SOLUTIONS!

MATH 324 B
Exam I
April 26, 2019

Name ______________________________

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

1	12	
2	14	
3	14	
4	10	
Bonus	4	
Total	54	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 4 problems. Try not to spend more than 10 minutes on each page.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. Recall that for a positive number a, the volume of the sphere S of radius a, where

$$S = \{(x,y,z) : x^2 + y^2 + z^2 \le a^2\},$$

is given by $\frac{4}{3}\pi a^3$. Verify this in three different ways:

(a) (4 pts) Using a triple integral and spherical coordinates. Set it up and EVALUATE.

$$\iiint_S dV = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^3}{3}\right]_0^a \sin\phi \, d\phi \, d\theta$$

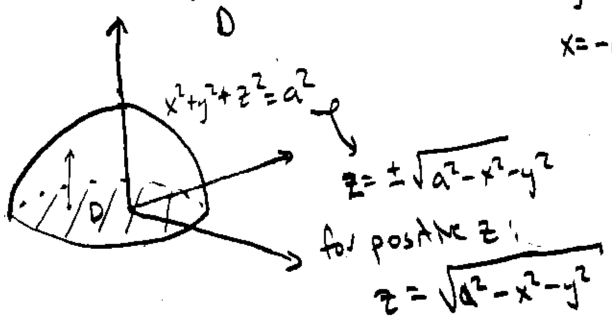
$$= \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta = \frac{a^3}{3} \int_0^{2\pi} [-\cos\phi]_0^{\pi} d\theta = \frac{a^3}{3} \int_0^{2\pi} 2 \, d\theta$$

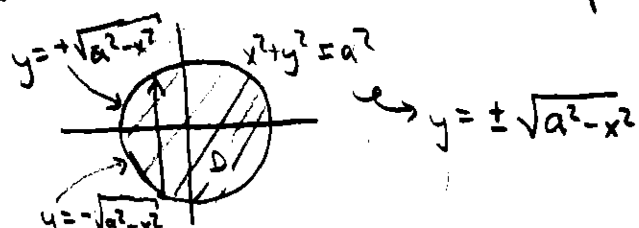
$$= \frac{4\pi a^3}{3}$$

(b) (4 pts) Use a double integral where you compute the volume under a function of the form $z = f(x,y)$ (You must find the function!). Set it up using *Cartesian Coordinates* BUT DO NOT EVALUATE IT! (Hint: Leveraging symmetry may be useful.)

$$\iint_D f(x,y) \, dA = 2 \cdot \int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} \, dy \, dx$$

(Many variations possible!)





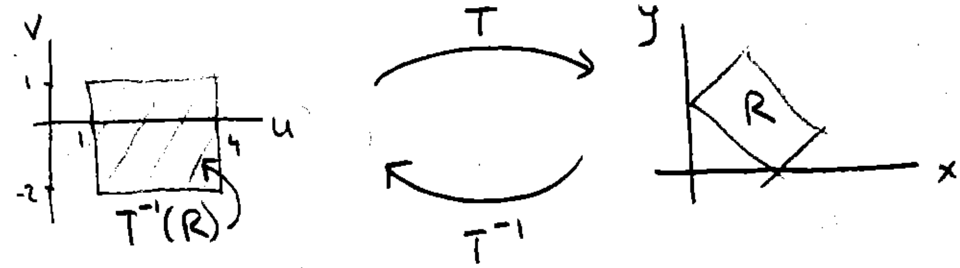
(c) (4 pts) Use a double integral where you compute the volume under a function of the form $z = f(x,y)$. Set it up using *polar coordinates* BUT DO NOT EVALUATE IT!

$$2 \cdot \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(\sqrt{a^2 - r^2}\right) r \, dr \, d\theta$$

$$\left(\sqrt{a^2 - x^2 - y^2} \Longrightarrow \sqrt{a^2 - r^2} \right)$$

2. (14 points) Evaluate $\iint_R (x+y)^3\,dx\,dy$ using a change of coordinates where R is the parallelogram bounded by the lines $x+y=1$, $x+y=4$, $x-2y=1$ and $x-2y=-2$.

Hint: Consider letting $u = x+y$ and $v = x-2y$.



$$T^{-1}: \begin{cases} ① \; u = x+y \\ ② \; v = x-2y \end{cases} \longrightarrow \quad 2\cdot① + ②: \; 2u+v = 3x$$

$$x = \tfrac{1}{3}(2u+v) = \tfrac{2}{3}u + \tfrac{1}{3}v$$

$$①-② : \; u - v = 3y$$

$$y = \tfrac{1}{3}u - \tfrac{1}{3}v$$

$$\Rightarrow T: \begin{cases} x = \frac{2}{3}u + \frac{1}{3}v \\ y = \frac{1}{3}u - \frac{1}{3}v \end{cases}$$

Jacobian: $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = -\frac{2}{9} - \frac{1}{9} = \frac{-3}{9} = \frac{-1}{3}.$$

Using Change of Variable Formula:

abs. value!

