INFINITE-TYPE TRANSLATION SURFACES

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CONTENTS

1. INTRODUCTION

In this article, we find a family of translation surfaces that are infinite-type finite-area surfaces whose Veech group is isomorphic to $\mathbb{Z}$. Additionally, we will focus on a particular direction on a smaller family of surfaces and study cylinder decompositions in that direction.

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2. A FAMILY OF INFINITE-TYPE FINITE AREA TRANSLATION SURFACES

We define a family of Güteltier Schwanz (armadillo tail, in German) surfaces, which we call GS surfaces from now. We place the first square, which we denote by $\square_1$, in the first quadrant so that the lower left vertex lies at the origin and all edges are parallel to the axes. For $k \geq 1$, glue the left side of $\square_{k+1}$ to the right side of $\square_k$ so that the bottom edge of all squares lie on the x-axis. We denote the side length of $\square_n$ by l_n, and assume that (l_n) is a strictly decreasing sequence. Bowman [1] and Degli Esposti–Del Magno–Lenci [2] allow rectangles instead of squares; in the former article these surfaces are called "stack of boxes".

We refer to the top (horizontal) edge of a square by the *roof* and the right (vertical) edge of a square that is identified to a segment on the y-axis by the *portal.*

In this article, we assume that all $\square_k$ are squares, and without loss of generality, we assume that $l_1 = 1$. Identify horizontal (vertical, resp.) edges via vertical (horizontal, resp.) translation, and the resulting translation surface is an infinite genus surface where all vertices are identified as one (wild) singularity that has infinite cone angle.

Randecker's notation: A translation surface $(X, \mathcal{A})$ is a connected surface X with a translation structure $\mathcal{A}$ on X. The translation surface $(X, \mathcal{A})$ is finite if the metric completion $\overline{X}$ is a compact surface and $\overline{X} \setminus X$ is discrete.

Definition 2.1 (Cone angle, infinite angle, and wild singularities [4])**.** *Let $(X, \mathcal{A})$ be a translation surface and σ a singularity of $(X, \mathcal{A})$.*

(1) The singularity σ is called a cone angle singularity of multiplicity $k > 0$ *if there exist*
- $\varepsilon > 0$,
- *an open neighborhood B of σ in $\overline{X}$, and*
- *a k-cyclic translation covering from $B \setminus \{\sigma\}$ to the once-punctured disk $B(0, \varepsilon) \setminus \{0\} \subset \mathbb{R}^2$.*

(2) The singularity σ is called an infinite angle singularity *or* cone angle singularity of multiplicity ∞ *if there exist*
- $\varepsilon > 0$,
- *an open neighborhood B of σ in $\overline{X}$, and*
- *an infinite cyclic translation covering from $B \setminus \{\sigma\}$ to the once-punctured disk $B(0, \varepsilon) \setminus \{0\} \subset \mathbb{R}^2$.*

(3) The singularity σ is called wild *if it is neither a cone angle nor an infinite angle singularity.*

The following are examples of finite-area GS surfaces.

Example 2.2. *(1) The GS surface where $l_n = r^{n-1}$, for $r \in (0, 1)$, which we call a* geometric Güteltier Schwanz surface. *Its area is $\frac{1}{1-r^2}$.*

(2) The harmonic Güteltier Schwanz surface *where $l_n = \dfrac{1}{n}$. While the surface is not bounded in the horizontal direction, its area is finite since area(harmonic GS) $=$ $\zeta(2) = \frac{\pi^2}{6}$.*

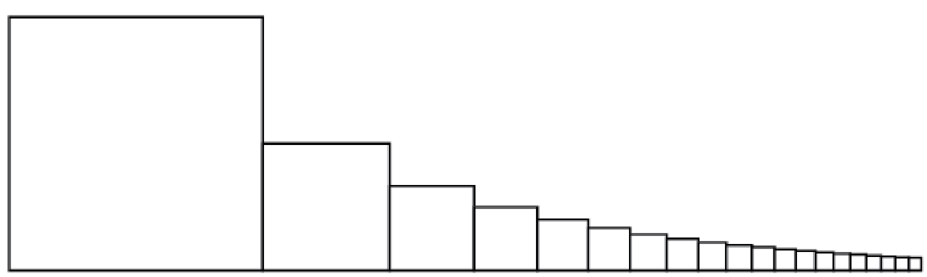

FIGURE 1. The harmonic Güteltier Schwanz surface

We denote by X_n the *truncated GS surface* $\bigcup\limits_{i=1}^{n} \square_i$.

3. VEECH GROUP OF GÜTELTIER SCHWANZ SURFACES IS $\mathbb{Z}$

Theorem 3.1. *Veech group of the Güteltier Schwanz surface is $\mathbb{Z}$*

Proof. The action of the Veech group must fix the line on the surface corresponding to the x-direction with infinite appearances of the one singularity. There is no other line on where this occurs on the surface with a corresponding sequence of saddle connections going to zero. Moreover, the convergence to zero of saddle connections implies that the eigenvalue in this direction must be 1. The only possible element in $SL_2(\mathbb{R})$ with such an eigenvalue must be either the identity or a parabolic element. $\square$

4. A PARTICULAR DIRECTION ON GEOMETRIC GÜTELTIER SCHWANZ SURFACES

In the following section, we construct a cylinder decomposition in a particular direction on any rational geometric Güteltier Schwanz surfaces. Does this direction correspond to a parabolic element or not?

A *cylinder* is closed subspace of the surface foliated by homotopic closed trajectories, whose boundary consists of saddle connections. A closed trajectory in a cylinder is called a *waist curve*. A *cylinder decomposition* is a decomposition of the surface into a possibly infinite number of cylinders. Each cylinder may only intersect with another cylinder at the boundary. A boundary component, which we call a *spine* may be a concatenation of possibly infinitely many saddle connections. If a spine is made of a single saddle connection, we call it a *rigid spine*. If a spine is comprised of multiple (possibly infinitely many) saddle connections, we call it a *flexible spine*.

As an example, consider the cylinder decomposition of a Güteltier Schwanz in the rather obvious horizontal way, given our choice of polygonal representation of the surface.

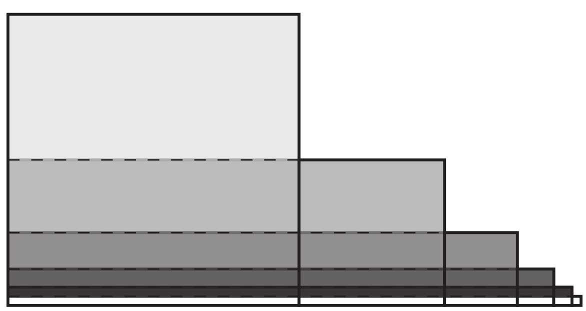

FIGURE 2. A cylinder decomposition of a GS surface in the horizontal direction

Observe that as we move from top to bottom, each cylinder in the cylinder decomposition becomes longer and thinner, eventually limiting to a flexible spine at the base of the polygonal representation (the closed trajectory comprised of an infinite concatenation of saddle connections).

There is another less obvious cylinder decomposition on the surface which does not appear in the orbit of this horizontal cylinder decomposition. In the same way that the above cylinder decomposition is comprised of an infinite number of cylinders limiting to a spine, so will this cylinder decomposition. However, this one limits to a rigid spine, not a flexible spine. In the remainder of this section, we will construct this cylinder decomposition via an induction argument.

