$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)} \quad \text{BAYES FORMULA}$$

RANDOM VARIABLES

X = RANDOM VARIABLE

$X : \Omega \to \mathbb{R}$ (REAL NUMBER LINE OR A SUBSPACE)

IF X TAKES FINITE OR COUNTABLY INFINITE NUMBER OF VALUES, IT IS DISCRETE. OTHERWISE, X IS CONTINUOUS.

$X = \{x_1, x_2, \dots, x_n, \dots\}$ $\qquad p_i \geq 0 \quad \forall i$

$p_i = P(X = x_i),\ i = 1, 2, \dots$ $\qquad \sum_{i=1}^{\infty} p_i = 1$

$F(x) = P(X \leq x)$ IS A CUMULATIVE DISTRIBUTION FUNCTION

$F(-\infty) = 0,\quad F(\infty) = 1$

$F(x)$ IS RIGHT CONTINUOUS, NONDECREASING

$P(X \in (a, b]) = F(b) - F(a)$

$P(X = x) = F(x) - F(x-0) = F(x) - \lim_{\delta \to 0^-} F(x+\delta)$

FOR A CONTINUOUS RANDOM VARIABLE, $F(x)$ IS DIFFERENTIABLE.

$f(x) = F'(x)$ IS THE PROBABILITY DENSITY FUNCTION

$f(x) \geq 0,\ f(-\infty) = f(+\infty) = 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1, \quad F(x) = \int_{-\infty}^{x} f(z)\,dz, \quad P(a \leq x \leq b) = \int_a^b f(x)\,dx$$

COVARIANCE IS A MEASURE OF LINEAR DEPENDENCE. * DEPENDS ON UNITS

IF X_1 AND X_2 ARE INDEPENDENT, THEN $\mathrm{COV}(X_1, X_2) = 0$. HOWEVER, ONE MAY HAVE $\mathrm{COV}(X_1, X_2) = 0$, BUT X_1 AND X_2 ARE FUNCTIONALLY DEPENDENT (OR DEPENDENT IN PROBABILISTIC SENSE).

CORRELATION COEFFICIENT * UNITLESS STATISTIC

$$\rho(X_1, X_2) = \frac{\mathrm{COV}(X_1, X_2)}{\sqrt{\mathrm{VAR}(X_1)} \cdot \sqrt{\mathrm{VAR}(X_2)}}, \quad 0 \le |\rho(X_1, X_2)| \le 1$$

$$\Updownarrow$$

$$-1 \le \rho(X_1, X_2) \le 1$$

IF $\rho(X_1, X_2) = \pm 1$, THEN $X_1 = aX_2 + b$ WHERE $a > 0$ IF $\rho = 1$, $a < 0$ IF $\rho = -1$.

THE CHARACTERISTIC FUNCTION. ~~THE MOME~~

THE MOMENT GENERATING FUNCTION

THE MOMENT GENERATING FUNCTION OF X : $M_X(t) = E[e^{tX}]$

IF X IS CONTINUOUS, $M_X(t) = \int_{-\infty}^{\infty} e^{tx} p(x)\,dx$ (PROVIDED THE INTEGRAL EXISTS)

$$M_X'(t) = \int_{-\infty}^{\infty} x e^{tx} p(x)\,dx \xrightarrow{t=0} M_X'(0) = \int_{-\infty}^{\infty} x p(x)\,dx = E[X]$$

$$E[X^k] = \frac{d^k}{dt^k} M_X(t) \Big|_{t=0}$$

THE CHARACTERISTIC FUNCTION OF X

$$\varphi_X(\omega) = E[e^{i\omega X}]$$

$$E[X] = \sum_{x=0}^{K} x \cdot \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}} = \sum_{x=0}^{K} \frac{\cancel{x} \cdot \frac{M}{\cancel{x}}\binom{M-1}{x-1}\binom{N-M}{K-x}}{\frac{N}{K}\binom{N-1}{K-1}}$$

$$= \frac{MK}{N} \sum_{x=0}^{K} \frac{\binom{M-1}{x-1}\binom{(N-1)-(M-1)}{(K-1)-(x-1)}}{\binom{N-1}{K-1}}$$

