Solutions

MATH 308 C
Exam I
July 15, 2019

Name ______________________

Student ID #______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________

1	16	
2	14	
3	10	
4	10	
Bonus	5	
Total	50	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 60 minutes to complete the exam and there are 4 problems. Try not to spend more than 12 minutes on each problem.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (16 Points) True / False and Short Answer.

Clearly indicate whether the statement is true or false.

(a) **TRUE** / **FALSE** The **span** of a set of vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is the set of all linear combinations of the vectors.

(b) **TRUE** / **FALSE** Let $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ be a set of linearly dependent vectors in $\mathbb{R}^3$. Then $\vec{v_1} \in \text{span}\{\vec{v_2}, \vec{v_3}, \vec{v_4}\}$.

e.g. $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

(c) **TRUE** / **FALSE** A system of equations with more variables than equations always has at least one solution.

It could still be inconsistent.

e.g. $\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \end{cases}$.

(d) **TRUE** / **FALSE** If one does Gauss-Jordan Elimination on an augmented matrix, the resulting augmented matrix is in reduced echelon form and is unique.

(e) **TRUE** / **FALSE** Given a set of **linearly dependent** vectors, one can **always** express one of the vectors as a linear combination of the others.

(This was a theorem in 2.3.)

$\Rightarrow$ let $\{\vec{v}_1, .., \vec{v}_m\}$ be lin. dep. Then $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = \vec{0}$ has a non-trivial solution, so some $x_i \neq 0$. Solve for $\vec{v}_i$.

Give an example of each of the following. If there is no such example, write NOT POSSIBLE.

(f) **Give an example** of 3 vectors in $\mathbb{R}^2$ that do not span $\mathbb{R}^2$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}.$$

(g) **Give an example** of a set of linearly dependent vectors in $\mathbb{R}^3$ such that when you remove any one vector, the set is linearly independent and spans $\mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

remove any one vector, and the remaining three are linearly independent.
(or, the remaining three span $\mathbb{R}^3$, so by the unifying thm, they are linearly independent.)

(h) **Give an example** of a matrix in echelon form with a pivot in every column where there are more columns than rows.

NOT POSSIBLE.

2. (14 Points)

   (a) For what values of a will the system of equations have one or more solutions? For what values of a will there be no solution?

$$2x_1 + x_2 - x_3 = 1$$
$$x_1 - x_2 + 5x_3 = 3$$
$$-2x_1 + 2x_2 + ax_3 = -6$$

[Possible solution]

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & -1 & 5 & 3 \\ -2 & 2 & a & -6 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 5 & 3 \\ 2 & 1 & -1 & 1 \\ -2 & 2 & a & -6 \end{array}\right]$$

$$\underset{\sim}{\overset{-2R_1+R_2 \to R_2}{2R_1+R_3 \to R_3}} \left[\begin{array}{ccc|c} 1 & -1 & 5 & 3 \\ 0 & 3 & -11 & -5 \\ 0 & 0 & 10+a & 0 \end{array}\right]$$

$\Rightarrow$ There is more than one solution when $\boxed{a=-10}$

Otherwise, there is exactly one solution.

$\Rightarrow$ There are no values of a where there is no solution.

(b) Now let $a = -10$. Solve the system of equations and **give your answer in vector form.**

When $a=-10$:

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 5 & 3 \\ 0 & 3 & -11 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{\frac{1}{3}R_2 \to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 5 & 3 \\ 0 & 1 & -\frac{11}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{R_2+R_1 \to R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{11}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$, parameter

$\Rightarrow x_2 = -\frac{5}{3} + \frac{11}{3}s$

$x_1 = \frac{4}{3} - \frac{4}{3}s$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{5}{3} \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{4}{3} \\ \frac{11}{3} \\ 1 \end{bmatrix}$$

(c) What is the solution to the following system of equations? Explain your answer.

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 + 5x_3 &= 0 \\ -2x_1 + 2x_2 - 10x_3 &= 0 \end{aligned}$$

$$\vec{x} = s \begin{bmatrix} -\frac{4}{3} \\ \frac{11}{3} \\ 1 \end{bmatrix}$$

This is the homogeneous part of the solution.