The following is a key theorem in which we identify a long closed saddle connection which turns out to be a (limiting) rigid spine of a cylinder decomposition on certain GS surfaces. Every GS surface is an infinite connected sum of tori; this particular saddle connection wraps around each torus in this decomposition. Note that the following theorem is very general and requires no assumption on the parameter r. By $\frac{1}{2-r}$-direction, we mean the direction with slope $\frac{1}{2-r}$ relative to our polygonal representation.

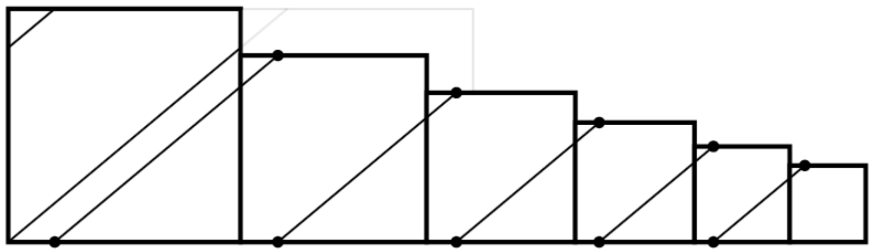

FIGURE 3. A geometric Güteltier Schwanz surface with a trajectory of slope $\frac{1}{2-r}$

Theorem 4.1. *For geometric Güteltier Schwanz surfaces for $r \in (0,1)$, there exists a closed saddle connection in the $\frac{1}{2-r}$-direction that goes through every torus.*

Proof. We start from the origin, the lower left vertex of $\square_1$. Since $r < \frac{1}{2-r} < 1$, the straight line through the origin of slope $\frac{1}{2-r}$ hits the portal of $\square_1$ at point $(1, \frac{1}{2-r})$. By identification with the y-axis (the left edge of $\square_1$), the trajectory continues and hits the roof of $\square_1$ at $(1-r, 1)$. Again by identification, the trajectory continues from $(1-r, 0)$ and hits the roof of $\square_2$ at $(1+r(1-r), r)$. The trajectory continues to hit the roof partitioning the roof with a fixed ratio. Hence, due to similarity, it hits every roof without hitting any vertex. In other words, it "wraps around every torus." $\square$

Alternatively, via renormalization (under the action of $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$), one can view the trajectory with slope $\dfrac{1}{2-r} - 1 = \dfrac{r-1}{2-r}$ direction. Starting from the upper left vertex $(0,1)$, the linear flow hits the portal on $\square_1$ at $\left(1, \dfrac{1}{2-r}\right)$. Continuing from $\left(0, \dfrac{1}{2-r}\right)$, the trajectory does not hit any roof nor portal and tends to $\left(\dfrac{1}{1-r}, 0\right)$, hence yields a rigid spine. In fact, given our polygonal representation, any trajectory starting from $(0,0)$ with slope $\dfrac{1}{2-r} + \mathbf{n}$, for any $\mathbf{n} \in \mathbb{N}$, (or $(0,1)$ with slope $\dfrac{1}{2-r} - \mathbf{n}$, for $\mathbf{n} \in \mathbb{N}$) yields a rigid spine.

In the following theorem, we show that for a specific family of geometric GS surfaces, i.e., when $r = \dfrac{1}{q}$ for $q \in \mathbb{N} \setminus \{1\}$, there exists a cylinder in the $\dfrac{1}{2-r}$-direction. We call this cylinder cyl_1, and the existence of this cylinder will be the base case for the induction that follows.

Theorem 4.2. *Given any geometric GS surface with parameter $r = \frac{1}{q}$, there exists a cylinder in the $\dfrac{1}{2-r}$-direction, that lies entirely in $X_2 = \square_1 \cup \square_2$.*

Proof. For $q = \mathbb{N} \setminus \{1\}$, the slope of the trajectory is $\frac{1}{2-r} = \frac{q}{2q-1}$.

First, we show that the trajectory from $(1,0)$ with slope $\frac{q}{2q-1}$ stays entirely in X_2.

Start from $(1,0)$ we hit the portal of $\square_2$ at $\left(1+\dfrac{1}{q}, \frac{1}{2q-1})\right)$, hence the first point at which the trajectory hits the vertical axis is at $\left(0, \dfrac{1}{2q-1}\right)$. We continue and hit $\left(1, \dfrac{1+q}{2q-1}\right)$, which through the portal is identified to $\left(0, \dfrac{1+q}{2q-1}\right)$. Note that the nth time the trajectory hits the vertical axis is at $\left(0, \dfrac{1}{2q-1} + \dfrac{(n-1)q}{2q-1} - \lfloor \dfrac{1}{2q-1} + \dfrac{(n-1)q}{2q-1} \rfloor\right)$. This is true because $\dfrac{1}{2q-1} < \dfrac{1}{q} < \dfrac{2}{2q-1}$, which means that we will always hit the portal, unless the numerator of $\dfrac{1}{2q-1} + \dfrac{(n-1)q}{2q-1} - \lfloor \dfrac{1}{2q-1} + \dfrac{(n-1)q}{2q-1} \rfloor$ is 1. Pick $n = 2q-2$, then the trajectory hits the singularity at $(1,1)$. In other words, the trajectory goes through $\square_2$ exactly once at the beginning and stays in $\square_1$.

Furthermore, this is the first time the trajectory hits the singularity. This follows from the fact that $\gcd(q, 2q-1) = 1$. The trajectory hits the y-axis at points $\left\{(0,y) : y \in \left\{\frac{(n-1)q+1}{2q-1} - \lfloor \frac{(n-1)q+1}{2q-1} \rfloor\right\}\right\}$. Note that the numerator of the y-coordinates in this set (not in the same order in which the trajectory hits the y-axis) is

$$\{1, 1+q, 1+2q, \ldots, 1+(2q-3)q \equiv 0 \mod (2q-1), q, 2q \equiv 1 \mod (2q-1)\}.$$

In other words, the trajectory wraps around $\square_1$ exactly $2q-3$ times.

The saddle connection that we constructed above acts as the "bottom" saddle connection of cyl_1. We can obviously take the width of the cylinder to be a sufficiently small number. However, we claim that the width of a maximal cylinder (or maximal width) is $\dfrac{q-1}{q(2q-1)}$. Below, we show that the trajectory starting from $\left(0, \dfrac{1}{q}\right)$ in the $\dfrac{q}{2q-1}$-direction is a closed trajectory. In other words, we define a circle rotation $T : [0,1]/\sim \to [0,1]/\sim$ by $T(x) = x + \dfrac{q}{2q-1}$, and track certain points on the y-axis.

Take the set of points where the bottom saddle connection hit the y-axis and add our desired maximal width $\dfrac{q-1}{q(2q-1)}$:

$$\left\{\frac{i}{2q-1}\right\}_{i=1,\neq q}^{2q-2} + \frac{q-1}{q(2q-1)},$$

where a set + number denotes adding the number to each element of the set. We have $\left\{\dfrac{(i+q)-1}{q(2q-1)}\right\}_{i=1,\neq q}^{2q-2}$ which we split into three groups 1) $i = 1, \ldots, q-2$, 2) $i = q-1$, and 3) $i = q+1, \ldots, 2q-2$. We show that T swaps 1) and 3)

$$T^{2q-3}\left(\frac{1}{q}\right) = \frac{1}{q} + (2q-3)\frac{q}{2q-1} = \frac{2q^3 - 3q + 2q - 1}{q(2q-1)} \mod \frac{q-1}{q(2q-1)}.$$

Since $\dfrac{q-1}{q(2q-1)} < \dfrac{1}{q}$ the trajectory is at $\left(1, \dfrac{q-1}{q(2q-1)}\right)$. We continue through $\square_2$ which sends $\left(1, \dfrac{q-1}{q(2q-1)}\right)$ to $\left(1+\dfrac{1}{q}, \dfrac{1}{1}\right)$. To ensure that the trajectory always hit the portal before its $2q-3$th iterate, we need to show that $T^i\left(\dfrac{1}{q}\right) > \dfrac{1}{q}$ for $i < 2q-3$.

