Lecture Notes: Week #1
Logical Reasoning

We begin our course by thinking about conditional logic, which is, in some very real sense, how we do math. This kind of logic shows up in every math course. Here are a few examples, with the conditional language boxed in red:

(From linear algebra, e.g. Math 308)

THM Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. $\boxed{\text{Then}}$ this set is linearly dependent $\boxed{\text{if and only if}}$ one of the vectors in the set is in the span of the other vectors.

THM Let T be a linear transformation. $\boxed{\text{Then}}$ T is one-to-one $\boxed{\text{if and only if}}$ the only solution to $T(\vec{x}) = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$.

(From vector calculus, e.g. Math 324)

THM $\int_{\tilde{C}} \vec{F}\cdot d\vec{r}$ is independent of path $\boxed{\text{if and only if}}$ $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C.

(From differential calculus, e.g. Math 124)

<u>THM</u> (L'Hopital's rule)

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). $\boxed{\text{Suppose}}$ that

$$\left[\lim_{x\to a} f(x) = 0 \quad \boxed{\text{and}} \quad \lim_{x\to a} g(x) = 0\right]$$

$\boxed{\text{or}}$ that

$$\left[\lim_{x\to a} f(x) = \pm\infty \quad \boxed{\text{and}} \quad \lim_{x\to a} g(x) = \pm\infty\right]$$

$\boxed{\text{then}}$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$\boxed{\text{if}}$ the limit on the right side exists.

You do not need to be familiar with what these theorems mean (except L'Hopital's rule), but take a moment and think carefully about the role of the boxed red language. How is it "conditional"? You may have also noticed some boxed green language in L'Hopital's rule. Take a moment to think carefully about that language as well - we will be discussing it soon enough.

We won't be proving statements quite as involved as those above, but by the end of this course you should be able to read proofs of these statements and follow the logic.

Our more immediate goal is to understand the red-boxed language, the conditional logic. Conditional logic is built around a conditional statement:

(*) $\boxed{\text{If [something], then [something else].}}$ (*)

Let's look at a concrete example.

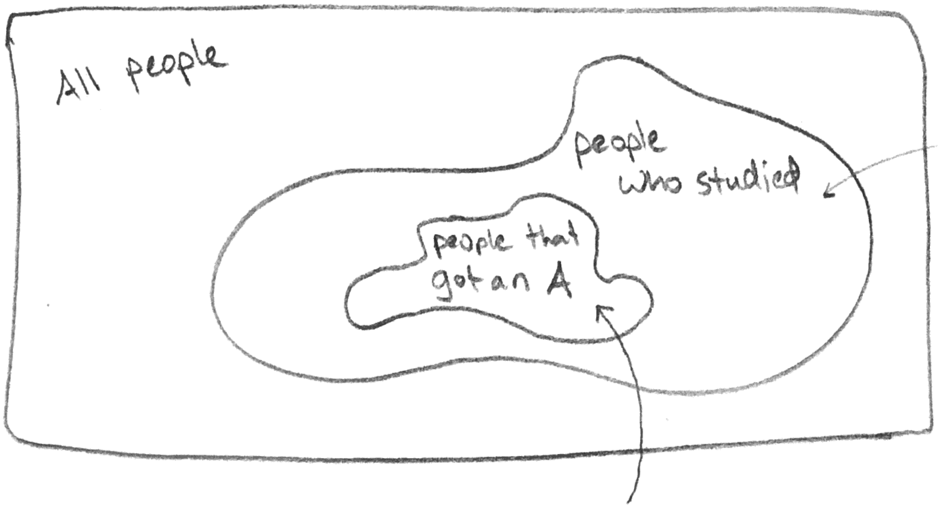
<u>EXAMPLE 1</u> "If you got an A on the exam, then you studied."

Our goal is to understand what this says, and perhaps more importantly, what it does not say.

First, let's try to understand what it *does* say. From the statement, we can say that if we *observe* that someone got an A on the exam, then we *know* that they studied. In other words:

"You got an A on the exam only if you studied."

We could also say it like this: the people who got an A on the exam are a "subset" of the people who studied. (We will come back to this word "subset" in a few weeks.) We can diagram this last interpretation:



maybe all!
some of "all the people" studied

maybe all!
Some of "people who studied" got an A.

Looking at the diagram, we can conclude the following:

"If you did not study, you did not get an A."

