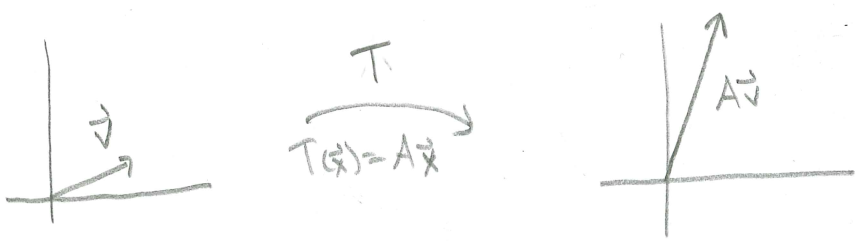
Lecture #14

6.1 Eigenvalues + Eigenvectors

Think for a moment about what a linear transformation does to a vector. For any linear transformation, we can find matrix representing that linear transformation such that $T(\vec{x}) = A\vec{x}$. Pick a particular vector $\vec{v}$, then $T(\vec{v}) = A\vec{v}$, where $A\vec{v}$ is another vector. If $T: \mathbb{R}^n \to \mathbb{R}^n$, then A is a square matrix $(n \times n)$, $\vec{v}$ is in $\mathbb{R}^n$, and $A\vec{v}$ is in $\mathbb{R}^n$. Thus, $A\vec{v}$ must be a stretch/compression of $\vec{v}$ coupled with a rotation. In other words, A can only change two properties of $\vec{v}$, its length and direction (especially since these two properties fully define vectors!).



Sometimes, depending on the linear transformation, there are vectors which do not rotate ... they only stretch or compress along a line. It turns out that these vectors, and the corresponding stretching ratio, play an important role in understanding how to categorize and classify linear transformations. We won't be going this far in our study of linear transformations, but it is still especially worthwhile to see. These special vectors and corresponding stretching ratio arise in many,

many applications, including image processing, machine learning, and even Google's page rank algorithm. We'll start with a simple example, find a few of these vectors, then give some definitions.

EXAMPLE 1

Let $A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$. Determine whether any of the following vectors rotate when multiplied on the left by A:

$$\vec{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = 3\vec{v}_1 = \begin{bmatrix} 15 \\ 12 \end{bmatrix}.$$

Compute $A\vec{v}_i$ for each vector listed:

$$A\vec{v}_1 = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 35 \\ 28 \end{bmatrix} = 7\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

does <u>not</u> rotate!

$$A\vec{v}_2 = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

rotates! (Not a scalar multiple of $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$.)

$$A\vec{v}_3 = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This one rotates, by $180°$. This is a special rotation though, since $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is <u>parallel</u> (or anti-parallel, if you took M126) to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$A\vec{v}_4 = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 15 \\ 12 \end{bmatrix} = \begin{bmatrix} 105 \\ 84 \end{bmatrix} = 7\begin{bmatrix} 15 \\ 12 \end{bmatrix}$$ ← does not rotate!

