SOLUTIONS

MATH 324 A
Exam I
February 1, 2019

Name ______________________________

Student ID #________________ Section ________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

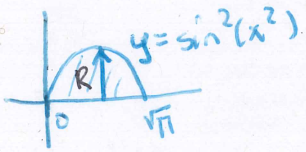
SIGNATURE:______________________________

1	8	
2	12	
3	16	
4	14	
Bonus	4	
Total	~~54~~ 50	

- Your exam should consist of this cover sheet, followed by 4 problems and a bonus question. Check that you have a complete exam.
- Pace yourself. You have 50 minutes to complete the exam and there are 5 problems. Try not to spend more than 10 minutes on each page.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (8 points) Compute the mass of the lamina that is the shape of the region in the plane bounded by the curves $y = 0$ and $y = \sin^2(x^2)$ in the upper half plane where $0 \leq x \leq \sqrt{\pi}$. Assume the density at a point (x, y) is 2 times the distance from the y-axis.



Density: $\rho(x,y) = 2x$

$$m = \iint_R \rho(x,y)\, dA = \int_0^{\sqrt{\pi}} \int_0^{\sin^2(x^2)} 2x \, dy \, dx$$

$$= \int_0^{\sqrt{\pi}} \Big[2xy \Big]_{y=0}^{y=\sin^2(x^2)} dx$$

$$= \int_0^{\sqrt{\pi}} \sin^2(x^2) \cdot 2x \, dx$$

$u = x^2$

$du = 2x\,dx$

$$= \int_{x=0}^{x=\sqrt{\pi}} \sin^2(u)\, du$$

$$= \int_{u=0}^{u=\pi} \frac{1}{2}(1 - \cos 2u)\, du$$

$$= \left[\frac{1}{2}u - \frac{1}{4}\sin(2u) \right]_{u=0}^{u=\pi}$$

$$= \left(\frac{\pi}{2} - 0\right) - (0 - 0)$$

$$= \frac{\pi}{2}$$

2. Recall that for a positive number a, the volume of the sphere S of radius a, where

$$S = \{(x,y,z) : x^2 + y^2 + z^2 \le a^2\},$$

is given by $\frac{4}{3}\pi a^3$. Verify this in three different ways:

(a) (4 pts) Using a triple integral and spherical coordinates. Set it up and EVALUATE.

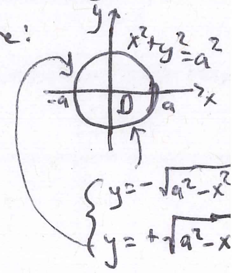
$$\iiint_S dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=a} \sin\phi \, d\phi\, d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} \underbrace{\left[-\cos\phi\right]_{\phi=0}^{\phi=\pi}}_{2} d\theta = \frac{2a^3}{3}\int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}.$$

(b) (4 pts) Use a double integral where you compute the volume under a function of the form $z = f(x,y)$ (You must find the function!). Set it up using *Cartesian Coordinates* BUT DO NOT EVALUATE IT! (Hint: Leveraging symmetry may be useful)

[Many acceptable answers!]

projection onto xy-plane:



$f(x,y) = \sqrt{a^2-x^2-y^2}$

$\left(\begin{array}{l} x^2+y^2+z^2 = a^2 \\ \Leftrightarrow z = \pm\sqrt{a^2-x^2-y^2} \end{array} \right)$

+: top half of sphere.

−: bottom " " "

both halves of the sphere!

$$\iint_D f(x,y)\, dy\, dx = \boxed{2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx}$$

OR

$$= \boxed{4\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx}$$

OR others!