This statement is logically equivalent to the first statement! (It carries the same meaning.) However, it does look significantly different. We have a special name for this statement: it is the (*) $\boxed{\text{contrapositive}}$ (*) of our first statement.

It comes from negating each part of the statement, and swapping the implication. To negate a statement, we add "not." We can diagram this out:

means "implies"

$$\text{got an A} \implies \text{studied}$$

Read this as, "If you got an A, then you studied." Or, you can read this as "You got an A implies that you studied."

Now, we can negate each part:

Left side: did not get an A

Right side: did not study

And then we swap the implication:

swapped the direction of implication!

$$\text{did } \underline{\text{not}} \text{ get an A} \Longleftarrow \text{did } \underline{\text{not}} \text{ study}$$

Or, if you would like, we can write it like this:

$$\text{did } \underline{\text{not}} \text{ study} \Longrightarrow \text{did } \underline{\text{not}} \text{ get an A}$$

In either case, we can read this as, "If you did <u>not</u> study, then you did <u>not</u> get an A." Or, "You did not study <u>implies</u> that you did not get an A."

Homework Question 1

Write the contrapositive of each statement in "if-then form."

a) If a is irrational, then a^2 and $2a$ are irrational.

b) r being a rational number implies r^2 is a rational number.

c) The candidate passes the driver's test only if the candidate can parallel park.

Now, let's think about this slightly differently. The statement "If you got an A, then you studied," means that it is $\boxed{\underline{\text{necessary}}}$ to study in order to get an A, but it is not $\boxed{\text{sufficient}}$ (in other words, studying does not guarantee an A). However, if we know someone got an A, that is sufficient to conclude that they studied. We can diagram this as follows:

$$\underbrace{\text{You got an A}}_{\substack{\text{sufficient to}\\ \text{conclude you}\\ \text{studied}}} \implies \underbrace{\text{you studied}}_{\substack{\text{necessary in order}\\ \text{to "get an A"}}}$$

Later in class, you may find it helpful to think about what is necessary and what is sufficient when trying to prove things.

Okay, so that is a lot. Let's do one more interpretation, then we will summarize. For some people, this next interpretation is the most natural, especially if you think about mathematics as a process of discovery. Namely, if we assume something, can we conclude another? Given our statement, we can think of the left side as an hypothesis or assumption, and think of the right side as the conclusion given this hypothesis/assumption:

$$\text{You got an A} \implies \text{you studied}$$

If you assume this, or take this as hypothesis, you can conclude this!

Later, when we begin proving mathematical statements like this, you will want to use the assumption to "exhibit" the conclusion.

Homework Question 2

Identify the hypothesis and conclusion in each statement below.

a) n^2 is odd whenever n is an odd integer.

b) If n is a positive integer, then the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

c) In order to pass the vision test, it is sufficient for the candidate to read the line Q Z S P M W 4.

Now, let's summarize. We'll add a few extra common ways to express a conditional statement. Here, we will do it symbolically, where "A" is a statement like "you got an A on the exam" and "B" is a statement like "you studied."

$$A \Longrightarrow B$$

1. If A, then B.
2. If A is true, then B is also true.
3. A is true only if B is true.
4. A implies B.
5. B is true whenever A is true.
6. For A to be true, it is necessary for B to be true.
7. For B to be true, it is sufficient for A to be true.

<u>Homework Question 3</u>

Write the following statements in the logically equivalent "If - then" form (form 1 above).

a) $(n-1)(n-2)=0$ whenever $n=1$.

b) $n=2$ is sufficient for $n^2-n-2=0$.

c) $n=2$ is necessary for $n^2-4n+4=0$.

Great. So now, let's talk about what our example conditional statement does <u>not</u> say.

Here's the example again:

<u>EXAMPLE 1</u> "If you got an A on the exam, then you studied."

This <u>DOES NOT SAY</u>: "If you studied, then you got an A on the exam."

In fact, this second statement has a special name. We call it the converse of the first statement.

<u>DEF</u> The converse of the conditional statement $A \Rightarrow B$ is $B \Rightarrow A$.

WARNING! A conditional statement and its converse are not logically equivalent. In other words, just because one of them is true DOES NOT mean the other is true.

It could be that both $A \Rightarrow B$ and $B \Rightarrow A$ are true, but knowing only $A \Rightarrow B$ DOES NOT mean we can conclude $B \Rightarrow A$.