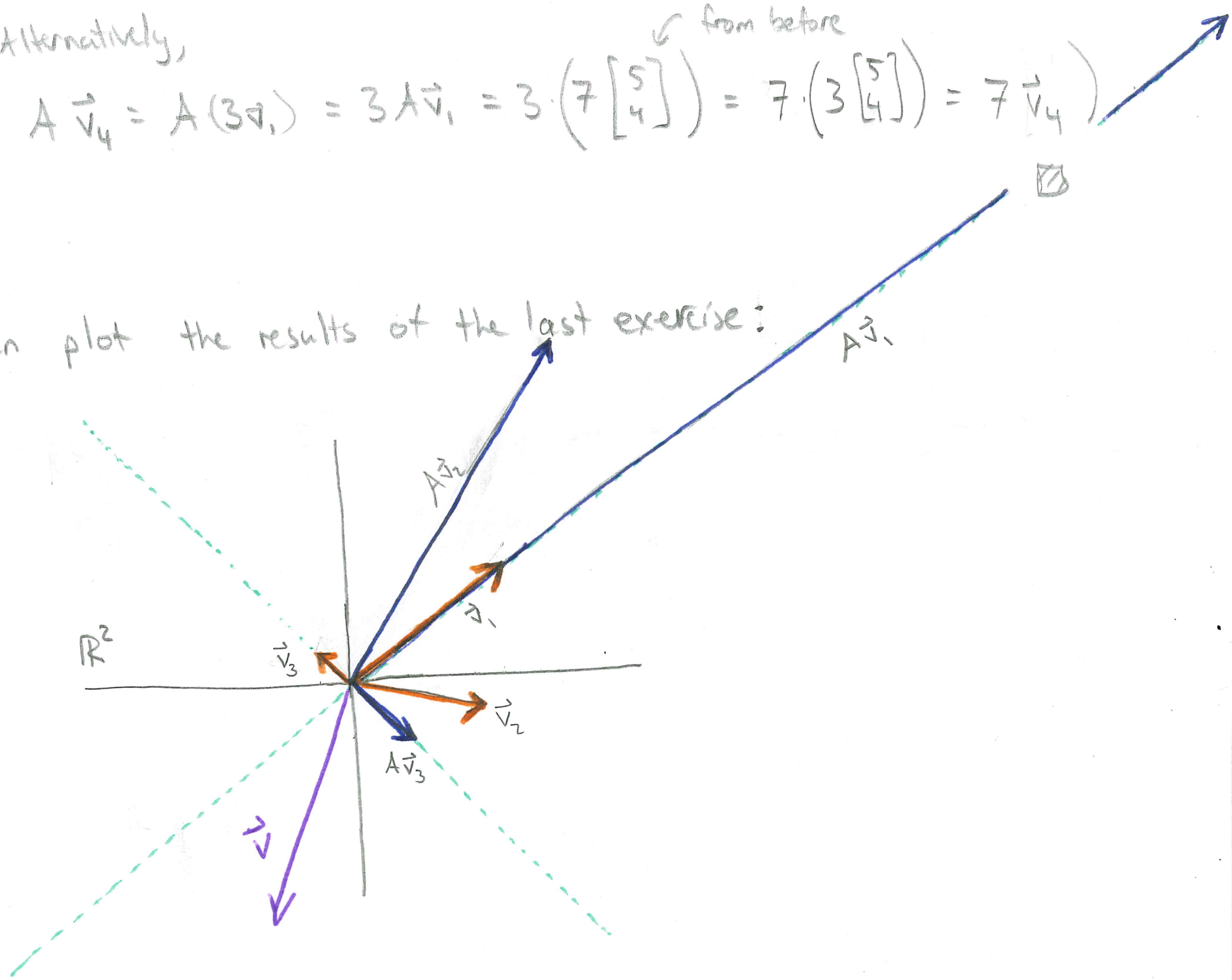
(Alternatively,

from before

$$A\vec{v}_4 = A(3\vec{v}_1) = 3A\vec{v}_1 = 3\cdot\left(7\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) = 7\cdot\left(3\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) = 7\vec{v}_4$$)

$\boxtimes$

We can plot the results of the last exercise:



We did not plot $\vec{v}_4$, however, we did plot $\vec{v}$. Don't worry about $\vec{v}$ yet, there is an exercise later in the (next set of) notes that asks you to do something with $\vec{v}$.

Notice a few things. First, $\vec{v}_4$ isn't unique in that it also stretches (like $\vec{v}_1$) by "7". Any vector along that particular line, the span $\left\{\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right\}$, stretches in exactly the same way.

In fact, any vector along the span of $\vec{v}_3$ behaves similarly. It will rotate 180° and stretch by "2" (i.e. double in length). These are the only two lines in $\mathbb{R}^2$ where this happens! We will see that this happens at most along two lines.

Now, look at $\vec{v}_2$. $A\vec{v}_2$ is a rotation and scaling. Can you look at $\vec{v}_2$, and see why $A\vec{v}_2$ should be where it is? We actually can ... but we'll get to that. [Hint: Can you decompose $\vec{v}_2$ in a nice way, as the sum of two ... convenient vectors, then apply A?]

