SUPERDENSITY AND BOUNDED GEODESICS IN MODULI SPACE

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ABSTRACT. We identify a condition on the vertical linear flow of a translation surface that is both necessary and sufficient to conclude that the associated geodesic in the moduli space of translation surfaces is bounded. This builds upon work of Masur, Kerckhoff, and Smillie and work of Beck and Chen.

1. INTRODUCTION

A *translation surface* is a polygon (or finite set of polygons) in the complex plane such that every side of a polygon is identified with a parallel side by translation. There is a natural notion of "north" on a translation surface being the positive imaginary direction coming from the ambient $\mathbb{C}$. Translation surfaces are flat, away from a finite set of singular points. The singular points are cone points whose angles are integer multiples of 2π [14], [15].

There is a dynamical system commonly studied on translation surfaces: the linear flow Φ_t on the surface. This is the usual geodesic flow on the translation surface with the singular points removed. If a trajectory hits a singular point, we stop.

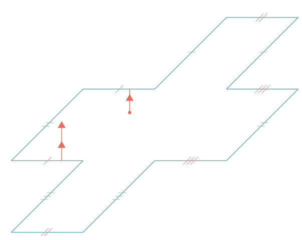

FIGURE 1. Linear flow segment on a translation surface

One motivation for studying such a system is its relationship to billiard trajectories on polygons with angles that are rational multiples of π. A billiard trajectory is a straight line path, until the trajectory hits an edge of the polygon. At that point, the trajectory bounces off the edge keeping the angle of incidence is equal to the angle of reflection. Such polygons can be "unfolded' to "straighten" the billiard path. Since the angles are rational multiples of π, the number of directions an billiard path can go is finite. After reflecting the polygon enough, we arrive at only a single direction. In other words, the unfolded polygon is a translation surface, and the corresponding dynamical systems is the linear flow [14], [15]. In fact, a linear flow trajectory hitting a singular point on the translation surface is equivalent to the billiard trajectory hitting a corner in the original billiard polygon.

Equivalently, a *translation surface* is a pair (X, ω) where X is a compact, connected Riemann surface without boundary and ω a non-zero holomorphic differential on X. If we fix the genus of the underlying Riemann surface, the moduli space of pairs (X, ω) forms a vector bundle over $\mathcal{M}_g$, the moduli space of genus g Riemann surfaces, whose fibers are $\mathbb{C}^g$. We will denote this $\mathcal{H}_g$ and refer to it as the moduli space of translation surfaces. Additionally, we will suppress the notation of the underlying Riemann surface and use the notation ω to denote a translation surface.

The moduli space of translation surfaces is equipped with an $SL_2(\mathbb{R})$ action, where the action is the usual linear action on the plane. Elements of the form $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ for any $t \in \mathbb{R}$ form a one-parameter subgroup which we will refer to as the (Teichmüller) *geodesic flow* and the corresponding flow segment a geodesic segment or trajectory.

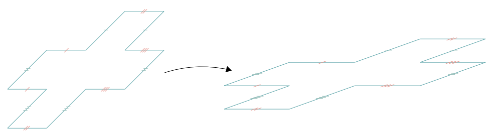

FIGURE 2. Translation surface ω and $g_t\omega$

There is a long history of interactions between the dynamical systems on individual translation surfaces and a dynamical system on the moduli space of translation surfaces, in particular, between the linear flow on a translation surface and geodesic flow on the moduli space. For example, Masur proved what is now known as Masur's criterion by building on of earlier work with Kerckhoff and Smillie [7], [13], [12]. Masur used it as a tool to give an upper bound on the Hausdorff dimension of quadratic differentials whose vertical linear flow is not uniquely ergodic. Since, Masur's criterion has become an important tool for dynamicists studying the $SL_2(\mathbb{R})$ action on translation surfaces.

Theorem 1.1 (Masur's Criterion)**.** Let g_t denote the geodesic flow in the moduli space on the space of translation surfaces and let ω be a translation surface. If $g_t \cdot \omega$ is recurrent, as in, returns to a compact set infinitely often, then the vertical straight line flow is uniquely ergodic.

