MATH 324 A

Exam II

March 1, 2019

Name ____________________

Student ID #____________ Section ____________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:____________________

(SOLUTIONS!)

| | | |
|---|---|---|
| 1 | 16 | |
| 2 | 7 | |
| 3 | 7 | |
| 4 | 12 | |
| 5 | 8 | |
| BONUS | 5 | |
| Total | 50 | |

- Your exam should consist of this cover sheet, followed by 5 problems and a Bonus. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 5 problems. Try not to spend more than 10 minutes on each page.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (16 points) Clearly indicate whether each statement is true or false.

(a) **TRUE** / **FALSE** A line integral over a vector field $\vec{F}$ on an open, connected domain $D$ is independent of path if and only if the vector field on $D$ is conservative.

(b) **TRUE** / **FALSE** Suppose $\vec{F} = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$ and $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ on an open, simply-connected domain $D$. Then line integrals over curves in $D$ are independent of path.

(c) **TRUE** / **FALSE** A line integral over a differentiable, conservative vector field only depends on the initial point and terminal point of the curve one is integrating over.

(d) **TRUE** / **FALSE** Suppose $\int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop $C$ in the domain $D = \{(x,y) : 1 < x^2 + y^2 < 4\}$. Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on $D$.

(e) **TRUE** / **FALSE** Let $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ be a smooth vector field defined on $D = \{(x,y) : x^2+y^2 < 1\}$. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then for every closed curve $C$, $\int_C P(x,y)dx + Q(x,y)dy = 0$.

(f) **TRUE** / **FALSE** A line integral with respect to arc length is dependent on the orientation (direction) of the curve.

(g) **TRUE** / **FALSE** Let $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ be a smooth vector field defined on an open, simply-connected domain $D$. Suppose $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ on $D$. Then $\vec{F}$ is not conservative, but $\vec{F}$ could be conservative on a slightly larger open, simply-connected domain $\tilde{D}$ that contains $D$.

(h) **TRUE** / **FALSE** Consider the vector field $\vec{F}(x,y) = \langle y^3 cos(x), -3y^2 sin(x)\rangle$. F is conservative on $\mathbb{R}^2$.

$$\frac{\partial P}{\partial y} = 3y^2 \cos(x)$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

$$\frac{\partial Q}{\partial x} = -3y^2 \cos(x)$$

2. (7 points) Suppose $f$ is a differentiable function of $x$ and $y$, and $g(u, v) = f(e^u + \sin(v), e^u + \cos(v))$. Use the table of values to calculate $\frac{\partial g}{\partial u}(0,0)$ and $\frac{\partial g}{\partial v}(0,0)$. Show ~~your~~ all of your work.

| | $f$ | $g$ | $\frac{\partial f}{\partial x}$ | $\frac{\partial f}{\partial y}$ |
|---|---|---|---|---|
| $(0,0)$ | 2 | 0 | 3 | 4 |
| $(1,2)$ | 0 | 2 | 7 | 9 |

$x(u,v) = e^u + \sin v$

$y(u,v) = e^u + \cos(v)$

$$\frac{\partial g}{\partial u}(0,0) = \left(\underbrace{\frac{\partial f}{\partial x}}_{(x,y)=(1,2)} \cdot \underbrace{\frac{\partial x}{\partial u}}_{(u,v)=(0,0)} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}\right)\Bigg|_{(0,0) \leftarrow (u,v)}$$

$$= 7 \cdot 1 + 9 \cdot 1 = \boxed{16}$$

* when $(u,v) = (0,0)$

$(x,y) = (1,2)$ *

$\frac{\partial x}{\partial u} = e^u$  $\frac{\partial y}{\partial u} = e^u$

$\frac{\partial x}{\partial v} = \cos v$  $\frac{\partial y}{\partial v} = -\sin(v)$

$$\frac{\partial g}{\partial v}(0,0) = \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}\right)\Bigg|_{(0,0)}$$

$$= 7 \cdot 1 + 9 \cdot 0 = \boxed{7}$$

3. (7 points) Let $f(x, y) = e^x \sin(y) + 2xy$.

