SOLUTIONS!

MATH 324 B

Exam II

May 22, 2019

Name ____________________

Student ID #____________________ Section ____________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:____________________

1	10	
2	7	
3	7	
4	10	
5	16	
BONUS	5	
Total	50	

- Your exam should consist of this cover sheet, followed by 5 problems and a Bonus. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 5 problems. Try not to spend more than 10 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (10 points) Clearly indicate whether each statement is true or false.

   (a) TRUE / **FALSE** A line integral over a vector field $\vec{F}$ on an open, connected domain D is independent of path if and only if the vector field on D is conservative.

   (b) **TRUE** / FALSE Consider the vector field $\vec{F}(x,y) = \langle y^3 cos(x), -3y^2 sin(x)\rangle$. F is conservative on $\mathbb{R}^2$.

   $\frac{\partial P}{\partial y} = 3y^2 \cos(x)$

   $\frac{\partial Q}{\partial x} = -3y^2 \cos(x)$

   $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

   (c) **TRUE** / FALSE A line integral with respect to arc length is dependent on the orientation (direction) of the curve.

   (d) TRUE / **FALSE** Suppose $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ and $\int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop C in the domain $D = \{(x,y) : 1 < x^2 + y^2 < 4\}$. Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

   open, connected $\Rightarrow$ conservative $\Rightarrow$

   (e) TRUE / **FALSE** Let $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ be a differentiable vector field (with continuous second derivatives) defined on $D = \{(x,y) : x^2 + y^2 < 1\}$. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then for every closed curve C, $\int_C P(x,y)dx + Q(x,y)dy = 0$.

2. (7 points) Suppose f is a differentiable function of x and y, and $g(u,v) = f(e^u + \sin(v), e^u + \cos(v))$. Use the table of values to calculate $\frac{\partial g}{\partial u}(0,0)$ and $\frac{\partial g}{\partial v}(0,0)$. Show your all of your work.

	f	g	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial y}$
$(1,2)$	2	0	3	4
$(0,0)$	0	2	7	9

$x(u,v) = e^u + \sin(v)$ ⇒ ① $\frac{\partial x}{\partial u} = e^u$, $\frac{\partial x}{\partial v} = \cos(v)$, $\frac{\partial y}{\partial u} = e^u$, $\frac{\partial y}{\partial v} = -\sin(v)$

$y(u,v) = e^u + \cos(v)$ ⇒ ② at $(u,v) = (0,0)$, $(x,y) = (e^0 + \sin(0), e^0 + \cos(0)) = (1,2)$

$$\frac{\partial g}{\partial u}(0,0) = \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}\right)\Big|_{(0,0)} = 3\cdot(e^0) + 4(e^0) = 7$$

$$\frac{\partial g}{\partial v}(0,0) = \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}\right)\Big|_{(0,0)} = 3\cdot(\cos(0)) + 4(-\sin(0)) = 3$$

3. (7 points) Let $f(x,y) = e^x\cos(y) + 2xy$.

(a) At the point $(0,\pi)$, in which direction is the slope of the surface $f(x,y)$ the largest?

$$\nabla f(x,y) = \langle e^x\cos(y) + 2y, -e^x\sin(y) + 2x\rangle$$
$$\nabla f(0,\pi) = \langle e^0\cos(\pi) + 2\pi, -e^0\sin(\pi) + 2\cdot 0\rangle$$
$$= \langle 2\pi - 1, 0\rangle$$

(b) What is the slope of the surface at the point $(0,\pi)$ in the direction of the unit vector $\vec{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$?

$$D_{\vec{u}} f(0,\pi) = \nabla f(0,\pi)\cdot\vec{u} = \langle 2\pi-1, 0\rangle\cdot\langle\frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$$
$$= \pi - \frac{1}{2}$$

4. (10 points) Compute

$$\int_C \cos(y)e^{x\cos(y)}\,dx - x\sin(y)e^{x\cos(y)}\,dy$$

where C is the line segment from $(1,\pi)$ to $(2,\pi)$ followed by the line segment from $(2,\pi)$ to $(2,2\pi)$. If you decide to use a Theorem, state why you can use the Theorem.

(several possible solutions!)

[NOT REQ'D FOR POINTS]

Notice that you can definitively say $\vec{F}(x,y) = \langle \cos(y)e^{x\cos(y)}, -x\sin(y)e^{x\cos(y)} \rangle$ is conservative:

① Partials cont. on $\mathbb{R}^2$ (simply-connected)

② $\dfrac{\partial P}{\partial y} = \cos(y)\cdot e^{x\cos(y)}\cdot(-x\sin(y)) + -\sin(y)e^{x\cos(y)}$

$\dfrac{\partial Q}{\partial x} = -x\sin(y)\cdot e^{x\cos(y)}\cdot(\cos(y)) + -\sin(y)e^{x\cos(y)}$

$\Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

METHOD 1: Find a potential function.