$\vec{v}_1$ and $\vec{v}_3$ above are special vectors, we call these eigen vectors. The stretching factor is called an eigenvalue.

<u>DEF</u> Let A be an $n \times n$ matrix. Then a * $\boxed{\text{non zero}}$ * vector $\vec{u}$ is an $\boxed{\underline{\text{eigenvector}}}$ of A if there exists a scalar λ such that

$$\boxed{A\vec{v} = \lambda\vec{v}}.$$

The scalar λ is called an $\boxed{\underline{\text{eigen value}}}$ of A.

<u>REMARK 1</u> Not every matrix has an eigenvector when you are thinking about linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$.

EXERCISE Can you think of a linear transformation that has **No** eigenvectors?

[HINT! Can you think of a linear transformation that ... rotates every vector?]

REMARK 2 Return to the example and think for a moment about the "alternative" computation for $A\vec{v}_4$. It actually didn't matter that $\vec{v}_4 = 3\vec{v}_1$ ... it only mattered that it was a scaling of $\vec{v}_1$. If we let $\vec{v}_4 = t\,\vec{v}_1$ for some real number (possibly negative), then we can show $A\vec{v}_4 = 7\vec{v}_4$. (Do this if you do not see it!) This is really just a restatement of an earlier claim: that all vectors in the span of $\vec{v}_1$ stretch by a factor of 7 along that line (7 is the eigenvalue!).

This leads us to our first fact:

THM Let A be a square matrix and suppose that $\vec{v}$ is an eigenvector of A associated with eigenvalue λ. Then for any scalar $\boxed{c \neq 0}$, $c\vec{u}$ is also an eigenvector associated with λ.

NOTE: We defined eigenvectors to be non-zero!

pf: $A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda c\vec{v}$.

Now, the obvious question is how do we find these eigenvectors for a given matrix? How do we even know if they exist for a given matrix? It turns out that to find eigenvectors, it is easiest if you already know the eigenvalues. So let's change our question: how would you find the eigenvalues of a matrix?

Let's do a thought experiment. We want to find all possible values of λ such that $A\vec{v} = \lambda\vec{v}$.

$$A\vec{v} = \lambda\vec{v} \rightsquigarrow A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\rightsquigarrow A\vec{v} - \lambda I_n\vec{v} = \vec{0}$$
$$\rightsquigarrow (A - \lambda I_n)\vec{v} = \vec{0}.$$

For λ to be an <u>eigenvalue</u>, we need this equation to admit <u>non-zero</u> solutions $\vec{v}$ so that we have an eigenvector! (If the only solution was the $\vec{0}$, then $\vec{v} = \vec{0}$ is not an eigenvector, which means the corresponding λ is <u>not</u> an eigenvalue.)

In order to have a non-zero solution to that equation, we need $\boxed{(A - \lambda I_n)}$ to have a <u>non-zero kernel</u>.

this is <u>just</u> a matrix!

If $(A - \lambda I_n)$ has a non-zero kernel (remember that A must be square, by definition of the eigenvector, so $A - \lambda I_n$ is also square) this means $A - \lambda I_n$ is <u>not</u> invertible!

We just learned an "easy" way to identify when a square matrix is not invertible - it is when the determinant of the matrix is 0. (!)

i.e., we want to find all λ such that

It is exactly these λ that have eigenvectors!

$$\det(A - \lambda I_n) = 0.$$

DEF We call the equation $\det(A-\lambda I_n)=0$ the $\boxed{\text{characteristic equation}}$.

Let's do an example.

EXAMPLE 2 Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Find all of the eigenvectors of A.

Let's use the characteristic equation:

$$A - \lambda I_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}.$$

$$\det(A - \lambda I_2) = (2-\lambda)(2-\lambda) - 1$$

$$= 4 - 4\lambda + \lambda^2 - 1$$

$$\boxed{= \lambda^2 - 4\lambda + 3}$$

(*) see definition on the following page.