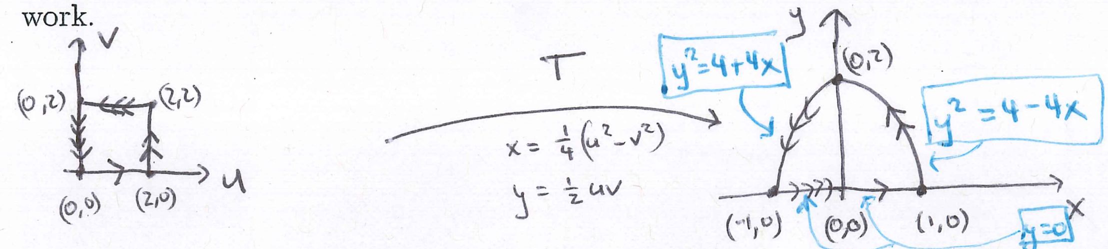
(c) (4 pts) Use a double integral where you compute the volume under a function of the form $z = f(x,y)$. Set it up using *polar coordinates* BUT DO NOT EVALUATE IT!

$r^2 = x^2+y^2$, $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\iint_D f(r\cos\theta, r\sin\theta)\, dA = 2\int_0^{2\pi}\int_0^{a} \sqrt{a^2-r^2}\cdot r\, dr\, d\theta.$$

3. In this problem, we will use the change of variables $x = \frac{1}{4}(u^2 - v^2)$, $y = \frac{1}{2}uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$.

(a) (6 points) What is the image of the square $S = \{(u, v) \mid 0 \le u \le 2,\ 0 \le v \le 2\}$ under the transformation given? To receive full credit, clearly draw and label each edge of the square, draw the image of the square, and clearly indicate which edge in the image corresponds to which edge in the square. Additionally, label each edge in the image with equations. Solve each equation used in your labeling for either y or y^2. Show your work.



$\rightarrow$: $0 \le u \le 2$, $v = 0$

plug in: $x = \frac14 u^2$, $y = 0$ $\Big\}$ at $u=0$, $x=0$; at $u=2$, $x=1$

$\leftleftarrows$: $0 \le u \le 2$, $v = 2$

at $(u,v) = (2,2)$, $(x,y) = (0,2)$; $(u,v) = (0,2)$, $(x,y) = (-1,0)$

plug in: $x = \frac14 u^2 - 1$, $y = u$ $\rightsquigarrow$ $x = \frac14 y^2 - 1$ $\rightsquigarrow$ $\boxed{y^2 = 4+4x}$

$\twoheaduparrow$: $u = 2$, $0 \le v \le 2$

at $(u,v) = (2,0)$, $(x,y) = (1,0)$; at $(u,v) = (2,2)$, $(x,y) = (0,2)$

plug in: $x = 1 - \frac14 y^2$, $y = v$ $\rightsquigarrow$ $x = 1 - \frac14 y^2$ $\rightsquigarrow$ $\boxed{y^2 = 4-4x}$

$\twoheaddownarrow$: $u = 0$, $0 \le v \le 2$

plug in: $x = -\frac14 v^2$, $y = 0$ $\Big\}$ at $v = 2$, $x = -1$; at $v = 0$, $x = 0$

(b) (4 points) Compute the Jacobian of the coordinate transformation. What is the absolute value of the Jacobian?

$\frac{\partial x}{\partial u} = \frac12 u \quad \frac{\partial x}{\partial v} = -\frac12 v$

$\frac{\partial y}{\partial u} = \frac12 v \quad \frac{\partial y}{\partial v} = \frac12 u$

$\rightsquigarrow J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac12 u & -\frac12 v \\ \frac12 v & \frac12 u \end{vmatrix} = \frac14 u^2 + \frac14 v^2 = \underline{\frac14(u^2+v^2) \ge 0}.$

$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac14(u^2+v^2)\right| = \frac14(u^2+v^2)$ since

(c) (6 points) Compute the integral $\iint_R y\,dA$ using the change of coordinates above. (If needed, complete the computation on the back of this page.)

$$\iint_R y\,dA = \iint_{T^{-1}(R)} \frac12 uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^2\int_0^2 \frac12 uv\left(\frac14(u^2+v^2)\right) du\,dv$$

