see previous lecture notes for precise statements!

REM (16.3)

We have the following implications:

① Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

where C is a smooth curve given by $\vec{r}(t)$, for $a \le t \le b$.

Notice, this means if $\vec{F} = \nabla f$ (is conservative), then the line integral is independent of path.

② "Independent of path $\iff \int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop C"

③ "Independent of path $\implies$ Conservative."

④ "Conservative $\implies \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$"

⑤ "$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and D simply connected $\implies$ Conservative"

16.4 Green's Theorem

DEF (positive orientation)

We say a positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.

THM (Green's Theorem)

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

REMARK It may not look like it, but this is a generalization of the Fundamental theorem of Calculus. Rem:

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

Notice that the left-hand side is evaluation along boundary points (which in this case, are just a and b). The right side is an integral of a derivative of that function. Compare that with Green's Theorem.

REMARK We will see two other generalizations of the Fundamental Theorem of Calculus, and in each one, we will see the same relationship between "boundary" and "derivatives". This is suggestive of a bigger picture just out of sight!

EXAMPLE 1

Evaluate $\oint_C \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$, where C is the circle $x^2+y^2=9$.

> **NOTE** $\oint_C$ just means that C is a <u>closed loop</u> that is <u>positively oriented</u>

SOLUTION

STEP 1: Do NOT parametrize! Instead, make sure $P(x,y)$ and $Q(x,y)$ are well-defined on the circle

$$\left(\underset{3y - e^{\sin(x)}}{\overset{P(x,y)}{\shortparallel}} \qquad \underset{7x+\sqrt{y^4+1}}{\overset{Q(x,y)}{\shortparallel}} \right)$$

KEY: $\rightsquigarrow$ <u>and</u> the <u>inside</u> of the circle. You need for partial derivatives to exist on the entire disk.

(They are!)

STEP 2: Apply Green's Theorem

$$\frac{\partial P}{\partial y} = 3, \quad \frac{\partial Q}{\partial x} = 7, \quad D = \{(x,y) : x^2+y^2 \le 9\}$$

$$\oint_C \left(3y - e^{\sin(x)}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy = \iint_D (7-3)\,dA$$

$$= 4 \iint_D dA$$

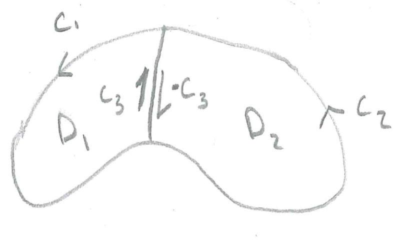
$$= 4 \cdot (\pi (3^2))$$

$$\boxed{= 36\pi}$$

(Extending Green's Theorem)

We can apply Green's Theorem to any set of finite simple regions.
(By simple region we just mean a region that can be integrated over with respect to x or to y first, and only yield one integral.
i.e. doesn't split into two integrals!)

How?

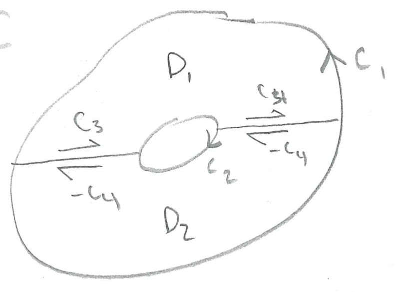


$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

and

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

or



"hole in the region"

If we add "C_3" and "C_4", we can break a region with a hole into two regions without a hole. We can apply Green's Theorem here.

EXERCISE Work EXAMPLES in 16.4.

REMARK We can also use Green's theorem to compute areas.
Notice

$$\iint_D 1 \cdot dA = A(D),$$

so if we pick $P(x,y)$, $Q(x,y)$ such that $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1\right)$, this computes an area!

EXAMPLE Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

SOLUTION Could compute $\iint_D dA$, where D is the ellipse, but it is actually (for me) a little quicker to use Green's Theorem. Notice, if we let

$$P(x,y) = \frac{-1}{2} y$$

$$Q(x,y) = \frac{1}{2} x$$

Then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{2} - \frac{-1}{2} = 1$. So,

$$A(D) = \iint_D 1 \cdot dA = \frac{1}{2} \oint_C x\,dy - y\,dx$$, where C is an ellipse!

$\Rightarrow$ parametrize:

$x(t) = a\cos t$

$y(t) = b\sin t$

$$= \frac{1}{2} \int_0^{2\pi} \Big((\underbrace{a\cos t}_{x(t)}) \cdot \underbrace{b\cos t}_{y'(t)} \Big) dt - \Big((\underbrace{b\sin t}_{y(t)}) (\underbrace{-a\sin t}_{x'(t)}) \Big) dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab\,(\cos^2 t + \sin^2 t)\,dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab\,dt$$

$$= \frac{1}{2} ab\,[2\pi] = \boxed{\pi ab}$$