MATH 235 HOMEWORK 11

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Problem 1

Question 0.1. *Let $\beta = \{v_1, ..., v_n\}$ be a basis for $\mathbb{R}^n$, and let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Show that A is invertible iff the set $\{Av_1, ..., Av_n\}$ is a basis.*

Let $\mathcal{L}_A$ be the left-multiplication transformation associated with A. Assume that A is invertible (iff $\mathcal{L}_A$ is invertible). By its surjectivity, $V = span(\mathcal{L}_A(\beta))$. Thus $\{Av_1, ..., Av_n\}$ generates V. Now, suppose that $\sum f_i\mathcal{L}_A(v_i) = 0$. Then $\sum f_i\mathcal{L}_A(v_i) = \mathcal{L}_A(\sum f_i v_i) = 0$ and it follows that $\sum f_i v_i \in ker(\mathcal{L}_A)$. By its injectivity, $\sum f_i v_i = 0$ and $f_1 = \cdots = f_n = 0$. Hence, there exists only the trivial representation of 0 as a linear combination of $Av_1, ..., Av_n$. Thus, $\{Av_1, ..., Av_n\}$ is a basis for V. Conversely, assume that $\{Av_1, ..., Av_n\}$ is a basis for V. We want to show that T is bijevtive. Since $\mathcal{L}_A$ is a linear operator on $\mathbb{R}^n$, it suffices to show that is is surjective. $Im(\mathcal{L}_A) = span(\mathcal{L}_A(\beta)) = span(\{Av_1, ..., Av_n\}) = V$(i.e., $\mathcal{L}_A$ is surjective). Thus, $\mathcal{L}_A$ is bijective and A is invertible.

Problem 2

Let $n \geq 2$ and $V = \mathcal{P}_n(\mathbb{R})$. Choose two distinct scalars $a, b \in \mathbb{R}$ and define the set $W_1 = \{p(x) \in V \mid p(a) = p(b) = 0\}$ and $W_2 = \{(x-a)(x-b)q(x) \in V \mid q(x) \in \mathcal{P}_{n-2}(\mathbb{R})\}$.

Question 0.2. *Show that W_1 and W_2 are subspaces of V.*

i) *Clearly, $0 \in W_1$. Let $p, q \in W_1$; let $c \in \mathbb{R}$. Then $p + cq \in W_1$ since $(p+cq)(a) = p(a) + cq(a) = 0 + c \cdot 0 = 0$ and $(p+cq)(b) = p(b) + cq(b) = 0 + c \cdot 0 = 0$. Thus, W_1 is a subspace of V.*

ii) *$(x-a)(x-b)0 = 0$ and $0 \in \mathcal{P}_{n-2}(\mathbb{R})$. Hence, $0 \in W_2$. Let $r, s \in W_2$; let $c \in \mathbb{R}$. Then $r = (x-a)(x-b)p(x)$ and $s = (x-a)(x-b)q(x)$, for some $p, q \in \mathcal{P}_{n-2}(\mathbb{R})$. Also, $r + cs \in W_2$ since $(p+cq)(x) = (x-a)(x-b)p(x) + c(x-a)(x-b)q(x) = (x-a)(x-b)p(x) + (x-a)(x-b)(cq)(x) = (x-a)(x-b)(p+cq)(x)$ and $p + cq \in \mathcal{P}_{n-2}(\mathbb{R})$ by the closure of $\mathcal{P}_{n-2}(\mathbb{R})$. Thus, W_2 is a subspace of V.*

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Question 0.3. *Show that $W_1 = W_2$.*
For $p(x) \in \mathcal{P}_n(\mathbb{R})$, $p(x) \in W_1$ iff a and b are roots of p iff $(x-a)$ and $(x-b)$ divide p iff $p(x) = (x-a)(x-b)q(x)$ for some $q(x) \in \mathcal{P}_{n-2}(\mathbb{R})$ iff $p(x) \in W_2$.

