Lecture Notes: Week 5

More Set Theory, Maps and functions

We start this weeks notes with a continuation of set theory. These notes will largely stick to "Definition, theorem, proof" style. Make sure you are watching lectures - you will occasionally find extra intuition there.

<u>THM 8</u> If A and B are finite sets, then

$$|A \cup B| + |A \cap B| = |A| + |B|.$$

<u>Sketch</u>: First notice that if A and B are <u>disjoint</u> finite sets, we have the following:

(1) $|A \sqcup B| = |A| + |B|$.

This is because each $x \in A \cup B$ is in A or B, but <u>not</u> both, meaning if we count the number of elements in A, none of those are in B, and vice-versa.

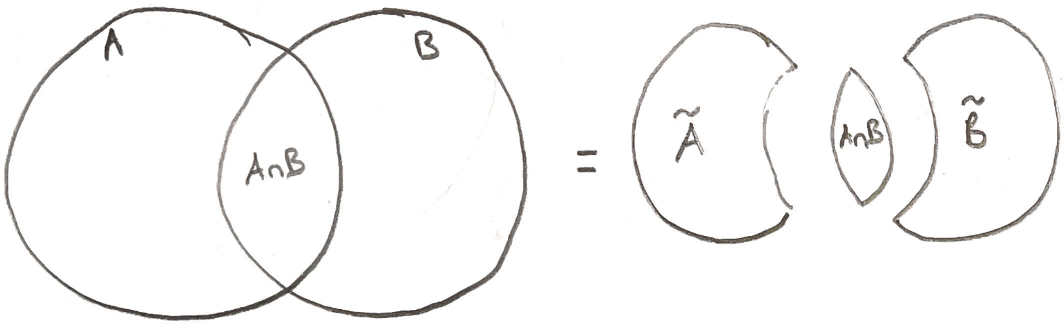
Now, notice that this is a simpler version of our theorem since $A \cap B = \emptyset$ for disjoint sets:

$$|A \sqcup B| + |A \cap B| = |A \sqcup B| + |\emptyset|$$
$$= |A \sqcup B|,$$

so really:

$$|A \sqcup B| + |A \cap B| = |A \sqcup B| = |A| + |B|.$$

Now, we will consider the general case and use property (1) to help us prove the statement. Since property (1) requires disjoint sets, we need to split our union into a union of disjoint pieces. (This is a common technique!)



$$\tilde{A} := A \setminus (A \cap B)$$
$$\tilde{B} := B \setminus (A \cap B)$$

Then, it is not hard to show:

$$A = \tilde{A} \sqcup (A \cap B)$$
$$B = \tilde{B} \sqcup (A \cap B).$$

Now, notice:

$$\begin{aligned}
|A \cup B| &= |\tilde{A} \sqcup (A \cap B) \sqcup \tilde{B} \sqcup (A \cap B)| \\
&= |\tilde{A} \sqcup (A \cap B) \sqcup \tilde{B}| \\
&= |\tilde{A}| + |(A \cap B) \sqcup \tilde{B}| \\
&= |\tilde{A}| + |A \cap B| + |\tilde{B}|
\end{aligned}$$

(but ... we don't really want $\tilde{A}$ or $\tilde{B}$ here .. so..)

$$\begin{aligned}
&= |\tilde{A} \sqcup A \cap B| + |\tilde{B}| \\
&= |A| \quad + |\tilde{B}|
\end{aligned}$$

And there is $|A|$. There is still a $|\tilde{B}|$, but that little maneuver (as goofy as it was) tells us how to turn $\tilde{B}$ into B ...

$$|A| + |\tilde{B}| = |A| + |\tilde{B}| + |A \cap B| - |A \cap B|$$

We need this!

So we correct for it with this

(This is called "adding a fancy 0"!)

$$= |A| + |\tilde{B} \sqcup A \cap B| - |A \cap B|$$

$$= |A| + |B| - |A \cap B|.$$

So, we showed that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In other words $|A \cup B| + |A \cap B| = |A| + |B|$, as desired ☑

Homework Question 1.

Use THM 8 several times (and other identities from last week's lectures) to prove that for finite sets A, B, and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Then state the analogous theorem (do not prove!) for $|A \cup B \cup C \cup D|$.