LET $x-1=z$

$$= \frac{MK}{N} \underbrace{\sum_{z=0}^{K} P(X=z \mid N-1, M-1, K-1)}_{=1}$$

"AVERAGE ~~PROPORTION OF SAMPLE WITH X RED BALLS~~" PROPORTION OF RED BALLS ~~NUMBER OF~~ IN SAMPLE" $= \frac{MK}{N}$

$$\rightarrow \frac{E[X]}{K} = \frac{M}{N}$$

$$VAR(X) = \frac{KM}{N} \cdot \frac{(N-M)(N-K)}{N(N-1)}$$

BINOMIAL DISTRIBUTION

CONSIDER n INDEPENDENT, IDENTICAL TRIALS. EACH TRIAL HAS TWO OUTCOMES, SUCCESS OR FAILURE.

PROBABILITY OF SUCCESS IS p.
PROBABILITY OF FAILURE IS $q = 1-p$

$X = \{$NUMBER OF SUCCESSES IN n TRIALS$\}$, $X = 0, \dots, n$

$$P(X=x \mid n, p) = \binom{n}{x} p^x q^{n-x}$$

$\xi_i = \{$NUMBER OF SUCCESSES IN i^{th} TRIAL$\}$, $i = 1, \dots, n$

$\xi_i = \{0, 1\}$ IS THE BERNOULLI VARIABLE. , $\xi_1, \xi_2, \dots$ ARE INDEPENDENT, IDENTICALLY DISTRIBUTED

DERIVATION OF POISSON DISTRIBUTION $\nearrow n = \frac{t_0}{\Delta t}$

t_0 IS FIXED TIME INTERVAL. $t_0 = n \cdot \Delta t$ WHERE $\Delta t = \{$ A SMALL TIME INTERVAL

$n = \{$ THE NUMBER OF TIME INTERVALS

WITHIN Δt EITHER ONE EVENT CAN OCCUR OR NONE (BY ASSUMPTION). TIME INTERVALS ARE INDEPENDENT.

p = THE PROBABILITY THAT EVENT OCCURS WITHIN TIME INTERVAL Δt

$X = \{$ THE NUMBER OF EVENTS WITH OCCURED WITHIN n TIME PERIODS $\}$

$X \sim$ BINOMIAL(n, p)

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E[X] = np = t_0 \cdot \frac{p}{\Delta t}$$

AS $\Delta t \to 0$, $n \to \infty$, $np \to \lambda$, $E[X] \to \lambda$

FIND $\varphi_n(t)$, CHARACTERISTIC FUNCTION OF X.

$$\varphi_n(t) = E[e^{itX}] = \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^{it})^x (1-p)^{n-x}$$

$$= [pe^{it} + 1 - p]^n$$

$$= [1 + p(e^{it} - 1)]^n$$

$$\lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} \left(1 + \frac{\lambda}{n}(e^{it}-1)\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{\lambda}{n}(e^{it}-1)\right)^{\frac{n}{\lambda(e^{it}-1)} \cdot \lambda(e^{it}-1)}$$

RECALL $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{\infty} = e$

~~$= \lim_{n\to\infty} [1 +$~~ $= e^{\lambda(e^{it}-1)} = \varphi_{POISSON}(t)$

GEOMETRIC DISTRIBUTION

NEGATIVE BINOMIAL WITH $r=1$

$X = \{$ THE NUMBER OF A TRIAL WHERE THE FIRST SUCCESS OCCURS $\}$

$$P(X=x \mid p) = p(1-p)^{x-1} \quad , \; x = 1,2,3,\ldots$$

OR

$$P(Y=y \mid p) = p(1-p)^{y} \quad , \; y = 0,1,2,\ldots$$

$$E[X] = \frac{1-p}{p} + 1 = \frac{1}{p} \quad , \; VAR(X) = \frac{1-p}{p^2}$$

MEMORYLESS PROPERTY

$$P(X>s \mid P>t) = P(X>s-t)$$

PROBABILITY OF GETTING $(s-t)$ ADDITIONAL FAILURES AFTER t FAILURES HAVE OCCURRED IS THE SAME AS TO GET $(s-t)$ FAILURES.

