

16.8 Stokes' TheoremREM Green's Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

In 16.5, we noticed that we could rewrite this expression using the curl operator, provided we think of \vec{F} as a "3-dim" vector field, in other words

$$\vec{F} = P\hat{i} + Q\hat{j} \xrightarrow{\text{"think of as" }} \vec{F} = P\hat{i} + Q\hat{j} + O\hat{k}$$

O in the \hat{k}
↓ direction!

The expression reduced to

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}(\vec{F}) \cdot \hat{k} dA.$$

Notice that the vector field \vec{F} is constrained to move only in the xy -plane (and planes parallel to the xy -plane). This means \hat{k} is the direction normal to the movement in the vector field.

This is actually true in a more general setting:

THM (Stokes' Theorem)

Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \vec{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \iint_S \operatorname{curl}(\vec{F}) \cdot \hat{n} dS.$$

EXERCISE Compare this to Green's Theorem. Derive Green's Theorem from Stokes' theorem. (What do C and S need to be? What does the vector field need to be?)

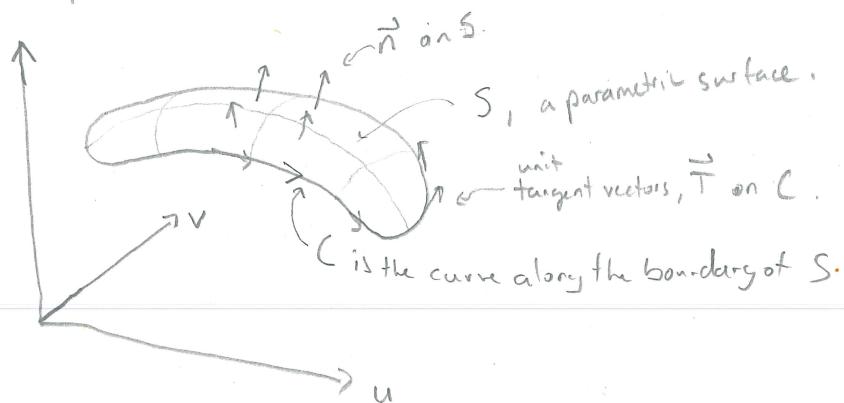
$\Rightarrow C$ needs to be a simple, closed, piecewise smooth curve in \mathbb{R}^2 .

$\Rightarrow S$ needs to be the region bounded by C in \mathbb{R}^2

$\Rightarrow \vec{F}$ needs to have no \hat{k} component

\Rightarrow This shows we can think of Green's Theorem as a consequence of Stokes' Theorem, or equivalently, as Stokes' Theorem being a generalization of Green's Theorem!

PICTURE



REMARK

The key idea to notice is that Stoke's Theorem is relating an integral over a region to an integral over a boundary.

Just like Green's Theorem, we need to recognize Stoke's Theorem as a generalization of the Fundamental theorem of Calculus:

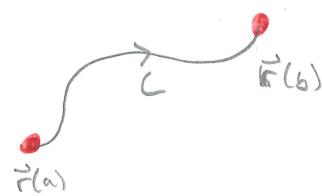
FTC

$$\int_a^b F'(x) dx = F(\underline{b}) - F(\underline{a})$$



Fundamental Theorem
of Line Integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\underline{b})) - f(\vec{r}(\underline{a}))$$



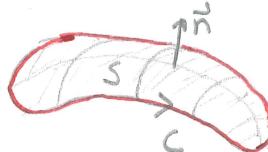
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P dx + Q dy$$



Stoke's Theorem

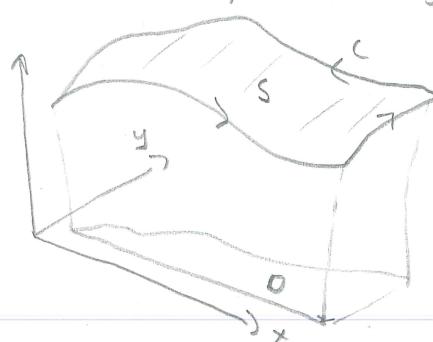
$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$



Proof of Stoke's Theorem (special case: S is the graph of a function)

Assume S is the graph of a function, so $\vec{r}(x,y) = x\hat{i} + y\hat{j} + z(x,y)\hat{k}$

and $(x,y) \in D$:



Also, assume $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$, where partial derivatives of P, Q, R are continuous.