DEF For arbitrary sets X and Y, the cartesian product is

$$X \times Y := \{ (x,y) : x \in X,\ y \in Y \}.$$

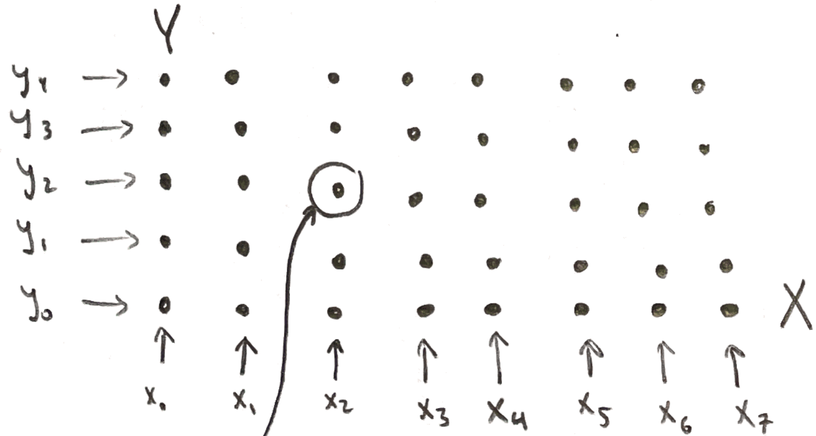
Here (x,y) is an "ordered pair," i.e. first x then y.

This is not the same as $\{x,y\}$ since $\{x,y\} = \{y,x\}$ but $(x,y) \neq (y,x)$ unless $x=y$.

THM 9 If X and Y are finite sets, then

$$|X \times Y| = |X||Y|.$$

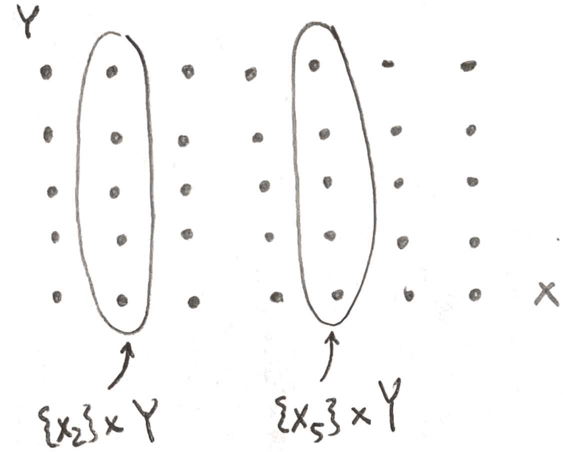
Sketch: Here's a picture we should keep in our head (a collection of ordered pairs...)



dot represents the ordered pair $(x_2,y_2) \in X \times Y$

Consider the sets $\{x_1\} \times Y$, $\{x_2\} \times Y$, etc. We can split $X \times Y$ into a disjoint union of such sets.

this requires proof! It is not hard to show.



For general finite sets, we have the following:

$$X \times Y = \bigsqcup_{x \in X} \{x\} \times Y$$

disjoint! (need to show: $\{x_i\} \times Y \cap \{x_j\} \times Y = \emptyset$)

So, we can compute (again using property (1) from the proof of THM 8 ... many times!)

$$|X \times Y| = \left| \bigsqcup_{x \in X} \{x\} \times Y \right|$$

property (1)

$$= \sum_{x \in X} |\{x\} \times Y|$$

the number of ordered pairs is the same as the number of elements in Y.

$$= \sum_{x \in X} |Y|$$

$$= |X||Y|,$$ as desired. $\boxtimes$

Homework Question 2

Define $X \times Y \times Z$ similarly and state (do not prove) the Theorem analogous to THM 9.

Homework Question 3

Prove that for A, B, and C any sets such that B and C are disjoint,

$$A \times (B \sqcup C) = (A \times B) \sqcup (A \times C).$$

Maps (Functions)

DEF Let X and Y be sets. A map f from X to Y (denoted $f: X \longrightarrow Y$) is a rule which associates to each $x \in X$ exactly one $y \in Y$, denoted by $f(x)$. We call X the domain of f, and Y the codomain of f.

EXAMPLES (and some DEFs)

① For $X = \mathbb{R}$ and $Y = \mathbb{R}$, $f(x) = x^2$ defines a map $f: \mathbb{R} \to \mathbb{R}$.

② **DEF** For any X and Y and $c \in Y$, $f(x) = c$ for all $x \in X$ defines a constant map from X to Y.

For $X = \mathbb{R}$, $Y = \mathbb{R}$, $c = 0$, $f(x) = 0$ for all $x \in \mathbb{R}$ defines a constant map.