$$\iint_R (x+y)^3\,dx\,dy = \iint_{T^{-1}(R)} u^3 \cdot \left|\frac{-1}{3}\right| du\,dv$$

$$= \frac{1}{3}\int_{-2}^{1}\int_{1}^{4} u^3\,du\,dv = \frac{1}{3}\int_{-2}^{1}\left[\frac{u^4}{4}\right]_1^4 dv$$

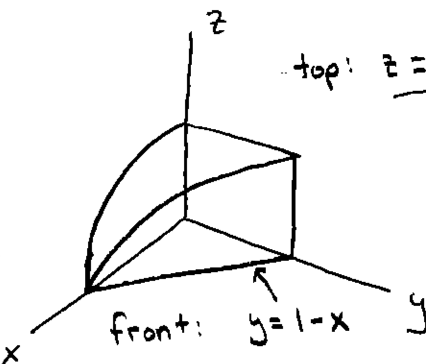
$$= \frac{1}{3}\int_{-2}^{1}\left(\frac{4^4}{4} - \frac{1}{4}\right) dv$$

$$= \frac{1}{3}\left(4^3 - \frac{1}{4}\right)\cdot 3 = \boxed{64 - \frac{1}{4}}$$

3. (14 points) Consider the following integral.

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

Rewrite the integral in the following two orders: $dz\,dy\,dx$ and $dx\,dz\,dy$.

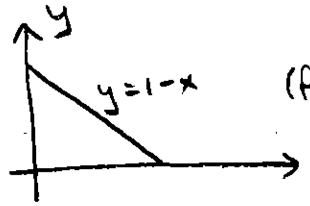


top: $\underline{z = 1 - x^2}$

$x = \pm\sqrt{1-z}$

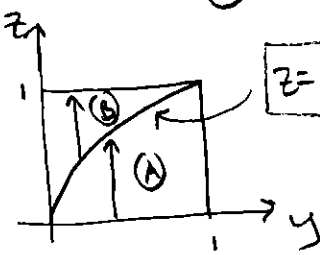
take positive $\underline{x}$

① $$\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x^2} f(x,y,z)\,dz\,dy\,dx$$



(Projection onto xy-plane.)

② $$\underbrace{\int_{y=0}^{y=1} \int_{z=0}^{z=2y-y^2} \int_{x=0}^{x=1-y} f(x,y,z)\,dx\,dz\,dy}_{Ⓐ} + \underbrace{\int_{y=0}^{y=1} \int_{z=2y-y^2}^{z=1} \int_{x=0}^{x=\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy}_{Ⓑ}$$



$\boxed{z = 2y - y^2}$ $\Rightarrow$ intersection between

and $\left.\begin{array}{l} z = 1 - x^2 \\ y = 1 - x \end{array}\right\}$

$\Downarrow$

$x = 1 - y$

$\Downarrow$ plug into

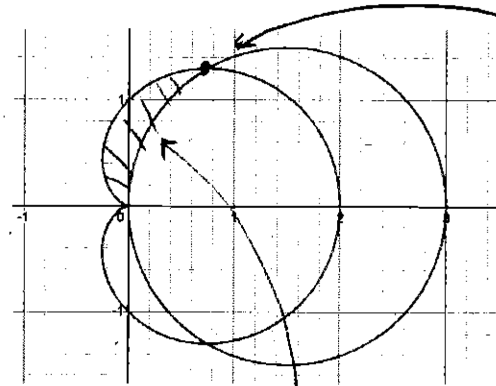
$z = 1 - (1-y)^2$

$z = 1 - 1 + 2y - y^2$

$\boxed{z = 2y - y^2}$

4. (10 points) Set-up an integral to compute the area inside the cardioid $r = 1 + \cos\theta$ and outside the circle $\boxed{r = 3\cos\theta}$ (*) DO NOT EVALUATE.

Hint: Make sure you integrate over regions with positive r values.

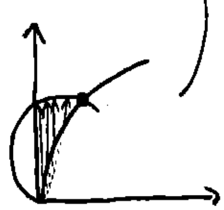


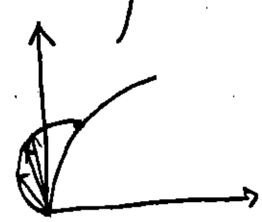
intersection:

$$1+\cos\theta = 3\cos\theta$$
$$1 = 2\cos\theta$$
$$\frac{1}{2} = \cos\theta$$
$$\theta = \frac{\pi}{3}$$

Use symmetry: $2\cdot$ (Area of top)

$$2\cdot\left(\underbrace{\int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=3\cos\theta}^{r=1+\cos\theta} r\,dr\,d\theta}\; + \;\underbrace{\int_{\theta=\pi/2}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta} r\,dr\,d\theta}\right)$$





BONUS: (4 points) Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $A(D)\sqrt{1 + a^2 + b^2}$, where a, b, and c are constants.

$$SA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_D \sqrt{1 + a^2 + b^2}\ dA$$

$$= \sqrt{1 + a^2 + b^2} \iint_D dA$$

$$= \left(\sqrt{1 + a^2 + b^2}\right) \cdot A(D) .$$

$\blacksquare$