Lecture #12

4.4 Change of Basis

Recall from 4.2 when we defined basis, we immediately recognized that there are many choices of basis for a subspace. Here, we will learn how to change from one basis to a different one. Let's start by considering $\mathbb{R}^n$.

DEF The $\boxed{\underline{\text{standard}}\ \underline{\text{basis}}}$ in $\mathbb{R}^n$ is $\{\vec{e}_1, \ldots, \vec{e}_n\}$.

(This should remind you of the beginning of the 4.2 lecture notes...)

Let $S = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis in $\mathbb{R}^2$. Let $\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Notice that

$$\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 3\vec{e}_1 + 5\vec{e}_2,$$

meaning we can describe $\vec{x}$ in terms of the given basis as $\begin{array}{c} \begin{bmatrix} 3 \\ 5 \end{bmatrix}_S \end{array} \begin{array}{l} \text{of } \vec{e}_1 \\ \text{of } \vec{e}_2 \end{array}$.

Notice ... the vector is "represented" in the same way because we have been implicitly using the standard basis to describe vectors! To see what this means, consider a different basis, say for example $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. (Check that this is a basis!). Now, notice that

$$4\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \vec{x}$$

So we could say

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$4\begin{bmatrix}1\\1\end{bmatrix}$, 4 of the first vector in the basis.

$1\begin{bmatrix}-1\\1\end{bmatrix}$, 1 of the second vector in the basis.

with respect to the basis $\mathcal{B}$

Like we mentioned in 4.2, this is a <u>unique solution</u>, so for any vector $\vec{x}$, we have a unique "representation" of the vector in the basis $\mathcal{B}$.

In fact ... it would be better to write $\vec{x} = [\vec{x}]_S$ when we mean the standard basis, because it is more precise. But, just writing $\vec{x} = \begin{bmatrix}3\\5\end{bmatrix}$ implies the standard basis.

> <u>Key Idea</u>: Given a vector $\vec{x}$, we have no way of representing that vector (with numbers) unless we first choose a basis. By default, we choose the standard basis so that we can actually describe what vector, but we could have chosen a different basis!

Above, we have two representations of the vector $\vec{x}$!

① $(\vec{x} =)\, [\vec{x}]_S = \begin{bmatrix}3\\5\end{bmatrix}$

(we've been writing this)

② $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}4\\1\end{bmatrix}$

<u>DEF</u> <u>Coordinate vector with respect to the basis $\mathcal{B}$</u>.

Now, let's focus on going between bases in $\mathbb{R}^n$. In particular, lets start by exploring how to from a given basis $\mathcal{B}$ to the standard basis S, and vice-versa, from the standard basis S to a given basis $\mathcal{B}$.

<u>EXAMPLE 1</u>

Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$ and let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 14 \\ -25 \end{bmatrix}$. Find $\vec{x}$ with respect to the standard basis.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 14 \\ -25 \end{bmatrix}$ means $14\vec{u}_1 - 25\vec{u}_2 = 14\begin{bmatrix} 2 \\ 7 \end{bmatrix} - 25\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

If we compute this,

$$14\begin{bmatrix} 2 \\ 7 \end{bmatrix} - 25\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix},$$

meaning $\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, i.e. $[\vec{x}]_S = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Notice, we could have written

$$14\vec{u}_1 - 25\vec{u}_2 = [\vec{u}_1\ \vec{u}_2]\begin{bmatrix} 14 \\ -25 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Let $U = [\vec{u}_1\ \vec{u}_2]$, then $U[\vec{x}]_{\mathcal{B}} = \vec{x}$.

DEF The matrix U, consisting of column vectors of a basis $\mathcal{B}$ is the $\boxed{\text{change of basis matrix}}$ from $\mathcal{B}$ to the standard basis S.