Assume there exists an f such that $\nabla f = \langle \cos(y)e^{x\cos(y)}, -x\sin(y)e^{x\cos(y)} \rangle$.

Then

$$\frac{\partial f}{\partial x} = \cos(y)e^{x\cos(y)} \rightarrow f(x,y) = e^{x\cos(y)} + g(y).$$

(proposed f)

Using the proposed f,

$$\frac{\partial f}{\partial y} = e^{x\cos(y)}\cdot(-x\sin(y)) + g'(y)$$

By assumption,

$$\frac{\partial f}{\partial y} = -x\sin(y)\cdot e^{x\cos(y)}, \quad \text{so} \quad g'(y) = 0 \Rightarrow g(y) = C.$$

Then $f(x,y) = e^{x\cos y} + C$ for any C constant we want.

Apply FTLI:

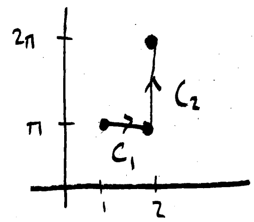
$$\int_C \cos(y)e^{x\cos(y)}dx - x\sin(y)e^{x\cos(y)}dy = \int_C \nabla f\cdot d\vec{r} = \underbrace{f(2,2\pi)}_{\text{endpoint}} - \underbrace{f(1,\pi)}_{\text{starting point}} = \boxed{e^2 - e^{-1}}$$

4. (10 points) Compute

$$\int_C \cos(y)e^{x\cos(y)}\,dx - x\sin(y)e^{x\cos(y)}\,dy$$

where C is the line segment from $(1,\pi)$ to $(2,\pi)$ followed by the line segment from $(2,\pi)$ to $(2,2\pi)$. If you decide to use a Theorem, state why you can use the Theorem.

METHOD 2 : Direct Computation



C_1 parametrization $\begin{cases} x(t) = t \\ y(t) = \pi \end{cases}, \quad 1 \le t \le 2$

C_2 parametrization $\begin{cases} x(t) = 2 \\ y(t) = t \end{cases}, \quad \pi \le t \le 2\pi$

$$\int_C = \int_{C_1} + \int_{C_2} = \int_1^2 \cos(\overset{y(t)}{\pi})\, e^{t\cos(\pi)} \cdot \overbrace{1}^{x'(t)}\, dt - \cancel{t\sin(\pi)\, e^{t\cos(\pi)}} \cdot \overbrace{0}^{y'(t)} \cdot dt$$

$$+ \int_\pi^{2\pi} \cancel{\cos(t)\, e^{2\cos(t)}} \cdot \overbrace{0}^{x'(t)} \cdot dt - 2\sin(t)\, e^{2\cos(t)} \cdot \overbrace{1}^{y'(t)}\, dt$$

$$= \int_1^2 -e^{-t}\, dt + \int_\pi^{2\pi} -2\sin(t)\, e^{2\cos t}\, dt$$

$u = 2\cos t$
$du = -2\sin t\, dt$

$$= \left[e^{-t}\right]_1^2 + \int_{-2}^{2} e^{u}\, du$$

$$= \left(e^{-2} - e^{-1}\right) + \left(e^{2} - e^{-2}\right)$$

$$= e^{-2} - e^{-1} + e^{2} - e^{-2} = \boxed{e^{2} - e^{-1}}$$

5. (16 points) In this problem, you will compute the following integral $\int_C \frac{x}{(x^2+y^2)^{\frac{3}{2}}}dx + \frac{y}{(x^2+y^2)^{\frac{3}{2}}}dy$ over different curves C. If you decide to use a Theorem, state why you can use the Theorem.

   (a) First, let C follow the circle of radius a, $x^2+y^2=a^2$, with a positive orientation.

$\boxed{\text{NOTE! CANNOT use Green's Theorem}}$

Let $\begin{cases} x(t) = a\cos t \\ y(t) = a\sin t \end{cases}$, $0 \le t \le 2\pi$.

$$\int_C \frac{x}{(x^2+y^2)^{3/2}}dx + \frac{y}{(x^2+y^2)^{3/2}}dy = \int_0^{2\pi} \frac{a\cos t}{[a^2(\cos^2 t+\sin^2 t)]^{3/2}}(-a\sin t)dt + \frac{a\sin t}{[a^2(\cos^2 t+\sin^2 t)]^{3/2}}(a\cos t)dt$$

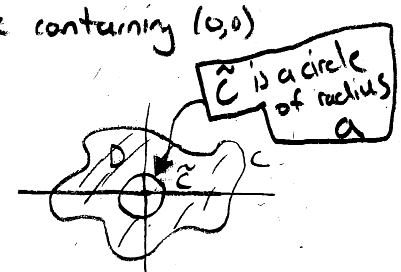
$$= \int_0^{2\pi} \frac{-a^2\cos t\sin t + a^2\cos t\sin t}{a^3}dt$$

$$= \int_0^{2\pi} 0\, dt$$

$$= 0.$$

   (b) Next, let C be **any** positively oriented simple, closed curve containing $(0,0)$. Continue on the back of this page if you need more space.