③ **DEF** For any set X, the identity map $i_X : X \to X$ (or $id_X : X \to X$) is defined by $i_X(x) = x$ for all $x \in X$.

For $X = \mathbb{R}$, $f(x) = x$ is the identity map $id_{\mathbb{R}}$ or $i_{\mathbb{R}}$.

④ **DEF** For any set X and any subset $A \subset X$, the inclusion map $i_{A,X} : A \to X$ is defined by $i_{A,X}(a) = a$.

$$\underset{a \in A}{i_{A,X}(a)} = \underset{a \in X \supset A}{a}.$$

For $X = \mathbb{R}$ and $A = \mathbb{Z}$, $\mathbb{Z} \subset \mathbb{R}$ and $i_{\mathbb{Z},\mathbb{R}} : \mathbb{Z} \to \mathbb{R}$ is $i_{\mathbb{Z},\mathbb{R}}(n) = n$.

DEF Two maps $f: X \to Y$ and $g: U \to V$ are **equal** (written $f = g$) if

① $X = U$ (Domains same!)

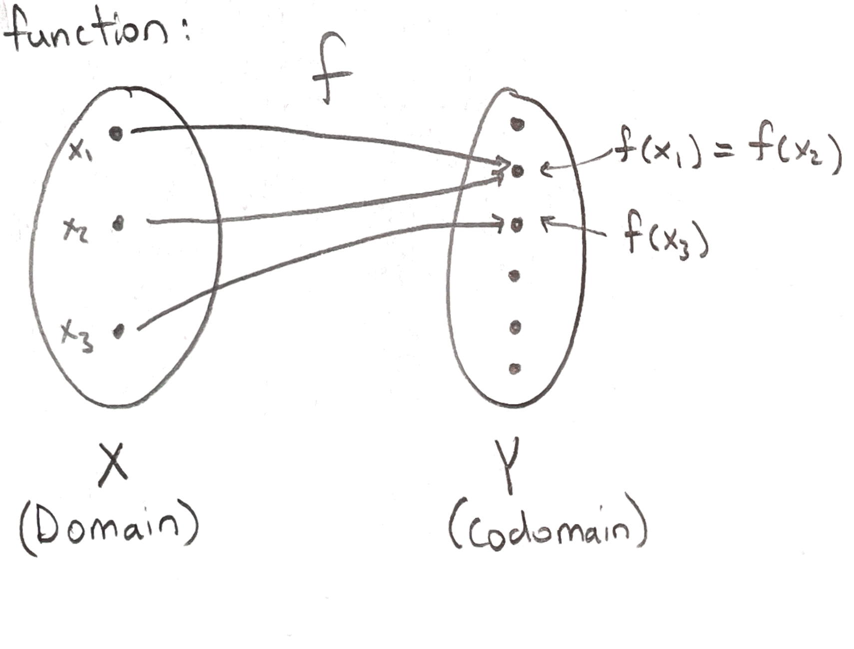
② $Y = V$ (Codomains same!)

③ $f(x) = g(x)$, for all $x \in X$.

EXAMPLE For $A \subset X$, $i_A = i_{A,X}$ only if $A = X$.

(Check!)

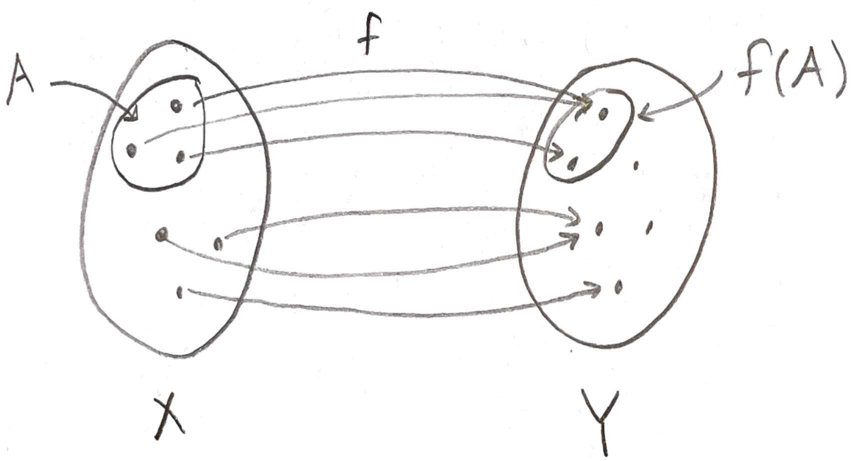
PIC of function:



DEF Let X and Y be sets, and let $f: X \to Y$ be a map.