In this note, we identify a quantitative density condition on the vertical flow of a translation surface ω that is both necessary and sufficient to guarantee that the associated geodesic is bounded in the moduli space. The condition is inspired by a paper of Beck and Chen [1], where they study billiard trajectories on similar objects.

Definition 1.1 (Superdensity)**.** Let ω be a translation surface. We say the linear flow Φ_t is *superdense* if there exists a constant $C > 0$ such that for every $r > 0$, the segment of the flow Φ_t for $t \in [0, Cr]$ gets $\frac{1}{r}$-close to every point on ω.

We show the following.

Theorem 1.2. Let $\omega \in \mathcal{H}_g$ be a translation surface. There exists a compact set $K \subset \mathcal{H}_g$ such that $g_t\omega \in K$ for all $t > 0$ if and only if the vertical (north or south) linear flow on ω is superdense.

The proof given in Section 2 identifies a geometric property, the maximum diameter of a translation surface in the compact set K, as the means by which we can control the quantitative density of the vertical linear flow.

As a corollary, we have the following:

Corollary 1.1. A superdense linear trajectory is uniquely ergodic, but the converse is not true.

1.1. **Some recent related results.** There have been a number of results that help explain the phenomenon described in Masur's criterion. For instance, Cheung and Masur constructed a half-translation surface (where we allow side identifications by translation and rotation of π) whose vertical flow is uniquely ergodic and the corresponding geodesic in the moduli space of Riemann surfaces diverges [4]. Not long after, Cheung and Eskin showed that if the geodesic diverges to infinity slowly enough, then the vertical linear flow is uniquely ergodic [3].

Chaika and Treviño found a different condition that implies unique ergodicity of the vertical linear flow on a translation surface [2]. Let $\delta(g_t\omega)$ be the systole on $g_t\omega$, which here we mean the shortest length of a non-contractible set of saddle connections. If $\int_0^\infty \delta^2(g_t\omega)\,dt$ diverges, then the vertical linear flow is uniquely ergodic. In short, the result says that the length of the shortest contractible set of saddle connections cannot get too short too quickly. The geodesic must stay sufficiently far from the boundary of the moduli space for sufficient time.

Our result differs in that it identifies a condition on the moduli space that is equivalent to a quantitative density condition on the linear flow. This is analogous to results in homogeneous dynamics seek to quantify the density of orbits. For example, the quantitative version of the Oppenheim conjecture seeks to give explicit quantitative information about the density of the orbits of unipotent flows [5], [6], [10]. Recently Lindenstrauss, Margulis, Mohammadi, and Shah gave effective bounds on time that the unipotent flow can spend near homogeneous subvarieties of an arithmetic quotient G/Γ [9]. Their motivation for this is to be able to prove quantitative density statements about unipotent flows in this setting.

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2. Proof of theorem

Lemma 2.1. Let $d(\omega)$ be the diameter of the translation surface ω. Then d is continuous. Furthermore, if d is bounded on a set $K \subset \mathcal{H}(\kappa)$, then K is compact.

Proof. Let ω_n be a sequence of translation surfaces whose diameter is bounded by D. Following Masur and Smillie, we take a Delauney triangulation of each surface. Then the edge lengths in the triangulation are bounded by twice the diameter of the surface [13], hence $2D$. Thus, since each edge length is bounded, we can construct a limiting surface ω_∞. ■

Lemma 2.2. Let ω denote a translation surface and g_t the geodesic flow. If there exists a compact set K such that $g_t\omega \in K$ for all $t > 0$, then the vertical (north or south) linear flow on ω is superdense.

Proof. Let ω denote a translation surface such that $g_t\omega$ is a subset of a compact set K for all $t \in \mathbb{R}$. Then the diameter has a maximum on K. Call it D.

