The Cyclic Lightbulb Game

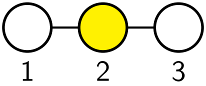
Claire Hickey, Justin Maier, Josh Wollenweber

University of Pittsburgh

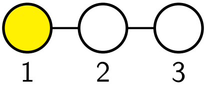
Math 1800,
January 31, 2019

The Lightbulb Game

We want to turn off the lightbulbs:

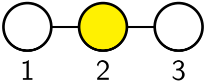


Pressing a bulb's switch changes the state of the bulb itself and the bulbs directly adjacent to it. For example, after pressing 1:

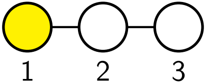


The Lightbulb Game

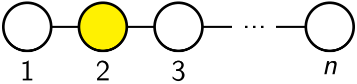
We want to turn off the lightbulbs:



Pressing a bulb's switch changes the state of the bulb itself and the bulbs directly adjacent to it. For example, after pressing 1:



We can extend this game to n lightbulbs:

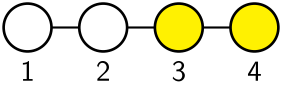


The central question: When can we always find a solution?

Converting the Lightbulb Game to a Problem in Linear Algebra

We can represent a lightbulb game like the one below with a system of linear equations:

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$



Converting the Lightbulb Game to a Problem in Linear Algebra

We can represent this in matrix form $Ac = b$, where b is starting configuration, and the matrix A is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Converting the Lightbulb Game to a Problem in Linear Algebra

We can represent this in matrix form $Ac = b$, where b is starting configuration, and the matrix A is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

To *determine* whether this system always has solution we will use the *determinant*.

Converting the Lightbulb Game to a Problem in Linear Algebra

We can represent this in matrix form $Ac = b$, where b is starting configuration, and the matrix A is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

To *determine* whether this system always has solution we will use the *determinant*.
Recall that the system $Ac = b$ has exactly one solution for every b if and only if $|A| \neq 0$.

Calculating the Determinant

To calculate the determinant we will make use of cofactor expansion. For example,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13}(-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Calculating the Determinant

We will also need the determinants of upper and lower triangular matrices:

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

Calculating the Determinant

We will also need the determinants of upper and lower triangular matrices:

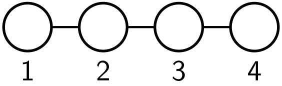
$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

This holds for n dimensional triangular matrices as well (can be shown by induction).

When is the Linear Lightbulb Game Solvable?

The matrix for the linear game on 4 lightbulbs looks like

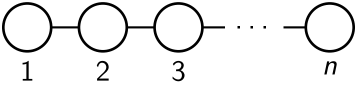
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$



When is the Linear Lightbulb Game Solvable?

On n lightbulbs, that is

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$



This is a tridiagonal matrix with 1s on the diagonal, which we denote by T_n.

When is the Linear Lightbulb Game Solvable?

The determinant of T_n is given by the recursive formula:

$$|T_n| = |T_{n-1}| - |T_{n-2}|$$

When is the Linear Lightbulb Game Solvable?

The determinant of T_n is given by the recursive formula:

$$|T_n| = |T_{n-1}| - |T_{n-2}|$$

Seeing that $|T_1| = 1$ and $|T_2| = 0$, we get

$$|T_n| = \begin{cases} 1 & n \equiv 0, 1 \pmod 6, \\ 0 & n \equiv 2, 5 \pmod 6, \\ -1 & n \equiv 3, 4 \pmod 6. \end{cases}$$

When is the Linear Lightbulb Game Solvable?

The determinant of T_n is given by the recursive formula:

$$|T_n| = |T_{n-1}| - |T_{n-2}|$$

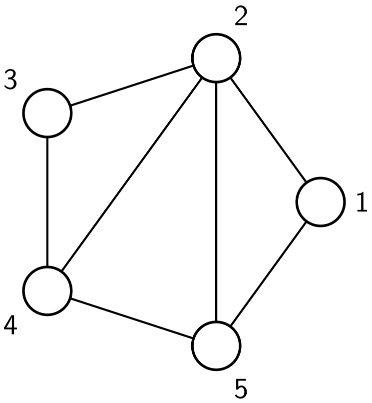
Seeing that $|T_1| = 1$ and $|T_2| = 0$, we get

$$|T_n| = \begin{cases} 1 & n \equiv 0, 1 \pmod 6, \\ 0 & n \equiv 2, 5 \pmod 6, \\ -1 & n \equiv 3, 4 \pmod 6. \end{cases}$$

So the linear game is always solvable, unless $n \equiv 2 \pmod 3$.

