Lecture Notes: Week 3

Quantifiers, Cases, and some Set Theory

Our goal this week is to introduce the first definitions and operations in Set theory. We begin by thinking about quantifiers.

Universal Statements (For all)

Consider the following statements:

1. For all even integers, n, n^2 is even.
2. For all non-zero, real numbers r, $r^2 > 0$.
3. For any odd integer n, n^2 is odd.

These are examples of universal statements, statements that claim a property (e.g. n^2 is even, $r^2 > 0$, or n^2 is odd) is true of an entire collection of objects (e.g. all even integers, all non-zero, real numbers, or all odd integers). We can describe these kinds of statements slightly better with some set theory.

<u>DEF</u> A <u>set</u> is any well-defined collection of objects.

We can think of a set as a "box containing certain objects." We will allow the notion of a "box with nothing in it," i.e. the <u>empty set</u>. We will come back to this later.

<u>EXAMPLES</u> (of sets)

$\mathbb{Z}$: denotes the set of all integers $\{\ldots,-2,-1,0,1,2,\ldots\}$

$\mathbb{N}$: denotes the set of all positive integers, not including 0 $\{1,2,3,\ldots\}$

$\mathbb{Q}$: denotes the set of all rational numbers.

$\mathbb{R}$: denotes the set of all real numbers

$\mathbb{C}$: denotes the set of all complex numbers.

Notation We often denote sets with capital letters, e.g. A, or with curly brackets if we want to explicitly denote what is in the set, e.g. $\{1,2,3\}$ is a set containing 1, 2, and 3.

DEF We call the objects in a set the **elements** in the set, or **members** of the set, or on occasion, **points** in the set.

Notation We often use lower case letters to denote arbitrary elements in a set, e.g. "a is an element of A."

Notation We have shorthand notation for the expression "a is an element of A":

$$a \in A$$

think "is in" or "is an element of"

We also have a conditional definition of a set:

we can take a set we understand, then find a collection of elements that satisfy some condition. For example, we can define a set B so that B contains the integers strictly between 0 and 6. We often write such sets in this way:

$$B = \{ n \in \mathbb{Z} \mid 0 < n < 6 \}$$

"B is the set of integers n such that $\underbrace{0 < n < 6}_{\text{condition}}$."

Now, consider the following sets (and compare them to the universal statements on the first page!):

EXAMPLE 1 $\{ n \text{ is an even integer} \mid n^2 \text{ is even} \}$

This is the set of even integers such that n^2 is even.

EXAMPLE 2 $\{ r \text{ is a non-zero number in } \mathbb{R} \mid r^2 > 0 \}$

This is the set of non-zero real numbers in $\mathbb{R}$ such that $r^2 > 0$.

EXAMPLE 3 $\{n \text{ is an odd integer} \mid n^2 \text{ is odd}\}$

This is the set of odd integers n such that n^2 is odd.

Now notice, the examples above are not universal statements, but we can make them universal statements as follows:

EXAMPLE 1

$$\{n \text{ is an even integer} \mid n^2 \text{ even}\} = \{n \text{ is an even integer}\}.$$

Here "=" means that every element in the set on the left is an element of the set on the right, and vice-versa. We will give a different definition later, but for now, this will suffice.

This equality is saying that the set of even integers such that n^2 is even is exactly the same as the set of even integers. In other words, all even integers have the property that their square is even, which is a universal statement.

Similarly,

EXAMPLE 2

$$\{r \text{ is a non-zero number in } \mathbb{R} \mid r^2 = 0\}$$

$$= \{r \text{ is a non-zero number in } \mathbb{R}\}$$

and,

EXAMPLE 3

$$\{n \text{ is an odd integer} \mid n^2 \text{ is odd}\} = \{n \text{ is an odd integer}\}.$$

are both universal statements. If you need, take a moment to think through this carefully.

Now, let's think about two questions:

1. How do we prove a universal statement?
2. How do we show a universal statement is false?

How to prove (true)

Let $E = \{n \in \mathbb{Z} \mid n \text{ is even}\}$. Now, let's rephrase our first example sentence:

"For all even integers n, n^2 is even."

set-theoretically:

$$\{n \in E \mid n^2 \in E\} = E$$

We need to show that every $x \in \{n \in E \mid n^2 \in E\}$ is also in E, and vice-versa, every $x \in E$ is also in $\{n \in E \mid n^2 \in E\}$.

Notice, if $x \in \{n \in E \mid n^2 \in E\}$, it is most definitely in E since every element in $\{n \in E \mid n^2 \in E\}$ is an element in E such that $n^2 \in E$. So, we do not need to worry about showing this. Instead, we need to show that every $x \in E$ is also in $\{n \in E \mid n^2 \in E\}$.

Now, notice that this is the same as

"If $x \in E$, then $x^2 \in E$."

So we have gone in a big circle... and arrived back where we started. With conditional logic. In other words, to prove, "For all even integers n, n^2 is even," we need only prove, "If $x \in E$, then $x^2 \in E$." To do this, we leverage one of the proof techniques from last week.