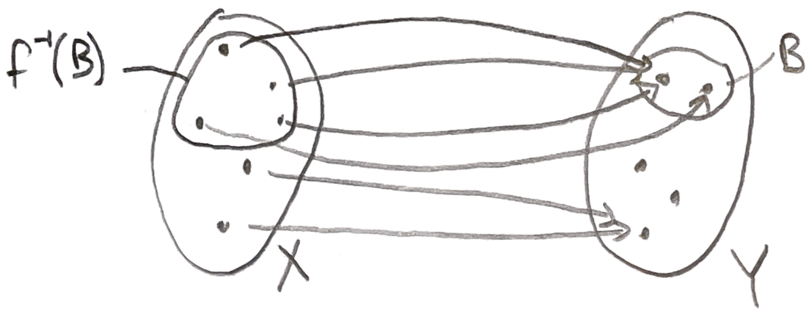
(a) For each $A \subset X$, the *image of A by f*, denoted *$f(A)$* is the subset of Y defined by

$$\boxed{y \in f(A) \text{ if and only if } y = f(x) \text{ for some } x \in A.}$$



(b) For each $B \subset Y$, the *inverse image* or *preimage of B by f*, denoted $f^{-1}(B)$ is the subset defined by

$$\boxed{x \in f^{-1}(B) \text{ if and only if } f(x) \in B.}$$



WARNING !

The preimage or inverse image is $\boxed{\text{NOT}}$ the inverse function, despite the notation appearing this way.

QUESTION How do functions interact with our set operations?

THM 1 For each map $f: X \longrightarrow Y$ and each $\mathcal{B} \subset \mathcal{P}(Y)$,

collection of subsets of Y

① $$f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right) = \bigcup_{B \in \mathcal{B}} f^{-1}(B)$$

② $$f^{-1}\left(\bigcap_{B \in \mathcal{B}} B\right) = \bigcap_{B \in \mathcal{B}} f^{-1}(B)$$

Also, for each $B \in \mathcal{P}(Y)$, B is a subset of Y

③ $$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B).$$

Think: "Preimages play well with unions, intersections, and complements."

Sketch: (1) We need to show $x \in f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right)$ if and only if $x \in \bigcup_{B \in \mathcal{B}} f^{-1}(B)$. (Why?)

Let $x \in X$.

$x \in f^{-1}\left(\bigcup_{B \in \mathcal{B}} B\right) \iff f(x) \in \left(\bigcup_{B \in \mathcal{B}} B\right)$, by def. of preimage,

$\iff f(x) \in B$ for some $B \in \mathcal{B}$, by def. of union of a collection of sets.

$\iff x \in f^{-1}(B)$ for some $B \in \mathcal{B}$, by def. of preimage.

$\iff x \in \bigcup_{B \in \mathcal{B}} f^{-1}(B)$ for some $B \in \mathcal{B}$, by def. of a union of a collection of sets.

✓

Homework Question 4

Prove (2) in THM 1 above similarly to how we have proven (1).

(3) Similarly, we need to show $x \in f^{-1}(Y \setminus B)$ if and only if $x \in X \setminus f^{-1}(B)$. (Why?)

Let $x \in X$.

$x \in f^{-1}(Y \setminus B) \iff f(x) \in Y \setminus B$ by definition of the preimage.

$\iff f(x) \notin B$, by definition of the complement.

$\iff x \notin f^{-1}(B)$, by definition of the preimage (negated).

$\iff x \in X \setminus f^{-1}(B)$, by definition of the complement.

$\boxtimes$

<u>THM 2</u> For each map $f: X \to Y$ and each $\mathcal{A} \subset \mathcal{P}(X)$,

① $f\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in \mathcal{A}} f(A)$

!

② $f\left(\bigcap_{A \in \mathcal{A}} A\right) \subset \bigcap_{A \in \mathcal{A}} f(A)$

Think: "Images play nicely with unions, but not intersections..."

Sketch: ① We want to show $y \in f\left(\bigcup_{A \in \mathcal{A}} A\right)$ if and only if $y \in \bigcup_{A \in \mathcal{A}} f(A)$. For $y \in Y$,

$$y \in f\Big(\bigcup_{A\in\mathcal{A}} A\Big) \iff y = f(x) \text{ for some } x \in \bigcup_{A\in\mathcal{A}} A,$$

by definition of the image.

$\iff y = f(x)$ for some $x \in A$, for some $A \in \mathcal{A}$,

by definition of a union of a collection.