Question 0.4. *Show that $dim(W_1) = n-1$ and find an explicit basis for it.*
Since $W_1 = W_2$, $dim(W_1) = dim(W_2)$. Define $T \in \mathcal{L}(\mathcal{P}_{n-2}(\mathbb{R}), \mathcal{P}_n(\mathbb{R}))$ by $T(p(x)) = (x-a)(x-b)p(x)$. Notice that $Im(T) = W_2$. We want to show that $ker(T) = \{0\}$ so that $dim(W_2) = dim(\mathcal{P}_{n-2}(\mathbb{R})) - dim(ker(T)) = n-1$. $\forall\, r(x) \in \mathcal{P}_{n-2}(\mathbb{R})$, $T(r(x)) = (x-a)(x-b)r(x) = 0$ iff $r(x) = 0$. Thus, $ker(T) = \{0\}$, as desired. Now, $Im(T) = W_2 = span(T(\beta))$, where β is the standard ordered basis for $\mathcal{P}_{n-2}(\mathbb{R})$. Then $S = \{T(1), T(x), T(x^2), ..., T(x^{n-2})\} = \{x^2 - (a+b)x + ab, x^3 - (a+b)x^2 + abx, ..., x^n - (a+b)x^{n-1} + abx^{n-2}\}$ generates W_2. Also, since T is injective, by Section 2.1 Exercise 14(b), S is linearly independent. Thus, $\{x^2 - (a+b)x + ab, x^3 - (a+b)x^2 + abx, ..., x^n - (a+b)x^{n-1} + abx^{n-2}\}$ is a basis.

Problem 3

Question 0.5. *Let $a \in \mathbb{R}$ be a fixed scalar and let $T \in \mathcal{L}(\mathcal{P}_n(\mathbb{R}), \mathbb{R})$ given by $T(p(x)) = p(a)$. Show that T is a linear transformation, find bases and dimension for $ker(T)$ and $Im(T)$.*
Let $p, q \in \mathcal{P}_n(\mathbb{R})$; let $c \in \mathbb{R}$. Then $T((p+cq)(x)) = (p+cq)(a) = p(a) + cq(a) = T(p(x)) + cT(q(x))$. Thus, T is linear. $r \in ker(T)$ iff $r(p) = 0$ iff $(x-a)$ divides $r(x)$. Similar to Problem 2, $ker(T) = \{(x-a)q(x) : q(x) \in \mathcal{P}_{n-1}(\mathbb{R})\}$ and $dim(ker(T)) = (n-1+1) - 0 = n$, from which it follows that $dim(Im(T)) = (n+1) - n = 1$. Define $S \in \mathcal{L}(\mathcal{P}_{n-1}(\mathbb{R}), \mathcal{P}_n(\mathbb{R}))$ by $S(p) = (x-a)p$. Then $ker(T) = Im(S) = span(S(\gamma))$, where $\gamma = \{1, x, ..., x^{n-1}\}$. Then $\{S(1), S(x), ..., S(x^{n-1})\} = \{x-a, x^2 - ax, x^3 - ax^2, ..., x^n - ax^{n-1}\}$ generates $ker(T)$. Notice that if $a = 0$, then it is immediate that the set is linearly independent. Suppose that $a \neq 0$ and that the set is linearly dependent. Then $\exists f_i \in \mathbb{R}$, not all zero, such that $f_1(x-a) + \cdots + f_{n-1}(x^n - ax^{n-1}) = 0$. Combining like terms, we have $-af_1 + (f_1 - af_2)x + \cdots + (f_{n-2} - af_{n-1})x^{n-1} + f_{n-1}x^n = 0$. It follows that $f_1 = f_{n-1} = 0$, which forces $f_2 = \cdots = f_{n-2} = 0$. Thus, $\{x-a, x^2-ax, x^3-ax^2, ..., x^n-ax^{n-1}\}$ is a basis for $ker(T)$. Now, $q \in Im(T)$ iff q is relatively prime to the vectors in $ker(T)$. That is, $\forall\, p \in ker(T)$, $\exists r, s$ such that $p(x)r(x) + q(x)s(x) = 1$. At $x = a$, $q(a)s(a) = 1$ and it follows that $s(x)$ is a multiplicative inverse of $q(x)$. The existence of inverse of $q(x)$ implies that $q(x)$ is of degree 0. Then $Im(T) = \mathbb{R}$ and a basis for $Im(T)$ is $\{1\}$.

