The Cyclic Lightbulb Game

Names

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Math 1800,
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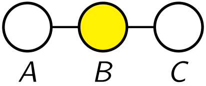
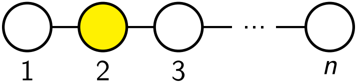
The Lightbulb Game

Suppose we have the lightbulbs. . .



Or more generally,



The Recursive Formula

We will show the adjacency matrix A_n of the cyclic graph C_n, has its determinant given by

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1}$$

The Recursive Formula

A_n has the form:

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}$$

The Recursive Formula

We preform cofactor expansion along the first row of A_n. This will yield 3 terms.

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix}$$

The first term

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix} \quad (-1)^{1+1} \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 1 \end{vmatrix}$$

This is a tridiagonal matrix with ones on each diagonal. So this term is just $|T_{n-1}|$.

The Recursive Formula

The second term

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix} \qquad (-1)^{1+2} \underbrace{\begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 1 & 1 \end{vmatrix}}_{|B_{n-1}|}$$

We denote this matrix by B_{n-1}.

The third term

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \dots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & 1 \end{pmatrix} \quad (-1)^{1+n} \underbrace{\begin{vmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 0 & 1 \end{vmatrix}}_{|C_{n-1}|}$$

We denote this matrix by C_{n-1}.

The Recursive Formula

So far we have

$$|A_n| = |T_{n-1}| - |B_{n-1}| + (-1)^{n+1}|C_{n-1}|$$

We now perform cofactor expansion on B_{n-1} and C_{n-1} as well.

The Recursive Formula

We expand on the first column of B_{n-1}, giving us two terms.

First term

$$B_{n-1} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 1 & 1 \end{pmatrix} \quad (-1)^{1+1} \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 1 \end{vmatrix}$$

This is the tridiagonal matrix T_{n-2}.

We expand on the first column of B_{n-1}, giving us two terms.

Second term

$$B_{n-1} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 1 & 1 \end{pmatrix} \quad (-1)^{n-1+1} \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 1 & 1 \end{vmatrix}$$

This is a lower triangular matrix with 1s along the diagonal. So its determinant is 1.

The Recursive Formula

We have obtained the following formula for $|B_{n-1}|$:

$$|B_{n-1}| = |T_{n-2}| + (-1)^n$$

We expand on the first column of C_{n-1}, giving us two terms.

First term

$$C_{n-1} = \begin{pmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 0 & 1 \end{pmatrix} \qquad (-1)^{1+1} \begin{vmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 \end{vmatrix}$$

This is an upper triangular matrix with 1s along its diagonal. So its determinant is 1.

We expand on the first column of C_{n-1}, giving us two terms.

Second term

$$C_{n-1} = \begin{pmatrix} 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 0 & 0 & 1 \end{pmatrix} \qquad (-1)^{n-1+1} \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 & 1 \end{vmatrix}$$

This is the tridiagonal matrix T_{n-2} again.

The Recursive Formula

For $|C_{n-1}|$ we have:

$$|C_{n-1}| = 1 + (-1)^n |T_{n-2}|$$

We substitute into the formula for $|A_n|$:

The Recursive Formula

We substitute into the formula for $|A_n|$:

$$\begin{aligned}
|A_n| &= |T_{n-1}| - |B_{n-1}| + (-1)^{n+1}|C_{n-1}| \\
&= |T_{n-1}| - (|T_{n-2}| + (-1)^n) + (-1)^{n+1}\left(1 + (-1)^n|T_{n-2}|\right) \\
&= |T_{n-1}| - |T_{n-2}| + 2(-1)^{n+1} + (-1)^{2n+1}|T_{n-2}| \\
&= |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1}
\end{aligned}$$

We substitute into the formula for $|A_n|$:

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But when is $|A_n| = 0$? We claim if and only if $n \equiv 0 \pmod 3$.

The Recursive Formula

Recall that $|T_n|$ is given by

$$|T_n| = \begin{cases} 1 & n \equiv 0, 1 \pmod{6}, \\ 0 & n \equiv 2, 5 \pmod{6}, \\ -1 & n \equiv 3, 4 \pmod{6}. \end{cases}$$

The Recursive Formula

$(\Rightarrow)$: If $|A_n| = 0$, then

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0$$

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But $|T_{n-1}|$ and $|T_{n-2}|$ only take on the values 0,1, -1 so...

$$|A_n| = \{1, -1, 0\} + \{2, -2, 0\} + \{2, -2\} = 0$$

The Recursive Formula

$(\Rightarrow)$: If $|A_n| = 0$, then

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So we must have that $|T_{n-1}| = 0$.

The Recursive Formula

$(\Rightarrow)$: If $|A_n| = 0$, then

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0$$

But $|T_{n-1}|$ and $|T_{n-2}|$ only take on the values 0,1, -1 so...

$$|A_n| = \{1, -1, 0\} + \{2, -2, 0\} + \{2, -2\} = 0$$

So we must have that $|T_{n-1}| = 0$.
But then $n \equiv 0, 3 \pmod 6$ i.e. $n \equiv 0 \pmod 3$.

The Recursive Formula

$(\Leftarrow)$: If $n \equiv 0 \pmod 3$ we have two possibilities for n mod 6:

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$(\Leftarrow)$: If $n \equiv 0 \pmod 3$ we have two possibilities for n mod 6:

1. $n \equiv 0 \pmod 6$, so $|T_{n-1}| = 0$, $|T_{n-2}| = -1$ and n is even.

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0 + 2 - 2 = 0.$$

The Recursive Formula

$(\Leftarrow)$: If $n \equiv 0 \pmod 3$ we have two possibilities for n mod 6:

1. $n \equiv 0 \pmod 6$, so $|T_{n-1}| = 0$, $|T_{n-2}| = -1$ and n is even.

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0 + 2 - 2 = 0.$$

2. $n \equiv 3 \pmod 6$, so $|T_{n-1}| = 0$, $|T_{n-2}| = 1$ and n is odd.

$$|A_n| = |T_{n-1}| - 2|T_{n-2}| + 2(-1)^{n+1} = 0 - 2 + 2 = 0.$$