For any $r > 0$ and any $\varepsilon > 0$, let $N = Dr$ and let Φ_s denote the vertical linear flow on ω. Consider the vertical straight line segment starting at any point on the surface $L_{\varepsilon N} := \{\Phi_s : s \in [0, \varepsilon N]\}$. Note that the length of $L_{\varepsilon N}$ is εN.

Apply $g_{\log(N)}$ to ω. See Figure 3.

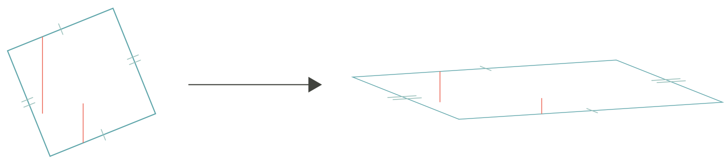

FIGURE 3. Apply $g_{\log(N)}$

Let $g_{\log(N)}L_{\varepsilon N}$ denote the image of $L_{\varepsilon N}$ under the action. Then $g_{\log(N)}L_{\varepsilon N}$ has length ε. Since $g_{\log(N)}\omega \in K$, the diameter of $g_{\log(N)}\omega$ is bounded by D. Let $U = \{x \in g_{\log(N)}\omega : \mathrm{dist}(x, g_{\log(N)}L_{\varepsilon N}) < D\}$ and notice that U covers $g_{\log(N)}\omega$. Apply $g_{-\log(N)}$ to U, and we have a set that covers ω. See Figure 4.

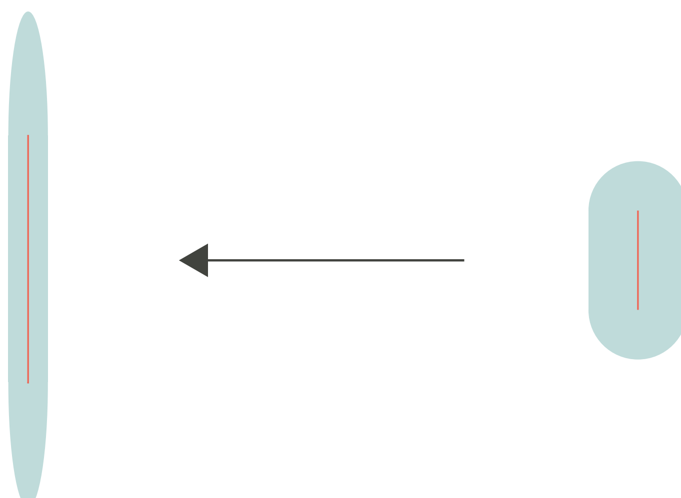

FIGURE 4. Apply $g_{-\log(N)}$ to U

Extend the vertical geodesic segment $L_{\varepsilon N}$ by DN in the positive and negative directions and reparametrize by $\tilde{s} = s + DN$. Let L_D be the reparametrized curve, so that $L_D = \{\Phi_{\tilde{s}} : \tilde{s} \in [0, \varepsilon N + 2DN]\}$. Notice that $[0, \varepsilon N + 2DN] = [0, (\varepsilon D + 2D^2)r]$.

Now, observe that for any $x \in \omega$, $\mathrm{dist}(x, L_D) < \frac{D}{N} = \frac{1}{r}$. For any $r > 0$, the vertical segment $[0, (\varepsilon D + 2D^2)r]$ is within $\frac{1}{r}$ of every point on the surface. The vertical linear flow is superdense as desired. ∎

Lemma 2.3. If the vertical linear (north or south) flow on ω is superdense, then there exists a compact set K such that $g_t\omega \in K$ for all $t > 0$.

Proof. Let ω be such that the vertical linear flow Φ_s is superdense. Then, for any intial point on the surface, there exists a constant C such that for any $r > 0$, the vertical (north or south) segment $L_{Cr} := \{\Phi_s : s \in [0, Cr]\}$ of length Cr is within $\frac{1}{r}$ of every point on ω. Let $U = \{x \in \omega : \mathrm{dist}(x, L_{Cr}) < \frac{1}{r}\}$ and note that U covers ω.