The Lightbulb Game on a Graph

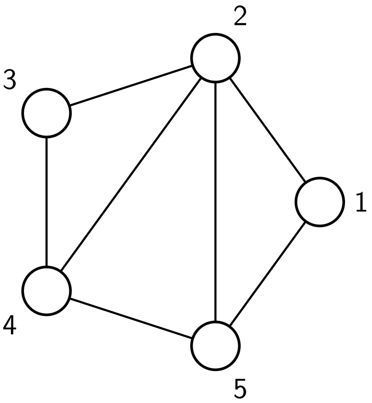
We can further generalize this game by using a graph to specify which switches affect which lightbulbs:



The Adjacency Matrix

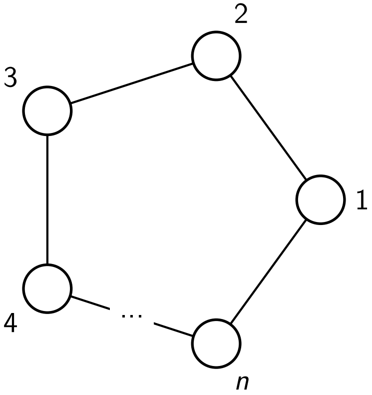
The matrix A in the linear system for this lightbulb game on a graph is represented by the adjacency matrix of the graph.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$



The Cyclic Lightbulb Game

We investigated the case where the graph is a cycle on n vertices.



The Recursive Formula

We will show the adjacency matrix A_n of the cyclic graph C_n, has its determinant given by

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1}$$

The Recursive Formula

A_n has the form:

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}$$

The Recursive Formula

We preform cofactor expansion along the first row of A_n. This will yield 3 terms.

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}$$

The Recursive Formula

First term

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix} \qquad (-1)^{1+1} \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 1 \end{vmatrix}$$

This is a tridiagonal matrix with ones on each diagonal. So this term is just $|T_{n-1}|$.

The Recursive Formula

Second term

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix} \qquad (-1)^{1+2} \underbrace{\begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 1 & 1 \end{vmatrix}}_{|B_{n-1}|}$$

We denote this matrix by B_{n-1}.

Third term

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix} \quad (-1)^{1+n} \underbrace{\begin{vmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 0 & 1 \end{vmatrix}}_{|C_{n-1}|}$$

We denote this matrix by C_{n-1}.

The Recursive Formula

So far we have

$$|A_n| = |T_{n-1}| - |B_{n-1}| + (-1)^{n+1}|C_{n-1}|$$

We now perform cofactor expansion on B_{n-1} and C_{n-1} as well.

We expand on the first column of B_{n-1}, giving us two terms.

First term

$$B_{n-1} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 1 & 1 \end{pmatrix} \qquad (-1)^{1+1} \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 1 \end{vmatrix}$$

This is the tridiagonal matrix T_{n-2}.

We expand on the first column of B_{n-1}, giving us two terms.

Second term

$$B_{n-1} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 1 & 1 \end{pmatrix} \quad (-1)^{n-1+1} \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 1 & 1 \end{vmatrix}$$

This is a lower triangular matrix with 1s along the diagonal. So its determinant is 1.

The Recursive Formula

We have obtained the following formula for $|B_{n-1}|$:

$$|B_{n-1}| = |T_{n-2}| + (-1)^n$$

We expand on the first column of C_{n-1}, giving us two terms.

First term

$$C_{n-1} = \begin{pmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 0 & 1 \end{pmatrix} \quad (-1)^{1+1} \begin{vmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 \end{vmatrix}$$

This is an upper triangular matrix with 1s along its diagonal. So its determinant is 1.

We expand on the first column of C_{n-1}, giving us two terms.