KEY IDEA: To go from $\mathcal{B}$ to S, use the formula

$$U[\vec{x}]_{\mathcal{B}} = \vec{x}.$$

So that wasn't too bad. Can we reverse the question? Can we start with a vector in standard form and write the vector in terms of a given basis?

EXAMPLE 2

Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\} = \left\{\begin{bmatrix}2\\7\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}\right\}$. Let $\vec{x} = \begin{bmatrix}3\\-2\end{bmatrix}$.

Find $[\vec{x}]_{\mathcal{B}}$.

So, we already know the answer (from Example 1), but let's work through it. From Example 1, we know that for $U = [\vec{u}_1, \vec{u}_2]$,

$$U[x]_{\mathcal{B}} = \vec{x}.$$

Notice, since the column vectors of U are a basis, and U is 2x2 (square), we can apply the unifying theorem!

This means U is <u>invertible</u>!

(*) <u>KEY IDEA</u>: A matrix whose column vectors form a basis is invertible!

So, then let U^{-1} be the inverse of U, and

$$U^{-1} U [\vec{x}]_{\mathcal{B}} = U^{-1}\vec{x}$$

$$[\vec{x}]_{\mathcal{B}} = U^{-1}\vec{x}.$$

Since U is 2×2, we can use our formula for computing the inverse:

$$U^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{where } U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ so}$$

$$U^{-1} = \frac{1}{8-7}\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}. \quad \leftarrow U = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}, \text{ see Ex 1.}$$

Then

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 14 \\ -25 \end{bmatrix}.$$

☒

<u>PROP</u> The matrix U^{-1}, where U consist of column vectors of a basis $\mathcal{B}$, is a <u>change of basis matrix</u> <u>from the standard basis S to $\mathcal{B}$</u>.

<u>KEY IDEA</u>: To go from S to $\mathcal{B}$, use the formula

$$[\vec{x}]_{\mathcal{B}} = U^{-1}\vec{x}.$$

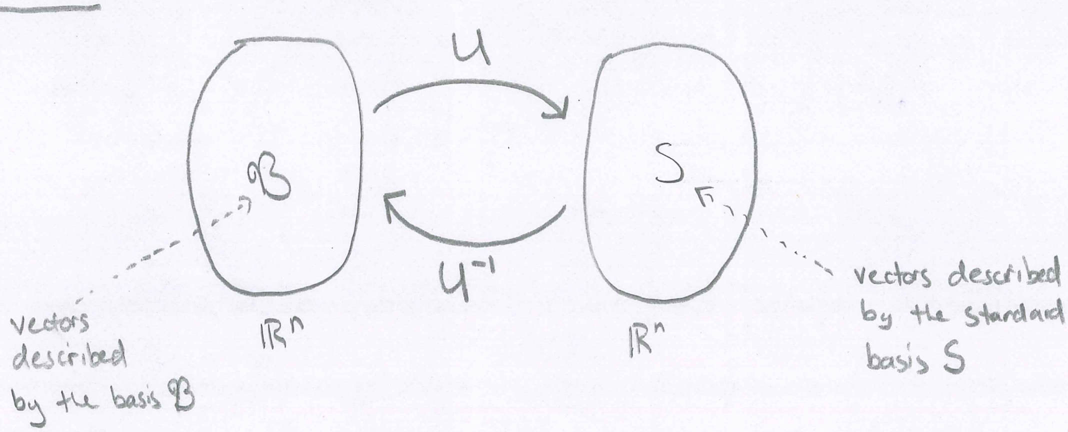
Let's summarize what we've done in a theorem:

<u>THM</u> Let $\vec{x}$ be expressed with respect to the standard basis, and let $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$ be any basis for $\mathbb{R}^n$. If $U = [\vec{u}_1 \cdots \vec{u}_n]_{n \times n}$, then

① $\vec{x} = U[\vec{x}]_{\mathcal{B}}$

② $[\vec{x}]_{\mathcal{B}} = U^{-1}\vec{x}$.

