Age of 911 Callers

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1. Given that the likelihood is

$$L(\Lambda) = exp\left\{-\sum_{a=0}^{110}\Lambda(a)\right\}\prod_{i=1}^{N}\Lambda(a_i)$$

If we factor $\Lambda(a) = \delta\lambda(a_i)$ then we can rewrite the likelihood function as

$$L(\Lambda) = \exp\{-\delta\}\,\delta^N\prod_{i=1}^{N}\lambda(a_i)$$

since $\sum_{a=0}^{110}\lambda(a) = 1$.

2. For this problem, I elected to use a Poisson(50) truncated at 110. In order to sum to 1, we need to divide the normal poisson by the sum of the densities from 0 to 110, and when I calculated this in R it turned out to just be 1, so we can just use a normal Poisson as our density for $\lambda(a)$.

With this new value for $\lambda(a)$ our likelihood function becomes:

$$L(\delta|a) = exp\{-\delta\}\,\delta^N\prod_{i=1}^{N}\frac{50^{a_i}e^{-50}}{a_i!}$$

3. If we treat $\prod_{i=1}^{N}\frac{50^{a_i}e^{-50}}{a_i!}$ as a constant then we can see that this likelihood appears to be the kernel of a Gamma($N+1$,1) so I propose we use a Gamma(5, 0.2) as the conjugate prior for δ.

If we multiply these together we get

$$exp\{-0.2\delta\}\,\delta^{5-1}exp\{-\delta\}\,\delta^N = exp\{-1.2\delta\}\,\delta^{N+5-1}$$

We recognize this as the kernel of the Gamma($N+5$, 1.2) distribution, and we see that the Gamma distribution is indeed conjugate to our likelihood. Therefore our posterior density is

$$f(\delta|N+5, 1.2) = \frac{1.2^{N+5}}{\Gamma(N+5)}exp\{-1.2\delta\}\,\delta^{N+5-1}$$

4. Now we let $\lambda(a)$ be the MN(1, p_0, p_1, ..., p_{110}). Then our likelihood becomes

$$L(\delta|a) = exp\left\{-\delta\right\}\delta^N \prod_{i=1}^{N} p_{a_i} = exp\left\{-\delta\right\}\delta^N p_0^{N_0} p_1^{N_1} \dots p_{110}^{N_{110}} \propto p_0^{N_0} p_1^{N_1} \dots p_{110}^{N_{110}}$$

   We recognize this as the kernel of the Dirichlet distribution and so we propose a symmetric Dirichlet distribution as a conjugate prior for this distribution. Let our density for our prior be

$$f(p_1, p_2, \dots, p_{110}|\alpha) = \prod_{i=1}^{110} p_i^{\alpha-1}$$

   Then multiplying this by our likelihood we get

$$\left(\prod_{i=1}^{110} p_i^{\alpha-1}\right) p_0^{N_0} p_1^{N_1} \dots p_{110}^{N_{110}} = p_0^{\alpha+N_0-1} p_1^{\alpha+N_1-1} \dots p_{110}^{\alpha+N_{110}-1}$$

   We can see that we are again left with the kernel of the Dirichlet distribution and so our posterior distribution for $p_1, \dots, p_{110}$ is

$$\frac{\Gamma\left(\sum_{i=1}^{110} \alpha + N_i\right)}{\prod_{i=1}^{110} \Gamma\left(\alpha + N_i\right)} \prod_{i=1}^{110} p_i^{N_i+\alpha-1}$$

5. The plot of the posterior density of δ and the posterior means of $p_0, \dots, p_{110}$ can be seen in Figures 1 and 2, respectively.