Idea: Show any positively oriented simple closed curve containing $(0,0)$ is equal to the line integral from part a.



Apply Green's Theorem to the following region:

$$\int_C \frac{x}{(x^2+y^2)^{3/2}}dx + \frac{y}{(x^2+y^2)^{3/2}}dy + \int_{-\tilde{C}} \frac{x}{(x^2+y^2)^{3/2}}dx + \frac{y}{(x^2+y^2)^{3/2}}dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA$$

$\leftarrow$ negative orientation

$$= 0$$

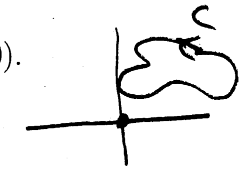
(Quotient) $\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)^{3/2}\cdot 0 - y\cdot\frac{3}{2}(x^2+y^2)^{1/2}\cdot 2x}{(x^2+y^2)^3} = \frac{-3xy}{(x^2+y^2)^{5/2}}$

(Product) $\frac{\partial P}{\partial y} = -\frac{3}{2}x(x^2+y^2)^{-5/2}\cdot 2y = \frac{-3xy}{(x^2+y^2)^{5/2}}$

$$\int_C + \int_{-\tilde{C}} = 0 \Rightarrow \int_C \frac{x}{(x^2+y^2)^{3/2}}dx + \frac{y}{(x^2+y^2)^{3/2}}dy = \int_{\tilde{C}} \frac{x}{(x^2+y^2)^{3/2}}dx + \frac{y}{(x^2+y^2)^{3/2}}dy \overset{\text{(part a)}}{=} 0.$$

$\tilde{C} \leftarrow$ positive orientation!

(c) Next, let C be **any** positively oriented simple, closed curve **not** containing $(0,0)$.



Apply Green's Theorem, In part b, we saw $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, so

$$\int_C \frac{x}{(x^2+y^2)^{3/2}}\,dx + \frac{y}{(x^2+y^2)^{3/2}}\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0.$$

(d) Is the vector field $\vec{F} = \left\langle \frac{x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2)^{\frac{3}{2}}} \right\rangle$ conservative? Why or why not? **Fully justify your answer.**

Yes. All line integrals over positively oriented closed curves are 0. Reversing the orientation, we see that all line integrals over negatively oriented closed curves are also 0, i.e. <u>all</u> <u>line integrals over closed curves are 0</u>. In addition, the domain of the vector field is $\mathbb{R}^2 \setminus \{(0,0)\}$ ($\mathbb{R}^2$ without the origin), which is <u>open</u> and <u>connected</u>. Thus, we may conclude $\vec{F}$ is conservative.

Bonus: (5 points) Consider the electric field generated by a single electron located at the origin in $\mathbb{R}^3$

$$\vec{E} = \overbrace{\frac{\epsilon Q x}{(\sqrt{x^2+y^2+z^2})^3}}^{P}\hat{\imath} + \overbrace{\frac{\epsilon Q y}{(\sqrt{x^2+y^2+z^2})^3}}^{Q}\hat{\jmath} + \overbrace{\frac{\epsilon Q z}{(\sqrt{x^2+y^2+z^2})^3}}^{R}\hat{\mathbf{k}}$$

where Q is the charge of the electron and ϵ a constant. Show that $\text{div}\vec{E} = 0$ for the electric field.

$$\frac{\partial P}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2}\,\varepsilon Q - \varepsilon Q x \cdot \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}\cdot \cancel{2}x}{(x^2+y^2+z^2)^3}$$

$$\frac{\partial Q}{\partial y} = \frac{(x^2+y^2+z^2)^{3/2}\,\varepsilon Q - \varepsilon Q y \cdot \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}\cdot \cancel{2}y}{(x^2+y^2+z^2)^3}$$

$$\frac{\partial R}{\partial z} = \frac{(x^2+y^2+z^2)^{3/2}\,\varepsilon Q - \varepsilon Q z \, \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}\cdot \cancel{2}z}{(x^2+y^2+z^2)^3}$$

$$\text{div } E = \frac{3\varepsilon Q(x^2+y^2+z^2)^{3/2} - 3\varepsilon Q x^2(x^2+y^2+z^2)^{1/2} - 3\varepsilon Q y^2(x^2+y^2+z^2)^{1/2} - 3\varepsilon Q z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

← factor out of each term.

$$= \frac{3\varepsilon Q(x^2+y^2+z^2)^{3/2} - 3\varepsilon Q(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3}$$

$$= \frac{3\varepsilon Q(x^2+y^2+z^2)^{3/2} - 3\varepsilon Q(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}$$

$$= 0.$$