<u>PICTURE</u>

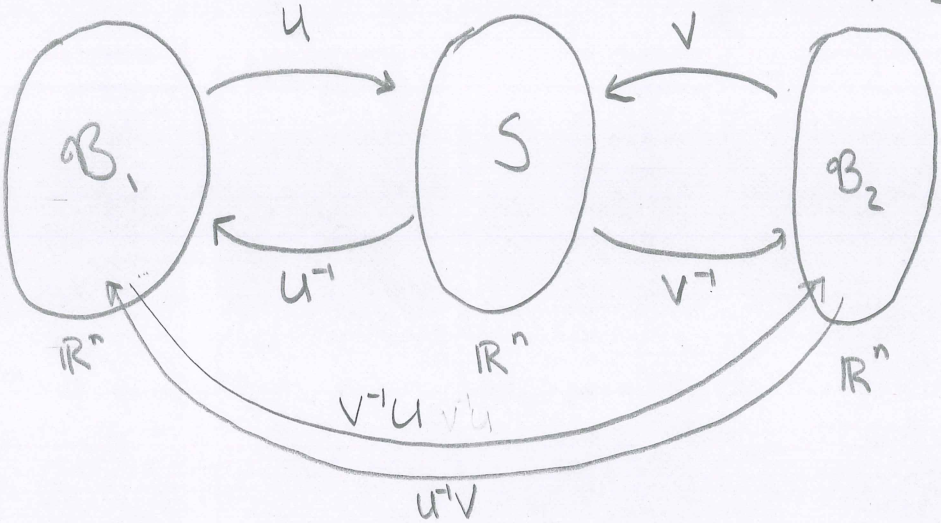


This theorem gives us a complete description of how to change basis from $\mathcal{B}$ to S, and back, for any basis $\mathcal{B}$.

What if we wanted to change basis between two non-standard bases $\mathcal{B}_1$ and $\mathcal{B}_2$? What would our change of basis matrix look like? We can actually use the theorem above to answer this question.

Let's start by trying to draw a diagram of the situation. To use the theorem, we will need to "pass through the standard basis."

PICTURE Let $\mathcal{B}_1 = \{\vec{u}_1, \ldots, \vec{u}_n\}$, $U = [\vec{u}_1 \cdots \vec{u}_n]_{n \times n}$, and $\mathcal{B}_2 = \{\vec{v}_1, \ldots, \vec{v}_n\}$, with $V = [\vec{v}_1 \cdots \vec{v}_n]$.



So, to change the basis from $\mathcal{B}_1$ to $\mathcal{B}_2$, we want to first change to S by applying U, then from S to $\mathcal{B}_2$, by applying V^{-1}. Thus, the change of basis matrix for $\mathcal{B}_1$ going to $\mathcal{B}_2$ is $\boxed{V^{-1}U}$. Similarly, if we go from $\mathcal{B}_2$ to $\mathcal{B}_1$, passing through the standard basis, we see that the change of basis matrix is $U^{-1}V$.

Let's summarize this in a theorem.

THM Let $\mathcal{B}_1 = \{\vec{u}_1, .., \vec{u}_n\}$ and $\mathcal{B}_2 = \{\vec{v}_1, .., \vec{v}_n\}$ be bases for $\mathbb{R}^n$.
If $U = [\vec{u}_1 \cdots \vec{u}_n]_{n\times n}$ and $V = [\vec{v}_1 \cdots \vec{v}_n]_{n\times n}$, then

① $[\vec{x}]_{\mathcal{B}_2} = V^{-1}U\,[\vec{x}]_{\mathcal{B}_1}$

② $[\vec{x}]_{\mathcal{B}_1} = U^{-1}V\,[\vec{x}]_{\mathcal{B}_2}$.