We can compute:

$$\begin{aligned}\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} &= \iint_D \text{curl}(\vec{F}) \cdot \vec{n} dS \\ &= \iint_D \text{curl}(\vec{F}(\vec{r}(x,y))) \cdot \vec{n} \cdot |\vec{r}_x \times \vec{r}_y| dA \\ &= \iint_D \text{curl}(\vec{F}(\vec{r}(x,y))) \cdot \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} \cdot |\vec{r}_x \times \vec{r}_y| dA \\ &= \iint_D \text{curl}(\vec{F}(\vec{r}(x,y))) \cdot (\vec{r}_x \times \vec{r}_y) dA\end{aligned}$$

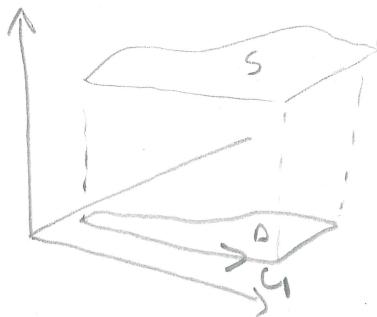
$$\text{curl}(\vec{F}(\vec{r}(x,y))) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

$$\begin{aligned}\vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = -\frac{\partial z}{\partial x} \hat{i} + \frac{\partial z}{\partial y} \hat{j} + \hat{k} \\ (\vec{r} &= x\hat{i} + y\hat{j} + z(x,y)\hat{k})\end{aligned}$$

Then $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \cdot \left\langle -\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, 1 \right\rangle dA$

$$= \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} + -\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dA$$

Next, we need to know about the boundary. Assume we can parametrize the boundary of D as follows ((call it C_1))



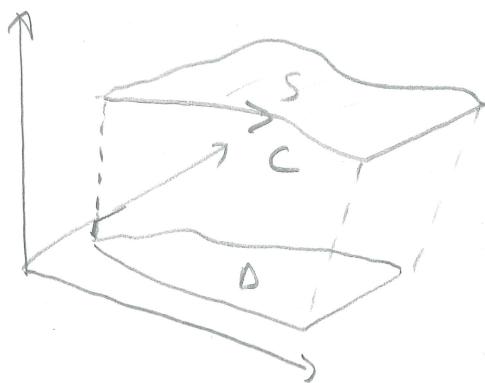
boundary of D : C_1

$$\begin{cases} x = x(t) \\ y = y(t), \quad a \leq t \leq b \end{cases}$$

Then the boundary of S , our curve C , can be parametrized by

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(x(t), y(t)), \quad a \leq t \leq b \end{cases}$$

$$\Rightarrow \vec{r}_2(t) = \langle x(t), y(t), z(x(t), y(t)) \rangle$$



Then, we can compute:

