MATH 308 C
Exam II
August 5, 2019

Name ____

Student ID #________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:__________________________________

SOLUTIONS!

1	16	
2	10	
3	12	
4	12	
Bonus	5	
Total	50	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 60 minutes to complete the exam and there are 4 problems. Try not to spend more than 12 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (16 Points) True / False and Short Answer.

Clearly indicate whether the statement is true or false.

(a) **TRUE** / **FALSE** If A and B are both invertible $n \times n$ matrices, then AB is invertible.

(b) **TRUE** / **FALSE** A linear transformation T is one-to-one if and only if $\text{Ker}(T) = \{\vec{0}\}$.

(c) **TRUE** / **FALSE** There exists a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that $\text{Range}(T) = \text{Ker}(T)$. [Hint: Consider the Rank-Nullity Theorem.]

Let A be such that $T(\vec{x}) = A\vec{x}$. Then $\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{range}(T))$

Similarly, $\text{nullity}(A) = \dim(\text{null}(A)) = \dim(\text{Ker}(T))$. (And A is 3×3.)

Since $\dim(\text{Ker}(T)) + \dim(\text{range}(T)) = 3$. Since the dimensions can never be the same, $\text{Range}(T) \neq \text{Ker}(T)$ ever!

(d) **TRUE** / **FALSE** Let A and B be square matrices. Then $(A+B)^2 = A^2 + 2AB + B^2$.

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B)$$
$$= A^2 + AB + BA + B^2.$$

(e) **TRUE** / **FALSE** Let A be an $n \times m$ matrix such that $A^T\vec{x} = \vec{b}$ is a consistent linear system for every $\vec{b}$ in $\mathbb{R}^m$. Then $n < m$.

$A^T\vec{x} = \vec{b}$ consistent means $\text{col}(A^T) = \mathbb{R}^m$, so $\text{rank}(A^T) = m$.

Since $\text{rank}(A^T)$ is the $\dim(\text{col}(A))$, which is the same as $\dim(\text{row}(A))$,

$\text{rank}(A^T) \leq \min\{\#\text{rows}, \#\text{columns}\} = \min\{m, n\}$. Thus,

$n \geq m$. So false.

Give an example of each of the following. If there is no such example, write NOT POSSIBLE.

(f) **Give an example** of a matrix A such that $A^8 = I_2$, but $A^k \neq I_2$ for $0 < k < 8$. (k must be an integer.)

$\Rightarrow$ We can achieve this with a 45° rotation:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \ \theta = \frac{\pi}{4} \Rightarrow \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Let $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

(g) **Give an example** of a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^4$, where $T(\vec{x}) = A\vec{x}$ for some matrix A, such that $\text{Rank}(A) = 2$ and $\text{Nullity}(A) = 2$.

NOT POSSIBLE.

(A is 4×3, so $\text{rank}(A) + \text{nullity}(A) = 3$.)

(h) **Give an example** of a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ where $m < n$ and T is not one-to-one.

$T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$. ($T: \mathbb{R}^2 \to \mathbb{R}^3$)

2. (10 Points) Let T be a linear transformation such that $T(\vec{x}) = A\vec{x}$ where A and its equivalent reduced row echelon form is given by

$$A = \begin{bmatrix} 3 & -2 & -1 & 3 \\ -1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 7 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the codomain of T?

$\mathbb{R}^3$

$$\left[\begin{array}{cccc|c} 3 & -2 & -1 & 3 & 0 \\ -1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 7 & 0 \\ 0 & 1 & 2 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) Give a basis for Null(A).

Let $x_3 = s$, $x_4 = t$. Then $\begin{cases} x_1 + s + 7t = 0 \\ x_2 + 2s + 9t = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = -s + -7t \\ x_2 = -2s + -9t \end{array}$

Thus,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s + -7t \\ -2s + -9t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7 \\ -9 \\ 0 \\ 1 \end{bmatrix}$$ linearly independent!

So basis for null(A) : $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -9 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Give a basis for Col(A)

$\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) Give a basis for Row(A).

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 9 \end{bmatrix} \right\}$

(e) Is T one-to-one? Onto? **Justify your answer.**

T is <u>not</u> one-to-one: $\ker(T) = \text{null}(A) \neq \{\vec{0}\}$.

T is <u>not</u> onto: $\text{range}(T) = \text{col}(A) \neq \mathbb{R}^3$.

