

Lecture Notes: Week #1
Logical Reasoning

We begin our course by thinking about conditional logic, which is, in some very real sense, how we do math. This kind of logic shows up in every math course. Here are a few examples, with the conditional language boxed in red:

(From linear algebra, e.g. Math 308)

THM Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ be a set of vectors in \mathbb{R}^n . Then this set is linearly dependent if and only if one of the vectors in the set is in the span of the other vectors.

THM Let T be a linear transformation. Then T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$.

(From vector calculus, e.g. Math 324)

THM $\int_{\tilde{C}} \vec{F} \cdot d\vec{r}$ is independent of path if and only if $\int_C \vec{F} \cdot d\vec{r} = \vec{0}$ for every closed path C .

(From differential calculus, e.g. Math 124)

THM (L'Hopital's rule)

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\left[\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0 \right]$$

Or that

$$\left[\lim_{x \rightarrow a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm \infty \right]$$

then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists.

You do not need to be familiar with what these theorems mean (except L'Hopital's rule), but take a moment and think carefully about the role of the boxed red language. How is it "conditional"? You may have also noticed some boxed green language in L'Hopital's rule. Take a moment

to think carefully about that language as well - we will be discussing it soon enough.

We won't be proving statements quite as involved as those above, but by the end of this course you should be able to read proofs of these statements and follow the logic.

Our more immediate goal is to understand the red-boxed language, the conditional logic. Conditional logic is built around a conditional statement:

(*) If [Something], then [Something else]. (*)

Let's look at a concrete example.

EXAMPLE 1 "If you got an A on the exam, then you studied."

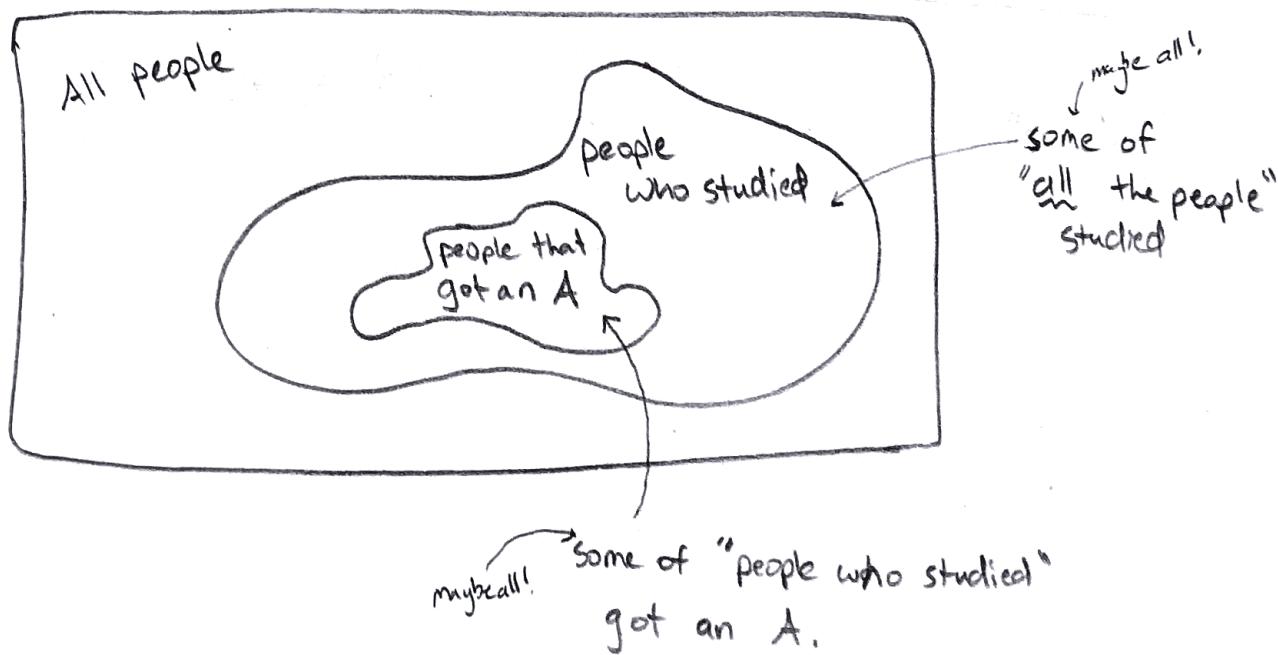
Our goal is to understand what this says, and perhaps more importantly, what it does not say.

First, let's try to understand what it does say. From the statement, we can say that if we observe that someone got an A on the exam, then we know that they studied. In other words:

"You got an A on the exam only if you studied."

We could also say it like this: the people who got an A on the exam are a "subset" of the people who studied. (We will come back to this word "subset" in a few weeks.)