So, if we want $\det(A - \lambda I_2) = 0$, we want to find the roots of a polynomial in λ:

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$(*) \quad \boxed{\lambda = 1, 3} \quad (*)$$

So we have two eigen values, $\lambda_1 = 1$, $\lambda_2 = 3$.

DEF Notice how computing the determinant of $A - \lambda I_2$ in the last example led to a polynomial equation in λ. This will always happen, $\det(A - \lambda I_n)$ will always generate a degree n polynomial. We call this polynomial the **characteristic polynomial**.

Now, given eigenvalues, how would we find corresponding eigenvectors? Well, since $A\vec{v} = \lambda\vec{v}$, we can solve for some $\vec{v}$ that satisfies $(A - \lambda I_n)\vec{v} = \vec{0}$. This means picking any non-zero vector in the null space of the matrix $(A - \lambda I_n)$.

Let's do an example.

EXAMPLE 3 Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Find an eigenvector v_1 corresponding to the eigenvalue 1, then find an eigenvector v_2 corresponding to the eigenvalue 3.

Let's start with $\lambda_1 = 1$.

$$(A - \lambda_1 I_2)\vec{v} = 0 \rightsquigarrow \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{v} = 0$$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\vec{v} = 0$$

So, since $(A - \lambda_1 I_2) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, and we need non-trivial solutions to this equation, we just use an augmented matrix and compute the way we have been computing all quarter! Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and we see that v_2 is a free variable. Let $v_2 = s$. Then $v_1 - v_2 = 0 \rightsquigarrow v_1 - s = 0 \rightsquigarrow v_1 = s$. Thus,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus, we can pick $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (Notice, for any value of $s \neq 0$, we get a different eigenvector!).

Next, we'll find an eigenvector corresponding to $\lambda_2 = 3$.

As before, we compute $(A - \lambda_2 I_2) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Next, we find a vector in the null space; Let $\vec{v}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we see v_2 is a free variable. Let $v_2 = s$. Then $v_1 = -s$.

Thus,

$$\vec{v}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

and letting $s=1$, we get a non-zero eigenvector $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\blacksquare$

rem: Eigenvectors are by def. non-zero!

Notice that in the last example, every vector (except $\vec{v}=\vec{0}$) in the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$. Every vector (except $\vec{v}=\vec{0}$) in the span of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$.

In other words,

1. any $\vec{v}$ in span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is an eigenvector with $\lambda_1 = 1$, except $\vec{v}=\vec{0}$,
2. any $\vec{v}$ in span $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is an eigenvector with $\lambda_2 = 3$, except $\vec{v}=\vec{0}$.

i.e., the set of eigenvectors corresponding to $\lambda_1 = 1$ is a subspace if you add to it the $\vec{0}$. Similarly, the set of vectors corresponding to $\lambda_3 = 2$ is a subspace if you include the $\vec{0}$.

__NOTE__ We can show that the collection of $\vec{v}$ satisfying $A\vec{v} = \lambda\vec{v}$ for some matrix $A_{n \times n}$ and scalar λ is __always__ a subspace.

__EXERCISE__ Show this! Recall that you (may) have already done this for the case where $\lambda = -2$ (see Midterm 2 solutions).

__THM__ Let A be an $n \times n$ matrix with eigenvalue λ. Let S denote the set of all eigenvectors associated with λ, together with the zero vector $\vec{0}$. Then S is a subspace of $\mathbb{R}^n$.

The proof of this follows from the exercise above. This enables us to make a definition (we have found another useful subspace!).

__DEF__ Let A be a square matrix with eigenvalue λ. The subspace of all eigenvectors associated with λ, together with the zero vector, is called the __eigenspace__ of λ.

EXAMPLE 4 Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. What is the eigenspace E_1 corresponding to the eigenvalue $\lambda_1 = 1$? What is the eigenspace E_2 corresponding to the eigenvalue $\lambda_2 = 3$?

We actually saw what these were in the last example:

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$E_2 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

$\square$

Next, notice that in Example 2, our characteristic polynomial is $\lambda^2 - 4\lambda + 3$, which nicely factored to become $(\lambda - 3)(\lambda - 1)$, and we saw that our eigenvalues were 1 and 3.

