

## Lecture #12

### 4.4 Change of Basis

Recall from 4.2 when we defined basis, we immediately recognized that there are many choices of basis for a subspace. Here, we will learn how to change from one basis to a different one. Let's start by considering  $\mathbb{R}^n$ .

DEF The standard basis in  $\mathbb{R}^n$  is  $\{\vec{e}_1, \dots, \vec{e}_n\}$ .

(This should remind you of the beginning of the 4.2 lecture notes...)

Let  $S = \{\vec{e}_1, \vec{e}_2\}$  be the standard basis in  $\mathbb{R}^2$ . Let

$\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$  be a vector in  $\mathbb{R}^2$ . Notice that

$$\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 3\vec{e}_1 + 5\vec{e}_2,$$

meaning we can describe  $\vec{x}$  in terms of the given basis as  $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ ,  
of  $\vec{e}_1$  of  $\vec{e}_2$ .

Notice ... the vector is "represented" in the same way because we have been implicitly using the standard basis to describe vectors! To see what this means, consider a different

basis ; say for example  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ . (Check that this is a basis!). Now, notice that if

$$4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \vec{x}$$

so we could say

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

with respect to  
the basis  $\mathcal{B}$

$4 [1]$ , 4 of the first vector in the basis.  
 $1 [-1]$ , 1 of the second vector in the basis.

Like we mentioned in 4.2, this is a unique solution, so for any vector  $\vec{x}$ , we have a unique "representation" of the vector in the basis  $\mathcal{B}$ .

In fact ... it would be better to write  $\vec{x} = [\vec{x}]_S$  when we mean the standard basis, because it is more precise. But, just writing  $\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$  implies the standard basis.

Key Idea: Given a vector  $\vec{x}$ , we have no way of representing that vector (with numbers) unless we first choose a basis. By default, we choose the standard basis so that we can actually describe what vector, but we could have chosen a different basis!

Above, we have two representations of the vector  $\vec{x}$ :

$$\textcircled{1} \quad (\vec{x} = ) [\vec{x}]_S = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(we've been  
writing this)

$$\textcircled{2} \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

DEF Coordinate vector with respect  
to the basis  $\mathcal{B}$ .

Now, let's focus on going between bases in  $\mathbb{R}^n$ . In particular, let's start by exploring how to go from a given basis  $B$  to the standard basis  $S$ , and vice-versa, from the standard basis  $S$  to a given basis  $gB$ .

### EXAMPLE 1

Let  $gB = \{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$  and let

$[\vec{x}]_{gB} = \begin{bmatrix} 14 \\ -25 \end{bmatrix}$ . Find  $\vec{x}$  with respect to the standard basis.

$$[\vec{x}]_B = \begin{bmatrix} 14 \\ -25 \end{bmatrix} \text{ means } 14\vec{u}_1 - 25\vec{u}_2 = 14 \begin{bmatrix} 2 \\ 7 \end{bmatrix} - 25 \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

If we compute this,

$$14 \begin{bmatrix} 2 \\ 7 \end{bmatrix} - 25 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix},$$

meaning  $\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ , i.e.  $[\vec{x}]_S = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ .

Notice, we could have written

$$14\vec{u}_1 - 25\vec{u}_2 = [\vec{u}_1 \vec{u}_2] \begin{bmatrix} 14 \\ -25 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Let  $U = [\vec{u}_1 \vec{u}_2]$ , then  $U[\vec{x}]_{gB} = \vec{x}$ .

DEF The matrix  $U$ , consisting of column vectors of a basis  $B$  is the change of basis matrix from  $B$  to the standard basis  $S$ .

KEY IDEA: To go from  ${}_{\mathcal{B}}$  to  $S$ , use the formula

$$U[\vec{x}]_B = \vec{x}.$$

So that wasn't too bad. Can we reverse the question?

Can we start with a vector in standard form and write the vector in terms of a given basis?

EXAMPLE 2

Let  $B = \{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ . Let  $\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ .

Find  $[\vec{x}]_{\mathcal{B}}$ .

So, we already know the answer (from Example 1), but let's work through it. From Example 1, we know that

for  $U = [\vec{u}_1 \ \vec{u}_2]$ ,

$$U[\vec{x}]_B = \vec{x}.$$

Notice, since the column vectors of  $U$  are a basis, and  $U$  is  $2 \times 2$  (square), we can apply the unifying theorem!

This means  $U$  is invertible!

KEY IDEA: A matrix whose column vectors form a basis is invertible!

