RESEARCH STATEMENT

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1. Introduction

I work at the intersection of harmonic analysis, geometry, and dynamics, and I study two primary objects: translation surfaces and metric graphs. I hope to connect the Laplacian to *dynamical* and *algebraic* properties of these objects.

A *translation surface* is a polygon or a set of polygons in the plane such that each side of the polygon(s) is identified to a parallel side by translation [41], [42]. Translation surfaces are flat away from a finite set of singular points. The singular points are cone points whose angles are integer multiples of 2π. Translation surfaces come with a natural action of $SL_2(\mathbb{R})$, where the action of a matrix is just the usual linear action. Since the linear action sends parallel lines to parallel lines, the action sends translation surfaces to translation surfaces. We call the stabilizer of this action the *Veech group* of the surface. For an example of a stabilizing element, consider the unit square with opposite sides identified by translation (a torus) and let $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

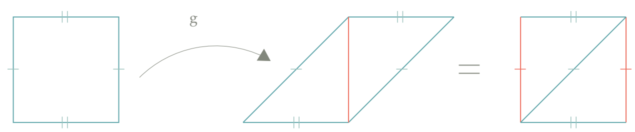

FIGURE 1. Stabilizing element of the Veech group

Using the cut-and-paste procedure pictured in Figure 1, we can reassemble the new polygon as the old, meaning the underlying topological space is the same. This example shows us that the Veech group is not always trivial. Visually, the action of the matrix appears related to linear maps on the surface, and in fact, this is true. We can identify the Veech group with the collection of derivatives of affine linear maps on the surface [39].

In this setting, I have been studying the relationship between the $SL_2(\mathbb{R})$ action, the geometry of the surface (e.g. simple closed geodesics, linear flow), and the properties of the associated flat Laplacian on the translation surface [34], [35]. This is discussed in Section 2.

Furthermore, I have been studying how dynamical and geometric properties of *the moduli space of translation surfaces* control the dynamical properties on individual translation surfaces. Consider, for example, Masur's Criterion which tells us that if we pick a translation surface in the moduli space, flow by the geodesic flow, and observe a trajectory that returns infinitely often to a compact set, then we are guaranteed to have a uniquely ergodic vertical (north or south) linear flow on the translation surface. Along these lines, I have showed that there is a condition on the linear flow of a translation surface, *superdensity*, that is both necessary and sufficient to know that the associated geodesic trajectory in the moduli space never leaves a compact set [36]. This is discussed in Section 2.3.

A *metric graph* is a compact, connected metric space such that any point has a neighborhood isometric to a star shaped set. In other words, it is a one-dimensional manifold, except at finitely many points which are isomorphic to stars with more than two branches.

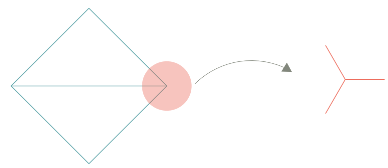

FIGURE 2. Chart to a star-shaped set

I am involved in ongoing work with Junaid Hasan and Farbod Shokrieh in which we study methods of computing analytic torsion on metric graphs with respect to a cohomology that arises from the theory of Chambert-Loir and Ducros [9]. This study has connections to both tropical and non-archimedean geometry, and touches on a much deeper question concerning mysterious analogies between tropical geometry and complex geometry. This is discussed in Section 3.

On both types of objects, translation surfaces and metric graphs, the Laplacian turns out to be difficult to study. The existence of singularities, despite how mild, creates difficulties in the spectral and representation theory.