$\square$

We claim that this saddle connection to be the top boundary of cyl_1 by showing that there are no saddle connections in between the top saddle connection and bottom saddle connection.

Claim 2. There are no saddle connections that hit the vertical axis between $\left(0, \frac{1}{2q-1}\right)$ and $\left(0, \frac{1}{q}\right)$.

Proof. (of claim). This follows from the fact that there is no integer n that satisfies $\frac{n}{2q-1} < \frac{1}{q} < \frac{n}{2q-1} + \frac{q-1}{q(2q-1)}$. $\square$

In conclusion, for a geometric GS surface with parameter $r = \frac{1}{q}$, cyl_1 lies entirely in X_2.

Remark 4.3. *The assumption that $r = \frac{1}{q}$ cannot be removed. For example, if $r = 2/3$, the saddle connection starting from $(1,0)$ is not contained in X_2 but in X_3.*

We now construct a sequence of cylinders in this direction. Our main tool is the following dilation map.

Definition 4.4. *The map $\tilde{f}_r : \mathbb{R}^2 \to \mathbb{R}^2$, where*

$$f_r : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} rx+1 \\ ry \end{pmatrix}.$$

Observe that $\tilde{f}_r$ is an injective map.

The map does not descend from $\mathbb{R}^2$ to a well-defined map on the quotient of the polygonal representation of a GS surface. The issue is that the map is not well-defined along the leftmost edge of the first square. The leftmost edge of the first square is mapped to the leftmost edge of the second square, but the rightmost edge of each square is mapped to the rightmost edge of the subsequent square. However, the map does descend to a partial quotient, where we only identify the top and bottom edges. That is the content of the following lemma, whose proof is elementary.

Lemma 4.5. *Let P_r be the polygonal representation of a GS surface X_r with parameter r such that the polygonal representation is embedded in $\mathbb{R}^2$ and the side identifications forgotten. Let X_r^{tb} be a quotient of P_r by identifying the top and bottom edges only. Let $\square_k^{tb}$ denote the k^{th}-square in X_r^{tb}. Then $\tilde{f}_r$ descends to a map f_r on X_r^{tb}. The image under f_r of $\square_k^{tb}$) is $(\square_{k+1}^{tb})$.*

Let $q : P_r \to X_r^{tb}$ be the quotient map identifying the top and bottom edges of the polygon. Let cyl_k be a cylinder, and lift cyl_k to X_r^{tb}. Call this lift cyl_k^{tb}. Let L and R denote the unidentified left and right edges of the polygon. Inductively define cyl_{k+1} as the closure of $q \circ f_r(\mathrm{cyl}_k^{tb} \setminus (L \cup R))$ with respect to the linear flow in the $\frac{1}{2-r}$-direction. Observe that cyl_{k+1} doesn't depend on the chosen lift of cyl_k.

The following lemma is the inductive step in the construction of the cylinder decomposition.

Lemma 4.6. *If cyl_k is a cylinder with skew width $r^k \frac{1-r}{2-r} = \frac{q}{q^k(2q-1)}$, then cyl_{k+1} is a cylinder with skew width $r^{k+1} \frac{1-r}{2-r} = \frac{q}{q^{k+1}(2q-1)}$.*

Proof. Let γ_n be a waist curve of cyl_k and lift the waist curve to X_r^{tb}. Call the lift γ_n^{tb}. Enumerate the points γ_n □

Theorem 4.7. *For a GS surface with parameter $r = \frac{1}{q}$, $q \in \mathbb{N} \setminus \{1\}$, there exists an infinite cylinder decomposition in the $\frac{1}{2-r}$-direction.*

Proof. Observe that the intersections of the cylinders constructed inductively in Lemma 4.6 intersect only along saddle connections. Moreover, the sequence of cylinders converges to a rigid spine, the closed trajectory identified in Theorem 4.1. □

Figure 4 shows the first few cylinders in this cylinder decomposition for $r = \frac{1}{2}$.

5. DAMI'S INDUCTION STRATEGY (ON FINDING THE BOTTOM SADDLE CONNECTION OF EACH CYL)

This section provides the construction of the bottom saddle connection of each cylinder, which leads to the proof of Theorem 4.7. Lemma 5.1 shows this for cyl_2, and Theorem 5.7 generalizes this for all cyl_k.

The strategy for the Lemma is to build cyl_2 "on top of" cyl_1. The top boundary of cyl_1 is a single closed saddle connection (by Theorem 4.2), which we denote by tsc_1, for **t**op **s**addle **c**onnection of cyl_1. We extend this saddle connection to construct bsc_2, for **b**ottom **s**addle **c**onnection of cyl_2. In the following lemma, we show the existence of a saddle connection which we denote by bsc_2' so that $\mathrm{bsc}_2 = \mathrm{tsc}_1\#\mathrm{bsc}_2'$, where # denotes concatenation of two saddle connections.

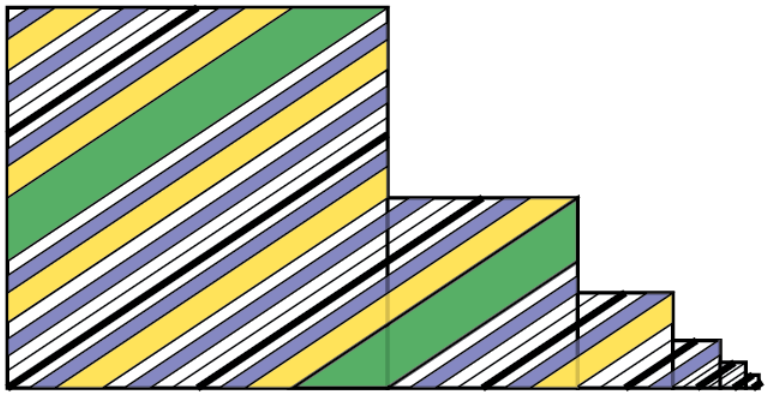

FIGURE 4. Cylinder decomposition on the geometric Güteltier Schwanz surface (parameter $r = \frac{1}{2}$)

Lemma 5.1. *Given any geometric GS surface with parameter $r = \frac{1}{q}$, $q \in \mathbb{N} \setminus \{1\}$, there exists a second cylinder whose bottom boundary saddle connection contains the top boundary saddle connection of cyl_1. or remove the cylinder part and replace to bsc*

Proof. We use the following notations:

- $\xrightarrow{x}$ indicates the linear flow where the horizontal displacement is x,
- $\xrightarrow{\text{portal}(*)}$ indicates that the flow hit a portal of $\square_*$, i.e., the x-coordinate is $1 + \cdots + \frac{1}{q^{*-1}}$ and the y-coordinate lies between $\frac{1}{q^*}$ and $\frac{1}{q^{*-1}}$, hence it is mapped to the corresponding point on the y-axis.
- $\xrightarrow{\text{mod } 1/q^*}$ indicates that the flow passed the roof of $\square_{*+1}$, i.e., the point is not on the polygonal representation of the surface, hence the y-coordinate is to be adjusted.