EXAMPLE 3

Suppose that

$$\mathcal{B}_1 = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\7\end{bmatrix}\right\} \quad \text{and} \quad \mathcal{B}_2 = \left\{\begin{bmatrix}3\\5\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}.$$

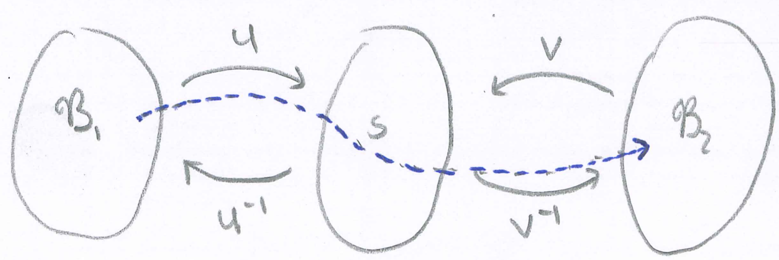
Find $[\vec{x}]_{\mathcal{B}_2}$ if $[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix}-1\\2\end{bmatrix}$. Find $[\vec{y}]_{\mathcal{B}_1}$ if $[\vec{y}]_{\mathcal{B}_2} = \begin{bmatrix}5\\-4\end{bmatrix}$.

column vectors are basis $\mathcal{B}_1$

Lets start with $[\vec{x}]_{\mathcal{B}_2}$. First, let $U = \begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}$ and let

column vectors are basis $\mathcal{B}_2$

$V = \begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$. Then, since we are given $[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix}-1\\2\end{bmatrix}$, we want to change basis <u>from $\mathcal{B}_1$ to $\mathcal{B}_2$</u>. Let's draw a picture:



So, we want $V^{-1}U$ as the change of basis matrix; Thus,

$$[x]_{\mathcal{B}_2} = V^{-1}U\,[\vec{x}]_{\mathcal{B}_1} \,.$$

Now, compute! Use the 2×2 inverse formula to see that

$$V^{-1} = \frac{1}{9-10}\begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}. \quad \text{Then}$$

$$V^{-1}U = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -4 & -11 \end{bmatrix}$$

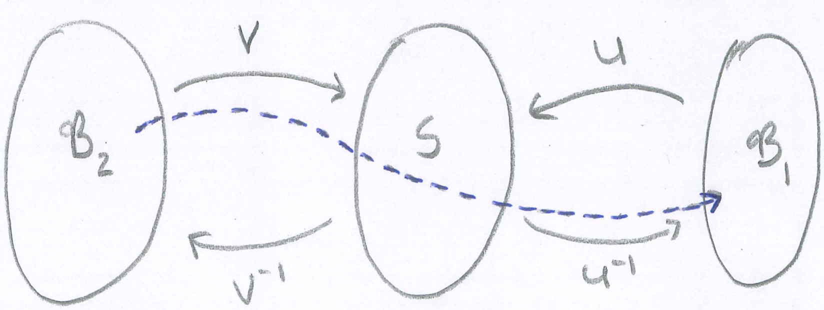
and we see

$$[\vec{x}]_{\mathcal{B}_2} = \begin{bmatrix} 3 & 8 \\ -4 & -11 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ -18 \end{bmatrix}.$$

Now, lets find $[\vec{y}]_{\mathcal{B}_1}$. Since we know $[\vec{y}]_{\mathcal{B}_2} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$,

to find $[\vec{y}]_{\mathcal{B}_1}$ we need to change basis from $\mathcal{B}_2$ to $\mathcal{B}_1$.