Whenever you have a conditional statement whose implication works in both direction, i.e.

$$A \Rightarrow B$$

and

$$B \Rightarrow A$$

then we often say

A is true (*) $\boxed{\text{if and only if}}$ (*) B is true

or

A if and only if B.

Here "A only if B" is the same as "If A, then B."
You can think of "A if B" as "If B, then A."
In other words, we really are saying that A implies B and B implies A when we say A if and only if B.

NOTATION Occasionally you may see "iff." This is shorthand notation for "if and only if."

To see why it is important that we recognize that a conditional statement and its converse are not logically equivalent, consider the following.

DEF We say a finite collection of integers $n_1, n_2, \dots, n_m$ is mutually relatively prime if their greatest common denominator is 1.

THM The integers $n_1, n_2, \dots, n_m$ are mutually relatively prime if and only if the equation

$$n_1 x_1 + n_2 x_2 + \cdots + n_m x_m = 1$$

has a solution in integers $x_1, x_2, \dots, x_m$.

If we were told to prove the theorem, we would need to first show

1. integers $n_1, \dots, n_m$ mutually relatively prime implies the equation has a solution in integers.

Once we have shown this, we have proven one implication.

Since the converse of ①:

> ② The equation has a solution in integers implies the integers $n_1, \dots, n_m$ are mutually relatively prime.

is <u>not</u> logically equivalent to ①, we must show this also. Once we have shown it, then we have proven the implication in both directions, hence, we have proven an if and only if statement.

<u>REMARK</u> You do not need to memorize the definition or theorem on the previous page! It is only there so that we can think about the conditional logic embedded in a mathematical statement.

<u>Homework Question 4</u>

Write the converse and contrapositive of each of the following:

a) If $\sqrt{2} < \sqrt{5}$, then $2 < 5$.

b) If $2 \geq 5$, then $\sqrt{2} \geq \sqrt{5}$

We now need to take a moment and review the words "and" and "or", from a purely logical perspective, then we will start using these terms with our conditional logic. (Actually, we have already used "and" in a crucial way, did you catch it?)

DEF

We say that a proposition (or statement) is a sentence that is either true or false.

Notice, we have been thinking about conditional statements so far — these a fairly complex statements. Right now, we want to consider simpler statements, such as

$$2+3=6 \quad \leftarrow \text{False!}$$

or

$$5-1=4 \quad \leftarrow \text{True!}$$

(*) $\boxed{\text{AND}}$ (*)

Let A and B be two statements. Then,

if

(1) A is true, B is true, then A and B is a true statement.

EXAMPLE $2+3=5$ ← true

$4-1=3$ ← true

then ($2+3=5$ and $4-1=3$) ← true!

(2) A is true, B is false, then A and B is a false statement

EXAMPLE Some dogs are brown ← true

Josh doesn't want dinner. ← false

then (some dogs are brown and Josh doesn't want dinner) is false!

(3) A is false, B is true, then A and B is a false statement.

EXAMPLE $2 \cdot 3 = 7$ ← False

4 divides 96. ← true

then ($2 \cdot 3 = 7$ and 4 divides 96) is false.

(4) A is false, B is false, then A and B is a false statement!

EXAMPLE $2 \cdot 3 = 5$ ← false

4 divides 101 ← false

then ($2 \cdot 3 = 5$ and 4 divides 101) is false.

We can summarize this in a "truth table"

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

Similarly,

(*) $\boxed{\text{OR}}$ (*)

Let A and B be two statements. Then, we summarize the "truth" of A or B with the following "truth table".

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

Homework Question 5

Provide examples of A and B for each line of the truth table for "or" (similar to what we did for "and" before we gave a truth table).

Homework Question 6

Explain how the following indeterminant forms can be deduced from the hypothesis of L'Hopital's rule on p. 2 of these notes.

Indeterminant forms : $\frac{0}{0}$, $\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{-\infty}{-\infty}$.

<u>Note</u> The expression:

" $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$ "

is unclear. You may take it to mean:

$$\left[\lim_{x\to a} f(x) = +\infty \text{ or } \lim_{x\to a} f(x) = -\infty\right] \text{ and } \left[\lim_{x\to a} g(x) = +\infty \text{ or } \lim_{x\to a} g(x) = -\infty\right]$$

Homework Question 7 (thank you LSAC!)