So, then let  $U^{-1}$  be the inverse of  $U$ , and

(\*)

$$U^{-1} U [\vec{x}]_B = U^{-1} \vec{x}$$

$$[\vec{x}]_B = U^{-1} \vec{x}.$$

Since  $U$  is  $2 \times 2$ , we can use our formula for computing the inverse:

$$U^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where } U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ so}$$

$$U^{-1} = \frac{1}{8-7} \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}. \quad \leftarrow U = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}, \text{ see Ex1.}$$

Then

$$[\vec{x}]_B = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 14 \\ -25 \end{bmatrix}.$$

□

PROP The matrix  $U^{-1}$ , where  $U$  consist of column vectors of a basis  $B$ , is a change of basis matrix from the standard basis  $S$  to  $B$ .

KEY IDEA: To go from  $S$  to  $B$ , use the formula

$$[\vec{x}]_B = U^{-1} \vec{x}.$$

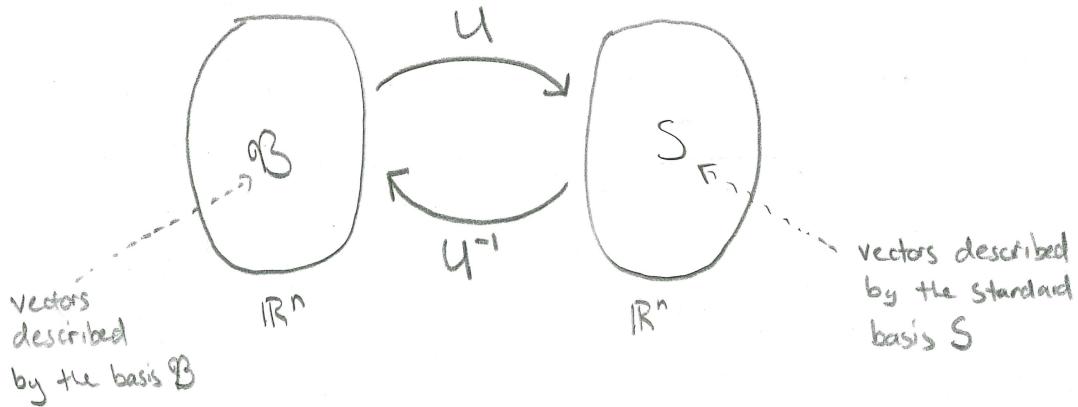
Let's summarize what we've done in a theorem:

THM Let  $\vec{x}$  be expressed with respect to the standard basis, and let  $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$  be any basis for  $\mathbb{R}^n$ . If  $U = [\vec{u}_1 \dots \vec{u}_n]_{n \times n}$ , then

$$\textcircled{1} \quad \vec{x} = U[\vec{x}]_{\mathcal{B}}$$

$$\textcircled{2} \quad [\vec{x}]_{\mathcal{B}} = U^{-1}\vec{x}$$

PICTURE

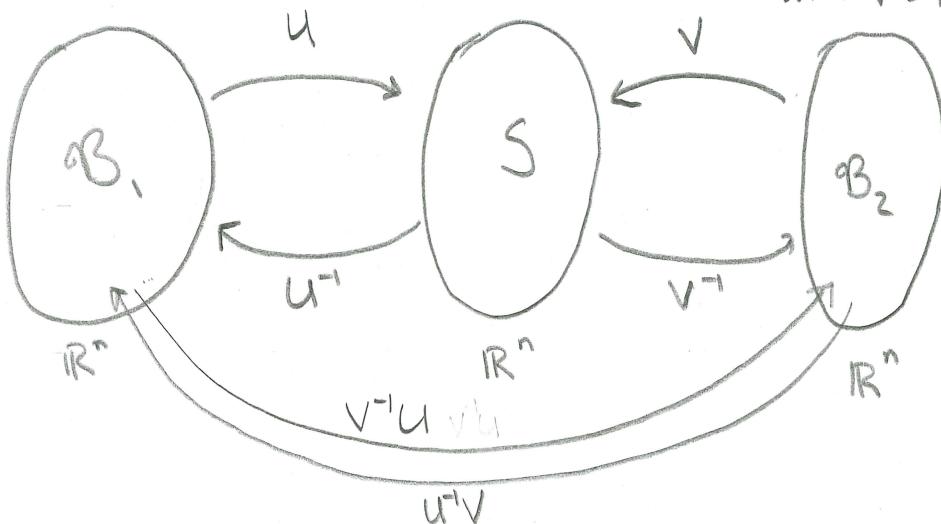


This theorem gives us a complete description of how to change basis from  $\mathcal{B}$  to  $S$ , and back, for any basis  $\mathcal{B}$ .

