Math 324 Winter 2019: Review Questions

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1. Review the True/False Questions from Midterm 2. What if, where you see $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ or $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, you saw instead $\text{curl}(\vec{F}) = \vec{0}$? Does this change the answers? Why or why not? Here, even though $\vec{F}$ may be a 2-dimensional vector field, you can assume it is 3-dimensional by letting the $\hat{\mathbf{k}}$ component be 0.

2. Use Green's Theorem to compute the area of the triangle with vertices $(0,0)$, (a,b), (c,d).

3. Review Question 5 from the second Midterm. Let $\vec{F}$ be the vector field from this problem. Let C be any positively oriented, simple, closed curve containing the origin. Show that $\int_C \vec{F} \cdot d\vec{r} = 2\pi$.

4. Let S be a parametric surface that is the graph of a function $z = f(x,y)$ over D, a domain in the xy-plane. Parametrize this graph as $\vec{r}(x,y)$ in the usual way and derive $\vec{r}_x$ and $\vec{r}_y$. Compute the following.

   (a) $\vec{r}_x \times \vec{r}_y$. Is this upward or downward relative to the z-direction? [**Hint:** Right hand rule.]

   (b) $|\vec{r}_x \times \vec{r}_y|$.

   (c) Simplify $\int_S dS$ using the Chapter 16 formula for surface area.

   (d) Simplify $\int_S f(x,y,z)dS$.

   (e) Simplify $\int_S \vec{F} d\vec{S}$ where $F = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$.

   Remark: On the exam, a question may say "if you know a formula, you may use it."

5. Let $\vec{F}$ be the same vector field as above. Let C be any positively oriented, simple, closed curve that ***does not*** contain the origin. Show that $\int_C \vec{F} \cdot d\vec{r} = 0$.

6. (a) Let $f\vec{F}$ denote $f(x,y,z)\vec{F}(x,y,z)$. Show that $\text{curl}(f\vec{F}) = f\text{curl}(\vec{F}) + \nabla f \times \vec{F}$.

   (b) Assume f has continuous second-order partial derivatives in $\mathbb{R}^3$, C is a simple closed loop with positive orientation that is the boundary of a parametric surface S. Note that S and C satisfy the conditions of Stokes' Theorem. Show that

$$\int_C f\nabla f \cdot d\vec{r} = 0.$$

7. Recall that heat flow through an three-dimensional object can be described by the vector field $\vec{F}(x,y,z) = -K\nabla u(x,y,z)$ where K is the conductivity of the object and $u(x,y,z)$ is the temperature. Gelila, an engineer, designs cooling systems for concentrated photovoltaic panels. One of the middle layers of the panel has a critical design constraint. This critical layer of her solar panel can be described by the parametric equation $\vec{r}(x,y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (0.05x^2 + 0.05y^2)\hat{\mathbf{k}}$ for (x,y) such that $x^2 + y^2 \leq 1$, where units are in meters. When she tests her cooling system, she notices the temperature settles at $u(x,y,z) = x^2 - y^2 + z + 78$ degrees Celsius in an open region around the critical layer.

   Compute the heat flux through the critical layer of Gelila's panel. You may assume the outward normal is the upward normal on the surface. You do not need to answer in the correct units, but note that your answer will depend on K.

8. Let $\vec{F}$ be an inverse square field, that is, $\vec{F} = \frac{cx}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{i}} + \frac{cy}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{j}} + \frac{cz}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{k}}$ for some constant c. Show that the flux of F across a sphere S with center the origin is independent of the radius of S. (Hint: You may use the fact that the outward normal of any sphere is $\hat{\mathbf{n}} = \frac{1}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, where r is the radius.)

9. Evaluate

$$\int_C (y + \sin(x))dx + (z^2 + \cos(y))dy + x^3 dz$$

   where C is the curve $\vec{r}(t) = \sin(t)\hat{\mathbf{i}} + \cos(t)\hat{\mathbf{j}} + \sin(2t)\hat{\mathbf{k}}$, for $0 \leq t \leq 2\pi$. (Hint: Observe that C lies on the surface $z = 2xy$.)

10. Let $\vec{F}(x,y,z) = z\tan^{-1}(y^2)\hat{\mathbf{i}} + z^3\ln(x^2+8)\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Find the flux of $\vec{F}$ across S, the part of the paraboloid $x^2 + y^2 + z = 20$ that lies above the plane $z = 4$ and is oriented upward.

11. Consider the electric field generated by a single electron located at the origin in $\mathbb{R}^3$

$$\vec{E} = \frac{\epsilon Q x}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{i}} + \frac{\epsilon Q y}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{j}} + \frac{\epsilon Q z}{(\sqrt{x^2+y^2+z^2})^3}\hat{\mathbf{k}}$$

   where Q is the charge of the electron and ϵ a constant dependent on the units used.

   (a) Show that $\text{div}\vec{E} = 0$ for the electric field.

   (b) Show that the flux of E through any closed surface that contains the origin is $4\pi\epsilon Q$.