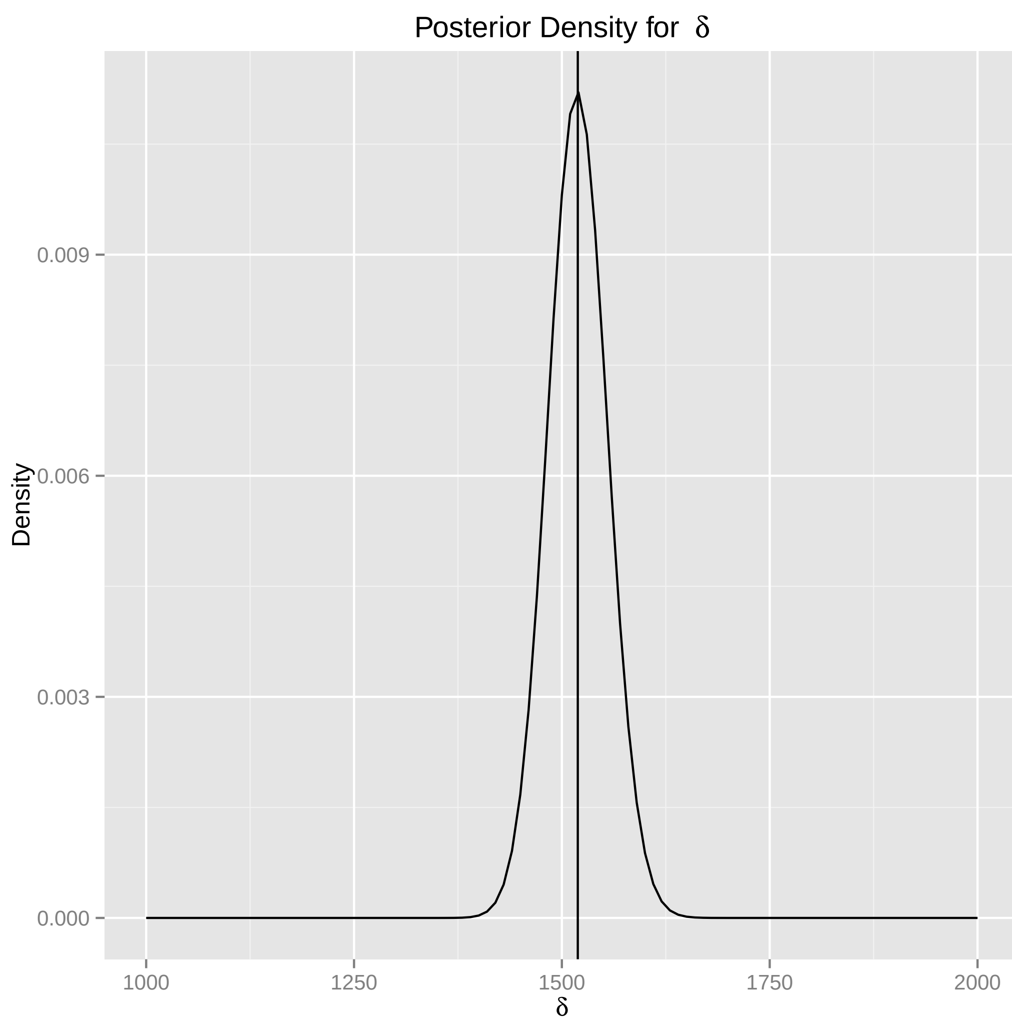

Figure 1: Posterior Density of δ

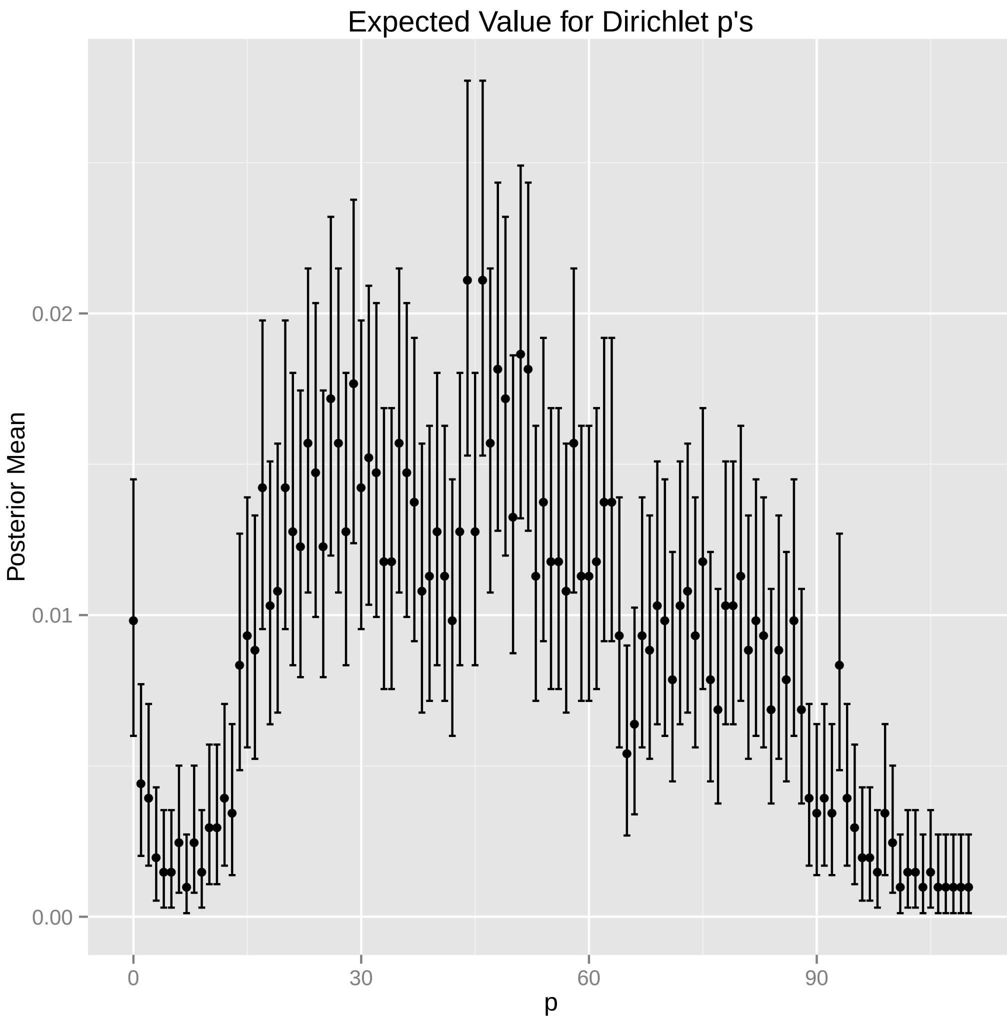

Figure 2: Means for $p_0, \ldots, p_{110}$