Draw a picture again:



So, we want $U^{-1}V$ to be our change of basis matrix, and

$$[\vec{y}]_{\mathcal{B}_1} = U^{-1}V[\vec{y}]_{\mathcal{B}_2}.$$

Let's compute. $U^{-1} = \frac{1}{7-6}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$. Then

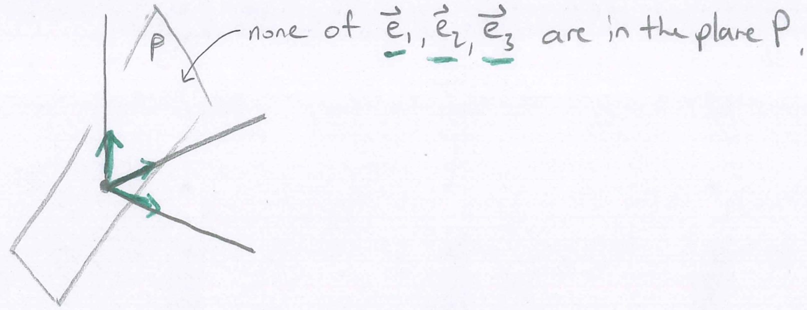
$$U^{-1}V = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ -4 & -3 \end{bmatrix}$$

and

$$[\vec{y}]_{\mathcal{B}_1} = \begin{bmatrix} 11 & 8 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 23 \\ -8 \end{bmatrix}.$$

<u>EXERCISE</u> Using U, V as in the last example. Show that $(V^{-1}U)^{-1} = U^{-1}V$. Is this true in general?

So far, we have only looked at changing bases in $\mathbb{R}^n$, but recall that we defined basis for <u>subspaces</u> of $\mathbb{R}^n$. This means that we can devise a method of changing basis in <u>subspaces</u>, not just in $\mathbb{R}^n$. However, this is a bit more delicate. Recall that a plane passing through the origin in $\mathbb{R}^3$ may or may not contain any of the usual vectors in the standard basis:

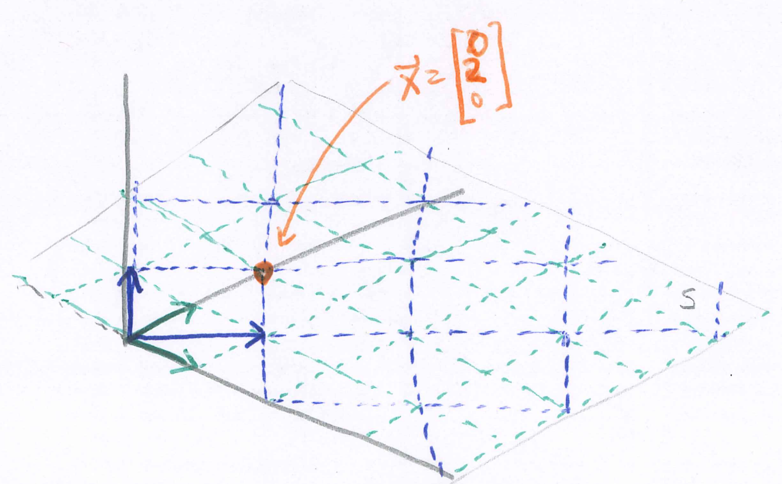


This means that we can't really use our technique of "passing through the standard basis". Let's try to construct an "easy" subspace to visualize, with a few "easy" bases $\mathcal{B}_1$ and $\mathcal{B}_2$ ... and see if we can come up with a new way to change the basis.

Let's take our subspace S to be the xy-plane in $\mathbb{R}^3$.

Then, let our two bases be

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$



Let's take the vector $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$ and write it in terms

of each bases. Starting with $\mathcal{B}_1$, we see that

$$0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix},$$

so,

$$[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{matrix} \leftarrow 0 \text{ of } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \\ \leftarrow 2 \text{ of } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \end{matrix}$$

Now, instead of looking at the picture and trying to figure out what linear combination of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is needed to produce $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ (remember, we are trying to <u>change the basis</u>), let's try to figure out what $[\vec{x}]_{\mathcal{B}_2}$ by using what we know about $\mathcal{B}_1$.