There are exactly three recycling centers in Rivertown: Center 1, Center 2, and Center 3. Exactly five kinds of material are recycled at these recycling centers: glass, newsprint, plastic, tin, and wood. Each recycling center recycles at least two, but no more than three of these kinds of material.

The following conditions must hold:

1. Any recycling center that recycles wood also recycles newsprint.
2. Every kind of material that Center 2 recycles is also recycled at Center 1.
3. Only one of the recycling centers recycles plastic, and that recycling center does not recycle glass.

(a) Which one of the following could be an accurate account of all the kinds of material recycled at each recycling center?

1. Center 1: newsprint, plastic, wood
   Center 2: newsprint, wood
   Center 3: glass, tin, wood

② Center 1: glass, newsprint, tin
Center 2: glass, newsprint, tin
Center 3: newsprint, plastic, wood

③ Center 1: glass, newsprint, wood
Center 2: glass, newsprint, tin
Center 3: plastic, tin

④ Center 1: glass, plastic, tin
Center 2: glass, tin
Center 3: newsprint, wood

⑤ Center 1: newsprint, plastic, wood
Center 2: newsprint, plastic, wood
Center 3: glass, newsprint, tin

(b) Which one of the following is a complete and accurate list of recycling centers in Rivertown any one of which could recycle plastic?

① Center 1 only

② Center 3 only

③ Center 1, Center 2

④ Center 1, Center 3

⑤ Center 1, Center 2, Center 3

(c) If center 2 recycles three kinds of material, then which one of the following kinds of material must center 3 recycle?

1. Glass
2. newsprint
3. plastic
4. tin
5. wood

(d) If each recycling center in Rivertown recycles exactly three kinds of materials, then which one of the following could be true?

1. Only center 2 recycles glass.
2. Only center 3 recycles newsprint
3. Only Center 1 recycles plastic
4. Only Center 3 recycles tin.
5. Only Center 1 recycles wood.

(e) If Center 3 recycles glass, then which one of the following kinds of material must Center 2 recycle?

1. glass
2. newsprint
3. plastic
4. tin
5. wood

(f) If Center 1 is the only recycling center that recycles wood, then which of the following could be a complete and accurate list of the kinds of material that one of the recycling centers recycles?

1. plastic, tin
2. newsprint, wood
3. newsprint, tin
4. glass, wood
5. glass, tin

One last remark, related to whether or not a statement is True or False... In the same way we made a "truth table" for "and" and "or," we can make one for an "implication":

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

? these are weird!

It is the top two lines of this truth table that we should be concerned about. The bottom two lines are a bit odd. The basically tell us:

If A, then B

is a true statement when A is false and B is true (or false).

For example:

A = cats can fly ← False

B = 2+2 = 4 ← true

then

"If cats can fly, then 2+2=4."

is a true statement. This is weird, but we can explain it like this: the statement is $\boxed{\text{vacuous}}$, i.e. the world in which cats can fly is empty, doesn't exist. Logically, then, the condition, "If cats can fly," applies to an "empty set," and for a set with nothing in it, everything is true! (You can't find a counterexample!)

You do not need to worry too much about this now, but you should be aware of this idea of a "vacuous truth." I will try to point it out when it becomes an issue for us.

Supplement: Negating statements with "and" and "or"

Occasionally, we may need to negate a statement of this form:

$$A \text{ and } B$$

or

$$A \text{ or } B.$$

For example, one way to prove an "If A, then B" type statement is to prove it's contrapositive (we will see examples of this next week). So, if the statement is of the form

"If (A and B) or (C and D), then E."

The contrapositive would be:

If not E, the not [(A and B) or (C and D)].

(Look back at L'Hopital's rule on p.2 for a concrete example!) It would be nice if we had a quick way to evaluate such a statement.

Let's look at a truth table to try and get a handle on what we should expect.

A	B	A and B	not (A and B)
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

here we just negate!

(rem

A	not A
T	F
F	T

!)

Notice, the negated values *almost* look like A or B (see the truth table for "or" on page 16). In fact, if we negated every value (T/F) for A and B on the left hand side, it would match the truth table for "or."

not A	not B	not A or not B	not (A and B)
F	F	F	F
F	T	T	T
T	F	T	T
T	T	T	T

Take a moment to think about this a bit more. For example, how might you deduce the need for an "or" by looking at lines 2 and 3:

A	B	A and B	not(A and B)
T	F	F	T
F	T	F	T

Similarly, lines 1 and 4 tell you about the need to negate ...