Question 0.6. *Let $T \in \mathcal{L}(V)$ such that T^2 is the zero transformation. Assume $dim(V) \geq 1$. Explain why T cannot be invertible. Show that the transformation $S \in \mathcal{L}(V)$ defined by $S(v) = v + T(v)$ is an isomorphism.*
Let $v \in Im(T)$ and suppose that $T(w) = v$, for some $w \in V$. Then $T(v) =$

$T(T(w)) = T^2(w) = 0$ and $v \in ker(T)$. Thus, $Im(T) \subseteq ker(T)$ and T is injective iff $Im(T) = \{0\}$. Hence, T is not invertible since T is injective iff T is the zero transformation, which is singular. Now, to show that S is an isomorphism, we need to show that it has an inverse. I claim that $S^{-1} \in \mathcal{L}(V)$ defined by $S^{-1}(v) = v - T(v)$, $\forall v \in V$ is the inverse of S. To verify the claim, $S^{-1}(S(v)) = S^{-1}(v + T(v)) = v + T(v) - T(v + T(v)) = v + T(v) - T(v) - T^2(v) = v + 0 - 0 = v$. Also, $\forall v \in V$, $SS^{-1}(v) = S(v - T(v)) = S(v) - ST(v) = v + T(v) - S(T(v)) = v + T(v) - T(v) + T^2(v) = v + 0 + 0 = v$. Thus, $S^{-1}S = SS^{-1} = I_V$ and the existence of S^{-1} implies that S is an isomorphism.

Problem 4

Question 0.7. *Suppose that $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ has two distinct eigenvalues, λ_1 and λ_2, and that $dim(E_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.*
By Theorem 5.7, $dim(E_{\lambda_2}) \geq 1$. But $dim(E_{\lambda_1}) + dim(E_{\lambda_2}) = (n-1) + dim(E_{\lambda_2}) \leq n$, from which it follows that $dim(E_{\lambda_2}) = 1$. Since E_{λ_1} and E_{λ_2} are T-invariant subspaces of V such that $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$ and $dim(V) = n = dim(E_{\lambda_1}) + dim(E_{\lambda_2})$, $V = E_{\lambda_1} \bigoplus E_{\lambda_2}$. Thus, V is the direct sum of the eigenspaces of T and by Theorem 5.11, T is diagonalizable.

Question 0.8. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Show that $p_{A^2}(t^2) = (-1)^n p_A(t)p_A(-t)$.*
By definition, $p_A(-t) = det(A - (-t)I) = det(A + tI)$. Then $p_{A^2}(t^2) = det(A - t^2I) = det(A - (tI)^2) = (-1)^n det((A - tI)(A + tI)) = (-1)^n det(A - tI)det(A + tI) = (-1)^n p_A(t)p_A(-t)$, as desired. (since the leading coefficient of $det(A - tI)det(A + tI)$ is $(-1)^n(-1)^n = (-1)^{2n} = 1^n$, we need to multiply $det(A - t^2I)$ by $(-1)^n$.)

Problem 5

Question 0.9. *Let V be an inner product space and let W be a subspace of V. For any vector $v \in V$ and $u \in W$, show that $||v - Proj_W(v)|| \leq ||v - u||$. If $d = ||v - Proj_W(v)||$, show that $||v - u|| = d$ only when $u = Proj_W(v)$.*
Let $u \in W$. Notice that $v = Proj_W(v) + (v - Proj_W(v))$ and $v - Proj_W(v) \in W^\perp$. Then $||v - u||^2 = ||(Proj_W(v) + (v - Proj_W(v))) - u||^2 = ||(Proj_W(v) - u) + (v - Proj_W(v))||^2 = ||Proj_W(v) - u||^2 + ||v - Proj_W(v)||^2 \geq ||v - Proj_W(v)||^2$. Thus $||v - Proj_W(v)|| \leq ||v - u||$. Also, $||v - u||^2 = ||Proj_W(v) - u||^2 + ||v - Proj_W(v)||^2$ implies that $||v - Proj_W(v)|| = ||v - u||$ iff $u = Proj_W(v)$.

Question 0.10. *Find the best polynomial $p(x)$ approximation of degree 3 to the function $f(x) = sin(x)$in the interval $[-\pi, \pi]$. Use a graphing software to provide a graph for $p(x)$ and $sin(x)$*
Define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{P}_3(\mathbb{R})$ by $\langle p, q\rangle = \int_{-\pi}^{\pi} p(x)q(x)\,dx$, for any $p, q \in \mathcal{P}_3(\mathbb{R})$. We want to find an orthonormal basis for the inner product space endowed

with the inner product defined above. Applying the Gram-Schmidt process to the standard ordered basis for $\mathcal{P}_3(\mathbb{R})$, we have