$\iff y \in f(A)$ for some $A \in \mathcal{A}$,

by definition of the image.

$\iff y \in \bigcup_{A\in\mathcal{A}} f(A)$ by definition of a union of a collection.

② We want to show that if $y \in f\Big(\bigcap_{A\in\mathcal{A}} A\Big)$, then $y \in \bigcap_{A\in\mathcal{A}} f(A)$. (Careful ... why?)

For $y \in Y$,

$$y \in f\Big(\bigcap_{A\in\mathcal{A}} A\Big) \iff y = f(x) \text{ for some } x \in \bigcap_{A\in\mathcal{A}} A,$$

by definition of the image.

$\iff y = f(x)$ for some x in <u>all</u> $A \in \mathcal{A}$. (*)

This is by definition. But it is too restrictive... we only need an implication of the definition to prove the statement...

$\Rightarrow y = f(x)$ for some $x \in A$, for all $A \in \mathcal{A}$.

$\iff y \in f(A)$ for all $A \in \mathcal{A}$, by definition of the image.

$\iff y \in \bigcap_{A\in\mathcal{A}} f(A)$, by definition of an intersection of a collection.

(*) The starred line, the same x works for all $A \in \mathcal{A}$.

In the subsequent line, the x is allowed to be different for different A.

$\boxtimes$

Why do we only get containment and not equality in THM 2, ② ? Here's a counterexample to equality.

COUNTEREXAMPLE

Let $X = \{1,2\}$, $Y = \{0\}$. Let $A_1 = \{1\}$, $A_2 = \{2\}$. Define $f : X \longrightarrow Y$ by $f(1) = 0$, $f(2) = 0$. So:

image, so set is the input!

$$f(\{1\}) = \{0\}$$
$$f(\{2\}) = \{0\},$$

and we see that $f(A_1) \cap f(A_2) = \{0\} \cap \{0\} = \{0\}$.

However, $f(A_1 \cap A_2) = f(\{1\} \cap \{2\}) = f(\emptyset) = \emptyset$.

So
$$\underset{\emptyset}{\underset{\parallel}{f(A_1 \cap A_2)}} \neq \underset{\{0\}}{\underset{\parallel}{f(A_1) \cap f(A_2)}}$$
$$\emptyset \subset \{0\}$$

Homework Question 5

Prove that for each map $f: X \to Y$

① $A \subset f^{-1}(f(A))$ for each $A \subset X$,

② $f(f^{-1}(B)) \subset B$ for each $B \subset Y$.

Here, $f^{-1}(f(A))$ is the inverse image (or preimage) of $f(A)$ by f. Similarly, $f(f^{-1}(B))$ is the image of $f(B)$ by f.

QUESTION Have you noticed ... when talking about an inverse image or an image, we a talking about where sets map (image) or come from (preimage), we are not talking about where elements map...? (see WARNING! a few pages back.)

DEF Given sets X, Y, Z and maps $f: X \to Y$, $g: Y \to Z$, the composite map (or composition of f and g), denoted $f \circ g$, is defined by

$$\boxed{g \circ f(x) = g(f(x)) \text{ for } x \in X.}$$

$$\begin{tikzcd} X \arrow[rr, dashed, "g \circ f"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

THM 3 Composition of maps is associative.

(In other words:

$$\begin{tikzcd}
Y \arrow[r, "h\circ g"] \arrow[dr, "g"] & W \\
X \arrow[u, "f"] \arrow[r, "g\circ f"'] \arrow[ur, dashed] & Z \arrow[u, "h"']
\end{tikzcd}$$

For all sets X, Y, Z, and W, and maps $f: X \to Y$, $g: Y \to Z$, and $h: Z \to W$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(Accordingly, we write simply $h \circ g \circ f$ for either of these two, indicated by the dashed map above.)

Sketch To show that two maps are equal, we must show three things.

1. Domains are the same.
2. Codomains are the same.
3. $h \circ (g \circ f)(x) = (h \circ g) \circ f(x)$ for all x in the domain.

To see 1 notice that by definition, the domain of $g \circ f$ is X, and the codomain is Z. Thus, the domain of $h \circ (g \circ f)$ is X, by definition. Now consider $(h \circ g) \circ f$. The domain of $h \circ g$ is Y, and the codomain W. Thus, the domain of $(h \circ g) \circ f$ is X.

To see ②, follow similar logic using only the definition of the composite map.