We show that a linear flow in the $\frac{1}{2-r}$-direction starting from $(1+r, 0)$ contains the top saddle connection of cyl_1. In terms of q, this is equivalent to flowing in the $\frac{q}{2q-1}$-direction from $\left(1 + \frac{1}{q}, 0\right)$.

$$\left(1+\frac{1}{q}, 0\right) \xrightarrow{\frac{1}{q^2}} \left(1+\frac{1}{q}+\frac{1}{q^2}, \frac{1}{q(2q-1)}\right) \xrightarrow{\text{portal}(3)} \left(0, \frac{1}{q(2q-1)}\right)$$
$$\xrightarrow{1} \left(1, \frac{1+q^2}{q(2q-1)}\right) \xrightarrow{\text{portal}(1)} \left(0, \frac{1+q^2}{q(2q-1)}\right) \xrightarrow{\text{mod } 1} \left(1, \frac{1+q}{q(2q-1)}\right).$$

All operations above apply to all $q \in \mathbb{N} \setminus \{1\}$. Observe that $\frac{1+q}{q(2q-1)} = \frac{1}{q}$ if $q = 2$, and $\frac{1+q}{q(2q-1)} < \frac{1}{q}$ if $q > 2$.

If $q = 2$, the linear flow in the $\frac{q}{2q-1}$-direction from $\left(1 + \frac{1}{q}, 0\right)$ to $\left(1, \frac{1+q}{q(2q-1)}\right)$ is a saddle connection, which we denote by bsc_2'.

If $q > 2$, we have $\frac{1+q}{q(2q-1)} < \frac{1}{q}$. Hence we do not yet have a saddle connection. Continuing from the last expression we have,

$$\left(1, \frac{1+q}{q(2q-1)}\right) \xrightarrow{\frac{1}{q}} \left(1+\frac{1}{q}, \frac{1+2q}{q(2q-1)}\right) \xrightarrow{\text{mod } 1/q} \left(1+\frac{1}{q}, \frac{2}{q(2q-1)}\right)$$
$$\xrightarrow{\text{portal}} \left(0, \frac{2}{q(2q-1)}\right) \xrightarrow{1} \left(1, \frac{2+q^2}{q(2q-1)}\right) \xrightarrow{\text{portal}} \left(0, \frac{2+q^2}{q(2q-1)}\right)$$
$$\xrightarrow{1} \left(1, \frac{2+2q^2}{q(2q-1)}\right) \xrightarrow{\text{mod } 1} \left(1, \frac{2+q}{q(2q-1)}\right).$$

Note that $\frac{2+q}{q(2q-1)} = \frac{1}{q}$ if $q = 3$, and $\frac{2+q}{q(2q-1)} < \frac{1}{q}$ if $q > 3$. Hence we have bsc_3'.

Induction hypothesis $\frac{n+q}{q(2q-1)} = \frac{1}{q}$ if $q = n+1$.

Inductive step We show that $\frac{n+1+q}{q(2q-1)} = \frac{1}{q}$ if $q = n+2$.

Assume that the linear flow has not yielded a saddle connection yet, i.e., $n > q-1$. Then,

$$\left(1, \frac{n+q}{q(2q-1)}\right) \xrightarrow{1/q} \left(1+\frac{1}{q}, \frac{n+2q}{q(2q-1)}\right) \xrightarrow{\text{mod } 1/q} \left(1+\frac{1}{q}, \frac{n+1}{q(2q-1)}\right)$$
$$\xrightarrow{\text{portal}} \left(0, \frac{n+1}{q(2q-1)}\right) \xrightarrow{1} \left(1, \frac{n+1+q^2}{q(2q-1)}\right) \xrightarrow{\text{portal}} \left(0, \frac{n+1+q^2}{q(2q-1)}\right)$$
$$\xrightarrow{1} \left(1, \frac{n+1+2q^2}{q(2q-1)}\right) \xrightarrow{\text{mod } 1} \left(1, \frac{n+1+q}{q(2q-1)}\right)$$

Since $\frac{n+1+q}{q(2q-1)} = \frac{1}{q}$ for $n+1 = q-1$, this concludes our proof that bsc_2' exists for all $q \in \mathbb{N} \setminus \{1\}$.

$\square$

Remark 5.2. *For any q, bsc_2' hits the y-axis $2q-2$ times at*

$$\left\{ \frac{1}{q(2q-1)}, \frac{1+q^2}{q(2q-1)}, \frac{2}{q(2q-1)}, \frac{2+q^2}{q(2q-1)}, \ldots, \frac{q-1}{q(2q-1)}, \frac{q-1+q^2}{q(2q-1)} \right\},$$

and tsc_1 hits the y-axis $2q-3$ times, hence bsc_2 hits the y-axis $4q-5$ times.

In the next theorem, we generalize the previous result to show that $\mathrm{bsc}_k = \mathrm{tsc}_{k-1} \# \mathrm{bsc}_k'$, where bsc_k' is the saddle connection in the $\frac{1}{2-r}$-direction starting from $(1 + r + \cdots + r^{k-1}, 0)$.

Theorem 5.3. *Given any geometric GS surface with parameter $r = \frac{1}{q}$, $q \in \mathbb{N} \setminus \{1\}$, cyl_k exists whose bottom boundary saddle connection contains the top boundary saddle connection of cyl_{k-1}.*

Proof. We use the same notations we did in the previous lemma.

$$\left(1+\cdots+\frac{1}{q^{k-1}},0\right)\xrightarrow{1/q^k}\left(1+\cdots+\frac{1}{q^k},\frac{1}{q^{k-1}(2q-1)}\right)$$
$$\xrightarrow{\text{portal}}\left(0,\frac{1}{q^{k-1}(2q-1)}\right)\xrightarrow{1}\left(1,\frac{1+q^k}{q^{k-1}(2q-1)}\right)$$
$$\xrightarrow{1}\left(1,\frac{1+2q^k}{q^{k-1}(2q-1)}\right)$$
$$\xrightarrow{\text{mod }1}\left(1,\frac{1+q^{k-1}}{q^{k-1}(2q-1)}\right).$$

The last point above is a singularity if $q=2$ and $k=2$, however, we assume k is large enough and continue.

$$\cdots\xrightarrow{1/q}\left(1+\frac{1}{q},\frac{1+2q^{k-1}}{q^{k-1}(2q-1)}\right)\xrightarrow{\text{mod }1/q}\left(1+\frac{1}{q},\frac{1+q^{k-2}}{q^{k-1}(2q-1)}\right).$$

Again, the last expression is a singularity if $q=2$ and $k=3$.
Induction hypothesis For l sufficiently less than k, assume

$$\left(1+\cdots+\frac{1}{q^l},\frac{1+q^{k-l-1}}{q^{k-1}(2q-1)}\right)$$

is a singularity if $q=2$.
Inductive step We show that

$$\left(1+\cdots+\frac{1}{q^{l+1}},\frac{1+q^{k-l-2}}{q^{k-1}(2q-1)}\right)$$

is a singularity if $q=2$.