Homework Question 1

Write the following universal statements in set-theoretic form by first defining a set, then identifying a condition on the set. Additionally, write the statements in the conditional form you would need to prove in order to prove the universal statement is true,

a) For all integers n, $|n| < n^2$.

b) For all odd integers n, n^{15} is odd.

c) For all even integers n, $n-1$ is odd.

Now, rewrite these conditional statements as a universal statement.

d) If n is a positive integer, $n+1$ is a positive integer.

e) If n is an even integer, $n+3$ is odd.

f) If n is an odd integer, $n+6$ is odd.

Great. That answers the question of how to prove. What about showing a universal statement is false?

Showing a universal statement is false

Consider the following (false) universal statement.

"For all $x \in \mathbb{R}$, $x^2 > 2$."

How might we show that this is false?

With a <u>counterexample</u>.

Here, let $x=1$, and notice that x is a real number.

Now, $x^2 = 1 \not> 2$, hence the statement is false.

In fact, this form of showing a statement is false works for the conditional form of the universal statement:

$$\text{"For all } x \in \mathbb{R},\ x^2 > 2\text{"} \longleftrightarrow \text{"If } x \in \mathbb{R}, \text{ then } x^2 > 2\text{."}$$

The conditional statement is false because we can find an element in $\mathbb{R}$ (something satisfying the sufficient condition or assumption) such that the necessary condition (or conclusion) is false.

Homework Question 2

Show by means of a counterexample that the following statements are false.

a) For all integers n, $n > |n|$.

b) For all odd integers n, $2n$ is odd.

c) For all even integers n, $n^2 > n$.

Now we are going to turn away from universal statements to talk about existential statements.

Existential Statements (there exists)

Consider the following statements

1. There exists an integer n such that $n^2 = 9$.
2. There exists an odd integer between 2π and 10.

These are examples of existential statements, statements that claim the existence of an element in a set (e.g. in the set of integers or in the set of odd integers) that satisfies some condition (e.g. $n^2 = 9$ or $2\pi < n < 10$).

We can also formulate such statements using sets. We need a definition to make this clear.

<u>DEF</u> The <u>empty set</u> is the unique set which has no elements at all. It is denoted by $\emptyset$.

With this definition, consider these two examples.

<u>EXAMPLE 1</u> (Let $\mathbb{Z}$ denote the integers.)

$$\{n \in \mathbb{Z} \mid n^2 = 9\} \neq \emptyset.$$

This reads: "The set of integers such that $n^2=9$ is <u>not</u> the empty set." In other words, there is at least one element in the set! Hence, this is the set-theoretic version of our existential statement: it claims the <u>existence</u> of at least one element in the set.

Similarly, notice that

EXAMPLE 2 (Let $\mathcal{O}$ denote the set of odd integers.)

$$\{ n \in \mathcal{O} \mid 2\pi < n < 10 \} \neq \emptyset$$

is an existential statement in set-theoretic form.

As before, we can ask how we might prove such a statement, and how we might show such a statement is false.

To prove (true)

We need only exhibit an element in the set, i.e., we find an example.

For example, to prove

$$\{ n \in \mathbb{Z} \mid n^2 = 9 \} \neq \emptyset,$$

we only need to notice that $-3 \in \mathbb{Z}$ and $-3^2 = 9$.

Similarly, noting that $7 \in \mathbb{O}$ and $2\pi < 7 < 10$ shows us that

$$\{ n \in \mathbb{O} \mid 2\pi < n < 10 \} \neq \emptyset$$

is true.

To show an existential statement is false

This requires a bit more thought. To show that an existential statement is false, we need to show that there does not exist some element, which is harder.

Consider the following (false) existential statement:

"There exists primes a, b, c such that $a^3 + b^3 = c^3$."

Proving this is false amounts to proving that its negation is true:

"There does not exist primes a, b, c such that $a^3 + b^3 = c^3$."

Now, if we think about this for a second, we can modify it as follows:

"For all primes a, b, c, $a^3 + b^3 \neq c^3$."

In other words, the negation of an existence statement is a universal statement.

So, to prove that an existence statement is false, we negate it and prove the corresponding universal statement is true.

Homework Question 3

For each of the following existential statements, write the negation in two ways: first, as a universal statement in "For all" form, then second, as the corresponding conditional statement in "If, then" form that you would need to prove in order to show that the existential statement is false.

a) There exists odd integers a, b, c such that $ax^2 + bx + c = 0$ has a rational solution.

b) There exists integers p, q, and r such that $p+q+r$ is odd and the number of odd elements in $\{p, q, r\}$ is even.

c) There exists a prime number p such that $p > q$ for any other prime number q.