Check out Example 6 on p. 254 in the textbook. For that 3x3 matrix, the characteristic polynomial is degree 3, and comes out to be $-\lambda^3 + 4\lambda^2 - 4\lambda$. We can factor $-\lambda$ out, which gives us $-\lambda(\lambda^2 - 4\lambda - 4) = -\lambda(\lambda - 2)(\lambda - 2)$. Here, we see that the polynomial has a repeated root! This turns out to have important consequences. Let's is $-\lambda^3 - 4\lambda^2 + 4\lambda = -\lambda(\lambda-2)^2$, we see that 0 is an eigenvalue of multiplicity 1 and 2 is an eigenvalue of multiplictty 2 ... and the eigenspace corresponding to 0 is 1 dimensional, whereas the eigenspace corresponding to 2 is 2 dimensional...

If you were to test whether or not this pattern continues with a few matrices, you might find that it does, but at some point, <u>it</u> <u>will</u> <u>break</u>! This happens to be one of those things that is ... <u>almost</u> ... true, but just isn't. The reason it is not true is because we are working in $\mathbb{R}^n$ ... and there is something broken about the structure of $\mathbb{R}^n$ as it relates to roots of polynomials. You may recall the fundamental theorem of algebra (if you have seen it) says that a degree n polymial has n roots when counted with multiplicity. However, some of those roots could be complex, which suggests some of our eigenvalues could be complex. This also means we may need to consider complex eigenvectors ...

rewrite our polynomial in a more suggestive way:

$$-\lambda(\lambda-2)(\lambda-2) = -(\lambda-0)^1(\lambda-2)^2.$$

When we set this to zero, we see the roots are 0 and 2, and 2 is repeated.

<u>DEF</u> We say a root α of a polynomial $P(x)$ $\boxed{\underline{\text{multiplicity } k}}$ if $P(x) = (x-\alpha)^k \cdot Q(x)$.

(Note, $Q(x)$ is just the rest of the factoring multiplied together. For $P(x) = -x(x-2)^2$, $\alpha = 2$, $Q(x) = -x$.)

<u>DEF</u> We say an $\boxed{\underline{\text{eigenvalue has multiplicity } k}}$ if the eigenvalue is a root of multiplicity k of the characteristic polynomial.

So, in Example 2, where the characteristic polynomial is $\lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$, we see that both 1 and 3 are eigenvalues of multiplicity 1. Notice that the corresponding eigenspaces are of dimension 1 also... In fact, if you look back at Example 6 in the textbook (p. 254) where the characteristic polynomial but we won't be doing that in this section. If you are interested, check out section 6.3 in the textbook. (You don't need to know anything about the fundamental theorem of algebra in our class!)

All that being said, the point is that the multiplicity does not tell you the dimension of the eigenspace when you are working with $\mathbb{R}^n$. However, it does tell us something. We'll get back to that. First, lets see an example of a matrix that has an eigenvalue with multiplicity differing from the dimension of the corresponding eigenspace.

EXAMPLE 5

Find the eigenvalues and a basis for each eigenspace of

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -2 & 0 \end{bmatrix}.$$

First, let's find the eigenvalues using $\det(A - \lambda I_3) = 0$.

$$(A - \lambda I_3) = \begin{bmatrix} 0-\lambda & 2 & -1 \\ 1 & -1-\lambda & 0 \\ 1 & -2 & 0-\lambda \end{bmatrix}, \quad \text{so}$$

$$\det(A - \lambda I_3) = -\lambda(-1-\lambda)(-\lambda) + 0 + (-1)(1)(-2) - (0) - 2(1)(-\lambda)$$
$$- (-1)(-1-\lambda)(1)$$

$$= -\lambda^2(\lambda+1) + 2 + 2\lambda + (-1-\lambda)$$
$$= -\lambda^3 - \lambda^2 + 2 + 2\lambda - 1 - \lambda$$
$$= -\lambda^3 - \lambda^2 + \lambda + 1$$
$$= -(\lambda-1)(\lambda+1)^2,$$

(*) see REMARK at end of lecture notes about factoring.

so we see that A has eigenvalues $\lambda_1 = -1$ (of multiplicity 2) and $\lambda_2 = 1$ (of multiplicity 1). Now, lets find the eigenspace.