$$v_1 = 1$$
$$v_2 = x - \frac{\langle x,1\rangle}{\langle 1,1\rangle}\cdot 1 = x - \frac{\int_{-\pi}^{\pi} x\,dx}{\int_{-\pi}^{\pi} 1\,dx} = x - \frac{0}{2\pi} = x$$
$$v_3 = x^2 - \frac{\langle x^2,1\rangle}{\langle 1,1\rangle} - \frac{\langle x^2,x\rangle}{\langle x,x\rangle}x = x^2 - \frac{\int_{-\pi}^{\pi} x^2\,dx}{\int_{-\pi}^{\pi} 1\,dx} - \frac{\int_{-\pi}^{\pi} x^3\,dx}{\int_{-\pi}^{\pi} x^2\,dx}x = x^2 - \frac{\frac{2}{3}\pi^3}{2\pi} - \frac{0}{\frac{2}{3}\pi^3}x = x^2 - \frac{1}{3}\pi^2$$
$$v_4 = x^3 - \frac{\langle x^3,1\rangle}{\langle 1,1\rangle} - \frac{\langle x^3,x\rangle}{\langle x,x\rangle}x - \frac{\langle x^3, x^2-\frac{1}{3}\pi^2\rangle}{\langle x^2-\frac{1}{3}\pi^2, x^2-\frac{1}{3}\pi^2\rangle}(x^2-\frac{1}{3}\pi^2) = x^3 - \frac{\int_{-\pi}^{\pi} x^3\,dx}{\int_{-\pi}^{\pi} 1\,dx} - \frac{\int_{-\pi}^{\pi} x^4\,dx}{\int_{-\pi}^{\pi} x^2\,dx}x -$$
$$\frac{\int_{-\pi}^{\pi} x^5 - \frac{1}{3}\pi^2x^3\,dx}{\int_{-\pi}^{\pi}(x^2-\frac{1}{3}\pi^2)^2\,dx}(x^2-\frac{1}{3}\pi^2) = x^3 - \frac{0}{2\pi} - \frac{\frac{2}{5}\pi^5}{\frac{2}{3}\pi^3}x - \frac{0}{\frac{8}{45}\pi^5}(x^2-\frac{1}{3}\pi^2) = x^3 - \frac{3}{5}\pi^2 x$$

Now, we need to normalize the basis vectors.

$$e_1 = \frac{1}{||1||}\cdot 1 = \frac{1}{\sqrt{2\pi}}$$
$$e_2 = \frac{1}{||x||}x = \frac{1}{\sqrt{\frac{2\pi^3}{3}}}\cdot x = \sqrt{\frac{3}{2\pi^3}}x$$
$$e_3 = \frac{1}{||x^2-\frac{1}{3}\pi^2||}(x^2-\frac{1}{3}\pi^2) = \frac{1}{\sqrt{\frac{8\pi^5}{45}}}(x^2-\frac{1}{3}\pi^2) = \sqrt{\frac{45}{8\pi^5}}(x^2-\frac{1}{3}\pi^2)$$
$$e_4 = \frac{1}{||x^3-\frac{3}{5}\pi^2x||}(x^3-\frac{3}{5}\pi^2x) = \frac{1}{\sqrt{\frac{8\pi^7}{175}}}(x^3-\frac{3}{5}\pi^2x) = \sqrt{\frac{175}{8\pi^7}}(x^3-\frac{3}{5}\pi^2x)$$

Thus, $\gamma = \{\frac{1}{\sqrt{2\pi}}, \sqrt{\frac{3}{2\pi^3}}x, \sqrt{\frac{45}{8\pi^5}}(x^2-\frac{1}{3}\pi^2), \sqrt{\frac{175}{8\pi^7}}(x^3-\frac{3}{5}\pi^2x)\}$ *is an orthonormal basis for* $\mathcal{P}_3(\mathbb{R})$. *The best polynomial approximation of* $sin(x)$ *of degree* 3 *is given by* $sin(x) \approx \sum\langle sin(x), p_i\rangle p_i$, *where* $p_i \in \gamma$. *Then* $sin(x) \approx \langle sin(x), \frac{1}{\sqrt{2\pi}}\rangle\cdot\frac{1}{\sqrt{2\pi}} + \langle sin(x), \sqrt{\frac{3}{2\pi^3}}x\rangle\cdot\sqrt{\frac{3}{2\pi^3}}x + \langle sin(x), \sqrt{\frac{45}{8\pi^5}}(x^2-\frac{1}{3}\pi^2)\rangle\cdot\sqrt{\frac{45}{8\pi^5}}(x^2-\frac{1}{3}\pi^2) + \langle sin(x), \sqrt{\frac{175}{8\pi^7}}(x^3-\frac{3}{5}\pi^2x)\rangle\cdot\sqrt{\frac{175}{8\pi^7}}(x^3-\frac{3}{5}\pi^2x) = \frac{1}{2\pi}(\int_{-\pi}^{\pi} sin(x)\,dx) + \frac{3}{2\pi^3}(\int_{-\pi}^{\pi} xsin(x)\,dx) + \frac{45}{8\pi^5}(\int_{-\pi}^{\pi}(x^2-\frac{1}{3}\pi^2)sin(x)\,dx) + \frac{175}{8\pi^7}(\int_{-\pi}^{\pi}(x^3-\frac{3}{5}\pi^2x)\,dx) = 0 + \frac{3}{2\pi^3}\cdot 2\pi\cdot x + 0 + \frac{175}{8\pi^7}\cdot\frac{4}{5}\pi(\pi^2-15)(x^3-\frac{3}{5}\pi^2x) = \frac{3}{\pi^2}x + \frac{35}{2\pi^6}(x^3-\frac{3}{5}\pi^2x)$, *since the integrands of the first and third integrals are odd. Thus,* $sin(x) \approx \frac{3}{\pi^2}x + \frac{35}{2\pi^6}(\pi^2-15)(x^3-\frac{3}{5}\pi^2x)$.