REMARK (on notation)

In math classes, you may ocassionally see someone write something like

$$\text{"} \forall n \in E,\ n^2 \in E \text{"}$$

or

$$\text{"} \exists n \in \mathbb{Z} \text{ such that } n^2 = 9. \text{"}$$

This is just shorthand. "$\forall$" means "for all", "$\exists$" means "there exists." Please do not use this notation in a proof! Use it when you are sketching ideas if you find it helpful.

One last remark about the universal and existential statements. "For all" and "there exists" are often called quantifiers since they are giving you explicit information about what (how much) you can find in the set. (Also! "For all" may be stated as "for any" or "given any.")

<u>Homework Question 4</u>

Let $f(x)$ be a real-valued function whose input is a real number. Negate the following statement.

"For all $x \in \mathbb{R}$, for **all** $\varepsilon > 0$, there exists a $\delta > 0$ such that for all y such that $|x - y| < \delta$, we have that $|f(x) - f(y)| < \varepsilon$."

<u>NOTE</u>: This will become your definition of continuity of a function later! The negation of this statement is a property that discontinuous functions have.

<u>HINT</u>: If the negation of an existential statement is a universal statement, what is the negation of a universal statement?

Now, let's use some of our new techniques to prove a universal statement.

<u>THM</u> For all odd integers n, there exists an integer k such that $n=2k+1$.

Sketch: We want to show:

$$n \text{ odd} \Rightarrow n=2k+1 \text{ for some integer } k$$

observe how the existential part of the statement manifests!

Assume n is odd. In fact, let's start with n positive, and if we can get that, maybe we will be able to do something similar for negative odd integers.

Consider

$$A = \{ k \in \mathbb{Z} \mid n \geq 2k \}.$$

Notice: $0 \in A$, so A is non-empty. Additionaly A is finite. For $k \in A$,

$2k \leq n$, which means $k \leq n$, hence there are only finitely many possible k. Therefore, A has a maximum element, an element such that all other elements are less than or equal to it. Let k_{max} be a maximum element in A.

Now, notice, we want to show

$$n = 2k_{max} + 1.$$

Then we will have proven the existence of k for an arbitrary n, meaning it will work for all n.

To show this, notice that

$$n \neq 2k_{max}$$

because n is odd, i.e. not even. That means that $n \geq 2k_{max} + 1$.

Now, if we can show that $n \leq 2k_{max} + 1$, we would be done (why?).

Let's try it. Since k_{max} is a maximum element, we know that $k_{max} + 1 \notin A$. That means that

$$n \ngeq 2(k_{max} + 1)$$ see the definition of the set A

so

$$n < 2k_{max} + 2.$$

Since $n < 2k_{max} + 2$, we know that $n \leq 2k_{max} + 2 - 1$, in other words, $n \leq 2k_{max} + 1$, as desired.

Combining ① $n \geq 2k_{max} + 1$
and ② $n \leq 2k_{max} + 1$

we can conclude $n = 2k_{max} + 1$. In other words, for all positive, odd integers n, there exists an integer k such that $n = 2k + 1$.

Now we have to do the same for negative, odd numbers.

However, since the argument will largely be the same, we might wonder if we can re-use what we have already done. Indeed! We can.

Homework Question 5

Prove the following statement.

"For all negative, odd integers n, there exists an integer k such that $n = 2k+1$."

HINT: Use what we have already proven about positive odd integers. Given a negative integer, how can you make it positive?

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With that homework problem, we will have shown that all odd integers are of the form $2k+1$ for some integer k.

Question Does that mean that we may now take as an alternative definition that odd integers are all of the integers of the form $2k+1$ for some k?

Unfortunately, no. In order to say this, we need to prove the following theorem:

THM An integer is odd if and only if it can be written in the form $2k+1$ for some integer k.

Fortunately, we have already proven the harder implication.

Homework Question 6

a) Prove the implication that we have not proven yet.

b) Use the theorem above to prove the following statement:

"The sum of two odd integers is even."

Homework Question 7

Prove the following statement. Let n be an integer.

"n is odd if and only if n^2 is odd."

REMARK (on Proof by Cases)

You may have noticed that ocassionally (as in the last proof we did and in the last contradiction proof that we did in the previous week's lectures) we resort to considering cases.

Sometimes this is an "obvious" approach that you will need to use. For example, if someone were to ask you to prove

"If $n=1$ or $n=2$, then $n^2-3n+2=0$,"

you would probably immediately break this into 2 cases: first, assume $n=1$, then show $n^2-3n+2=0$. Second, assume $n=2$, then show $n^2-3n+2=0$.

However, other times, when you are faced with a more challenging proof, consider various cases so as to simplify the proof (e.g. considering only positive odd numbers in the last proof). Similarly, if you are proving a statement about all integers, you may want to consider positive integers first, then deal with negative integers. Or, you may find it useful to deal with even integers first, then odd integers, etc. Food for thought!

Homework Question 8

Prove the following statement is false.

"There exists integers p, q, and r such that $p+q+r$ is odd and the number of odd elements in $\{p, q, r\}$ is even."