Since we know we need 0 of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and 2 of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, why don't we try to write the basis $\mathcal{B}_1$ <u>in terms of</u> the basis $\mathcal{B}_2$. As in, can we find:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad y_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \;?$$

We can definitely solve that! We get

$$\begin{cases} x_1 = \frac{1}{2} \\ x_2 = -\frac{1}{2} \end{cases}$$

⟸ You have to work two systems of equations to get this!

and

$$\begin{cases} y_1 = \frac{1}{2} \\ y_2 = \frac{1}{2} \end{cases}.$$

(*) (Notice: this means that $\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]_{\mathcal{B}_2} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$

and $\left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]_{\mathcal{B}_2} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. In other words, we have rewritten the basis vectors for $\mathcal{B}_1$ in terms of $\mathcal{B}_2$! To do it, we had to work a few systems of equations ...)

Then, since

$$\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{-1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

we can plug into our formula that gave

us coordinates of $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ in terms of $\mathcal{B}_1$:

$$0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$0 \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{-1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) + 2 \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

which reduces to

$$1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

so we see

1 of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$[\vec{x}]_{\mathcal{B}_2} = \left[\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right]_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1 of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

So what was the key step here? It was writing one of the bases in terms of the other! Let's streamline and summarize this procedure in a theorem, then work an example.

THM Let S be a subspace of $\mathbb{R}^n$ with bases $\mathcal{B}_1 = \{\vec{u}_1, \dots, \vec{u}_k\}$ and $\mathcal{B}_2 = \{\vec{v}_1, \dots, \vec{v}_k\}$. If we define

the columns are one basis written in terms of the other!

$$C = \Big[\, [\vec{u}_1]_{\mathcal{B}_2} \; [\vec{u}_2]_{\mathcal{B}_2} \; \cdots \; [\vec{u}_k]_{\mathcal{B}_2} \,\Big]_{k\times k}$$

then

Change of basis matrix!

$$[\vec{x}]_{\mathcal{B}_2} = C\,[\vec{x}]_{\mathcal{B}_1}.$$

EXERCISE Review the procedure we just developed. Do you see why the theorem works? (In our procedure, you should find that $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.)

Summarizing: To change a vector $\vec{x}$ from a basis $\mathcal{B}_1$ to a basis $\mathcal{B}_2$, rewrite the basis $\mathcal{B}_1$ in terms of $\mathcal{B}_2$, put the new vectors into a matrix C, then compute

$$[\vec{x}]_{\mathcal{B}_2} = C\,[\vec{x}]_{\mathcal{B}_1}.$$

EXAMPLE 4

Let $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix} \right\}$ and $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

be two bases of a subspace S of $\mathbb{R}^3$. Find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$, and find $[\vec{x}]_{\mathcal{B}_2}$ if $[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

To find the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$, we need to write the basis $\mathcal{B}_1$ in terms of $\mathcal{B}_2$.

① $$\begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -3 & 2 & -5 \\ 2 & 1 & 8 \end{array}\right] \overset{\text{(check!)}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$$

② $$\begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ -3 & 2 & -8 \\ 2 & 1 & 3 \end{array}\right] \overset{\text{(check!)}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} y_1 = 2 \\ y_2 = -1 \end{cases}$$

So, if we call $\mathcal{B}_1 = \{\vec{u}_1, \vec{u}_2\}$, then

$$[\vec{u}_1]_{\mathcal{B}_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad [\vec{u}_2]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then, the change of basis matrix is given by

$$C = \left[[\vec{u}_1]_{\mathcal{B}_2} \; [\vec{u}_2]_{\mathcal{B}_2} \right]_{2\times 2} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}.$$

Now, we can compute:

$$[\vec{x}]_{\mathcal{B}_2} = C\,[\vec{x}]_{\mathcal{B}_1}$$

$$[\vec{x}]_{\mathcal{B}_2} = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 7 \end{bmatrix}.$$

$\boxtimes$

<u>EXERCISE</u> Review the Chapter 4 conceptual problems posted on the website (under "other materials"). Also, check out the videos pertaining to Chapter 4 (also under "other materials").