What if we wanted to change basis between two non-standard bases  $\mathcal{B}_1$  and  $\mathcal{B}_2$ ? What would our change of basis matrix look like? We can actually use the theorem above to answer this question.

Let's start by trying to draw a diagram of the situation. To use the theorem, we will need to "pass through the standard basis."

PICTURE Let  $\mathcal{B}_1 = \{\vec{u}_1, \dots, \vec{u}_n\}$ ,  $\mathcal{U} = [\vec{u}_1, \dots, \vec{u}_n]_{n \times n}$ , and  $\mathcal{B}_2 = \{\vec{v}_1, \dots, \vec{v}_n\}$ , with  $V = [\vec{v}_1, \dots, \vec{v}_n]$ .



So, to change the basis from  $\mathcal{B}_1$  to  $\mathcal{B}_2$ , we want to first change to  $S$  by applying  $U$ , then from  $S$  to  $\mathcal{B}_2$ , by applying  $V^{-1}$ . Thus, the change of basis matrix for  $\mathcal{B}_1$  going to  $\mathcal{B}_2$  is  $[V^{-1}U]$ . Similarly, if we go from  $\mathcal{B}_2$  to  $\mathcal{B}_1$ , passing through the standard basis, we see that the change of basis matrix is  $U^{-1}V$ .

Let's summarize this in a theorem.

THM Let  $\mathcal{B}_1 = \{\vec{u}_1, \dots, \vec{u}_n\}$  and  $\mathcal{B}_2 = \{\vec{v}_1, \dots, \vec{v}_n\}$  be bases for  $\mathbb{R}^n$ . If  $U = [\vec{u}_1 \cdots \vec{u}_n]_{n \times n}$  and  $V = [\vec{v}_1 \cdots \vec{v}_n]_{n \times n}$ , then

$$\textcircled{1} \quad [\vec{x}]_{\mathcal{B}_2} = V^{-1} U [\vec{x}]_{\mathcal{B}_1}$$

$$\textcircled{2} \quad [\vec{x}]_{\mathcal{B}_1} = U^{-1} V [\vec{x}]_{\mathcal{B}_2}.$$

### EXAMPLE 3

Suppose that

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$

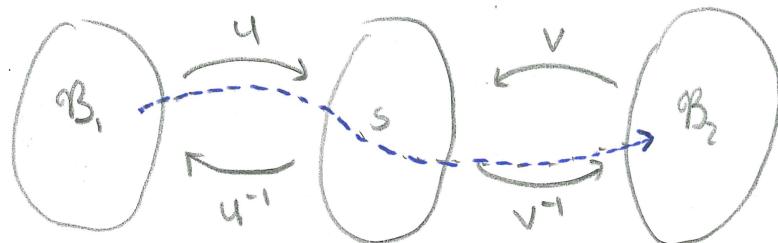
Find  $[\vec{x}]_{\mathcal{B}_2}$  if  $[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ . Find  $[\vec{y}]_{\mathcal{B}_1}$  if  $[\vec{y}]_{\mathcal{B}_2} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ .

Let's start with  $[\vec{x}]_{\mathcal{B}_2}$ . First, let  $U = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$  and let

$V = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ . Then, since we are given  $[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ , we

want to change basis from  $\mathcal{B}_1$  to  $\mathcal{B}_2$ . Let's draw

a picture:



so, we want  $V^{-1}U$  as the change of basis matrix; Thus,

$$[\vec{x}]_{\mathcal{B}_2} = V^{-1}U [\vec{x}]_{\mathcal{B}_1}.$$

Now, compute! Use the  $2 \times 2$  inverse formula to see that

$$V^{-1} = \frac{1}{9-10} \begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}. \quad \text{Then}$$

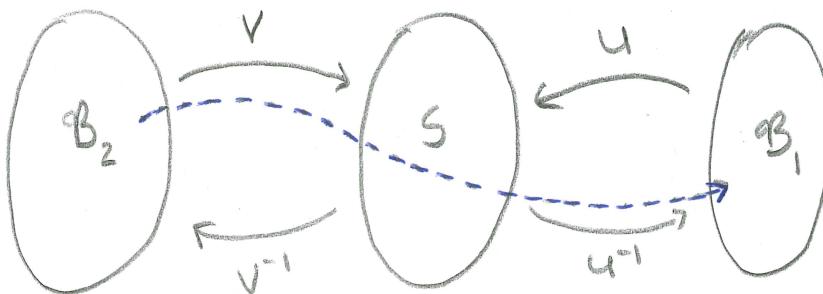
$$V^{-1}U = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -4 & -11 \end{bmatrix}$$

and we see

$$[\vec{x}]_{\mathcal{B}_2} = \begin{bmatrix} 3 & 8 \\ -4 & -11 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ -18 \end{bmatrix}.$$