Second term

$$C_{n-1} = \begin{pmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 0 & 1 \end{pmatrix} \qquad (-1)^{n-1+1} \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 1 \end{vmatrix}$$

This is the tridiagonal matrix T_{n-2} again.

The Recursive Formula

For $|C_{n-1}|$ we have:

$$|C_{n-1}| = 1 + (-1)^n |T_{n-2}|$$

The Recursive Formula

We substitute into the formula for $|A_n|$:

The Recursive Formula

We substitute into the formula for $|A_n|$:

$$
\begin{aligned}
|A_n| &= |T_{n-1}| - |B_{n-1}| + (-1)^{n+1}|C_{n-1}| \\
&= |T_{n-1}| - (|T_{n-2}| + (-1)^n) + (-1)^{n+1}\left(1 + (-1)^n|T_{n-2}|\right) \\
&= |T_{n-1}| - |T_{n-2}| + 2(-1)^{n+1} + (-1)^{2n+1}|T_{n-2}| \\
&= |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1}
\end{aligned}
$$

The Recursive Formula

We substitute into the formula for $|A_n|$:

$$\begin{aligned}|A_n| &= |T_{n-1}| - |B_{n-1}| + (-1)^{n+1}|C_{n-1}| \\ &= |T_{n-1}| - (|T_{n-2}| + (-1)^n) + (-1)^{n+1}(1 + (-1)^n|T_{n-2}|) \\ &= |T_{n-1}| - |T_{n-2}| + 2(-1)^{n+1} + (-1)^{2n+1}|T_{n-2}| \\ &= |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1}\end{aligned}$$

But when is $|A_n| = 0$? We claim if and only if $n \equiv 0 \pmod 3$.

The Recursive Formula

Recall that $|T_n|$ is given by

$$|T_n| = \begin{cases} 1 & n \equiv 0, 1 \pmod 6, \\ 0 & n \equiv 2, 5 \pmod 6, \\ -1 & n \equiv 3, 4 \pmod 6. \end{cases}$$

The Recursive Formula

$(\Rightarrow)$: If $|A_n| = 0$, then

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0$$

($\Rightarrow$) : If $|A_n| = 0$, then

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0$$

But $|T_{n-1}|$ and $|T_{n-2}|$ only take on the values 0,1, -1 so...

$$|A_n| = \{1, -1, 0\} + \{2, -2, 0\} + \{2, -2\} = 0$$

$(\Rightarrow)$: If $|A_n| = 0$, then

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0$$

But $|T_{n-1}|$ and $|T_{n-2}|$ only take on the values 0,1, -1 so...

$$|A_n| = \{1, -1, 0\} + \{2, -2, 0\} + \{2, -2\} = 0$$

So we must have that $|T_{n-1}| = 0$.

$(\Rightarrow)$: If $|A_n| = 0$, then

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0$$

But $|T_{n-1}|$ and $|T_{n-2}|$ only take on the values 0,1, -1 so...

$$|A_n| = \{1, -1, 0\} + \{2, -2, 0\} + \{2, -2\} = 0$$

So we must have that $|T_{n-1}| = 0$.
But then $n \equiv 0, 3 \pmod 6$ i.e. $n \equiv 0 \pmod 3$.

The Recursive Formula

$(\Leftarrow)$: If $n \equiv 0 \pmod 3$ we have two possibilities for n mod 6:

The Recursive Formula

$(\Leftarrow)$: If $n \equiv 0 \pmod 3$ we have two possibilities for n mod 6:

1. $n \equiv 0 \pmod 6$, so $|T_{n-1}| = 0$, $|T_{n-2}| = -1$ and n is even.

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0 + 2 - 2 = 0.$$

The Recursive Formula

$(\Leftarrow)$: If $n \equiv 0 \pmod 3$ we have two possibilities for n mod 6:

1. $n \equiv 0 \pmod 6$, so $|T_{n-1}| = 0$, $|T_{n-2}| = -1$ and n is even.

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0 + 2 - 2 = 0.$$

2. $n \equiv 3 \pmod 6$, so $|T_{n-1}| = 0$, $|T_{n-2}| = 1$ and n is odd.

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0 - 2 + 2 = 0.$$