Apply $g_{\log(t)}$ to U and notice that for every $t > 1$, we have a cover of $g_{\log(t)}\omega$. The diameter of $g_{\log(t)}\omega$ is bounded by either $D = \sqrt{\left(\frac{Cr}{t}\right)^2 + \left(\frac{2t}{r}\right)^2}$ or $D' = \frac{Cr}{t} + \frac{2}{rt}$. See Figure 5.

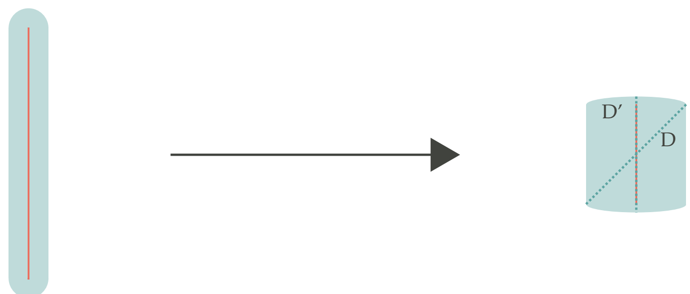

FIGURE 5. Apply $g_{\log(t)}$ to U

If $D \geq D'$, we can pick $r = \frac{\sqrt{2t}}{\sqrt{C}}$ and note that the diameter is bounded by $4C$. If $D' > D$, we can find an $r > 0$ such that the diameter is bounded by $C + 2$.

Thus, for any $t > 0$, there exists an $r > 0$ such that the diameter is bounded. The forward time trajectory is contained in a compact set. ∎

Theorem 1.2 follows immediately from Lemma 2.2 and Lemma 2.3.

REFERENCES

[1] J. Beck and W. Chen, *Superdensity of non-integrable billiards and Diophantine approximation*, preprint (2021).

[2] J. Chaika and R. Treviño, *Logarithmic laws and unique ergodicity*, Journal of Modern Dynamics **11 (1)** (2017), 563-588.

[3] Y. Cheung and A. Eskin, *Unique Ergodicity of Translation Flows*, Fields Institute Communications **51** (2007), 213-222.

[4] Y. Cheung and H. Masur, *A divergent Teichmüller geodesic with uniquely ergodic vertical foliation*, Israel Journal of Mathematics **152(1)** (2006), 1-15.

[5] A. Eskin, G. Margulis, and S. Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, Annals of Mathematics **147** (1998), 93-141.

[6] ———, *Quadratic forms of signature (2,2) and eigenvalue spacings on regular 2-tori*, Annals of Mathematics **1661** (2005), 679-725.

[7] S. Kerckhoff, H. Masur, and J. Smillie, *Interval exchange transformations and measured foliations*, Annals of Mathematics **124** (1986), 293-311.

[8] D. Kleinbock and B. Weiss, *Bounded geodesics in moduli space*, International Mathematics Research Notices **30** (2004), 1551-1560.

[9] E. Lindenstrauss, G. Margulis, A. Mohammadi, and N. Shah, *Quantitative behavior of unipotent flows and an effective avoidance principle*, preprint (2019).

[10] G. Margulis and A. Mohammadi, *Quantitative version of the Oppenheim conjecture for inhomogeneous quadratic forms*, Duke Mathematical Journal **158(1)** (2011), 121-160.

[11] H. Masur, *Interval exchange transformations and measured foliations*, Annals of Mathematics **115** (1982), 169-200.

[12] ______, *Hausdorff dimension of the set of nonergodic folitations of a quadratic differential*, Duke Mathematical Journal **66, no.3** (1992), 387-442.
[13] H. Masur and J. Smillie, *Hausdorff Dimension of sets of nonergodic measured foliations*, Annals of Mathematics **134** (1991), 455-543.
[14] A. Wright, *Translation Surfaces and their Orbit Closures*: *An Introduction for a Broad Audience*, EMS Surv. Math. Sci. (2015).
[15] A. Zorich, *Flat Surfaces*, Frontiers in Number Theory, Physics, and Geometry **Vol. 1** (2006).

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