3. (12 points)

(a) Produce a 2×2 matrix that reflects $\mathbb{R}^2$ over the x-axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \right.$$

(b) Produce a 2×2 matrix that rotates $\mathbb{R}^2$ by 270 degrees ($\frac{3\pi}{2}$ radians) counter-clockwise.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \ \theta = \frac{3\pi}{2} \Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(c) Compute the matrix that represents a reflection of $\mathbb{R}^2$ over the x-axis then a rotation by 270 degrees counter-clockwise, in that order. Call this matrix A.

$\Rightarrow$ Composition of linear transformations is multiplication of the corresponding matrices!

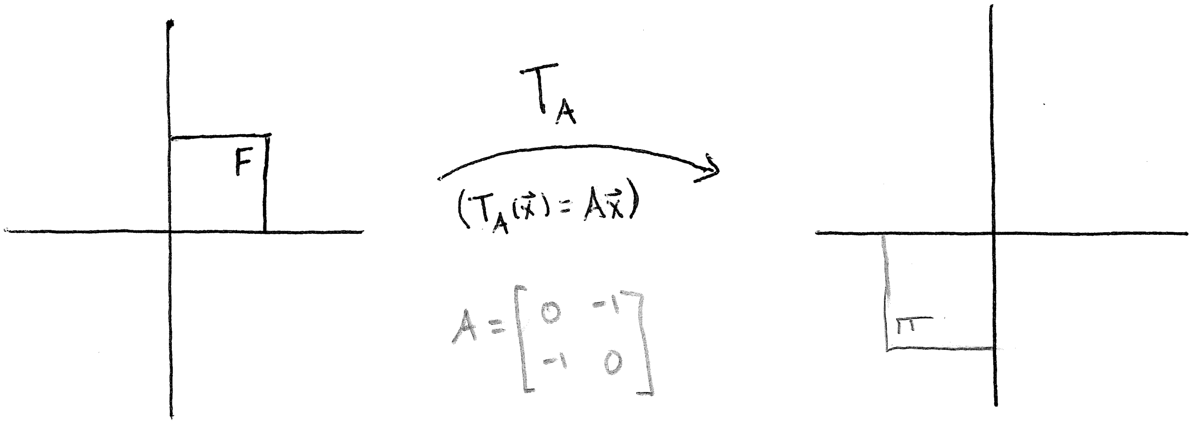
$$A = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{reflect}}}_{\text{then rotate}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

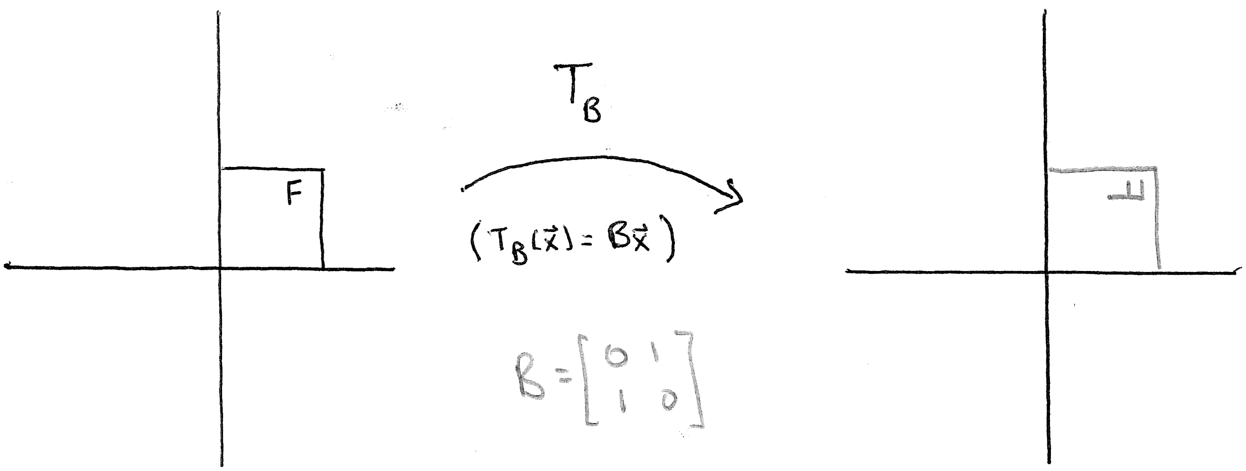
$(A\vec{x} = [\ ]\cdot[\ ]\vec{x})$

(d) Compute the matrix that represents a rotation of $\mathbb{R}^2$ by 270 degrees counter-clockwise, then a reflection over the x-axis, in that order. Call this matrix B.