Finally, to see ③, note that

$$h\circ(g\circ f)(x) = h(g(f(x))), \text{ by definition, for any } x\in X.$$

Similarly,

$$(h\circ g)\circ f(x) = h(g(f(x))), \text{ by definition, for any } x\in X.$$

$\blacksquare$

<u>REMARK</u> This property may seem silly, but it is a crucial property underpinning much of mathematics. This is often the best we can hope for. Notice that if $f: X\to X$ and $g: X\to X$, we could consider whether maps commute ... i.e. $f\circ g = g\circ f$. In general, this is <u>not</u> true. Consider $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$ and $g(x)=x+1$, respectively. Then

$$g\circ f(x) = g(f(x)) = g(x^2) = x^2+1$$

whereas

$$f\circ g(x) = f(g(x)) = f(x+1) = (x+1)^2.$$

DEF A map $f: X \to Y$ is

(a) Injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$;

(*) $\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$

or equivalently

(*) $\boxed{x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)}$

(b) Surjective if $f(X) = Y$ (the image of X is Y);

i.e.,

(*) $\boxed{\text{for all } y \in Y, \text{ there exists an } x \in X \text{ such that } f(x) = y}$

(c) bijective if f is both injective and surjective,

i.e.

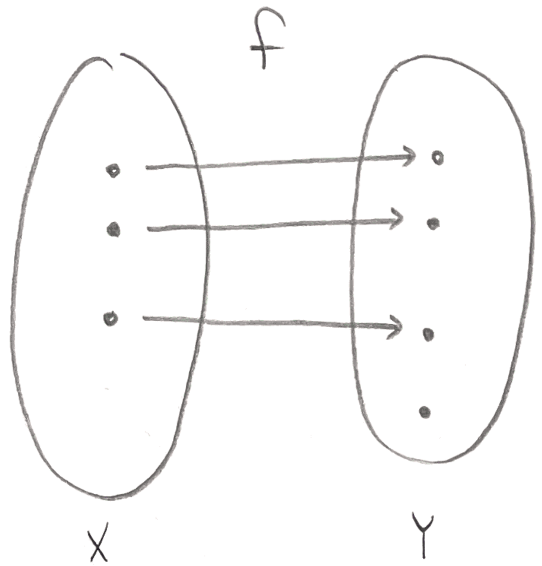
(*) $\boxed{\text{for all } y \in Y, \text{ there is exactly one } x \in X \text{ such that } f(x) = y}$.

If $f: X \to Y$ is bijective, we define the inverse map / function $f^{-1}: Y \to X$ by $f^{-1}(y) = x$ if and only if $f(x) = y$.

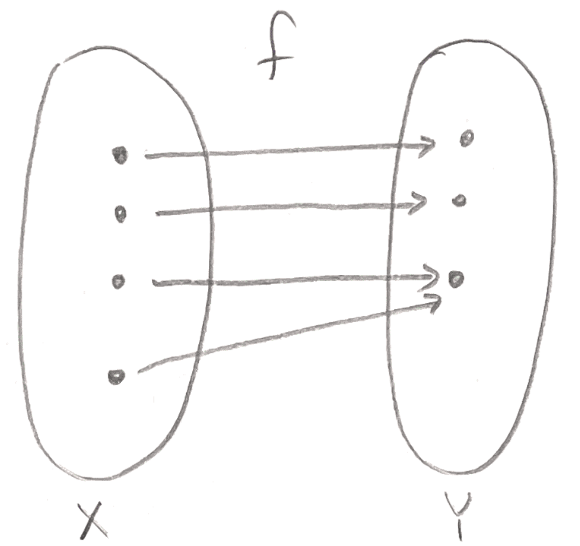
$$\boxed{f^{-1}(y) = x \iff f(x) = y}$$

REMARK Careful! Do not confuse or conflate the inverse function with the inverse image of a set!

