(15) <u>Method 1</u> $\boxed{\vec{F} = \langle x, -z, y \rangle}$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(r(\phi,\theta)) \cdot \overbrace{(\vec{r}_\theta \times \vec{r}_\phi)}^{\text{Careful!}} \, dA = -\iint_D \vec{F}(r(\phi,\theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) \, dA$$

$$S: \vec{r}(\phi,\theta) = \langle r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi \rangle$$

$$D = \{(\phi,\theta) : 0 \le \phi \le \tfrac{\pi}{2}, 0 \le \theta \le \tfrac{\pi}{2}\}$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2} \langle r\sin\phi\cos\theta, -r\cos\phi, r\sin\phi\sin\theta \rangle \cdot \langle r^2\sin^2\phi\cos\theta, r^2\sin^2\phi\sin\theta, r^2\sin\phi\cos\phi \rangle \, (d\phi\, d\theta)$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2} (r^3\sin^3\phi\cos^2\theta + \cancel{-r^3\sin^2\phi\sin\theta\cos\phi + r^3\sin^2\phi\sin\theta\cos\phi}) \, d\phi\, d\theta$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2} (r^3 \underline{\sin^3\phi}\cos^2\theta) \, d\phi\, d\theta$$

$$= -r^3 \int_0^{\pi/2}\int_0^{\pi/2} (1-\cos^2\phi)\sin\phi \cos^2\theta \, d\phi\, d\theta$$

$$= -r^3 \int_0^{\pi/2} \cos^2\theta \left(\int_0^{\pi/2} (\sin\phi - \underline{\cos^2\phi\sin\phi}) \, d\phi \right) d\theta$$

$u = \cos\phi$
$du = -\sin\phi$

$$= -r^3 \int_0^{\pi/2} \cos^2\theta \left[-\cos\phi + \frac{\cos^3\phi}{3} \right]_0^{\pi/2} d\theta$$

$$= -r^3 \int_0^{\pi/2} \cos^2\theta \left[(0+0) - (-1 + \tfrac{1}{3}) \right] d\theta$$

$$= -\frac{2r^3}{3} \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{-2r^3}{3} \int_0^{\pi/2} \frac{1}{2}(1 + \cos(2\theta)) \, d\theta$$

$$= \frac{-r^3}{3} \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{-r^3}{3} \left[\theta + \frac{1}{2}\sin 2\theta \right]_0^{\pi/2}$$

$$= \frac{-r^3}{3} \left[\left(\frac{\pi}{2} + 0 \right) - (0+0) \right]$$

$$= \frac{-\pi r^3}{6}$$

<u>METHOD 2</u> (Graph of a function)

$$S: \ \vec{r}(x,y) = \langle x, y, \sqrt{r^2-x^2-y^2} \rangle$$

$$D = \{ (x,y) : x \geq 0,\ y \geq 0, \text{ and } x^2+y^2 \leq r^2 \}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x,y))\cdot \overbrace{(\vec{r}_y \times \vec{r}_x)}^{\text{careful!}}\, dA = -\iint_D \vec{F}(\vec{r}(x,y))\cdot(\vec{r}_x \times \vec{r}_y)\, dA$$

$$\vec{r}_x \times \vec{r}_y = \left\langle \frac{x}{\sqrt{r^2-x^2-y^2}}, \frac{y}{\sqrt{r^2-x^2-y^2}}, 1 \right\rangle$$

$$= -\iint_D \langle x, -\sqrt{r^2-x^2-y^2}, y \rangle \cdot \left\langle \frac{x}{\sqrt{r^2-x^2-y^2}}, \frac{y}{\sqrt{r^2-x^2-y^2}}, 1 \right\rangle dA$$

$$= -\iint_D \left(\frac{x^2}{\sqrt{r^2-x^2-y^2}} + \cancel{-y} + \cancel{y} \right) dA$$

$$= -\iint_D \frac{x^2}{\sqrt{r^2-x^2-y^2}}\, dA$$

use polar, but let "r" be α since r is already being used.

$$= -\int_0^{\pi/2}\int_0^r \frac{\alpha^2 \cos^2\theta}{\sqrt{r^2-\alpha^2}}\, \alpha\, d\alpha\, d\theta$$

$$\begin{cases} x = \alpha\cos\theta \\ y = \alpha\sin\theta \\ dA = \alpha\, d\alpha\, d\theta \end{cases}$$

$$= -\int_0^{\pi/2}\int_0^{r} \frac{\alpha^3 \cos^2\theta}{\sqrt{r^2-\alpha^2}}\, d\alpha\, d\theta$$

trig sub

$\alpha = r\sin\phi$

$d\alpha = r\cos\phi\, d\phi$ and $\begin{bmatrix} \alpha = 0 \Rightarrow \phi = 0 \\ \alpha = r \Rightarrow \phi = \pi/2 \end{bmatrix}$

$$= -\int_0^{\pi/2}\int_0^{\pi/2} \frac{r^3\sin^3\phi\cos^2\theta}{\sqrt{r^2 - r^2\sin^2\phi}} (r\cos\phi\, d\phi)\, d\theta$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2} \frac{r^4\sin^3\phi\cos\phi\cos^2\theta}{\sqrt{r^2\underbrace{(1-\sin^2\phi)}_{\cos^2\phi}}}\, d\phi\, d\theta$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2} \frac{r^4\sin\phi\cos\phi\cos^2\theta}{r\sqrt{\cos^2\phi}}\, d\phi\, d\theta$$

$$= -\int_0^{\pi/2}\int_0^{\pi/2} \frac{r^3\sin^3\phi\,\cancel{\cos\phi}\,\cos^2\theta}{\cancel{\cos\phi}}\, d\phi\, d\theta$$

$$= \underbrace{-r^3\int_0^{\pi/2}\int_0^{\pi/2} \sin^3\phi\cos^2\theta\, d\phi\, d\theta}$$

see Method 1

$$\boxed{= \frac{-\pi r^3}{6}}$$

u-sub

$u = r^2 - \alpha^2 \rightsquigarrow \alpha^2 = r^2 - u$

$du = -2\alpha\, d\alpha$ and $\begin{bmatrix} \alpha = 0 \Rightarrow u = r^2 \\ \alpha = r \Rightarrow u = 0 \end{bmatrix}$

$$= \frac{1}{2}\int_0^{\pi/2}\int_{r^2}^{0} \frac{(r^2-u)\cos^2\theta}{\sqrt{u}} \cdot du\, d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2} \cos^2\theta \left(\int_0^{r^2} \left(\frac{r^2-u}{\sqrt{u}}\right) du\right) d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2} \cos^2\theta \left(\int_0^{r^2} (r^2\cdot u^{-1/2} - u^{1/2})\, du\right) d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2} \cos^2\theta \cdot \left[r^2\cdot(2u^{1/2}) - \frac{2}{3}u^{3/2}\right]_0^{r^2} d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2} \cos^2\theta \left(2r^3 - \frac{2}{3}r^3\right) d\theta$$

$$= \frac{-1}{2}\cdot\frac{4}{3}r^3 \underbrace{\int_0^{\pi/2} \cos^2\theta\, d\theta}_{\text{(see Method 1)}}$$

$$= -\frac{2r^3}{3}\left(\frac{\pi}{2}\right)$$

$$= -\frac{2r^3\pi}{6}$$

$$\boxed{= \frac{-\pi r^3}{6}}$$