2. TRANSLATION SURFACES: LAPLACIAN AND DYNAMICS

In 1989, Veech discovered a class of translation surfaces that have "large" stabilizers, specifically, stabilizers that are lattices in $SL_2(\mathbb{R})$ [39]. Such lattices are necessarily non-cocompact, finite covolume discrete subgroups of $SL_2(\mathbb{R})$ [20]. We call these surfaces *lattice surfaces*. Veech groups of lattice surfaces contain a hyperbolic element, which can be represented as a matrix with expanding and contracting eigenspaces. The corresponding linear action of this element, after several applications, sufficiently "mixes" the points on the surface. In fact, the map will be *ergodic* (with respect to the Lebesgue measure on the surface). For example, consider Arnold's cat map shown in Figure 3. One can imagine that after several iterations, the cat in Figure 3 will be quite blurred, illustrating that the points are moving around substantially.

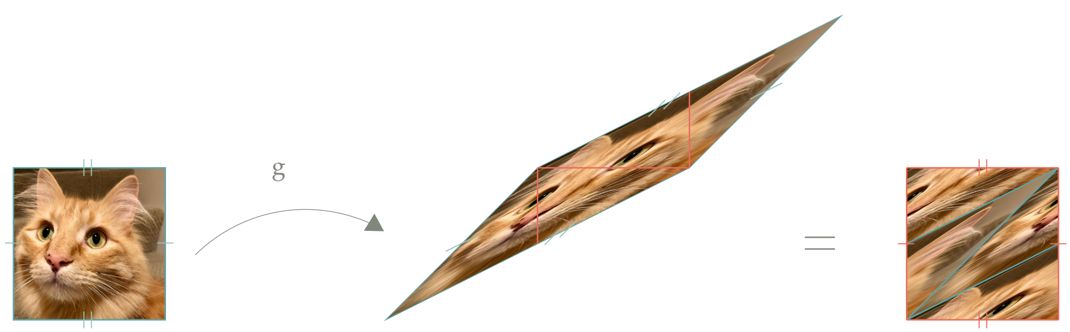

FIGURE 3. Arnold's Cat Map

Since the Veech group of a lattice surface contains a hyperbolic element, the action of the Veech group is *ergodic*. Hence, we can ask questions about the *density* of the orbits. One way to answer this is by proving that the action of the Veech group exhibits a *shrinking target property*. Fix a lattice surface S with Veech group Γ, and pick any $y \in S$. Let $B_g(y)$ denote the open ball of radius $\phi(||g||)$ (a decreasing function of the usual matrix norm). Does almost every $x \in S$ have the property that $g \cdot x \in B_g(y)$ for infinitely many $g \in \Gamma$? How fast can ϕ decrease (the target shrink) before this no longer holds?

2.1. The torus. Consider the linear action of $SL_2(\mathbb{Z})$ on the unit torus. The torus is an example of a translation surface and $SL_2(\mathbb{Z})$ is its Veech group. In fact, $SL_2(\mathbb{Z})$ is a lattice subgroup of

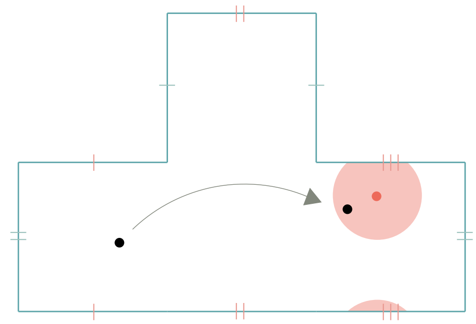

FIGURE 4. Hitting the target

$SL_2(\mathbb{R})$, so the torus is an example of a lattice surface. Our interest is in connecting the dynamics of the Veech group action to the Laplacian on the torus. The action of the Veech group on the surface induces a group representation, the *Koopman representation*, $\pi : SL_2(\mathbb{Z}) \to \mathcal{B}(L^2(\mathbb{T}^2))$, where $\pi(g)f(x) = f(g^{-1}x)$. Recall that the eigenfunctions of the Laplacian, $\Delta = -(\partial_x^2 + \partial_y^2)$, are solutions to $\Delta f = \lambda f$. We can compute eigenfunctions: $e^{2\pi imx}e^{2\pi iny}$, where $(m,n) \in \mathbb{Z}^2$. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, then

$$\pi(g)e^{2\pi imx}e^{2\pi iny} = e^{2\pi i(dm-cn)x}e^{2\pi i(an-bm)y}.$$

This is significant: the Koopman representation sends eigenspaces of the Laplacian to eigenspaces. In other words, *the action of the Veech group plays nicely with the spectral properties of the Laplacian.* In fact, we can say precisely how the eigenspaces are permuted by noting how (m,n) is permuted: by multiplying on the left by the inverse transpose of g.