$$P(\underbrace{X>s}_{A} \mid \underbrace{X>t}_{B}) = \frac{P(A \cap B)}{P(B)} = \frac{P(X>s)}{P(X>t)} = \frac{\sum_{x=s}^{\infty} p(1-p)^{x-1}}{\sum_{x=t}^{\infty} p(1-p)^{x-1}} = \frac{\frac{(1-p)^s p}{1-(1-p)}}{\frac{(1-p)^t p}{1-(1-p)}}$$

$$\rightarrow \; = \frac{(1-p)^s}{(1-p)^t} = (1-p)^{s-t} = P(X>s-t)$$

CONTINUOUS DISTRIBUTIONS

① UNIFORM DISTRIBUTION

$$f(x|a,b) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0 & \text{OTHERWISE} \end{cases}$$

$$E[X] = \int_a^b x f(x|a,b)\, dx = \frac{1}{b-a}\int_a^b x\, dx = \frac{b+a}{2}$$

$$Var(X) = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \cdot \frac{1}{b-a}\, dx = \frac{(b-a)^2}{12}$$

$$\phi_X(t) = \int_a^b e^{itx} \cdot \frac{1}{b-a}\, dx = \frac{1}{it(b-a)} \cdot \left(e^{itb} - e^{ita}\right)$$

② GAMMA DISTRIBUTION

RELIABILITY, SURVIVAL ANALYSIS

RECALL GAMMA FUNCTION:

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx, \quad \alpha > 0$$

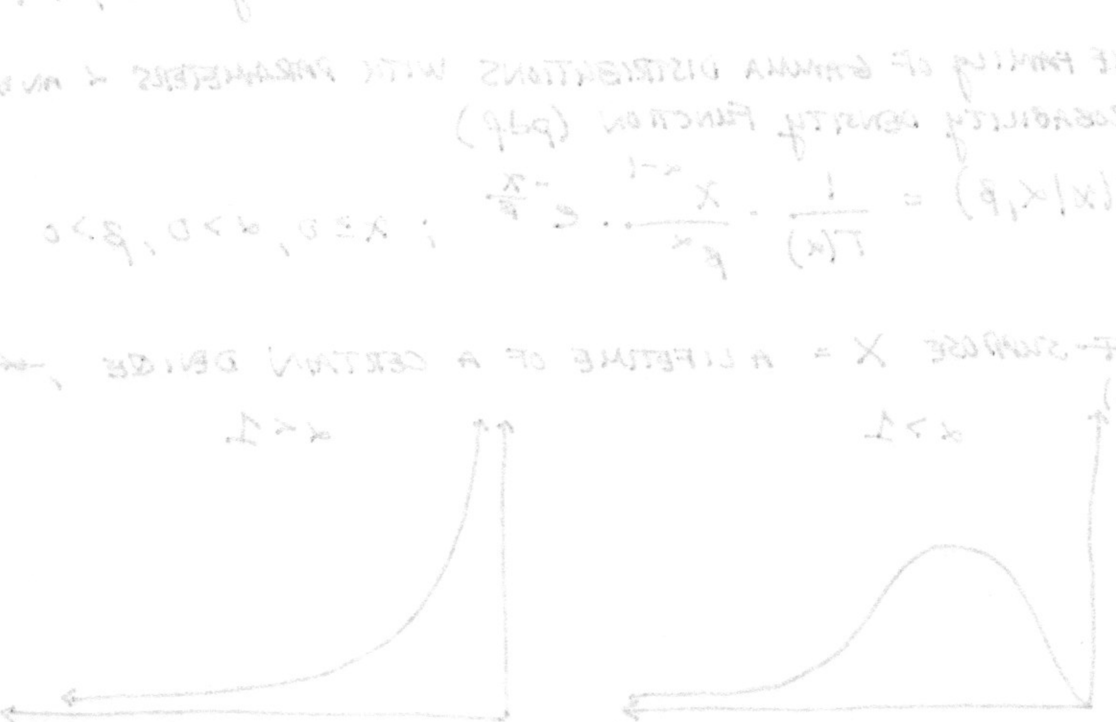
IF $\alpha = n \in \mathbb{Z}$, $\Gamma(n) = (n-1)!$