$$\begin{aligned}
 \oint_C \vec{F} \cdot d\vec{r} &= \int_{t=a}^{t=b} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}'_2(t) dt \\
 &= \int_a^b \langle P, Q, R \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \right\rangle dt \\
 &= \int_a^b P \frac{dx}{dt} + Q \frac{dy}{dt} + R \left(\frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \right) dt \\
 &= \int_a^b \left(\left(P + R \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left(Q + R \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right) dt \\
 &\quad \left\{ \begin{array}{l} \text{"un" parametrize. (look at this step} \\ \text{backwards! } x(t) \rightarrow dx = x'(t)dt \\ x'(t) = \frac{dx}{dt} \end{array} \right. \\
 &= \int_C \left(P + R \frac{\partial z}{\partial x} \right) dx + \left(Q + R \frac{\partial z}{\partial y} \right) dy \\
 &\quad \left\{ \begin{array}{l} \text{why did we do that? To apply Green's} \\ \text{Theorem!} \end{array} \right. \\
 &\quad \text{this is } C_1, \text{ not } C! \\
 &\quad \text{(why?)} \\
 &= \iint_D \left(\frac{\partial}{\partial x} \left(Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial z}{\partial x} \right) \right) dA \\
 &\quad \left\{ \begin{array}{l} \text{chain rule!} \\ \text{rem that } Q(x, y, z) = Q(x, y, \underline{z(x, y)}) \text{ (same for } P, R!) \end{array} \right. \\
 &= \iint_D \left[\underbrace{\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} \right)}_{\frac{\partial}{\partial x}(Q)} + \underbrace{\frac{\partial R}{\partial x} \frac{\partial z}{\partial y}}_{\frac{\partial}{\partial x}(R) \cdot \frac{\partial z}{\partial y}} + \underbrace{\frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}}_{R \cdot \frac{\partial}{\partial x}(\frac{\partial z}{\partial y})} + R \frac{\partial^2 z}{\partial x \partial y} \right] dA \\
 &\quad \text{lots of cancellations} \\
 &\quad - \left(\underbrace{\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y}}_{-} + \underbrace{\frac{\partial R}{\partial y} \frac{\partial z}{\partial x}}_{\frac{\partial}{\partial y}(R) \cdot \frac{\partial z}{\partial x}} + \underbrace{\frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x}}_{R \cdot \frac{\partial^2 z}{\partial y \partial x}} + R \frac{\partial^2 z}{\partial y \partial x} \right] dA
 \end{aligned}$$

$$= \iint_D \left[\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) + \left(\frac{\partial R}{\partial z} - \frac{\partial P}{\partial x} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} \right] dA$$

↗
matches our computation for $\iint_S \operatorname{curl}(\vec{F}) \cdot \vec{dS}$!

\Rightarrow This proves Stokes' Theorem for graphs of functions!

EXAMPLE 1 Evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = -y^2 \hat{i} + x \hat{j} + z^2 \hat{k}$

and C is the curve of the intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. (Orient C to be counterclockwise when viewed from above.)

$$\oint_C \vec{F} \cdot d\vec{r} = \dots \quad \text{use Stokes' Theorem}$$

- Notice, first order partials of P, Q, R are defined and continuous in all of \mathbb{R}^3
- The curve C has the correct orientation, plus it is simple and closed!

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot \vec{n} dS$$

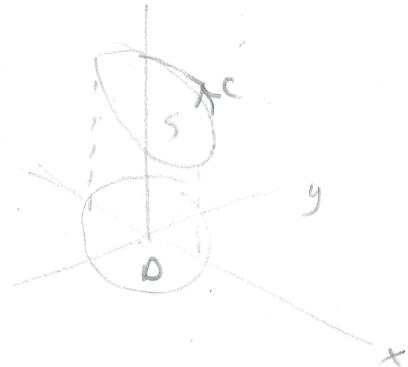
STEP1 Compute the curl

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (1+2y)\hat{k}$$

$$= (1+2y)\hat{k}$$

STEP2 Parametrize S (Notice it is the graph of a function!)

$$\begin{aligned} S: \quad x &= x \\ y &= y \\ z &= 2-y \end{aligned}$$



$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + (2-y)\hat{k}$$

$$D = \{(x,y) : x^2 + y^2 \leq 1\}$$

STEP3 Notice:

$$\iint_S \text{curl}(\vec{F}) \cdot \vec{n} dS = \iint_D \text{curl}(\vec{F}) \cdot (\vec{r}_x \times \vec{r}_y) \cdot |\vec{r}_x \times \vec{r}_y| dA$$

and $\vec{r}_x \times \vec{r}_y = -\frac{\partial z}{\partial x}\hat{i} + -\frac{\partial z}{\partial y}\hat{j} + \hat{k}$ (computed this a few times now!)

