14.5   Chain Rule

REM   Single variable case:

$$f(x) = g(h(x))$$

$$\frac{df}{dx} = \frac{dg}{d(h(x))} \cdot \frac{dh}{dx}$$

or $$\boxed{f'(x) = g'(h(x)) \cdot h'(x)}$$

We can write this as follows. Let $y = y(x)$, and $x = x(t)$. Then, provided these are differentiable functions,

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

EXAMPLE $$\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \cdot 2x$$

i.e. $$\frac{d}{dh}(\sin(h)) = \cos(h) \quad , \quad h(x) = x^2$$

and $$\frac{dh}{dx} = 2x$$

so, $$\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \cdot 2x$$

$$\left(= \frac{d}{dh}(\sin(h)) \cdot \frac{dh}{dx} \right).$$

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Now, assume $z = z(x,y)$. What if x and y are both functions of t?

Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

This is saying "the rate of change of z with respect to time" is "how fast x changes with time multiplied by how fast z changes with x" plus "how fast y changes with time multiplied by how fast z changes with y."

EXAMPLE 1 $z = x^3 y^2 + 7y^3 x$, where $x = \sin(t)$, $y = \cos(t)$.

Find $\frac{dz}{dt}$ when $t = 0$.

STEP 1: Use the chain rule to find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (3x^2 y^2 + 7y^3) \cos(t) + (2x^3 y + 21y^3 x)(-\sin t)$$

STEP 2: Plug in! $t = 0 \Rightarrow x = \sin(0) = 0, \ y = \cos(0) = 1$

$$\frac{dz}{dt}\bigg|_{t=0} = (0 + 7) \cdot 1 + (0) \cdot (0)$$

$$= \boxed{7}$$

Notice: If we take a derivative with respect to a variable, and the function really only depends on that one variable, it is a normal derivative, and we would write "$\frac{dz}{dt}$".

If we take a derivative with respect to a variable, and the function depends on multiple variables, then it is a partial derivative, and we would write "$\frac{\partial z}{\partial t}$" or "$z_t$".
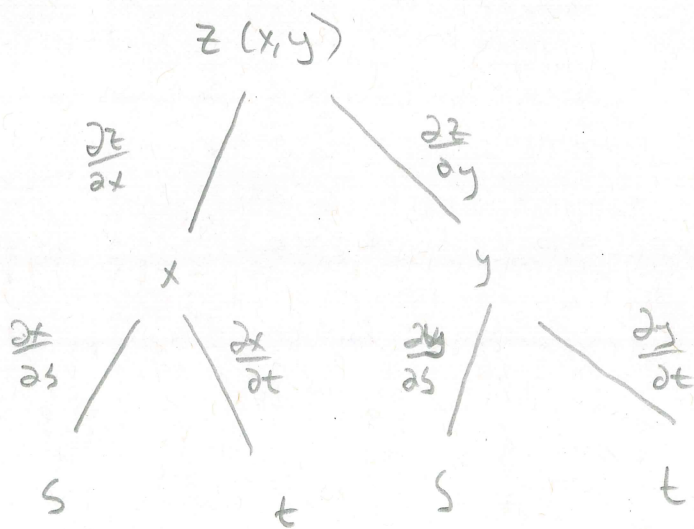
Exercise Explain the use of "normal" derivatives and partial derivatives in the version of the chain rule above. Hint: in the example, plug in the functions of $x(t)$ and $y(t)$ directly into z. Then see that $z(x,y) = z(t)$. .... Do you see how it is a matter of "perspective"?

Now, what if $z = z(x,y) \left(= f(x,y)\right)$, and both x and y are functions of two variables, s and t? We end up with two possible derivatives for z:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Notice : We can think about this similarly to the first version of the chain rule : "the rate of change of z with respect to s" is "how fast z changes with x times how fast x changes with s" plus "how fast z changes with y times how fast y changes with s", and similarly for t.

There is also a nice picture in the text:

$$z(x,y)$$

$$\frac{\partial z}{\partial x} \qquad\qquad \frac{\partial z}{\partial y}$$

$$x \qquad\qquad\qquad y$$

$$\frac{\partial t}{\partial s} \quad \frac{\partial x}{\partial t} \qquad \frac{\partial y}{\partial s} \quad \frac{\partial y}{\partial t}$$

$$s \qquad t \qquad s \qquad t$$

In fact, we can generalize this case to n-variables with m-variables each, i.e. Let $z = z(x_1, \ldots, x_n)$, where $x_1 = x_1(t_1, \ldots, t_m), \ldots, x_n = x_n(t_1, \ldots, t_m)$.

Then,

$$\frac{\partial z}{\partial t_1} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_1} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial z}{}$$

$$\vdots$$

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_m}$$

$$\vdots$$

$$\frac{\partial z}{\partial t_m} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_m} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_m} .$$

This is the __general__ version of the chain rule. In most cases in our class, we will be working with z as a function of x and y, and x and y functions of possibly two more variables. But you should understand how to do this with any number of variables. We will do a few exercises with higher numbers of variables so you get used to it.

EXAMPLE 2

$$u = x^4 y + y^2 z^3, \quad \text{where} \quad x = rse^t, \; y = rs^2 e^{-t}, \text{ and}$$
$$z = r^2 s \sin t.$$

Find $\frac{\partial u}{\partial s}$ when $r = 2, \; s = 1, \; t = 0.$

STEP 1 : Notice $u(x,y,z)$, $x(r,s,t)$, $y(r,s,t)$, $z(r,s,t)$.

STEP 2 ; Use general chain rule.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$\frac{\partial u}{\partial z}$

$$= (4x^3 y) \cdot (re^t) + (x^4 + 2yz^3) \cdot (2rs\, e^{-t}) + (3y^2 z^2)(r^2 \sin t)$$

at $r=2$, $s=1$, $t=0$: $\quad x=2$, $y=2$, $z=0$

$$\frac{\partial u}{\partial s}\Big|_{(2,1,0)} = (4 \cdot 2^3 \cdot 2)(2) + (2^4 + 2 \cdot 1 \cdot 0^3)(\underline{2 \cdot 2 \cdot 1 \cdot 1}) + (3 \cdot 2^2 \cdot 0^2) \cdot (0)$$

$$= 128 + 64 + 0$$

$$\boxed{= 192}$$