We can diagram this last interpretation:



Looking at the diagram, we can conclude the following:

"If you did not study, you did not get an A."

This statement is logically equivalent to the first statement!
(It carries the same meaning.) However, it does look significantly different. We have a special name for this statement: it is the ^(*) contrapositive ^(*) of our first statement.

It comes from negating each part of the statement, and swapping the implication. To negate a statement, we add "not." We can diagram this out:

means "implies"
{}
got an A \Rightarrow studied

Read this as, "If you got an A, then you studied." Or, you can read this as "You got an A implies that you studied."

Now, we can negate each part:

Left side: did not get an A

Right side: did not study

And then we swap the implication:

$$\text{did } \underline{\text{not}} \text{ get an A} \leftarrow \text{did } \underline{\text{not}} \text{ study}$$

{swapped the direction of implication!}

Or, if you would like, we can write it like this:

$$\text{did } \underline{\text{not}} \text{ study} \Rightarrow \text{did } \underline{\text{not}} \text{ get an A}$$

In either case, we can read this as, "If you did not study, then you did not get an A." Or, "You did not study implies that you did not get an A."

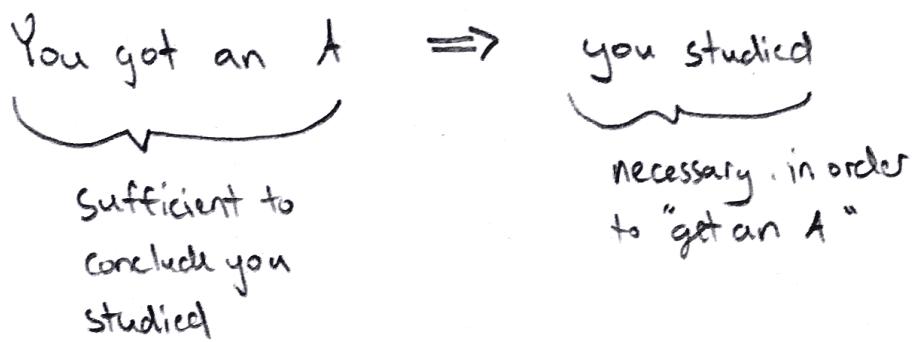
Homework Question 1

Write the contrapositive of each statement in "if-then form."

- If a is irrational, then a^2 and $2a$ are irrational.
- r being a rational number implies r^2 is a rational number.
- The candidate passes the driver's test only if the candidate can parallel park.

Now, let's think about this slightly differently. The statement "If you got an A, then you studied," means that it is necessary to study in order to get an A,

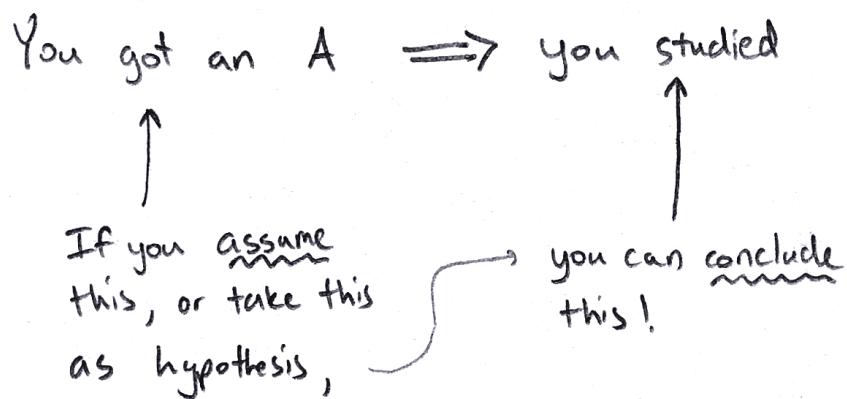
but it is not sufficient (in other words, studying does not guarantee an A). However, if we know someone got an A, that is sufficient to conclude that they studied. We can diagram this as follows:



Later in class, you may find it helpful to think about what is necessary and what is sufficient when trying to prove things.

Okay, so that is a lot. Let's do one more interpretation, then we will summarize. For some people, this next interpretation is the most natural, especially if you think

about mathematics as a process of discovery. Namely, if we assume something, can we conclude another? Given our statement, we can think of the left side as an hypothesis or assumption, and think of the right side as the conclusion given this hypothesis/assumption:



Later, when we begin proving mathematical statements like this, you will want to use the assumption to "exhibit" the conclusion.

Homework Question 2

Identify the hypothesis and conclusion in each statement below.

- n^2 is odd whenever n is an odd integer.
- If n is a positive integer, then the sum of the first n positive integers is $\frac{n(n+1)}{2}$.
- In order to pass the vision test, it is sufficient for the candidate to read the line Q Z S P M W H.