FOR ANY $\alpha > 0$, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

THE FAMILY OF GAMMA DISTRIBUTIONS WITH PARAMETERS α AND β HAS THE PROBABILITY DENSITY FUNCTION (pdf)

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)} \cdot \frac{x^{\alpha-1}}{\beta^\alpha} \cdot e^{-\frac{x}{\beta}} \; ; \; x \geq 0, \alpha > 0, \beta > 0$$

~~LET~~ SUPPOSE X = A LIFETIME OF A CERTAIN DEVICE, ~~...~~



IF X_1 AND X_2 ARE INDEPENDENT AND THEIR CHARACTERISTIC FUNCTIONS ARE $\varphi_1(t)$ AND $\varphi_2(t)$, THEN THE CHARACTERISTIC FUNCTION $\varphi(t)$ OF $X = X_1 + X_2$ IS $\varphi(t) = \varphi_1(t) + \varphi_2(t)$. THE CONVERSE IS TRUE; I.E. IF $X = X_1 + X_2$ AND $\varphi(t) = \varphi_1(t) \cdot \varphi_2(t)$, THEN X_1 AND X_2 ARE INDEPENDENT.

$X_1 \sim \text{GAMMA}(\alpha_1, \beta) \qquad \varphi_1(t) = (1 - i\beta t)^{-\alpha_1}$

$X_2 \sim \text{GAMMA}(\alpha_2, \beta) \qquad \varphi_2(t) = (1 - i\beta t)^{-\alpha_2}$

$\Rightarrow X = X_1 + X_2$

X_1, X_2 ARE INDEPENDENT $\longrightarrow \varphi(t) = (1 - i\beta t)^{-(\alpha_1 + \alpha_2)}$

$\Rightarrow X \sim \text{GAMMA}(\alpha_1 + \alpha_2, \beta)$ THE SUM OF INDEPENDENT GAMMA VARIABLES (WITH THE SAME PARAMETER β) IS GAMMA.

PARTICULAR CASES

① $\alpha = 1$ EXPONENTIAL DISTRIBUTION

$$f(x|\beta) = \frac{1}{\beta} \cdot e^{-x/\beta}, \quad x > 0, \ \beta > 0$$

HAS THE MEMORYLESS PROPERTY: $P(X > s \mid X > t) = P(X > s - t)$, $s > t$

NOTE THAT $P(X > a) = \int_a^\infty \frac{1}{\beta} e^{-x/\beta} dx = e^{-a/\beta}$

$$\rightarrow P(X > s \mid X > t) = \frac{P(X > s \cap X > t)}{P(X > t)} = \frac{P(X > s)}{P(X > t)} = \frac{e^{-s/\beta}}{e^{-t/\beta}} =$$

$$e^{-(s-t)/\beta} = P(X > s - t)$$

$$X \sim N(\mu, \sigma^2) \quad \longleftrightarrow \quad Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$P(Z \le z) = P\left(\frac{X-\mu}{\sigma} \le z\right) = P(X \le \sigma \cdot z + \mu)$$

$$= \int_{-\infty}^{\sigma z + \mu} \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

$$\downarrow \; t = \frac{x-\mu}{\sigma}$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t/2} \, dt$$

$$\rightarrow X = \mu + \sigma Z$$

$$E[Z] = \int_{-\infty}^{\infty} z \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}}_{\text{P.D.F. OF } N(0,1)} \, dz$$

$$= 0.$$

$$\text{VAR}(Z) = \int_{-\infty}^{\infty} (z-0)^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2} \, dz$$

$$= 2\int_{0}^{\infty} z^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2} \, dz$$

$$\frac{z^2}{2} = t \quad \Leftrightarrow \quad z = (2t)^{1/2}$$

$$dz = \frac{dt}{\sqrt{2t}}$$

$$= 2\int_{0}^{\infty} 2t \cdot \frac{e^{-t}}{\sqrt{2\pi}} \cdot \frac{dt}{\sqrt{2t}}$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} t^{1/2} e^{-t} \, dt$$