In 2016, Finkelshtein used this to decompose the Koopman representation and show that the L^2-norm of an averaged Koopman representation (averaged over a measure on a subgroup of $SL_2(\mathbb{Z})$) is equivalent to the L^2-norm of a Markov operator associated to a random walk on the subgroup. This, coupled with spectral estimates of an induced representation on the visual boundary of $SL_2(\mathbb{Z})$, enables one to prove a shrinking target property on the torus with respect to the action of *any* subgroup of the Veech group $SL_2(\mathbb{Z})$ [11].

2.2. **Spectral properties in dynamics.** My objective is to leverage a similar technique to prove a shrinking target property for lattice surfaces beginning with *square-tiled surfaces*, translation surfaces that are finitely branched covers of the unit square torus. This problem is challenging for the following reason: the action of the Veech group on a translation surface does not, in general, respect the eigenspaces of the Laplacian. In fact, the problem is worse. If we define the domain of the Laplacian to be the set of smooth functions with compact support away from the singularities, the domain of the adjoint of the Laplacian will include a set of functions with poles at the singularities [18]. A calculation shows that the Koopman representation sends these functions to functions outside the domain of the adjoint operator. This is a significant challenge to overcome if one wishes to use spectral methods to explore the action of the Veech group on the surface.

To address this for primitive square-tiled surfaces, I have pushed the dynamics on a square-tiled surface to a measurably conjugate dynamical system on a discrete set of labeled tori [35]. To make the argument work, one needs to show that the L^2-norm of an averaged Koopman operator is equal to the L^2-norm of an operator associated to a random walk on the acting group. However, only a bound between the L^2-norms can be achieved with the current machinery, not equality. This is currently being updated, but there is reason to believe that the conjectured result is true.

Conjecture 2.1 (S., 21+)**.** Let (X, ω) be a primitive square-tiled surface, and let Γ be a subgroup of the Veech group $SL(X, \omega)$. For any $y \in X$, for Lebesgue a.e. $x \in X$, the set

$$\{g \in \Gamma : |g.x - y| < ||g||^{-\alpha}\}$$

is

(1) finite for every $\alpha > \delta_\Gamma$
(2) infinite for every $\alpha < \delta_\Gamma$

where δ_Γ is the critical exponent of the subgroup Γ, and $||\cdot||$ is the operator norm.

Question 2.1. Can the same technique be used to solve a shrinking target problem for lattice surfaces? Translation surfaces?

If the technique works for square-tiled surfaces, the answer to this question is almost surely yes. The technique will work for other surfaces that can be decomposed into rectangles, but with one technical caveat concerning the critical exponent. The critical exponent δ_Γ is the exponential growth rate of the number of points in the orbit of Γ acting on the upper half-plane [37].

Question 2.2. For arbitrary non-cocompact lattice subgroups G of $SL_2(\mathbb{R})$, is it true that for any subgroup $\Gamma \subset G$ and any $\varepsilon > 0$ we can find a convex, cocompact subgroup $\Gamma' \subset \Gamma$ such that $\delta_{\Gamma'} > \delta_\Gamma - \epsilon$?

Studying this shrinking target problem has revealed interesting spectral questions concerning the interaction of the Laplacian with the action of the Veech group that have yet to be addressed in the literature. The relationship is problematic in the sense that they are not playing nicely together as described above, however, the shrinking target result suggests that there is value in exhaustively describing the relationship. In doing this work, we can also probe spectral questions on the translation surface. For instance, a question inspired by one that Luc Hillairet asked me:

Question 2.3. Given a group action on a square-tiled surface, can we use operators to identify when this action is the action of the Veech group or some subgroup of the Veech group?