For $q>2$, we then have

$$\xrightarrow{1/q^{l+1}}\left(1+\cdots+\frac{1}{q^{l+1}},\frac{1+2q^{k-l-1}}{q^{k-1}(2q-1)}\right)$$
$$\xrightarrow{\text{mod }1/q^{l+1}}\left(1+\cdots+\frac{1}{q^{l+1}},\frac{1+q^{k-l-2}}{q^{k-1}(2q-1)}\right),$$

and the last expression is a singularity if $q=2$, and this proves our hypothesis.

Let $l=k-3$, then the last expression becomes $\left(1+\cdots+\frac{1}{q^{k-2}},\frac{1+q}{q^{k-1}(2q-1)}\right)$, which is a singularity for $q=2$. For $q>2$, we have

$$\left(1+\cdots+\tfrac{1}{q^{k-2}},\tfrac{1+q}{q^{k-1}(2q-1)}\right)\xrightarrow{1/q^{k-1}}\left(1+\cdots+\tfrac{1}{q^{k-1}},\tfrac{1+2q}{q^{k-1}(2q-1)}\right)$$
$$\xrightarrow{\text{mod }1/q^{k-1}}\left(1+\cdots+\tfrac{1}{q^{k-1}},\tfrac{2}{q^{k-1}(2q-1)}\right)\xrightarrow{\text{portal}}\left(0,\tfrac{2}{q^{k-1}(2q-1)}\right)$$
$$\xrightarrow{1}\left(1,\tfrac{2}{q^{k-1}(2q-1)}\right).$$

We claim that the rest follows from the induction technique used in the Lemma 5.1.

□

Corollary 5.4. *The skew width of cyl_k in the $\frac{q}{2q-1}$-direction is $\frac{q-1}{q^k(2q-1)}$.*

Remark 5.5. *Given a geometric GS surface with parameter $r=\frac{1}{q}$, there exists an infinite cylinder decomposition in the $\frac{1}{2-r}$-direction. The number of times cyl_k intersects the y-axis is $2kq-(2k+1)$. Hence the sum of all skew widths is:*

$$\begin{aligned}\sum_{k=1}^{\infty}(2kq-(2k+1))\cdot\frac{q-1}{q^k(2q-1)} &= \frac{q-1}{2q-1}\sum_{k=1}^{\infty}\left(\frac{2k}{q^{k-1}}-\frac{2k}{q^k}-\frac{1}{q^k}\right)\\ &= \frac{q-1}{2q-1}\left(\frac{2q^2}{(q-1)^2}-\frac{2q}{(q-1)^2}-\frac{1}{q-1}\right)=1\end{aligned}$$

for all $q\in\mathbb{N}\setminus\{1\}$.

Corollary 5.6. *The "actual" width of cyl_k in the $\frac{q}{2q-1}$-direction is $\frac{q-1}{q^k\sqrt{q^2+(2q-1)^2}}$.*

Theorem 5.7. *The trajectory starting from $\left(1+\frac{1}{q}+\cdots+\frac{1}{q^{k-1}}\right)$ in the $\frac{q}{2q-1}$-direction, i.e., bsc_k hits $\{([0,y):0<y<1\}$ at*

$$\begin{aligned}&\left\{\left\{\frac{1}{2q-1},\cdots,\widehat{\frac{q}{2q-1}},\cdots\frac{2q-2}{2q-1}\right\}+\sum_{i=1}^{k-1}\frac{q-1}{q^i(2q-1)}\right\}\\ \cup\quad&\left\{\left\{\frac{i}{q(2q-1)},\frac{i+q^2}{q(2q-1)}\right\}_{i=1}^{q-1}+\sum_{i=2}^{k-1}\frac{q-1}{q^i(2q-1)}\right\}\\ &\vdots\\ \cup\quad&\left\{\left\{\frac{i}{q^{j-1}(2q-1)},\frac{i+q^j}{q^{j-1}(2q-1)}\right\}_{i=1}^{q-1}+\sum_{i=j}^{k-1}\frac{q-1}{q^i(2q-1)}\right\}\\ &\vdots\\ \cup\quad&\left\{\frac{i}{q^{k-1}(2q-1)},\frac{i+q^k}{q^{k-1}(2q-1)}\right\}_{i=1}^{q-1}\end{aligned}$$

where "set + number" indicates that the number is added to every element in the set.

Proof. This set can be obtained by adding appropriate skew widths. A simplified version is below we can switch this expression with the one in the statement

$$\begin{aligned}&\left\{\left\{\frac{i}{2q-1}\right\}_{i=1,\neq q}^{2q-2}+\frac{q^{k-1}-1}{q^{k-1}(2q-1)}\right\}\\ \cup\quad&\left\{\left\{\frac{i}{q(2q-1)},\frac{i+q^2}{q(2q-1)}\right\}_{i=1}^{q-1}+\frac{q^{k-2}-1}{q^{k-1}(2q-1)}\right\}\\ &\vdots\\ \cup\quad&\left\{\left\{\frac{i}{q^{j-1}(2q-1)},\frac{i+q^j}{q^{j-1}(2q-1)}\right\}_{i=1}^{q-1}+\frac{q^{k-j}-1}{q^{k-1}(2q-1)}\right\}\\ &\vdots\\ \cup\quad&\left\{\left\{\frac{i}{q^{k-1}(2q-1)},\frac{i+q^k}{q^{k-1}(2q-1)}\right\}_{i=1}^{q-1}\right\}\end{aligned}$$

or in terms of r,

$$\begin{aligned}
&\left\{\left\{\frac{ir}{2-r}\right\}_{i=1,\neq q}^{2q-2}+\frac{r-r^k}{2-r}\right\}\\
\cup\quad&\left\{\left\{\frac{ir^2}{2-r},\frac{ir^2+1}{2-r}\right\}_{i=1}^{q-1}+\frac{r^2-r^k}{2-r}\right\}\\
&\vdots\\
\cup\quad&\left\{\left\{\frac{ir^j}{2-r},\frac{ir^j+1}{2-r}\right\}_{i=1}^{q-1}+\frac{r^j-r^k}{2-r}\right\}\\
&\vdots\\
\cup\quad&\left\{\frac{ir^k}{2-r},\frac{ir^k+1}{2-r}\right\}_{i=1}^{q-1}.
\end{aligned}$$

□

Now we finally show that these saddle connections are actual boundaries of cylinders by finding a top boundary saddle connection (and nothing in between).

In Claim 2 of Theorem 4.2, we explicitly write down the top boundary. We will denote the length of the segment on the portal of $\square_2$ that intersects cyl_1 by *skew width* of cyl_1.

6. Linear flow on Güteltier Schwanz surfaces

Take cyl_k for an arbitrary k on a geometric GS surface with parameter $r = 1/q$. Then $f_r(\mathrm{cyl}_k)$ is a disconnected set of "cylinders." We will show that there is a circle rotation on $\{0\} \times [0,1]$ that fills in $f_r(\mathrm{cyl}_k)$ at the points of discontinuity and that this yields a cylinder, namely cyl_{k+1}.

In this section, we will define where the circle rotation is defined, and prove that waist curves of cylinders are periodic points under the circle rotation.