A	B	A and B	not(A and B)
T	T	T	F
F	F	F	T

Okay, that's all nice ... but it is also <u>all</u> abstract. This is what we can conclude:

$$\text{not}(A \text{ and } B) = (\text{not } A) \text{ or } (\text{not } B)$$

but, we should wonder if this really works.

Let's try a few examples.

EXAMPLE 2

Alexandra is 27 years old and Steffen is eating cake.

Negation: Alexandra is <u>not</u> 27 years old or Steffen is <u>not</u> eating cake.

Okay. But we should check a bit more.

(1) Assume Alexandra is 27 years old.
Assume Steffen is eating cake.

Then the example statement is <u>true</u>, so we would want the negation to be false:

- Alexandra is <u>not</u> 27 years old is <u>false</u>
- Steffen is <u>not</u> eating cake is <u>false</u>,

So, the negated statement

"Alexandra is <u>not</u> 27 years old or Steffen is <u>not</u> eating cake."

is also false!

Let's check another scenario.

② Assume Alexandria is <u>not</u> 27 years old.
Assume Steffen is eating cake.

Then the example statement (with "and") is false, so we would want the negation to be true.

- Alexandria is <u>not</u> 27 years old is <u>true</u>.
- Steffen is <u>not</u> eating cake is <u>false</u>.

So, the negated statement

"Alexandria is <u>not</u> 27 years old or
Steffen is <u>not</u> eating cake."

is true!

We have two other scenarios to check – you should try to check them!

<u>TRY</u>: ③ Assume Alexandria is 27 years old.
Assume Steffen is <u>not</u> eating cake.

④ Assume Alexandra is <u>not</u> 27 years old.
Assume Steffen is <u>not</u> eating cake.

Hopefully you are more convinced that our formula for negating an "and" statement works:

<u>Formula</u>: not (A and B) = (not A) or (not B)

<u>EXAMPLE 3</u> Getika likes basketball and Ginny does not play tennis.

<u>Negation</u>: Getika does <u>not</u> like basketball or Ginny <u>does</u> play tennis.

<u>EXAMPLE 4</u> The sky is blue and the eggplant is not purple.

<u>Negation</u>: The sky is not blue or the eggplant is purple.

Great. So now, let's turn our attention to "or."

Let's start again by looking at a truth table:

A	B	A or B	not(A or B)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

here we just negate!

Similar to before, how might lines 2 and 3 indicate we should consider an "and" statement? And how might lines 1 and 4 suggest negating? Thinking about this along the same lines as before (write it out again if you need!) we can abstractly see the following:

$$\boxed{\underline{\text{Formula}}: \ \text{not}(A \text{ or } B) = \text{not } A \text{ and not } B}$$

Let's try a few concrete examples.

EXAMPLE 5

A lexandra is 27 years old or Steffen is eating cake.

Negation: Alexandra is not 27 years old and Steffen is not eating cake.

As before, we can check 4 scenarios corresponding to the 4 scenarios on the truth table:

1. Assume Alexandra is 27 years old.
   Assume Steffen is eating cake.

2. Assume Alexandra is not 27 years old.
   Assume Steffen is eating cake.

3. Assume Alexandra is 27 years old.
   Assume Steffen is not eating cake.

4. Assume Alexandra is not 27 years old.
   Assume Steffen is not eating cake.

For each scenario, does the negation have the opposite result as the original statement (as in, when the original statement is true, is the negation false, and vice-versa)? Check this!

Couple more examples:

EXAMPLE 6

Getika likes basketball or Ginny does not play tennis.

Negation: Getika does not like basketball and Ginny does play tennis.

EXAMPLE 7

The sky is blue or the eggplant is not purple.

Negation: The sky is not blue and the eggplant is purple.

And ... one more, for fun:

EXAMPLE 8

Evaluate $\text{not}\left[(A \text{ and } B) \text{ or } (C \text{ and } D)\right]$

To do this, start on the outside and work your way in:

$$\text{not}\left[(A \text{ and } B) \text{ or } (C \text{ and } D)\right] = \left[\text{not } (A \text{ and } B)\right] \text{and} \left[\text{not } (C \text{ and } D)\right]$$

$$= \left[(\text{not } A) \text{ or } (\text{not } B)\right] \text{ and } \left[(\text{not } C) \text{ or } (\text{not } D)\right].$$