Now, lets find  $[\vec{y}]_{\mathcal{B}_1}$ . Since we know  $[\vec{y}]_{\mathcal{B}_2} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ , to find  $[\vec{y}]_{\mathcal{B}_1}$  we need to change basis from  $\mathcal{B}_2$  to  $\mathcal{B}_1$ .

Draw a picture again:



So, we want  $U'V$  to be our change of basis

matrix, and

$$[\vec{y}]_{B_1} = U^{-1}V [\vec{y}]_{B_2}.$$

Let's compute.  $U^{-1} = \frac{1}{7-6} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$ . Then

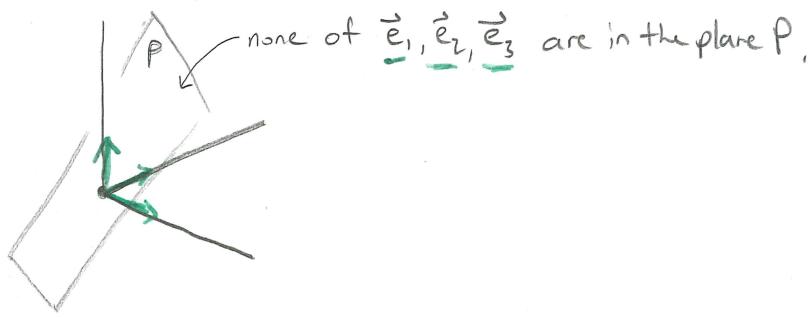
$$U^{-1}V = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ -4 & -3 \end{bmatrix}$$

and

$$[\vec{y}]_{B_1} = \begin{bmatrix} 11 & 8 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 23 \\ -8 \end{bmatrix}.$$

EXERCISE Using  $U, V$  as in the last example. Show that  
 $(V^{-1}U)^{-1} = U^{-1}V$ . Is this true in general?

So far, we have only looked at changing bases in  $\mathbb{R}^n$ ,  
but recall that we defined basis for subspaces of  $\mathbb{R}^n$ .  
This means that we can devise a method of changing  
basis in subspaces, not just in  $\mathbb{R}^n$ . However, this  
is a bit more delicate. Recall that a plane passing  
through the origin in  $\mathbb{R}^3$  may or may not contain  
any of the usual vectors in the standard basis!

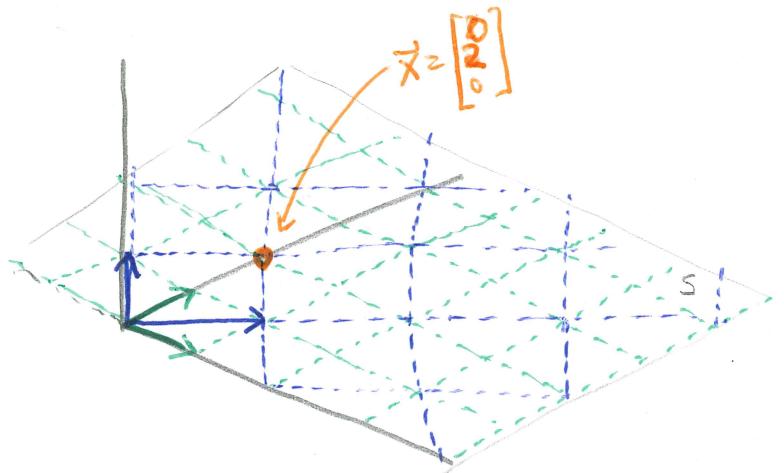


This means that we can't really use our technique of "passing through the standard basis". Let's try to construct an "easy" subspace to visualize, with a few "easy" bases  $\mathcal{B}_1$  and  $\mathcal{B}_2$  ... and see if we can come up with a new way to change the basis.

Let's take our subspace  $S$  to be the  $xy$ -plane in  $\mathbb{R}^3$ .