Now, let's summarize. We'll add a few extra common ways to express a conditional statement. Here, we will do it symbolically, where "A" is a statement like "you got an A on the exam" and "B" is a statement like "you studied."

$$A \Rightarrow B$$

1. If A, then B.
2. If A is true, then B is also true.
3. A is true only if B is true.
4. A implies B.
5. B is true whenever A is true.
6. For A to be true, it is necessary for B to be true.
7. For B to be true, it is sufficient for A to be true.

Homework Question 3

Write the following statements in the logically equivalent "If-then" form (form 1 above).

- a) $(n-1)(n-2) = 0$ whenever $n=1$.
- b) $n=2$ is sufficient for $n^2-n-2 = 0$.
- c) $n=2$ is necessary for $n^2-4n+4 = 0$.

Great. So now, let's talk about what our example conditional statement does not say.

Here's the example again:

EXAMPLE 1 "If you got an A on the exam, then you studied."

This DOES NOT SAY: "If you studied, then you got an A on the exam."

In fact, this second statement has a special name.

We call it the converse of the first statement.

DEF The converse of the conditional statement

$A \Rightarrow B$ is $B \Rightarrow A$.

(1) **WARNING!** (2)

A conditional statement and its converse

are not logically equivalent. In other words, just because one of them is true DOES NOT mean the other is true.

It could be that both $A \Rightarrow B$ and $B \Rightarrow A$ are true, but knowing only $A \Rightarrow B$ DOES NOT mean we can conclude $B \Rightarrow A$.

Whenever you have a conditional statement whose implication works in both direction, i.e.

$$A \Rightarrow B$$

and

$$B \Rightarrow A$$

then we often say

A is true $\overset{(*)}{\boxed{\text{if and only if}}}$ $\overset{(**)}{B}$ is true

or

A if and only if B.

Here "A only if B" is the same as "If A, then B."

You can think of "A if B" as "If B, then A."

In other words, we really are saying that A implies B and B implies A when we say A if and only if B.

NOTATION Occasionally you may see "iff." This is shorthand notation for "if and only if."

To see why it is important that we recognize that a conditional statement and its converse are not logically equivalent, consider the following.

DEF We say a finite collection of integers n_1, n_2, \dots, n_m is mutually relatively prime if their greatest common denominator is 1.

THM The integers n_1, n_2, \dots, n_m are mutually relatively prime if and only if the equation

$$n_1x_1 + n_2x_2 + \dots + n_mx_m = 1$$

has a solution in integers x_1, x_2, \dots, x_m .

If we were told to prove the theorem, we would need to first show

① integers n_1, \dots, n_m mutually relatively prime implies the equation has a solution in integers.

Once we have shown this, we have proven one implication.
Since the converse of ①:

② The equation has a solution in integers
implies the integers n_1, \dots, n_m are mutually
relatively prime.

is not logically equivalent to ①, we must show
this also. Once we have shown it, then we have
proven the implication in both directions, hence, we
have proven an if and only if statement.

REMARK You do not need to memorize the definition or
theorem on the previous page! It is only there
so that we can think about the conditional
logic embedded in a mathematical statement.

Homework Question 4

Write the converse and contrapositive of each
of the following:

a) If $\sqrt{2} < \sqrt{5}$, then $2 < 5$.

b) If $2 \geq 5$, then $\sqrt{2} \geq \sqrt{5}$

We now need to take a moment and review the words "and" and "or", from a purely logical perspective, then we will start using these terms with our conditional logic. (Actually, we have already used "and" in a crucial way, did you catch it?)

DEF

We say that a proposition (or statement) is a sentence that is either true or false.

Notice, we have been thinking about conditional statements so far — these are fairly complex statements. Right now, we want to consider simpler statements, such as

$$2+3=6 \leftarrow \text{False!}$$

or

$$5-1=4 \leftarrow \text{True!}$$

(*) **AND** (*)

Let A and B be two statements. Then,

if

① A is true, B is true, then A and B is a true statement.

EXAMPLE $2+3=5 \leftarrow$ true

$4-1=3 \leftarrow$ true

then $(2+3=5 \text{ and } 4-1=3) \leftarrow$ true!

② A is true, B is false, then A and B is a false statement

EXAMPLE Some dogs are brown. \leftarrow true

Josh doesn't want dinner. \leftarrow false

then (Some dogs are brown and Josh doesn't want dinner) is false!

③ A is false, B is true, then A and B is a false statement.

EXAMPLE $2 \cdot 3 = 7 \leftarrow$ False

4 divides 96. \leftarrow true

then $(2 \cdot 3 = 7 \text{ and } 4 \text{ divides } 96)$ is false.