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{rotate}}}_{\text{then reflect}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(e) Complete the following drawings. Show where the unit square gets mapped and draw F with the correct orientation on the new square.





(f) **Fill in the blank.** This is a geometric proof of the fact that matrix multiplication does not commute.

or any description of the fact that for general A, B, order of multiplication matters.

4. (12 points)

(a) Let A be any square matrix A. Show that the set S consisting of the vectors $\vec{v}$ such that $A\vec{v} = -2\vec{v}$ is a subspace.

① $\vec{0}$ is in S: $A\cdot\vec{0} = \vec{0} = -2\cdot\vec{0}$.

② If $\vec{u}, \vec{v}$ are in S, then: $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} \overset{\text{b/c } \vec{u},\vec{v} \text{ in } S}{=} -2\vec{u} + -2\vec{v} = -2(\vec{u}+\vec{v})$,

so we see that $\vec{u}+\vec{v}$ is in S.

③ If $\vec{u}$ is in S, then $A(r\vec{u}) = rA\vec{u} = r(-2\vec{u}) = -2r\vec{u} = -2(r\vec{u})$,

for any real number r, so we see that $r\vec{u}$ is in S.

$\Rightarrow$ S is a subspace.

(b) Now, let

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

Give a basis for S as given in part (a). You may use the back of this page if you need more space.

SOLUTION!: Want all vectors of the form $A\vec{v} = -2\vec{v}$. Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then solve:

(Many ways to do this!)

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & -2v_1 \\ 3 & -5 & 3 & -2v_2 \\ 6 & -6 & 4 & -2v_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 3 & -2v_1 \\ 0 & 4 & -6 & -2v_2+6v_1 \\ 0 & 12 & -14 & -2v_3+12v_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 3 & -2v_1 \\ 0 & 4 & -6 & -2v_2+6v_1 \\ 0 & 0 & 4 & -2v_3+\underbrace{12v_1+6v_2-18v_1}_{-6v_1} \end{array}\right]$$

Associated equations:

$$\begin{cases} v_1 - 3v_2 + 3v_3 = -2v_1 \\ 4v_2 - 6v_3 = -2v_2 + 6v_1 \\ 4v_3 = -2v_3 + 6v_2 - 6v_1 \end{cases} \Longrightarrow \begin{cases} 3v_1 - 3v_2 + 3v_3 = 0 \\ -6v_1 + 6v_2 - 6v_3 = 0 \\ 6v_1 - 6v_2 + 6v_3 = 0 \end{cases}$$

Make the system an augmented matrix again:

$$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ -6 & 6 & -6 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then, we see 2 free variables: $v_2 = s, v_3 = t$.

Then

$$v_1 - v_2 + v_3 = 0 \rightarrow v_1 = s - t$$

so,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

linearly independent!

Basis for S: $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

SOLUTION 2:

$$A\vec{v} = -2\vec{v} \rightarrow A\vec{v} = -2I_3\vec{v} \rightarrow A\vec{v} + 2I_3\vec{v} = \vec{0}$$
$$(A + 2I_3)\vec{v} = \vec{0}$$

using algebraic properties from 3.2.

Find a basis for null$(A + 2I_3)$.

just a matrix!

BONUS: (5 points) Recall from M126 that the Taylor series for the exponential function is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. We can use this to define the exponential of a square $n \times n$ matrix A:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

where we use the convention that $A^0 = I_n$.

(a) Compute e^A where A is the matrix $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$.

$A^0 = I_2$

$A^1 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\vdots$

$A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\vdots$

So, $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{1} \frac{A^n}{n!} + \cancel{\sum_{n=2}^{\infty} \frac{A^n}{n!}}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}.$

(b) Compute e^A where A is the matrix $\begin{bmatrix} 7 & 0 \\ 0 & 9 \end{bmatrix}$.

rem For a diagonal matrix, $A^n = \begin{bmatrix} 7^n & 0 \\ 0 & 9^n \end{bmatrix}$

So, $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}\begin{bmatrix} 7^n & 0 \\ 0 & 9^n \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} \frac{7^n}{n!} & 0 \\ 0 & \frac{9^n}{n!} \end{bmatrix}$

$= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{7^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{9^n}{n!} \end{bmatrix}$

$= \begin{bmatrix} e^7 & 0 \\ 0 & e^9 \end{bmatrix}.$