$$= \frac{2}{\sqrt{\pi}} \cdot \Gamma(3/2) = 1.$$

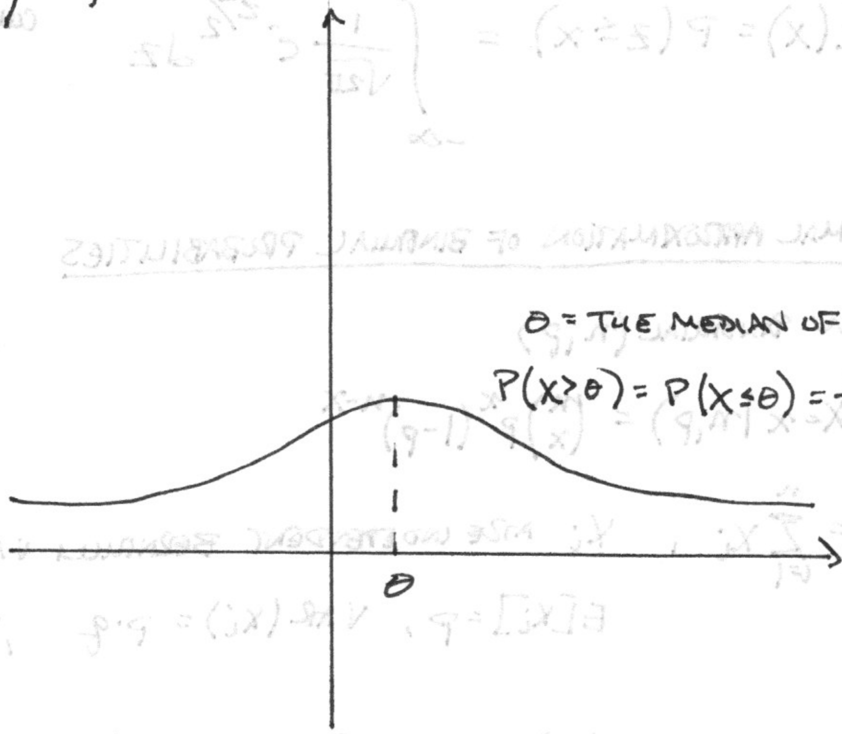
CAUCHY DISTRIBUTION

$$f(x\mid\theta,\sigma) = \frac{1}{\pi}\left(1+\frac{(x-\theta)^2}{\sigma^2}\right)^{-1}, \quad -\infty < x < \infty$$

$$\int_{-\infty}^{\infty} f(x\mid\theta,\sigma^2)\,dx = 1$$

$E[\cancel{x}] = \infty$, DOES NOT EXIST

$VAR(X)$ = DOES NOT EXIST



θ = THE MEDIAN OF X

$P(X>\theta) = P(X\le\theta) = \frac{1}{2}$

THE BETA DISTRIBUTION

$$f(x\mid\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\cdot x^{\alpha-1}(1-x)^{\beta-1}, \quad 0\le x<1;\ \alpha,\beta>0$$

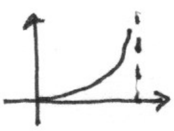
$B(\alpha,\beta)$ IS THE BETA FUNCTION

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

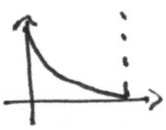
$$E[X] = \frac{\alpha}{\alpha+\beta}, \quad VAR(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

P.D.F.S

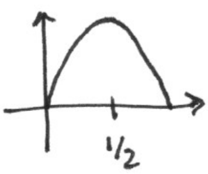
$\alpha>1,\ \beta=1$



$\alpha=1,\ \beta>1$

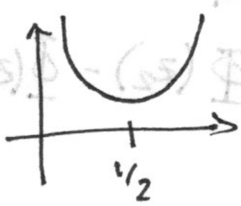


$\alpha=\beta>1$

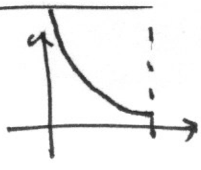


SYMMETRIC, UNI-MODAL

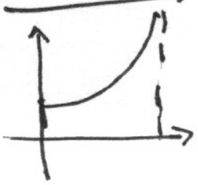
$\alpha=\beta<1$



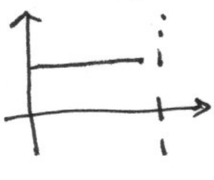
$\alpha<1,\ \beta=1$



$\alpha=1,\ \beta<1$



$\alpha=\beta=1$



UNIFORM(0,1)