We define the *generation zone* in $\square_1$ as

$$\text{generation zone} = \left\{(0,y) : 0 < y < \frac{q-1}{2q-1}\right\} \cup \left\{(1,y) : \frac{q}{2q-1} < y < 1\right\}.$$

Given the set of points a waist curve of cyl_k intersects $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$, we remove the points that lie in the generation zone. Take the remaining points and define sets S_1 and S_2 as the image of these points under f_r. That is,

$$S_1 = P_y \circ f_r\left(\gamma \cap \left\{(0,y) : 0 < y < \frac{q}{2q-1}\right\}\right)$$

and

$$S_2 = \left\{f_r\left(\gamma \cap \left\{(0,y) : \frac{q-1}{2q-1} < y < 1\right\}\right)\right\}$$

where $P_y(x,y) = (0,y)$ is the projection onto the y-axis.

Let $T : [0,1] \sim\longrightarrow [0,1]/\sim$ be a circle rotation where $T(x) = x + \frac{q}{2q-1}$.

Figure 5 illustrates the setting.

Theorem 6.1. *The circle rotation $T : [0,1]/\sim\longrightarrow [0,1]/\sim$ where $T(x) = x + \frac{q}{2q-1}$ maps S_1 to S_2 defined above. In other words, we "fill in" a waist curve of $f_r(\mathrm{cyl}_k)$ at the points of discontinuity to construct a waist curve of cyl_{k+1}.*

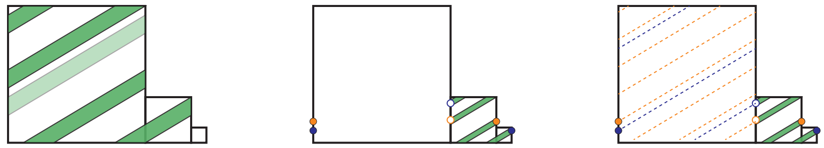

FIGURE 5. $\mathrm{cyl}_1 \setminus$ generation zone(left), $f_r(\mathrm{cyl}_1 \setminus \text{generation zone})$ (center), connecting S_1 and S_2 via the circle rotation T. (right)

To illustrate the simplest case ($k = 1$) before we prove the general case for all k, take γ to be the waist curve of cyl_k slightly above the trajectory starting from $(1, 0)$, i.e., bsc_1. Since bsc_1 intersects $\{(0, y) : 0 < y < 1\}$ at

$$\left\{ \frac{1}{2q-1}, \dots, \widehat{\frac{q}{2q-1}}, \dots, \frac{2q-2}{2q-1} \right\},$$

we have

$$S_1 = \left\{ (0, y) : y = \frac{1}{q(2q-1)}, \dots, \frac{q-1}{q(2q-1)} \right\}, \quad S_2 = \left\{ (1, y) : y = \frac{q-1}{q(2q-1)}, \dots, \frac{2q-2}{q(2q-1)} \right\}.$$

Take $i = 1, \dots, q-2$, then

$$\frac{i}{q(2q-1)} \xrightarrow{T} \frac{i+q^2}{q(2q-1)} \xrightarrow{T} \frac{i+2q^2}{q(2q-1)} \equiv \frac{i}{q(2q-1)},$$

where $\frac{1}{q} < \frac{i+q^2}{q(2q-1)} < 1$ for $i \in \{1, \dots, q-2\}$. Note that for the waist curve γ, we just need too add a sufficiently small ε to all values that appear above.

When $i = q-1$,

$$\frac{q-1}{q(2q-1)} \xrightarrow{T} \frac{q-1+q^2}{q(2q-1)} \xrightarrow{T} \frac{q-1+2q^2}{q(2q-1)} \equiv \frac{2q-1}{q(2q-1)} \xrightarrow{T} \cdots.$$

First note that while $\left(1, \dfrac{2q-1}{q(2q-1)}\right)$ is a singularity, γ is slightly above the singularity, hence we carry on by iterating T. Note that $T^{2q-1}\left(\frac{q-1}{q(2q-1)}\right) = \frac{q-1+(2q-1)q^2}{q(2q-1)} \equiv \frac{q-1}{q(2q-1)}$. We will need to show that for any $m < 2q-1$, $\frac{1}{q} < T^m(\frac{q-1}{q(2q-1)}) = \frac{q-1+mq^2}{q(2q-1)} < 1$. We will show that this holds for arbitrary k in the proof of the theorem below.

Below, we prove for the general case where γ is the waist curve of cyl_k, i.e., slightly above the trajectory starting from $\left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{k-1}}\right)$, bsc_k.

Proof. Theorem 5.7 tells us exactly where bsc_k intersects $\{[0, y] : 0 < y < 1\}$. Recall that S_1 consists of points on $\{0\} \times [0, 1]$ whose y-coordinates are

$$\frac{1}{q} \left(\bigcup_{j=1}^{k-1} \left\{ \left\{ \frac{i}{q^{j-1}(2q-1)} \right\}_{i=1}^{q-1} + \sum_{i=j}^{k-1} \frac{q-1}{q^i(2q-1)} \right\} \cup \left\{ \frac{i}{q^{k-1}(2q-1)} \right\}_{i=1}^{q-1} \right)$$

and S_2 consists of points on $\{1\} \times [0,1]$ whose y-coordinates are

$$\begin{aligned}
&\frac{1}{q}\left\{\left\{\frac{q-1}{2q-1}, \frac{q+1}{2q-1}, \cdots \frac{2q-2}{2q-1}\right\} + \sum_{i=1}^{k-1} \frac{q-1}{q^i(2q-1)}\right\} \\
\cup \quad &\bigcup_{j=2}^{k-1} \frac{1}{q}\left\{\left\{\frac{i+q^j}{q^{j-1}(2q-1)}\right\}_{i=1}^{q-1} + \sum_{i=j}^{k-1} \frac{q-1}{q^i(2q-1)}\right\} \\
\cup \quad &\frac{1}{q}\left\{\frac{i+q^k}{q^{k-1}(2q-1)}\right\}_{i=1}^{q-1}.
\end{aligned}$$

These can be simplified to

$$S_1 = \left\{(0,y) : y \in \bigcup_{j=1}^{k-1} \left\{\frac{(i+1)q^{k-j}-1}{q^k(2q-1)}\right\}_{i=1}^{q-1} \cup \left\{\frac{i}{q^k(2q-1)}\right\}_{i=1}^{q-1}\right\}$$

and

$$S_2 = \left\{(1,y) : y \in \left\{\frac{(1+i)q^{k-1}+q^k-1}{q^k(2q-1)}\right\}_{i=-1, i\neq 0}^{q-2} \cup \bigcup_{j=2}^{k-1} \left\{\frac{(1+i)q^{k-j}+q^k-1}{q^k(2q-1)}\right\}_{i=1}^{q-1} \cup \left\{\frac{i+q^k}{q^{k-1}(2q-1)}\right\}_{i=1}^{q-1}\right\}.$$

We will show that under some iterate of T, S_1 maps onto S_2. We will break this down into three cases. In each case, we show that a point in S_1 maps to a point in S_2 under T^2 (or T^{2q-1}) but any fewer iterate of T maps it to the complement of S_2 on $\{1\} \times [0,1]$, i.e., $\left\{(1,y) : \frac{q}{2q-1} < y < 1\right\}$.

Case 1. First we have $T\left(\frac{i}{q^k(2q-1)}\right) = \frac{i+q^k}{q^k(2q-1)}$ for any $i \in \{1, \ldots, q-1\}$.