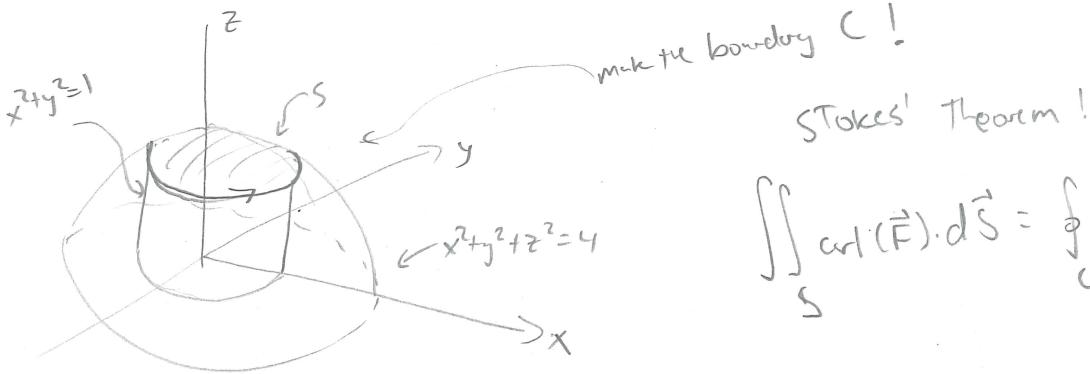
$$= 0\hat{i} + -1\hat{j} + 1\hat{k}$$

STEP4 Compute!

$$\iint_S \text{curl}(\vec{F}) \cdot \vec{dS} = \iint_D \text{curl}(\vec{F}) \cdot (\vec{r}_x \times \vec{r}_y) dA$$

$$\begin{aligned}
 &= \iint_D (1+2y) dA \\
 &\quad \text{D is a disk!} \\
 &\quad \text{use polar coordinates!} \\
 &= \int_0^{2\pi} \int_0^1 (1+2r\sin\theta) \cdot r dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 (r + 2r^2\sin\theta) dr d\theta \\
 &= \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{2r^3}{3} \sin\theta \right]_0^1 d\theta \\
 &= \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3} \sin\theta \right] d\theta \\
 &= \left[\frac{1}{2}\theta - \frac{2}{3}\cos\theta \right]_0^{2\pi} \\
 &= \left(\pi - \frac{2}{3} \right) - \left(0 - \frac{2}{3} \right) \\
 &= \boxed{\pi}
 \end{aligned}$$

EXAMPLE 2 Use Stokes' Theorem to compute $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x,y,z) = xz\hat{i} + yz\hat{j} + xy\hat{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy -plane.



$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

STEP 1 Parametrize the boundary:

Find the intersection of the cylinder and the sphere

$$\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 + z^2 = 4 \end{cases} \Rightarrow 1 + z^2 = 4 \\ z^2 = 3$$

$$z = \pm \sqrt{3}$$

We are looking for positive z , so $\boxed{z = \sqrt{3}}$.

Then, our parametrization follows $x^2 + y^2 = 1$ at a height of $\sqrt{3}$:

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = \sqrt{3} \end{cases}, \quad 0 \leq t \leq 2\pi$$

$$\text{So, } \vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + \sqrt{3} \hat{k}$$

$$\text{and } \vec{r}'(t) = -\sin t \hat{i} + \cos t \hat{j} + 0 \hat{k}$$

STEP 2 Plug parametrization: $\vec{F}(\vec{r}(t)) = (\cos t)\sqrt{3} \hat{i} + (\sin t)\sqrt{3} \hat{j} + \cos t \sin t \hat{k}$

STEP 3 Compute!

$$\begin{aligned}
 \oint_C \vec{F} \cdot d\vec{r} &= \int_{t=0}^{t=2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \\
 &= \int_0^{2\pi} \langle \sqrt{3} \cos t, \sqrt{3} \sin t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt \\
 &= \int_0^{2\pi} (-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t) dt \\
 &= -\int_0^{2\pi} 0 dt \\
 &= 0
 \end{aligned}$$

eg. $x^2 + y^2 + z^2 = 1$

QUESTION If S is a sphere and \vec{F} is a vector field that satisfies the conditions of Stokes' Theorem, what is $\iint_S \text{curl}(\vec{F}) \cdot \vec{n} dS$?

EXERCISES #1; #2, 5; #7, 9; #15, #16, #18-20. (16.8)