3. (10 points) Assume a system of equations represented by $A\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ has the solution

$$\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

where s, t are parameters (real numbers). Write a matrix equivalent to A in reduced row echelon form. Then clearly indicate which columns in the matrix are pivot columns and circle the pivots.

A matrix in reduced echelon form gives the solution, so we only need to reverse engineer the matrix:

$$x_4 = s, \quad x_2 = t$$

and $\quad x_1 = -2 + s - 5t$

$$x_3 = 3$$

so

constants

$$\left[\begin{array}{cccc|c} 1 & 5 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]$$

← this line says: $x_1 + 5\underbrace{x_2}_{t} - \underbrace{x_4}_{s} = -2$

← this line says: $x_3 = 3$

pivot column, free variable, pivot column, free variable

and

$$\boxed{A \sim \begin{bmatrix} 1 & 5 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$$

4. (10 points)

(a) Find all values of a such that the set of vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is **linearly dependent**.

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -10 \\ a \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 2 \\ a \\ -5 \end{bmatrix}.$$

Need non-trivial solutions to $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$. In other words, we need an infinite number of solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & -10 & a & 0 \\ -3 & a & -5 & 0 \end{array}\right] \underset{\substack{-2R_1+R_2 \to R_2 \\ 3R_1+R_3 \to R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -12 & a-4 & 0 \\ 0 & a+3 & 1 & 0 \end{array}\right]$$

To have an infinite number of solutions, we need a free variable. Since we have 3 variables and 3 rows, this will only happen if we have a 0 row! We need be able to multiple the second row by a number (scalar) and add it to the third row and get a 0 row.

in other words: $-12 \cdot x = a+3$ and $(a-4)x = 1$

$$\left\{ \begin{array}{l} x = \frac{a+3}{-12} \Rightarrow (a-4)\left(\frac{a+3}{-12}\right) = 1 \\ \Rightarrow (a-4)(a+3) = -12 \end{array} \right.$$

then multiply out and solve for a.

$a^2 - a = 0 \Rightarrow \underline{\underline{a = 0 \text{ or } 1}}$

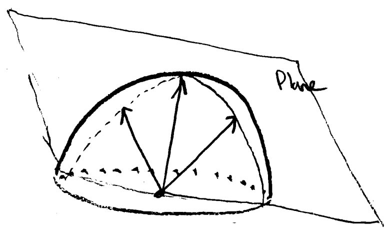
(b) Can you use part (a) to give the values of a such that the set spans $\mathbb{R}^3$? If so, what are they? Explain your thinking.

If $a \neq 0$ and $a \neq 1$, then the only solution will be the trivial solution. (Remember, for a homogeneous system, the trivial solution is <u>always</u> a solution.) This means the set will be linearly independent when $a \neq 0$ and $a \neq 1$. Since we have three linearly independent vectors in $\mathbb{R}^3$, by the unifying theorem, this means they <u>will</u> span $\mathbb{R}^3$, So a can be any real number except 0 or 1.

BONUS: (5 points) Consider the upper-half hemisphere, $H = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$. Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be vectors starting at the origin, each pointing to a unique point on H.

(a) Is the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ necessarily linearly independent? Explain your answer.

No, Intersect any plane passing through the origin with H, and pick three vectors on this plane.



(eg., use the yz-plane)

(b) Remove **one** vector (any one of them) from the set. Is the remaining set linearly independent? Explain your answer.

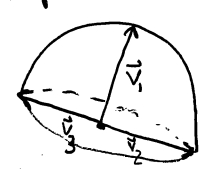
Yes, since $z>0$, we cannot pick vectors in the xy-plane. This means all three vectors will be unit vectors pointing in different directions. When we remove one, the remaining two do not point in the same direction, so they will span a plane. The two vectors cannot be written as a multiple of another, so the vectors must be linearly dependent. (Not true if we let $z=0$! see part c).)

(c) Repeat part (b), but modify the set H: let $H = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$.

If we remove one vector (any of them), then it will no longer be necessarily linearly independent. For example:

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Remove $\vec{v}_1$, and $\{\vec{v}_2, \vec{v}_3\}$ are not linearly independent.



($\vec{v}_2, \vec{v}_3$ span a line)