$S!$

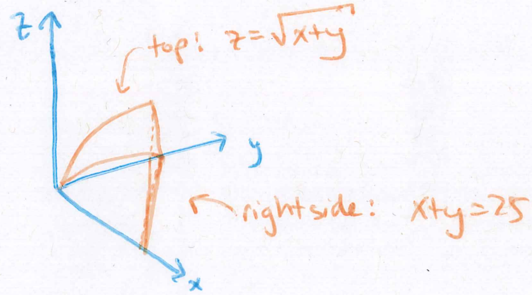
$$= \frac18 \int_0^2\int_0^2 (u^3v + uv^3)\,du\,dv$$

$$= \frac18 \int_0^2 \left[\frac{u^4}{4}v + \frac{u^2}{2}v^3\right]_0^2 dv$$

$$= \frac18 \int_0^2 (4v + 2v^3)\,dv = \frac12\left[2v^2 + \frac{2v^4}{4}\right]_{v=0}^{v=2}$$

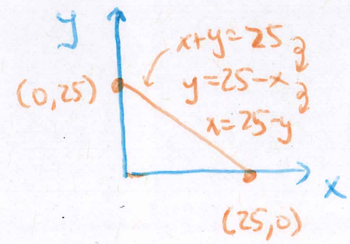
$$= \frac18\big((8+8) - (0+0)\big) = \cancel{2} = \boxed{2}$$

4. (14 points) Let E be the set of points bounded by $x = 0$, $y = 0$, $z = 0$, $x + y = 25$, and $z = \sqrt{x+y}$. Write $\iiint_E f(x, y)\, dV$ as an iterated integral in three different ways. Do not solve.



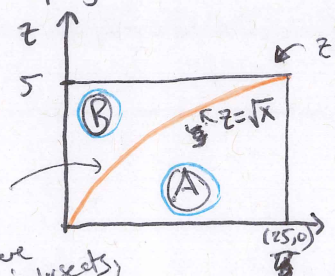
① $$\int_{x=0}^{x=25}\int_{y=0}^{y=25-x}\int_{z=0}^{z=\sqrt{x+y}} f(x,y)\,dz\,dy\,dx$$

projection onto xy-plane!



② $$\int_{y=0}^{y=25}\int_{x=0}^{x=25-y}\int_{z=0}^{z=\sqrt{x+y}} f(x,y)\,dz\,dx\,dy$$

projection onto xz-plane:



$z=5$ is the intersection of $z=\sqrt{x+y}$ and $x+y=25$ projected onto xz-plane.

(*) ③ $$\boxed{\int_{x=0}^{x=25}\int_{z=0}^{z=\sqrt{x}}\int_{y=0}^{y=25-x} f(x,y)\,dy\,dz\,dx \quad \text{(A)} \\ + \int_{x=0}^{x=25}\int_{z=\sqrt{x}}^{z=5}\int_{y=z^2-x}^{y=25-x} f(x,y)\,dy\,dz\,dx \quad \text{(B)}}$$

this line is where $z=\sqrt{x+y}$ intersects xz-plane

If you integrate y in region (A), the y-bounds are from

$y=0$ to $y=25-x$ (plane)

If you integrate y in region (B), the y-bounds are from

$z=\sqrt{x+y} \Rightarrow z^2=x+y \longrightarrow$ $y=z^2-x$ (curved surface) to $y=25-x$ (plane)

BONUS: (4 points) Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $A(D)\sqrt{1 + a^2 + b^2}$, where a, b, and c are constants.

$$SA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \quad , \quad \frac{\partial z}{\partial x} = a \quad \frac{\partial z}{\partial y} = b$$

$$= \iint_D \sqrt{1 + a^2 + b^2}\, dA$$

$$= \left(\sqrt{1 + a^2 + b^2}\right) \cdot \iint_D dA$$

$$= \left(\sqrt{1 + a^2 + b^2}\right) \cdot A(D).$$