(a) At the point $(0, \pi)$, in which direction is the slope of the surface $f(x, y)$ the largest?

$$\nabla f(x,y) = \langle e^x \sin(y) + 2y,\ e^x \cos(y) + 2x \rangle$$

$$\nabla f(0,\pi) = \langle e^0 \sin(\pi) + 2\pi,\ e^0 \cos(\pi) + 0 \rangle$$

$$= \langle 2\pi, -1 \rangle$$

(b) What is the slope of the surface at the point $(0, \pi)$ in the direction of the unit vector $\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$?

$$D_{\vec{u}} f(0,\pi) = \nabla f(0,\pi) \cdot \vec{u} = \langle 2\pi, -1 \rangle \cdot \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle$$

$$= \sqrt{3}\,\pi + \frac{-1}{2}$$

$$= \boxed{\sqrt{3}\,\pi - \frac{1}{2}}$$

4. (12 points) Compute

$$\int_C \sin(y)e^{x\sin(y)}dx + x\cos(y)e^{x\sin(y)}dy$$

where $C$ is the line segment from $(1, \pi)$ to $(2, \pi)$ followed by the line segment from $(2, \pi)$ to $(2, 2\pi)$. If you decide to use a Theorem, state why you can use the Theorem.

Notice, $\frac{\partial P}{\partial y} = \cos(y)\, e^{x\sin(y)} + \sin(y)\cdot e^{x\sin(y)} \cdot x\cos(y)$

$\frac{\partial Q}{\partial x} = \cos(y)\, e^{x\sin(y)} + x\cos(y)\, e^{x\sin(y)} \cdot \sin(y)$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and $P, Q$ are defined on all of $\mathbb{R}^2$.

Let $\vec{F} = \langle P, Q\rangle$, and we have that $\vec{F}$ is conservative.

Not needed, but shows some $f$ s.t. $\vec{F} = \nabla f$ exists!

Apply Fundamental Theorem of Line Integrals, need to find $f$ such that $\vec{F} = \nabla f$ first!

WANT: $P = \frac{\partial f}{\partial x} = \sin(y)\, e^{x\sin(y)}$ "antiderivative"

$f(x,y) = e^{x\sin(y)} + g(y)$

$\frac{\partial f}{\partial y} = e^{x\sin(y)} \cdot x\cos(y) + g'(y)$

WANT: $Q = \frac{\partial f}{\partial y} = x\cos(y)\, e^{x\sin(y)}$

$\Rightarrow g'(y) = 0.$

So $f = e^{x\sin(y)} + C$, for some constant $C$!

Then, $\int_C \sin(y)e^{x\sin(y)}dx + x\cos(y)e^{x\sin(y)}dy = \int_C \vec{F}\cdot d\vec{r}$

(FTLI) $= f(2,\pi) - f(1,\pi) = e^{2\sin(\pi)} - e^{\sin(\pi)} = 1 - 1 = \underline{0}.$

4. (10 points) Compute

Alternative Solution.

$$\int_C \sin(y)e^{x\sin(y)}\,dx + x\cos(y)e^{x\sin(y)}\,dy$$

where $C$ is the line segment from $(1,\pi)$ to $(2,\pi)$ followed by the line segment from $(2,\pi)$ to $(2,2\pi)$. If you use a Theorem, state why you can use the Theorem.