Then, let our two bases be

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$



Let's take the vector  $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$  in  $\mathbb{R}^3$  and write it in terms of each bases. Starting with  $\mathcal{B}_1$ , we see that

$$0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

so,

$$[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \begin{matrix} \text{0 of } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \text{2 of } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{matrix}.$$

Now, instead of looking at the picture and trying to figure out what linear combination of  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is needed to produce  $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

(remember, we are trying to change the basis), let's try to figure out what  $[\vec{x}]_{\mathcal{B}_2}$  by using what we know about  $\mathcal{B}_1$ .

Since we know we need 0 of  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and 2 of  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ;

why don't we try to write the basis  $\mathcal{B}_1$  in terms of the basis  $\mathcal{B}_2$ . As in, can we find:

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad y_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}?$$

We can definitely solve that! We get

$$\begin{cases} x_1 = \frac{1}{2} \\ x_2 = -\frac{1}{2} \end{cases}$$

← You have to work two systems of equations to get this!

and

$$\begin{cases} y_1 = \frac{1}{2} \\ y_2 = \frac{1}{2} \end{cases}.$$

(Notice: this means that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{QB_2} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

and  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{QB_2} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ . In other words, we

have rewritten the basis vectors for  $QB_1$  in terms of  $B_2$ ! To do it, we had to work a few systems of equations...)

Then, since

$$\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + -\frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

we can plug into our formula that gave

US coordinates of  $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$  in terms of  $B_1$ :

$$0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$0 \left( \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + 2 \left( \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

which reduces to

$$1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

so we see

$$[\vec{x}]_{B_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}_{B_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \underbrace{\text{1st } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{1st } \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}.$$

So what was the key step here? It was writing one of the bases in terms of the other! Let's streamline and summarize this procedure in a theorem, then work an example.

THM Let  $S$  be a subspace of  $\mathbb{R}^n$  with bases

$B_1 = \{\vec{u}_1, \dots, \vec{u}_k\}$  and  $B_2 = \{\vec{v}_1, \dots, \vec{v}_k\}$ . If we define

$$C = \left[ [\vec{u}_1]_{B_2} \quad [\vec{u}_2]_{B_2} \quad \cdots \quad [\vec{u}_k]_{B_2} \right]_{k \times k}$$

then

$$[\vec{x}]_{B_2} = C [\vec{x}]_{B_1}.$$

the columns are one basis written in terms of the other!

EXERCISE Review the procedure we just developed. Do you see why the theorem works? (In our procedure, you should find that  $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ .)

Summarizing: To change a vector  $\vec{x}$  from a basis  $B_1$  to a basis  $B_2$ , rewrite the basis  $B_1$  in terms of  $B_2$ , put the new vectors into a matrix  $C$ , then compute

$$[\vec{x}]_{B_2} = C [\vec{x}]_{B_1}.$$

EXAMPLE 4

Let  $B_1 = \left\{ \begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix} \right\}$  and  $B_2 = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

be two bases of a subspace  $S$  of  $\mathbb{R}^3$ . Find the change of basis matrix from  $B_1$  to  $B_2$ , and find  $[\vec{x}]_{B_2}$  if  $[\vec{x}]_{B_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ .

To find the change of basis matrix from  $\mathcal{B}_1$  to  $\mathcal{B}_2$ , we need to write the basis  $\mathcal{B}_1$  in terms of  $\mathcal{B}_2$ .

①

$$\begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 1 \\ -3 & 2 & -5 \\ 2 & 1 & 8 \end{array} \right] \xrightarrow{\text{(check!)}} \left[ \begin{array}{ccc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$$

②

$$\begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 3 \\ -3 & 2 & -8 \\ 2 & 1 & 3 \end{array} \right] \xrightarrow{\text{(check!)}} \left[ \begin{array}{ccc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow \begin{cases} y_1 = 2 \\ y_2 = -1 \end{cases}$$

So, if we call  $\mathcal{B}_1 = \{\vec{u}_1, \vec{u}_2\}$ , then

$$[\vec{u}_1]_{\mathcal{B}_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad [\vec{u}_2]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then, the change of basis matrix is given by

$$C = \begin{bmatrix} [\vec{u}_1]_{B_2} & [\vec{u}_2]_{B_2} \end{bmatrix}_{2 \times 2} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}.$$

Now, we can compute:

$$[\vec{x}]_{B_2} = C [\vec{x}]_{B_1}$$

$$[\vec{x}]_{B_2} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 7 \end{bmatrix}.$$

□

### EXERCISE

Review the Chapter 4 conceptual problems posted on the website (under "other materials").

Also, check out the videos pertaining to chapter 4 (also under "other materials").