t},$$

where $f^n(u) = f(u, n\Delta x)$.

The equation can be rewritten as the following iterative equation:

$$\begin{aligned} & f^{n+1}(u) - f^{n}(u) \\ = & \left[\mathcal{K}_w^{+}(f^n(u))\|(\nabla_w^{+} f^n)(u)\|_\infty - \mathcal{K}_w^{-}(f^n)(u)\|(\nabla_w^{-} f^n(u))\|_\infty\right]\Delta t \end{aligned}$$

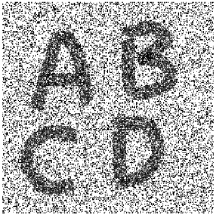
Algorithm (I)

Algorithm Mean curvature flow

Initialize $f = f_0$
for $n = 1$ to maxstep **do**
  $\mathcal{K}_w f =$ **MeanCurvature**(f)
  $f \leftarrow f + [(\mathcal{K}_w^+ f) \otimes \|\nabla^+ f\|_\infty - (\mathcal{K}_w^- f) \otimes \|\nabla^- f\|_\infty] * dt$
end for
return f

Numerical result (I)

noise image, psnr: 6.358824e+00



After 120 iter, psnr: 1.849059e+01

After 140 iter, psnr: 1.839319e+01

After 400 iter, psnr: 1.701532e+01

Salt and pepper with noise density 0.5

Numerical result (II)

Noise image, psnr: 2.011317e+01

After 15 iteration, psnr: 2.612567e+01

distance ≤ 1

After 15 iteration, psnr: 2.814363e+01

distance ≤ 2

After 10 iteration, psnr: 2.800570e+01

distance ≤ 3

- Gaussian noise: $\sigma^2 = 0.01$
- $\Delta t = 0.1$

Numerical result (III)

Noise image, psnr: 2.011317e+01

After 35 iteration, psnr: 2.809275e+01

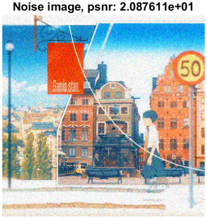
After 15 iteration, psnr: 2.814363e+01

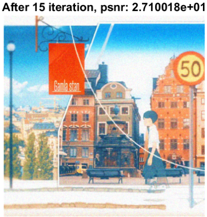
After 7 iteration, psnr: 2.812040e+01

$\Delta t = 0.05$ $\Delta t = 0.1$ $\Delta t = 0.2$

- Gaussian noise: $\sigma^2 = 0.01$
- distance ≤ 2

Numerical result (IV)





distance ≤ 1

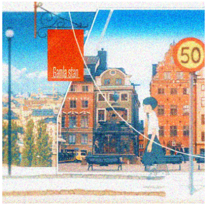
distance ≤ 2

distance ≤ 3

- Gaussian noise: $\sigma^2 = 0.01$
- $\Delta t = 0.1$

Numerical result (V)

Noise image, psnr: 2.087611e+01



After 23 iteration, psnr: 2.764967e+01

$\Delta t = 0.05$

After 10 iteration, psnr: 2.768178e+01

$\Delta t = 0.1$

After 6 iteration, psnr: 2.786991e+01

$\Delta t = 0.2$

- Gaussian noise: $\sigma^2 = 0.01$
- distance ≤ 2

Non-local Erosion and Dilation

- **Non-local Dilation:**

$$NLD(f(u)) = f(u) + \|(\nabla_w^+ f)(u)\|_\infty$$

- **Non-local Erosion:**

$$NLE(f(u)) = f(u) - \|(\nabla_w^- f)(u)\|_\infty$$

NLD and NLE

Notice that

$$
\begin{aligned}
& f^{n+1}(u) - f^{n}(u) \\
= & \left[\mathcal{K}_w^{+}(f^n(u))\|(\nabla_w^{+} f^n)(u)\|_\infty - \mathcal{K}_w^{-}(f^n(u))\|(\nabla_w^{-} f^n)(u)\|_\infty\right]\Delta t
\end{aligned}
$$

It can be rewritten using the NLD and NLE definitions as

$$
\begin{aligned}
f^{n+1}(u) = & f^n(u) + \left[\mathcal{K}_w^{+}(f^n(u)) NLD(f^n(u)) - \mathcal{K}_w^{+}(f^n(u)) f^n(u)\right]\Delta t \\
& - \left[\mathcal{K}_w^{-}(f^n(u)) f^n(u) - \mathcal{K}_w^{-}(f^n(u)) NLE(f^n(u))\right]\Delta t \\
\\
= & f^n(u)[1 - \Delta t(\mathcal{K}_w^{+}(f^n(u)) + \mathcal{K}_w^{-}(f^n(u)))] \\
& + \mathcal{K}_w^{+}(f^n(u)) NLD(f^n(u))\Delta t + \mathcal{K}_w^{-}(f^n(u)) NLE(f^n(u))\Delta t
\end{aligned}
$$

NLD and NLE

- If $\mathcal{K}_w f(u) \neq 0$, we can take $\Delta t = \frac{1}{|\mathcal{K}_w f(u)|}$ (depend on u), then the equation becomes

$$f^{n+1}(u) = \begin{cases} NLD(f^n(u)), & \text{if } \mathcal{K}_w f(u) > 0 \\ NLE(f^n(u)), & \text{if } \mathcal{K}_w f(u) < 0 \end{cases}$$