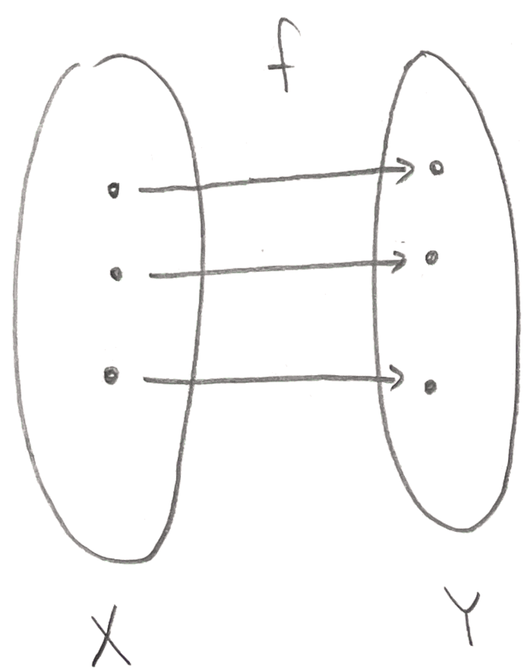
PICS



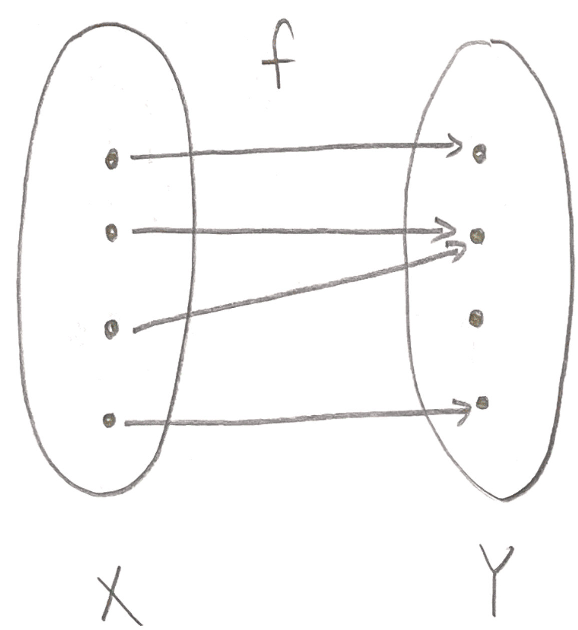
injective, not surjective



surjective, not injective



injective and surjective
(bijective)



neither injective
nor surjective

EXAMPLES The maps

$$\left\{\begin{array}{l} f:\mathbb{R}\to\mathbb{R},\quad f(x)=e^x \\[2ex] g:\mathbb{R}\to\mathbb{R},\quad g(x)=x^3-3x \\[2ex] h:\mathbb{R}\to\mathbb{R},\quad h(x)=x+1 \end{array}\right\} \text{ are } \left\{\begin{array}{l} \text{injective but not surjective} \\[2ex] \text{surjective but not injective} \\[2ex] \text{bijective.} \end{array}\right.$$

REMARK There are some commonly used synonyms:

1. "one-to-one" means injective,
2. "onto" means surjective,
3. "one-to-one and onto" means bijective.

Homework Question 6

Prove if $f: X \to Y$ is bijective, so is $\overbrace{f^{-1}: Y \to X}^{\text{denotes an inverse function}}$, and

1. $(f^{-1})^{-1} = f$,
2. $f^{-1} \circ f = i_X$,
3. $f \circ f^{-1} = i_Y$.

$$\begin{tikzcd} X \arrow[loop left, "i_X"] \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "f^{-1}"] \arrow[loop right, "i_Y"] \end{tikzcd}$$

THM 4 Given maps $f: X \to Y$, $g: Y \to Z$,

both f and g!

① f and g injective implies $g \circ f$ is injective, which implies f is injective,

② f and g surjective implies $g \circ f$ is surjective, which implies g is surjective,

③ f and g bijective implies $g \circ f$ is bijective, which implies f is injective and g is surjective.

Sketch: Before we begin the proof, notice that each statement above is really two statements:

① a) f and g injective $\Rightarrow$ $g \circ f$ is injective

and

b) $g \circ f$ injective $\Rightarrow$ f is injective.

The first statement is certainly something we would hope to be true: given two injective maps, the composition of these two maps is injective.

Let's prove this for ① a).