Hillairet's paper can be a guide for a particular choice of Laplacian on square-tiled surfaces [17]. Furthermore, a literature review leads us to ask the following.

Question 2.4. There are two types of eigenfunctions on a square-tiled surface: those that push down to eigenfunctions on the torus, and those that do not. For those that do not, how often do they occur, and can one show that the associated eigenvalue is simple?

There is reason to believe this property holds: a similar statement was recently proven in the case of metric graphs [1], and this property is similar to a result of Hillairet and Judge [19].

Answers to these questions would yield new spectral information about square-tiled surfaces, which may shed light on a question inspired by Veech's work. In 1989, Veech discovered an amazing fact about lattice surfaces. He packaged the lengths of simple closed geodesics on the surface into a zeta function (taking only one simple closed geodesic in a homotopy class) and showed that the zeta function has very close ties to the cusps of the $SL_2(\mathbb{R})$ orbit. Namely, the zeta function can be expressed in terms of the Eisenstein series arising from the cusps on the $SL_2(\mathbb{R})$ orbit [39]. He then used this expression to count these geodesics, showing that the number of geodesics with length less than R grows quadratically.

Question 2.5. Is there a connection between the cusps of the Veech group and the *eigenvalues* of the flat Laplacian on lattice surfaces? If so, does this provide a new pathway for thinking about Veech's question: extending the prime geodesic theorem to a larger class of translation surfaces?

A short computation shows that there is a connection on the torus, which can be viewed as a consequence of Veech's work. It is a multiplicative analogue of Poisson summation:

$$\zeta_{\mathrm{eig}}(s) = (2\pi)^{-2s}\zeta_R(2s)\zeta_{\mathrm{length}}(s)$$

In 1998, Veech published a paper in which he defined the Siegel-Veech transform and used it to give an answer to his question about a prime geodesic theorem [40]. Gutkin and Judge answered the question completely for all translation surfaces using dynamical methods [15], but there is no known spectral argument.

2.3. **Dynamics in the Moduli Space.** There is another dynamical system commonly studied on translation surfaces: the linear flow Φ_t on the surface. This is the usual geodesic flow on the translation surface with the singular points removed. If a trajectory hits a singular point, we stop.

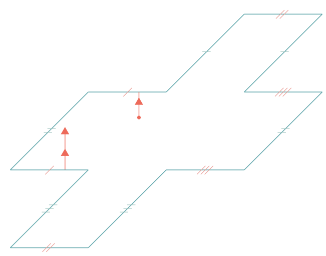

FIGURE 5. Linear flow segment on a translation surface

There is an equivalent definition of a translation surface, which is helpful in describing the moduli space of translation surfaces. A *translation surface* is a pair (X, ω) where X is a compact, connected Riemann surface without boundary and ω a non-zero holomorphic differential on X. If we fix the genus of the underlying Riemann surface, the moduli space Ω_g of pairs (X, ω) forms a vector bundle over $\mathcal{M}_g$, the moduli space of genus g Riemann surfaces, where the fiber over $X \in \mathcal{M}_g$ is the g-complex dimensional vector space $\Omega(X)$ of holomorphic 1-forms on X. We will suppress the notation of the underlying Riemann surface and use the notation ω to denote a translation surface for the remainder of this section.

In light of this new definition, the $SL_2(\mathbb{R})$ action defined in the introduction is an action on the moduli space of translation surfaces. Notice that elements of the form $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ for any $t \in \mathbb{R}$ form a one-parameter subgroup. We will refer to this as the (Teichmüller) *geodesic flow*.