- If $\mathcal{K}_w f(u) = 0$, then $f^{n+1}(u) = f^n(u)$

Description to dilation

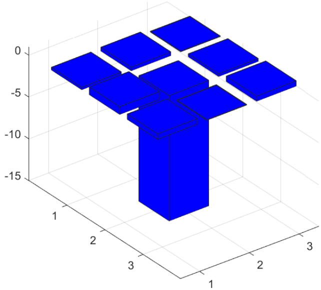
Assumed that

$$w(u,v) = \begin{cases} 1, & \text{if } u \sim v \\ 0, & \text{otherwise} \end{cases}$$

For example, let $u = (2,2)$. Then

$$f(v) > f(u) \text{ for } u \sim v$$

Hence we can get $K_w(f(u)) > 0$.



Recall:

$$\mathcal{K}_w(f(u)) = \frac{\sum_{v \in V} w(u,v)\,\mathrm{sign}(f(v) - f(u))}{\mu(u)}$$

Description to erosion

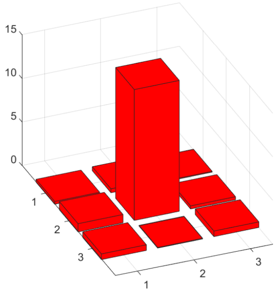
Assumed that

$$w(u,v) = \begin{cases} 1, & \text{if } u \sim v \\ 0, & \text{otherwise} \end{cases}$$

For example, let $u = (2,2)$. Then

$$f(v) < f(u) \text{ for } u \sim v$$

Hence we can get $\mathcal{K}_w(f(u)) < 0$.

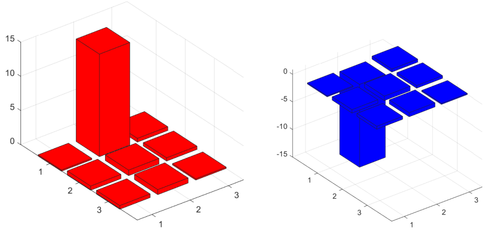


Recall:

$$\mathcal{K}_w(f(u)) = \frac{\sum_{v \in V} w(u,v)\,\mathrm{sign}(f(v) - f(u))}{\mu(u)}$$

NLD and NLE

However, good pixel may become noise.



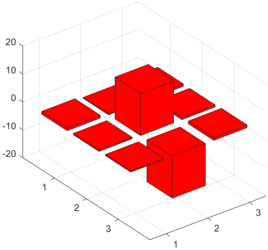
So, We add one term η before the infinity norm to reduce the step length.

$$NLD(f(u)) = f(u) + \eta \|(\nabla_w^+ f)(u)\|_\infty,$$
$$NLE(f(u)) = f(u) - \eta \|(\nabla_w^- f)(u)\|_\infty.$$

It is clear that if we fixed $\eta \in (0, 1)$, Erosion and Dilation will be slowed down.

η selection

- If u is a noise pixel, we hope η is large enough. From the picture at right hand side, we choose $\eta_{noise} = 0.5$.
- If u is NOT a noise pixel, we hope η is small enough. We choose $\eta_{noise} = 0.1$.
- If we choose $\eta_{noise} = 1$, $\|(\nabla_w^-)(f(u))\|_\infty$ will brings $f(u)$ from heaven to hell.
- We hope $\eta_{noise} \gg \eta_{good}$.



Algorithm (II)

Algorithm Nonlocal mean curvature flow

Initialize $f = f_0$
for $n = 1$ to maxstep **do**
  $\mathcal{K}_w f =$ **MeanCurvature**(f)
  $NLD = f + \eta\|\nabla^+ f\|_\infty$
  $NLE = f - \eta\|\nabla^- f\|_\infty$
  $f(\mathcal{K}_w f > 0) = NLD(\mathcal{K}_w f > 0)$
  $f(\mathcal{K}_w f < 0) = NLE(\mathcal{K}_w f < 0)$
end for
return f

Numerical result (I)

Noise image, psnr: 2.011317e+01

After 5 iteration, psnr: 2.766219e+01

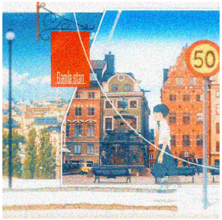
After 6 iteration, psnr: 2.786725e+01

After 7 iteration, psnr: 2.785649e+01

- Gaussian noise: $\sigma = 0.01$
- distance ≤ 2

Numerical result (II)

Noise image, psnr: 2.087611e+01



After 3 iteration, psnr: 2.638630e+01

After 5 iteration, psnr: 2.727393e+01

After 7 iteration, psnr: 2.700683e+01

- Gaussian noise: $\sigma = 0.01$
- distance ≤ 2

References

- El Chakik Abdallah, Elmoataz Abderrahim, Sadi Ahcene, On the Mean Curvature Flow on Graphs with Applications in Image and Manifold Processing.
- Abdallah El Chakik, Abddlerahim Elmoataz, Xavier Desquesnes, Mean curvature flow on graphs for image and manifold restoration and enhancement.

THE END

Thanks for listening!