Case 2-1. Consider the cases where $j = 2, \ldots, k-1$, and $i = 1, \ldots, q-1$. Then

$$T^2\left(\frac{(i+1)q^{k-j}-1}{q^k(2q-1)}\right) = \frac{(i+1)q^{k-j}-1+2q^{k+1}}{q^k(2q-1)} \equiv \frac{(i+1)q^{k-j}-1+q^k}{q^k(2q-1)}.$$

We show that $T\left(\frac{(i+1)q^{k-j}-1}{q^k(2q-1)}\right)$ does not hit any point in S_2, i.e.,

$$\frac{1}{q} < \frac{(i+1)q^{k-j}-1+q^{k+1}}{q^k(2q-1)} < 1.$$

The left inequality holds since it is equivalent to the inequalities below:

$$\begin{aligned}
q^{k-1}(2q-1) &< (i+1)q^{k-j}-1+q^{k+1} \\
1 &< (i+1)q^{k-j}+q^{k+1}-q^{k-1}(2q-1) \\
&= q^{k-1}\left((i+1)q^{1-j}+(q-1)^2\right)
\end{aligned},$$

and the right inequality holds since it is equivalent to

$$\begin{aligned}
q^k(q-1) &> (i+q)q^{k-j}-1 \\
q^k\left(q-1-(i+1)q^{-j}\right)+1 &\geq q^k\left(q-1-\frac{q}{q^j}\right)+1 > 0.
\end{aligned}$$

Case 2-2. Next, the cases where $j = 1$ and $i = 1, \ldots, q-2$ can be shown with the same technique as in Case 2-1: we have

$$T^2\left(\frac{(i+1)q^{k-1}-1}{q^k(2q-1)}\right) = \frac{(i+1)q^{k-1}-1+2q^k}{q^k(2q-1)} \equiv \frac{(i+1)q^{k-1}-1+q^k}{q^k(2q-1)},$$

and $\frac{1}{q} < T\left(\frac{(i+1)q^{k-1}-1}{q^k(2q-1)}\right) < 1$. The left inequality holds since

$$\begin{aligned} q^{k-1}(2q-1) &< (i+1)q^{k-1}-1+q^{k+1} \\ 1 &< q^{k-1}\left(i+2+2q+q^2\right). \end{aligned}$$

However, the right inequality holds only for $i = 1, \ldots, q-2$:

$$\begin{aligned} (i+1)q^{k-1}-1+q^{k+1} &< 2q^{k+1}-q^k \\ -1 &< q^{k+1}-(i+2)q^k = q^k\left(q-(i+2)\right). \end{aligned}$$

Case 3. Lastly, we deal with $j = 1$ and $i = q-1$.

After $2q-1$-iterates, $\frac{q^k-1}{q^k(2q-1)}$ is mapped to itself. We need to show that for any $m < 2q-1$, $T^m\left(\frac{q^k-1}{q^k(2q-1)}\right)$ falls between $\frac{1}{q}$ and 1, hence does not hit any other point in S_2.

If $m = 2l$, $(l = 1, \ldots, q-1)$, then

$$T^m\left(\frac{q^k-1}{q^k(2q-1)}\right) = \frac{q^k-1+2lq^k}{q^k(2q-1)} \equiv \frac{(l+1)q^k-1}{q^k(2q-1)}.$$

We show that

$$\frac{1}{q} < \frac{(l+1)q^k-1}{q^k(2q-1)} < 1.$$

The left-hand inequality is equivalent to

$$\begin{aligned} q^{k-1}(2q-1) &< (l+1)q^k-1 \\ 1 &< (l+1)q^k-q^{k-1}(2q-1) = q^{k-1}\left((l+1)q-(2q-1)\right) = \left((l-1)q+1\right), \end{aligned}$$

and the right-hand inequality is equivalent to

$$\begin{aligned} (l+1)q^k-1 &< q^k(2q-1) \\ -1 &< q^k(2q-2-l). \end{aligned}$$

If $m = 2l+1$, $(l = 1, \ldots, q-2)$, then

$$T^m\left(\frac{q^k-1}{q^k(2q-1)}\right) = \frac{q^k-1+(2l+1)q^{k+1}}{q^k(2q-1)} \equiv \frac{(l+1)q^k-1+q^{k+1}}{q^k(2q-1)}.$$

Again, we show that this does not hit any point in S_2. First it is greater than $\frac{1}{q}$ since

$$\begin{aligned} q^{k-1}(2q-1) &< (l+1)q^k-1+q^{k+1} \\ 1 &< q^{k-1}\left((l+1)q+q^2+(1-2q)\right) = q^{k-1}\left(q^2+(l-1)q+1\right), \end{aligned}$$

and less than 1 since

$$\begin{aligned} (l+1)q^k-1+q^{k+1} &< q^k(2q-1) \\ -1 &< q^k\left(2q-1-(l+1)-q\right) = q^k\left(q-(l+2)\right). \end{aligned}$$

Take γ to be ε above bsc_k. We have thus connected the disconnected segments of $f_r(\mathrm{cyl}_k)$ to construct a waist curve of cyl_{k+1}.

□

7. AREA OF CYLINDERS

In Corollary 5.6, we found the width of each cylinder. In this section, we will show the length of the waist curve for each cylinder to find the area of cyl_k as a function of q.

First, we will find the horizontal displacement of each waist curve. The table below lists the side lengths of each square and the number of times a waist curve of cyl_k goes through each square. This follows from the circle rotation defined in the previous section.

	$\square_1$	$\square_2$	$\cdots$	$\square_i$	$\cdots$	$\square_k$	$\square_{k+1}$
side length	1	$1/q$		$1/q^{i-1}$		$1/q^{k-1}$	$1/q^k$
#	$k(2q-2)-1$	$(k-1)(q-1)$		$(k-i+1)(q-1)$		$q-1$	1

Then the horizontal displacement of a waist curve of cyl_k is

$$\begin{aligned} & k(2q-2)-1+\frac{1}{q}(k-1)(q-1)+\frac{1}{q^2}(k-2)(q-1)+\cdots+\frac{1}{q^{k-1}}(q-1)+\frac{1}{q^k} \\ = \;& k(2q-2)-1+\sum_{i=1}^{k-1}\frac{(k-i)(q-1)}{q^i}+\frac{1}{q^k} \\ = \;& k(2q-2)-1+\frac{q-1}{q^k}\sum_{i=1}^{k-1}(k-i)q^{k-i}+\frac{1}{q^k} \\ = \;& k(2q-2)-1+\frac{(k-1)q^{k+1}-kq^k+q}{q^k(q-1)}+\frac{1}{q^k}. \end{aligned}$$

For the last equality, we refer to the remark below.