Problem 6

Let V *be an inner product space. A linear transformation* $T \in \mathcal{L}(V)$ *is called isometry if for any* $u, v \in V$, $\langle u,v\rangle = \langle T(u), T(v)\rangle$.

Question 0.11. *Show that a linear transformation* T *is an isometry if and only if for any orthonormal basis* $\beta, T(\beta)$ *is orthonormal.*

Suppose that T *is an isometry and that* β *is an orthonormal basis for* V. *Then for any* $v_i, v_j \in \beta, \langle T(v_i), T(v_j)\rangle = \langle v_i, T^*T(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij}$, *since* T *is an isometry iff* $T^*T = I$. *That is,* $T(\beta)$ *is orthonormal. Conversely, suppose that* β *is an orthonormal basis and* $T(\beta)$ *is an orthonormal subset of* V. *Then* $\forall\, x \in V$, $x = \sum\langle x, v_i\rangle v_i$ *and* $\langle x, x\rangle = \langle\sum\langle x, v_i\rangle v_i, \sum\langle x, v_j\rangle v_j\rangle = \sum\langle x, v_i\rangle\sum\langle x, v_j\rangle\delta_{ij} = \sum_i |\langle x, v_i\rangle|^2$. *Also, since* $\langle T(v_i), T(v_j)\rangle = \delta_{ij}$, $\langle T(x), T(x)\rangle = \langle T(\sum\langle x, v_i\rangle v_i), T(\sum\langle x, v_j\rangle v_j)\rangle = \sum\langle x, v_i\rangle\sum\langle x, v_j\rangle\langle T(v_i), T(v_j)\rangle = \sum_j |\langle x, v_j\rangle|^2$. *Thus,* $||x||^2 = ||T(x)||^2$ *iff* T *is an isometry, by Theorem* 6.18.

Question 0.12. *Show that an isometry is an isomorphism.*
Let $v \in ker(T)$. Then $\langle v, v\rangle = \langle T(v), T(v)\rangle = \langle 0, 0\rangle = 0$. By the property of inner products, it implies that $v = 0$. Thus, T is injective. Now, we need to show that T is surjective. For any orthonormal basis β for V, $Im(T) = span(T(\beta))$, but $span(T(\beta)) = V$, as shown in the preceding question. Thus, T is surjective, from which it follows that T is bijective iff T is invertible.

Question 0.13. *If T is an isometry, show that $det(T) = \pm 1$.*
By definition, if T is an isometry, then $TT^* = T^*T = I$. Then $det(I) = det(TT^*)$ $= det(T)det(T^*) = det(T)det(T) = (det(T))^2 = 1$, from which it follows that $det(T) = \pm 1$.

Question 0.14. *If T is an isometry over a real vector space and $\lambda \in \mathbb{R}$ is an eigenvalue, show that $\lambda = \pm 1$*
Let λ be an eigenvalue of an isometry T; let v be an eigenvector corresponding to λ. Then $\langle T(v), T(v)\rangle = \langle \lambda v, \lambda v\rangle = \lambda\bar{\lambda}\langle v, v\rangle = |\lambda|^2\langle v, v\rangle$. But also $\langle T(v), T(v)\rangle = \langle v, v\rangle$. We have $|\lambda|^2\langle v, v\rangle = \langle v, v\rangle$. Thus, $\lambda = \pm 1$.