For $\lambda_1 = -1$,

$$A - (-1)I_3 = A + I_3 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

so, to solve for all $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that $(A + I_3)\vec{v} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

thus, v_3 is a free variable, so

$$v_3 = s$$

$$v_2 = \frac{1}{2}s$$

$$v_1 = 0$$

and we have

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}s \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Thus, the eigenspace for $\lambda_1 = -1$ is

$$E_1 = \text{span}\left\{\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}\right\}.$$

(*) (Notice, -1 had multiplicity 2, but the dimension of the eigenspace is only 1.) (*)

Next, let's find the eigenspace for $\lambda_2 = 1$.

$$A - 1 \cdot I_3 = A - I_3 = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 0 \\ 1 & -2 & -1 \end{bmatrix}.$$

Then, to find all $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that $(A - I_3)\vec{v} = 0$:

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and we see that $v_3 = 0$ and v_2 is a free variable. Let $v_2 = s$, then:

$$v_3 = 0$$
$$v_2 = s$$
$$v_1 = 2s$$

So,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

This means that the eigenspace associated to $\lambda_2 = 1$ is

$$E_2 = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Now, it should be clear that the multiplicity of the eigenvalue, in our setting, does not determine the dimension of the associated eigenspace. However, we do have:

THM Let A be a square matrix with eigenvalue λ. Then the dimension of the associated eigenspace is less than or equal to the multiplicity of λ.

Now, go back to example 6 in the textbook one more time, and notice that 0 is an eigenvalue (of multiplicity 1).

What does it mean that 0 is an eigen value? It means there are non-zero vectors satisfying

$$A\vec{v} = 0 \cdot \vec{v}$$

for the matrix. In other words

$$A\vec{v} = \vec{0}$$

has non-trivial solutions! Thus A is not invertible!

Conversely, if A is not invertible, $A\vec{v} = \vec{0}$ has non-trivial solutions, thus 0 is an eigen value ($A\vec{v} = \vec{0} = 0 \cdot \vec{v}$).

We can use this fact to update our Unifying Theorem:

__THM__ (Unifying Theorem, version 8)

Let $S = \{\vec{a}_1, \dots, \vec{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let $A = [\vec{a}_1 \cdots \vec{a}_n]$ and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent.

(a) S spans $\mathbb{R}^n$

(b) S is linearly independent

(c) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$.

(d) T is onto.

(e) T is one-to-one.

(f) A is invertible.

(g) $\ker(T) = \{0\}$.

(h) S is a basis for $\mathbb{R}^n$.

(i) $\mathrm{col}(A) = \mathbb{R}^n$

(j) $\mathrm{row}(A) = \mathbb{R}^n$

(k) $\mathrm{rank}(A) = n$

(l) $\det(A) \neq 0$

(m) $\lambda = 0$ is not an eigen value of A.

Let's do one last example, using this last bit of information.

EXAMPLE 6

Show that $\lambda=0$ is an eigenvalue for the matrix

$$A = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \\ 4 & 1 & 2 \end{bmatrix}.$$

We have so many ways of doing this now! i.e. show that any of (a)–(l) are not true (in the unifying theorem), and that shows that $\lambda=0$ is an eigenvalue!

Here's an easy one:

$$\det(A) = 6 + 0 + 10 - 20 - (-4) - 0$$
$$= 0.$$

Thus A is not invertible, thus $\lambda=0$ is an eigenvalue.

EXERCISE

Suppose that A is a square matrix with characteristic polynomial $-\lambda(\lambda-1)^3(\lambda+2)^3$. What are the dimensions of A? What are the eigenvalues of A? Is A invertible? What is the largest possible dimension for an eigenspace of A?

Lastly, you may remember from p. of these notes that we can actually say what will happen to $\vec{v}$... We need a bit of information from 6.2 before we can do this, so you will find that exercise in next set of lecture notes.