Remark 7.1. *The previous computation follows since:*

$$\begin{aligned} \sum_{i=1}^{k-i}(k-i)q^{k-i} &= q+2q^2+3q^3+\cdots+(k-1)q^{k-1} \\ &= q+2q^2+3q^3+\cdots+(k-1)q^{k-1}+\left(q+\cdots+q^{k-1}\right)-\left(q+\cdots+q^{k-1}\right) \\ &= 2q+3q^2+\cdots+kq^{k-1}-\frac{q(q^{k-1}-1)}{q-1} \\ &= \left(q^2+\cdots+q^k\right)'-\frac{q^k-q}{q-1} \\ &= \left(\frac{q^2(q^{k-1}-1)}{q-1}\right)'-\frac{q^k-q}{q-1} \\ &= \frac{(k-1)q^{k+1}-kq^k+q}{(q-1)^2}. \end{aligned}$$

Proposition 7.2. *The horizontal displacement of the waist curve of* cyl_k *is*

$$(2q-1)\left(k-\frac{q^k-1}{q^k(q-1)}\right)$$

and the actual length of the waist curve, i.e., the circumference of cyl_k is

$$\left(k - \frac{q^k - 1}{q^k(q-1)}\right) \sqrt{(2q-1)^2 + q^2}.$$

Furthermore, the modulus of cyl_k is given as

$$\frac{\text{circumference}}{\text{width}} = \frac{q^2 + (2q-1)^2}{(q-1)^2} \left(kq^{k+1} - (k+1)q^k + 1\right),$$

and the area of cyl_k is given by

$$\text{area}\,(cyl_k) = \left(k - \frac{q^k - 1}{q^k(q-1)}\right) \frac{q-1}{q^k}.$$

Next, we verify that given $r = 1/q$, the infinite sum of $\text{area}(\text{cyl}_k)$ is equal to $\dfrac{1}{1-r^2}$, hence there exists an infinite cylinder decomposition in the $\dfrac{1}{2-r}$-direction.

Proposition 7.3. *Given a geometry GS surface with parameter $r = \dfrac{1}{q}$, $q \in \mathbb{N} \setminus \{1\}$, we show that*

$$\sum_{k=1}^{\infty} \text{area}\,(cyl_k) = \frac{1}{1-r^2} = \frac{q^2}{q^2-1},$$

where the cylinders lie in the $\dfrac{1}{2-r}$-direction.

Proof. We write $\text{area}\,(\text{cyl}_k) = \dfrac{k(q-1)}{q^k} - \dfrac{1}{q^k} + \dfrac{1}{q^{2k}}$. Following the same spirit as a previous remark, we use $\sum_{i=1}^{\infty} ir^i = \dfrac{r}{(1-r)^2}$, for $|r| < 1$. The sum of the first terms is

$$\sum_{k=1}^{\infty} \frac{k(q-1)}{q^k} = \frac{q}{q-1}.$$

The second and third terms are geometric sequences, hence we have

$$\sum_{k=1}^{\infty} \text{area}\,(\text{cyl}_k) = \frac{q}{q-1} + \frac{1}{q-1} + \frac{1}{q^2-1} = \frac{q^2}{q^2-1},$$

our desired result. $\square$

8. No parabolic element

Consider the horizontal cylinder decomposition of the armadillo tail seen in Figure 2. The *perpendicular* cylinder decomposition is comprised of exactly the squares. The element $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is in the Veech group of the surface and this parabolic element corresponds to the perpendicular cylinder decomposition: indeed, the affine map associated with this Veech group element twists these cylinders, but preserves them as a set.

This phenomenon is understood in the finite translation surface setting, where the existence of a cylinder decomposition with rationally related moduli implies a parabolic element in the Veech group and vice-versa. Here, we see that in the perpendicular cylinder decomposition, the modulus of each cylinder is 1 since each cylinder is a square. However, the moduli of the cylinders in the horizontal cylinder decomposition in Figure 2 goes to infinity, and there is no parabolic element in that direction.

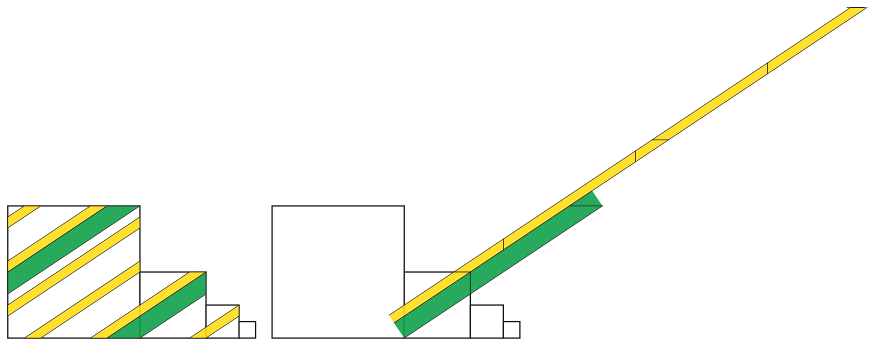

FIGURE 6. Cylinder decomposition $\mathcal{C}$ in the $\dfrac{1}{2-r}$ direction

Lemma 8.1. *Let $\mathcal{C}$ be a cylinder decomposition on a finite area infinite translation surface. Then if the moduli of the cylinders tend to ∞, there is no parabolic element in the Veech group corresponding to an affine map that preserves the cylinder decomposition.*

Remark 8.2. *The lemma allows for rationally related moduli, hence distinguishes the finite translation surface setting from the infinite translation surface setting.*

Proof. □

Corollary 8.3. *Let $\mathcal{C}$ be the cylinder decomposition constructed in the previous sections of this paper. There is no parabolic element in the Veech group corresponding to this cylinder decomposition.*

Proof. We observe that the modulus of cyl_k goes to ∞ as k goes to infinity. □

Lemma 8.4. *Let $\mathcal{C}$ be a cylinder decomposition on a finite area infinite translation surface. Then if the moduli of the cylinders tend to 0, there is no parabolic element in the Veech group corresponding to an affine map that preserves the cylinder decomposition.*

Proof. □

Merge these two lemmas! Compare to Thurston's construction: when two parabolic elements exist. Then extend it to show when 0 versus 1 versus 2 exist.

Remark 8.5. *In work of Hooper and Trevino [3], they observe that the golden ladder has a cylinder decomposition whose moduli are all equal, and the corresponding perpendicular cylinders are symmetric. They are able to find two parabolics, one in each direction. The construction of these parabolics was described by Thurston.*

Corollary 8.6. *Let $\mathcal{C}$ be the cylinder decomposition constructed in the previous sections of this paper. Then, on $\mathcal{C}^{\perp}$, there is no parabolic element in the Veech group corresponding to this cylinder decomposition.*

Proof. We observe that the modulus of $\mathrm{cyl}_k^\perp$ goes to 0 as k goes to infinity. From Corollary 5.6 and Proposition 7.2 we have,

$$\begin{aligned}
\text{circumference of } \mathrm{cyl}_k^\perp &= \sum_{i=k}^{\infty} \mathrm{width}(\mathrm{cyl}_i) \\
&= \sum_{i=k}^{\infty} \frac{q-1}{q^i\sqrt{q^2+(2q-1)^2}} \\
&= \frac{q-1}{q^k\sqrt{q^2+(2q-1)^2}}\frac{1}{1-1/q} \\
&= \frac{1}{q^{k-1}\sqrt{q^2+(2q-1)^2}},
\end{aligned}$$

$$\begin{aligned}
\text{width of } \mathrm{cyl}_k^\perp &= (\text{circumference of } \mathrm{cyl}_k) - (\text{circumference of } \mathrm{cyl}_{k-1}) \\
&= \left(\left(k - \frac{q^k-1}{q^k(q-1)} - \left(k-1-\frac{q^{k-1}-1}{q^{k-1}(q-1)}\right)\right)\right)\sqrt{q^2+(2q-1)^2} \\
&= \frac{q^k(q-1)+1-q}{q^k(q-1)}\sqrt{q^2+(2q-1)^2},
\end{aligned}$$

hence the modulus of $\mathrm{cyl}_k^\perp$ is equal to $\dfrac{q(q-1)}{(q^2+(2q-1)^2)\left(q^k(q-1)+1-q\right)}$ □

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