$C_1$ : $\begin{cases} x(t) = t \\ y(t) = \pi \end{cases}$, $1 \le t \le 2$ ⇐ first line segment.

$$\int_1^2 \underbrace{\sin(\pi)\cdot e^{t\sin(\pi)}\cdot \underbrace{1}_{x'(t)}\,dt}_{\downarrow} + \underbrace{t\cdot\cos(\pi)\cdot e^{t\sin(\pi)}\cdot\underbrace{0}_{y'(t)}\,dt}_{0}$$

$$= \int_1^2 0\cdot e^0\,dt = \int_1^2 0\,dt = 0$$

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$C_2$ : $\begin{cases} x(t) = 2 \\ y(t) = t\pi \end{cases}$, $1 \le t \le 2$ ⇐ ~~first~~ second line segment.

$$\int_1^2 \sin(\pi t)\, e^{2\sin(\pi t)}\cdot\underbrace{0}_{x'(t)=0}\,dt + 2\cos(\pi t)\, e^{2\sin(\pi t)}\cdot\pi\,dt$$

$$= \int_1^2 2\pi\cos(\pi t)\cdot e^{2\sin(\pi t)}\,dt$$

$u = 2\sin(\pi t)$
$du = 2\pi\cos(\pi t)\,dt$

$$= \int_{t=1}^{t=2} e^u\,du = \left[e^{2\sin(\pi t)}\right]_1^2 = \left[e^0 - e^0\right] = \underline{0}.$$

$$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} = \boxed{0}$$

5. (8 points) In this problem, you will compute the following integral $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ over two (possibly three!) curves $C$. If you decide to use a Theorem, state why you can use the Theorem.

$= \int \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle P, Q \rangle$

(a) First, let $C$ follow the unit circle $x^2 + y^2 = 1$ with a positive orientation. ← Cannot use Green's Theorem! ($\vec{F}$ not defined at $(0,0)$, which is in the curve!)

parametrize: $\begin{cases} x(t) = \cos t & \rightsquigarrow x'(t) = -\sin t \\ y(t) = \sin t & \rightsquigarrow y'(t) = \cos t \\ 0 \le t \le 2\pi \end{cases}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi} \left\langle \frac{-\sin t}{\underbrace{\cos^2 t + \sin^2 t}_{1}}, \frac{\cos t}{\underbrace{\cos^2 t + \sin^2 t}_{1}} \right\rangle \cdot \langle -\sin t, \cos t\rangle\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} dt = \boxed{2\pi}$$

(b) Second, let $C$ follow the circle $x^2 + y^2 = 25$ with a positive orientation. ← same problem!

$\begin{cases} x(t) = 5\cos t & \rightsquigarrow x'(t) = -5\sin t \\ y(t) = 5\sin t & \rightsquigarrow y'(t) = 5\cos t \\ 0 \le t \le 2\pi \end{cases}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-5\sin t}{\underbrace{25\cos^2 t + 25\sin^2 t}_{25}}, \frac{5\cos t}{\underbrace{25\cos^2 t + 25\sin^2 t}_{25}} \right\rangle \cdot \langle -5\sin t, 5\cos t\rangle\,dt$$

$$= \int_0^{2\pi} \frac{25(\sin^2 t + \cos^2 t)}{25}\,dt = \int_0^{2\pi} dt = \boxed{2\pi}$$

(c) (Extra Credit! 2 points) Third, let $C$ follow the curve $(x-5)^2 + (y-5)^2 = 1$ with a positive orientation. (You may work on the back of this page if you need more space.)

Notice!

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)\cdot(-1) - (-y)\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)\cdot(1) - (x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\Bigg\}\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$\vec{F}$ is defined inside $(x-5)^2 + (y-5)^2 = 1$ ! Use Green's Theorem: $\int_C \vec{F}\cdot d\vec{r} = \boxed{0}$

**Bonus:** (3 points) Assume $f(x,y,z)$ is a differentiable function with a continuous gradient and $x(t)$,$y(t)$, and $z(t)$ all have continuous derivatives. Show the following equality holds. **Justify each step.**

$$\int_a^b \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\right) dt = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$

$$\int_a^b \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}\right) dt$$

Chain rule (backwords)

$$= \int_a^b \frac{d}{dt}\left(f(x(t), y(t), z(t))\right) dt$$

Fundamental Theorem of Calculus

$$= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))$$