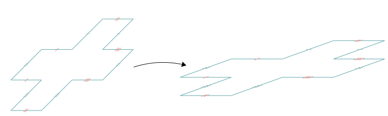

FIGURE 6. Translation surface ω and $g_t\omega$

One often finds that geometric and dynamical properties in the moduli space of all translation surfaces control the behavior of dynamical systems on individual translation surfaces. I have identified a condition on the vertical flow of a translation surface ω that is equivalent to boundedness of the associated geodesic in moduli space. The condition is inspired by a paper of Beck and Chen [6], where they study billiard trajectories on similar objects.

Definition 2.1 (Superdensity)**.** Let ω be a translation surface. We say the linear flow Φ_t is *superdense* if there exists a constant $c > 0$ such that for every $T > 0$, the segment of the flow Φ_t for $t \in [0, cT]$ gets $\frac{1}{T}$-close to every point on ω.

Beck and Chen show that the linear flow is superdense on a lattice surface if and only if the slope is a badly approximable number [6]. I show the following generalization [36].

Theorem 2.1 (S., 2021+)**.** Let $\omega \in \Omega_g$ be a translation surface. The linear flow on ω is superdense if and only if the associated Teichmüller geodesic $\{g_t\omega\}_{t>0}$ is bounded.

I prove this result using the *diameter* of the translation surface to control the quantitative density of the vertical linear flow. As a corollary, we have the following:

Corollary 2.1. If the linear flow on ω is superdense, it is uniquely ergodic. However, uniquely ergodic flows need not be superdense.

It is known that the Lyapunov exponents of the Kontsevich-Zorich cocycle control the rate of convergence (of the linear flow on the translation surfaces) to the ergodic mean [12], [3]. It would be desirable to understand this for the class of translation surfaces that exhibit a superdense flow. Furthermore, in ongoing discussions with Chen, we seek to quantify the superdense directions on a translation surface.

3. Metric Graphs and Tropical Curves

The following is joint work with Junaid Hasan and Farbod Shokrieh.

In 2006, Mikhalkin and Zharkov proved that there is a one-to-one correspondence between metric graphs and compact tropical curves, where *tropical curves* are connected topological spaces homeomorphic to a locally finite 1-dimensional simplicial complex equipped with a tropical structure, and the tropical structure is an atlas of maps into the semifield $\mathbb{R} \cup \{-\infty\}$ satisfying certain transition conditions [28]. The semifield here is the tropical semifield, where "addition" is $\max\{x, y\}$ and "multiplication" is $x + y$.

The field of tropical geometry is analogous to complex geometry in some surprising ways. For example, there is a rather mysterious Riemann-Roch theorem for graphs that was proven by Baker and Norine [4], [13], [28]. The only known proof of this theorem relies on Dhar's burning algorithm, which is purely combinatorial. We ask whether or not an analytic proof of Riemann-Roch could be constructed. Classically, such proofs in complex geometry leverage properties of the Laplacian. However, in the case of the tropical Riemann-Roch theorem, it is unclear *what functions* the Laplacian would be acting on, leading to a deeper question about what sheaf we should be studying.

My focus is the role that harmonic analysis can play in helping answer these kinds of questions, starting with metric graphs.

3.1. **Regularized determinants, heights, and analytic torsion on metric graphs.** In 1973, Ray and Singer computed analytic torsion on complex manifolds [32]. Given the similarities between complex geometry and tropical geometry, it is a natural question to ask if analytic torsion can be computed on metric graphs, and whether or not this gives us any insight into other open questions in tropical geometry.

Analytic torsion is an invariant on a Riemannian manifold that depends on the underlying topology, a chain complex of forms on the surface, and a representation of the fundamental group of the surface. It is computed by taking alternating sums (up to a constant) of the logarithm of the determinant of the Hodge Laplacian. The alternating sum is over the determinant of the Laplacian on each piece of a twisted (by the representation) chain complex of smooth k-forms over the surface [31]. The representation must be picked in a way that guarantees that the cohomology associated to the twisted chain complex is acyclic (each homology group collapses).