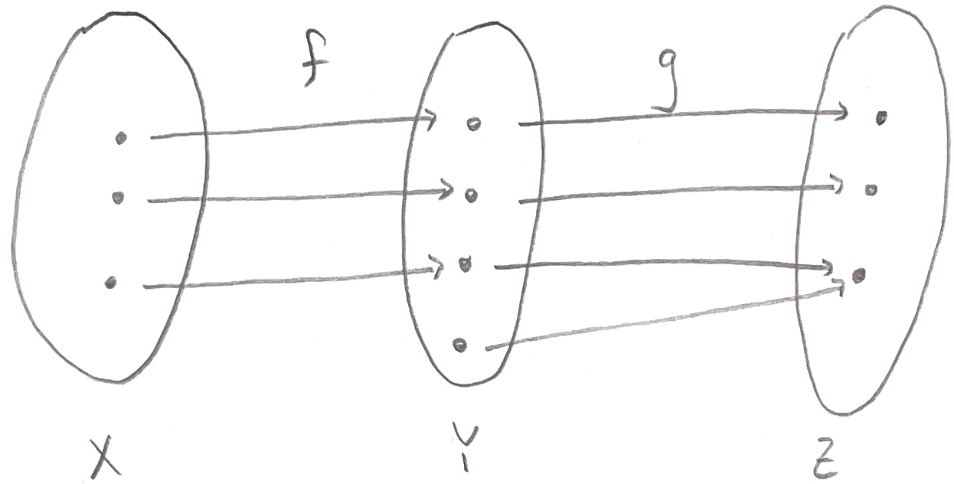
Assume f and g are injective. Let $x_1 \neq x_2$ be two points in the domain X of f (and $g \circ f$, by definition).

Consider $g \circ f(x_1)$ and $g \circ f(x_2)$. Since f is injective, by definition $f(x_1) \neq f(x_2)$. Notice $f(x_1)$ and $f(x_2)$ are in Y, the domain of g. Using the definition of composite maps, $g \circ f(x_1) = g(f(x_1))$ and $g \circ f(x_2) = g(f(x_2))$. Since $f(x_1) \neq f(x_2)$ and g is injective, by definition $g(f(x_1)) \neq g(f(x_2))$. Hence, by definition (again!) $g \circ f$ is injective.

Great. Now that we have proven that the composition of two injective maps is injective, we can ask a natural follow-up question - is the converse true? That is, if a composition of two maps is injective, are each of the maps injective?

It turns out that this is not true, in general.

Consider the two maps:



Here, f is injective, g is not injective, but $g \circ f$ is injective. Notice, however, that f is injective.

In fact, we will never be able to find an example of an injective $g \circ f$ such that f is not injective... because anything in the image of X by f is mapped by g into Y.

This is the content of 1 b):

$$g \circ f \text{ injective} \Rightarrow f \text{ is injective.}$$

We'll show this now.

Assume f is not injective. (We are using the contrapositive.)

Then, by negation of the definition, there exists $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. (Notice, the definition in this form is a universal statement: for all x_1 and x_2 such that $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.) Now, notice x_1 and x_2 are in the domain of $g \circ f$, and $g \circ f(x_1) = g(f(x_1))$ and $g \circ f(x_2) = g(f(x_2))$, by definition. Thus, since we picked x_1 and x_2 such that $f(x_1) = f(x_2)$, we have:

$$g \circ f(x_1) = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2).$$

Since $x_1 \neq x_2$, and $g \circ f(x_1) = g \circ f(x_2)$, we see that $g \circ f$ is not injective.

(Notice, since we used the contrapositive method, there was no need to point out a contradiction!)

Now let's prove 2. We will start with showing that if f and g is surjective, then $g \circ f$ is surjective.

Assume f and g are both surjective. Now, let $z \in Z$, the codomain of g (which is also the codomain of $g \circ f$ by definition of the composite map).

Since g is surjective, by definition we know that there exists a $y \in Y$ such that $g(y) = z$. Since f is surjective, we know (by definition, again) that there exists a $x \in X$ such that $f(x) = y$. Thus, by definition of the composition,

$$g \circ f(x) = g(f(x)) = g(y) = z.$$

We found an $x \in X$ such that $g \circ f(x) = z$. Since z was arbitrary, we have shown that $g \circ f$ is surjective.

So, that completes the direct proof. Now, we prove the second statement.

Assume g is not surjective. (we are using the contrapositive method again!) Then, by negation of the definition, we know that there exists a $z \in Z$ such that for all $y \in Y$, $g(y) \neq z$. Now, let $x \in X$ and consider $g \circ f(x)$. By definition of the composite map, $g \circ f(x) = g(f(x))$, where $f(x) \in Y$. Since g is not surjective, we know $g(f(x)) \neq z$. Since x was an arbitrary element its domain X, we have shown that for all $x \in X$, $g \circ f(x) \neq z$, i.e. there exists an $z \in Z$ such that for all $x \in X$, $g \circ f(x) \neq z$. Hence, $g \circ f$ is not surjective by negation of the definition of surjective, as desired.

That completes the proof of 2. 3 follows from 1 and 2 since our definition of bijective is that it is both injective and surjective. (Check this!)

$\boxtimes$