Math 324 Winter 2019: Review Questions

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1. Review the Question 4 from the first midterm. Make sure you understand the solution.
2. Re-work Quiz 1 (Triple Integral Set-up Practice).
3. Evaluate the integral by using an appropriate change of variables.

   $$\iint_R 5(x+y)e^{x^2-y^2}dA, \text{ where } R \text{ is enclosed by the lines } x-y=0,\ x-y=3,\ x+y=0,\ x+y=10.$$

   Hint: Can you factor x^2-y^2?
   You should get $\frac{5}{6}(e^{30}-31)$.
4. Evaluate the integral by using an appropriate change of variables.

   $$\iint_R 4\sin(81x^2+100y^2)dA, \text{ where } R \text{ is in first quadrant bounded by the ellipse } 81x^2+100y^2=1.$$

   Hint: Can you make the quarter-ellipse a circle?
   You should get $\frac{1}{90}\pi(1-\cos(1))$.
5. Review Question 3 on the first midterm.
6. Review the True/False Questions from midterm 2 and 16.3 Lecture notes. What if, where you see $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ or $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, you see instead $\text{curl}(\vec{F})=\vec{0}$? Does this change the answers? Why or why not? Here, even though $\vec{F}$ may be a 2-dimensional vector field, you can assume it is 3-dimensional by letting the $\hat{\mathbf{k}}$ component be 0.
7. Find the mass of a wire in the shape of a helix $x=t$, $y=\cos t$, $z=\sin t$ $0\le t\le 2\pi$, if the density at any point is equal to the square of the distance from the origin.
8. Compute $\int_C \sin(y)dx+(x\cos(y)-\sin(y))dy$ where C is *any* path from $(2,0)$ to $(1,\pi)$. **Hint:** The problem is well-defined, i.e. there is a solution! How can that be the case?
9. *Use Green's Theorem* to compute the area of the triangle with vertices $(0,0)$, (a,b), (c,d). You may assume that the vertices are ordered counter-clockwise.

10. Review Question 5 from the second Midterm. Let $\vec{F}$ be the vector field from this problem. Let C be any positively oriented, simple, closed curve containing the origin. Show that $\int_C \vec{F} \cdot d\vec{r} = 2\pi$.

11. Let $\vec{F}$ be the same vector field as above. Let C be any positively oriented, simple, closed curve that ***does not*** contain the origin. Show that $\int_C \vec{F} \cdot d\vec{r} = 0$.

12. Let S be a parametric surface that is the graph of a function $z = f(x, y)$ over D, a domain in the xy-plane. Parametrize this graph as $\vec{r}(x, y)$ in the usual way and derive $\vec{r}_x$ and $\vec{r}_y$. Compute the following.

    (a) $\vec{r}_x \times \vec{r}_y$. Is this upward or downward relative to the z-direction? [**Hint:** Right hand rule.]

    (b) $|\vec{r}_x \times \vec{r}_y|$.

    (c) Simplify $\int_S dS$ using the Chapter 16 formula for surface area.

    (d) Simplify $\int_S f(x, y, z) dS$.

    (e) Simplify $\int_S \vec{F} \cdot d\vec{S}$ where $F = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

    Remark: On the exam, a question may say "if you know a formula, you may use it." These are (some of the) formulas you may use.

13. Recall that heat flow through an three-dimensional object can be described by the vector field $\vec{F}(x, y, z) = -K\nabla u(x, y, z)$ where K is the conductivity of the object and $u(x, y, z)$ is the temperature. Gelila, an engineer, designs cooling systems for concentrated photovoltaic panels. One of the middle layers of the panel has a critical design constraint. This critical layer of her solar panel can be described by the parametric equation $\vec{r}(x, y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (0.05x^2 + 0.05y^2)\hat{\mathbf{k}}$ for (x, y) such that $x^2 + y^2 \leq 1$, where units are in meters. When she tests her cooling system, she notices the temperature settles at $u(x, y, z) = x^2 - y^2 + z + 78$ degrees Celsius in an open region around the critical layer.

    Compute the heat flux through the critical layer of Gelila's panel. You may assume the outward normal is the upward normal on the surface. You do not need to answer in the correct units, but note that your answer will depend on K.

14. Evaluate the flux of the vector $\vec{F} = xy\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$ across the surface S where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, and has upward orientation.

15. Evaluate the surface integral $\int\int_S \vec{F} \cdot d\vec{S}$ for the vector field $F(x, y, z) = \langle x, -z, y\rangle$ where S is the part of the sphere $x^2 + y^2 + z^2 = r^2$ in the first octant, with orientation towards the origin. In other words, compute the flux of $\vec{F}$ across S. Be clear about your choice of normal vector. **Solution:** $\frac{-\pi r^3}{6}$.

16. Use the Divergence Theorem to evaluate

$$\int\int_S (2x + 2y + z^2) dS$$

    where S is the sphere $x^2 + y^2 + z^2 = 1$.

17. Prove the following identities. You may assume S is the boundary surface for a region E in $\mathbb{R}^3$ such that S and E satisfy the conditions for the Divergence Theorem. Additionally, you may assume scalar functions and the component functions of the vector fields have continuous second-order partial derivatives. $\vec{n}$ denotes the outward unit normal vector to the surface S and $\Delta f(x,y,z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

    (a) $\iint_S curl(\vec{F}) \cdot d\vec{S} = 0$.

    (b) $\iint_S D_{\vec{n}} f dS = \iiint_E \Delta f dV$.

    (c) $Volume(E) = C \iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle x, y, z \rangle$. Find C.

18. Let $\vec{F}(x,y,z) = z\tan^{-1}(y^2)\hat{\mathbf{i}} + z^3 \ln(x^2+8)\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Find the flux of $\vec{F}$ across S, the part of the paraboloid $x^2 + y^2 + z = 20$ that lies above the plane $z = 4$ and is oriented upward.

19. Let $\vec{F}$ be an inverse square field, that is, $\vec{F} = \frac{cx}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{i}} + \frac{cy}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{j}} + \frac{cz}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{k}}$ for some constant c. Show that the flux of F across a sphere S with center the origin is independent of the radius of S. (Hint: You may use the fact that the outward normal of any sphere is $\hat{\mathbf{n}} = \frac{1}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, where r is the radius.)

20. Consider the electric field generated by a single electron located at the origin in $\mathbb{R}^3$

$$\vec{E} = \frac{\epsilon Q x}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{i}} + \frac{\epsilon Q y}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{j}} + \frac{\epsilon Q z}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{k}}$$

    where Q is the charge of the electron and ϵ a constant dependent on the units used.

    (a) Show that div$\vec{E} = 0$ for the electric field.

    (b) Show that the flux of E through any closed surface that contains the origin is $4\pi\epsilon Q$.