④ A is false, B is false, then A and B is a false statement!

EXAMPLE $2 \cdot 3 = 5 \leftarrow$ false

4 divides 101 \leftarrow false

then $(2 \cdot 3 = 5 \text{ and } 4 \text{ divides } 101)$ is false.

We can summarize this in a "truth table."

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

Similarly,

(*) **OR** (*)

Let A and B be two statements. Then, we summarize the "truth" of A or B with the following "truth table".

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

Homework Question 5

Provide examples of A and B for each line of the truth table for "or" (similar to what we did for "and" before we gave a truth table).

Homework Question 6

Explain how the following indeterminant forms can be deduced from the hypothesis of L'Hopital's rule on p. 2 of these notes.

Indeterminant forms : $\frac{0}{0}$, $\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{-\infty}{-\infty}$.

Note The expression:

$$\lim_{x \rightarrow a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm \infty$$

is unclear. You may take it to mean:

$$\left[\lim_{x \rightarrow a} f(x) = +\infty \text{ or } \lim_{x \rightarrow a} f(x) = -\infty \right] \text{ and } \left[\lim_{x \rightarrow a} g(x) = +\infty \text{ or } \lim_{x \rightarrow a} g(x) = -\infty \right]$$

Homework Question 7 (thank you LSAC!)

There are exactly three recycling centers in Rivertown: Center 1, Center 2, and Center 3. Exactly five kinds of material are recycled at these recycling centers: glass, newsprint, plastic, tin, and wood. Each recycling center recycles at least two, but no more than three of these kinds of materials.

The following conditions must hold:

- ① Any recycling center that recycles wood also recycles newsprint.
- ② Every kind of material that Center 2 recycles is also recycled at Center 1.
- ③ Only one of the recycling centers recycles plastic, and that recycling center does not recycle glass.

(a) Which one of the following could be an accurate account of all the kinds of material recycled at each recycling center?

- ① Center 1: newsprint, plastic, wood
Center 2: newsprint, wood
Center 3: glass, tin, wood

② Center 1: glass, newsprint, tin
Center 2: glass, newsprint, tin
Center 3: newsprint, plastic, wood

③ Center 1: glass, newsprint, wood
Center 2: glass, newsprint, tin
Center 3: plastic, tin

④ Center 1: glass, plastic, tin
Center 2: glass, tin
Center 3: newsprint, wood

⑤ Center 1: newsprint, plastic, wood
Center 2: newsprint, plastic, wood
Center 3: glass, newsprint, tin

(b) Which one of the following is a complete and accurate list of recycling centers in Rivertown any one of which could recycle plastic?

- ① Center 1 only
- ② Center 3 only
- ③ Center 1, Center 2
- ④ Center 1, Center 3
- ⑤ Center 1, Center 2, Center 3

(C) If center 2 recycles three kinds of material, then which one of the following kinds of material must center 3 recycle?

- ① glass
- ② newsprint
- ③ plastic
- ④ tin
- ⑤ wood

(d) If each recycling center in Rivertown Recycles exactly three kinds of materials, then which one of the following could be true?

- ① Only center 2 recycles glass.
- ② Only center 3 recycles newsprint
- ③ Only center 1 recycles plastic
- ④ Only center 3 recycles tin.
- ⑤ Only center 1 recycles wood.

(e) If Center 3 recycles glass, then which one of the following kinds of material must Center 2 recycle?

- ① glass ④ tin
- ② newsprint ⑤ wood
- ③ plastic

(f) If Center 1 is the only recycling center that recycles wood, then which of the following could be a complete and accurate list of the kinds of material that one of the recycling centers recycles?

- ① plastic, tin
 - ② newsprint, wood
 - ③ newsprint, tin
 - ④ glass, wood
 - ⑤ glass, tin
-

One last remark, related to whether or not a statement is True or False... In the same way we made a "truth table" for "and" and "or", we can make one for an "implication":

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

? these are weird!

It is the top two lines of this truth table that we should be concerned about. The bottom two lines are a bit odd. They basically tell us:

If A, then B

is a true statement when A is false and B is true (or false).

For example:

A = Cats can fly ← false

B = $2+2=4$ ← true

then

"If cats can fly, then $2+2=4$."

is a true statement. This is weird, but we can explain it like this: the statement is **vacuous**, i.e. the world in which cats can fly is empty, doesn't exist.

Logically, then, the condition, "If cats can fly," applies to an "empty set," and for a set with nothing in it, everything is true! (You can't find a counterexample!)

You do not need to worry too much about this now, but you should be aware of this idea of a "vacuous truth." I will try to point it out when it becomes an issue for us.