The definition of analytic torsion relies on the following definition of the determinant of the Laplacian, which we call a *regularized determinant*.

$$\det(\Delta) = e^{-\zeta'(0)}$$

where $\zeta(s) = \sum_{\lambda \in \sigma} \frac{1}{\lambda^s}$ is the zeta function packaging the spectrum of the Laplacian, σ. Here, we should note that we are using an analytic continuation of the zeta function to a meromorphic function on $\mathbb{C}$, and the function takes a value at 0, hence is holomorphic in a neighborhood of zero, so $\zeta'(0)$ makes sense.

It is not hard to check that this definition generalizes the usual determinant of a linear transformation on a finite dimensional vector space. The usual determinant is the product of the (finitely many) eigenvalues λ_i and

$$\prod_{i=1}^{n} \lambda_i = e^{-\frac{d}{ds}\left(\sum_{i=1}^{n} \frac{1}{\lambda_i^s}\right)\Big|_{s=0}}.$$

The derivative of the zeta function, or the logarithm of the determinant, is of independent interest since it connects to the *theory of heights* [33].

3.2. **The circle.** Consider the circle $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, which we think of as a simple example of a graph. We can compute eigenvalues of the operator directly by solving $\Delta f = \lambda f$. The non-zero spectrum σ is $\{n^2 : n \in \mathbb{Z}\}$.

We package the non-zero spectrum in a zeta-function, $\zeta(s)$, which has a nice connection to the Riemann zeta function $\zeta_R(s)$:

$$\begin{aligned}
\zeta(s) &= \sum_{\lambda \in \sigma} \frac{1}{\lambda^s} \\
&= \sum_{n \in \mathbb{Z}} \frac{1}{n^{2s}} \\
&= 2 \sum_{n \in \mathbb{N}} \frac{1}{n^{2s}} \\
&= 2\zeta_R(2s)
\end{aligned}$$

This means $\zeta'(s) = 4\zeta_R'(2s)$, hence $\zeta'(0) = 4(-\frac{1}{2}\log 2\pi) = -2\log 2\pi$. This gives the following value for the regularized determinant:

$$\det(\Delta) = e^{-\zeta'(0)} = (2\pi)^2 = 4\pi^2$$

To compute the analytic torsion, we pick a representation of the fundamental group $\mathbb{Z}$ and use the deRham cohomology on S^1. However, this is where our curiosities diverge from the classical set-up: what forms should we be using on the graph?

3.3. **General metric graphs.** For the case of a general graph, there is an additional complication in the spectral theory. Similar to translation surfaces, a generic metric graph may have more than one self-adjoint Laplacian. The choice of self-adjoint Laplacian is intimately tied to the sheaf we are interested in. For example, Kurasov and Sarnak have recently computed trace formulae for metric graphs. They define the Laplacian on a set of sufficiently differentiable functions which are continuous at vertices and the sum of the incoming slopes at vertices is zero [24].

Our interest is in a similar set of functions (or rather forms): we plan to use the theory of Chambert-Loir and Ducros [9] on tropical forms and currents. Our objective is to compute the analytic torsion on a general metric graph associated with the chain of superforms. This may not be possible, in full generality, so we are starting with simpler computations, for instance, on chains of loops where torsion computation is more easily linked to the Riemann Zeta function.

There is another question lingering for general metric graphs. Ray and Singer first defined analytic torsion with the suspicion that it was an analytic version of R-Torsion, an invariant defined on topological spaces, the difference being that R-Torsion is an invariant defined through the homology whereas analytic torsion is defined through the de Rham cohomology [31]. Several years later, Cheeger and Müller independently proved that this is true [10], [29].

Question 3.1. Is R-Torsion equivalent to analytic torsion on metric graphs? For what sheaf? Does this give any indication as to the kinds of duality to be expected between homology and cohomology in tropical geometry?

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