MATH 324 A
Exam I
July 9, 2018

Name ____________________________________

Student ID #_________________ Section _________________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE:______________________________

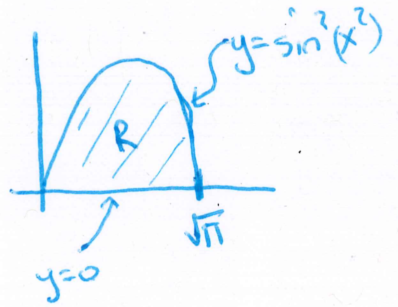
1	8	
2	12	
3	14	
4	16	
5	10	
Total	60	

- Your exam should consist of this cover sheet, followed by 5 problems. Check that you have a complete exam.
- Pace yourself. You have 60 minutes to complete the exam and there are 5 problems. Try not to spend more than 10 minutes on each page.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (8 points) Compute the mass of the lamina that is the shape of the region in the plane bounded by the curves $y = 0$ and $y = \sin^2(x^2)$ in the first quadrant where $x \leq \sqrt{\pi}$ if the density at a point (x, y) is 9 times the distance from the y-axis.

Region



$$\rho(x,y) = 9 \cdot \underbrace{x}_{}$$

x-coordinate is the distance

$$m = \iint_R \rho(x,y)\,dA = \int_{x=0}^{x=\sqrt{\pi}} \int_{y=0}^{y=\sin^2(x^2)} (9x)\,dA = \int_0^{\sqrt{\pi}} 9x \cdot \sin^2(x^2)\,dx$$

$u = x^2$
$du = 2x\,dx$

$$= \int_{x=0}^{x=\sqrt{\pi}} \frac{9}{2} \sin^2(u)\,du$$

$$= \frac{9}{2} \int_{x=0}^{x=\sqrt{\pi}} \sin^2(u)\,du$$

$$= \frac{9}{2} \int_{x=0}^{x=\sqrt{\pi}} \frac{1}{2}(1 - \cos(2u))\,du$$

$$= \frac{9}{4} \left[u - \frac{1}{2}\sin(2u) \right]_{x=0}^{x=\sqrt{\pi}}$$

$$= \frac{9}{4} \left[x^2 - \frac{1}{2}\sin(2x^2) \right]_{x=0}^{x=\sqrt{\pi}}$$

$$= \frac{9}{4} \left[(\pi - 0) - (0 - 0) \right]$$

$$= \frac{9\pi}{4}$$

2. (a) (2 points) Let $I = \int_{-\infty}^{\infty} e^{-(x^2)}dx$. Use a Theorem from class to write

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-(x^2)}dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-(y^2)}dy\right)$$

as a double integral.

$e^{-(x^2)} e^{-(y^2)} = e^{-(x^2)+-(y^2)} = e^{-(x^2+y^2)}$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy$$

(b) (8 points) Compute I^2.

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-(r^2)} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \lim_{b\to\infty} \int_{r=0}^{r=b} e^{-(r^2)} r\,dr\,d\theta$$

$u = r^2$, $\frac{1}{2}du = r\,dr$

$$= \int_0^{2\pi} \left(\lim_{b\to\infty} \int_{r=0}^{r=b} \frac{1}{2} e^{-u} du\right) d\theta$$

$$= \int_0^{2\pi} \left(\lim_{b\to\infty} \frac{1}{2}\left[-e^{-u}\right]_{r=0}^{r=b}\right) d\theta = \int_0^{2\pi}\left(\lim_{b\to\infty} \frac{1}{2}\left[-e^{-(r^2)}\right]_0^b\right) d\theta$$

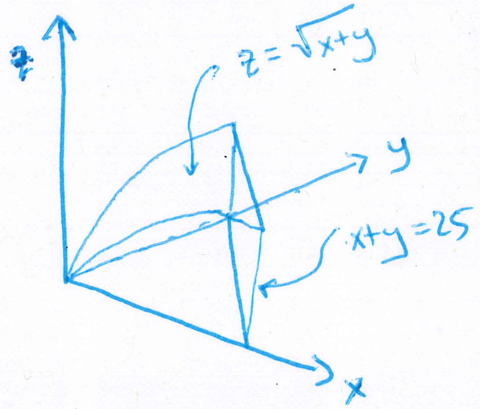
$$= \int_0^{2\pi}\left(\lim_{b\to\infty} \frac{1}{2}\left[\cancel{-e^{-(b^2)}} + \underbrace{e^0}_{1}\right]\right) d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{1}{2}[\theta]_0^{2\pi} = \pi$$

$-e^{-(b^2)} \to 0$ as $b \to \infty$

(c) (2 points) What is I? Justify your answer.

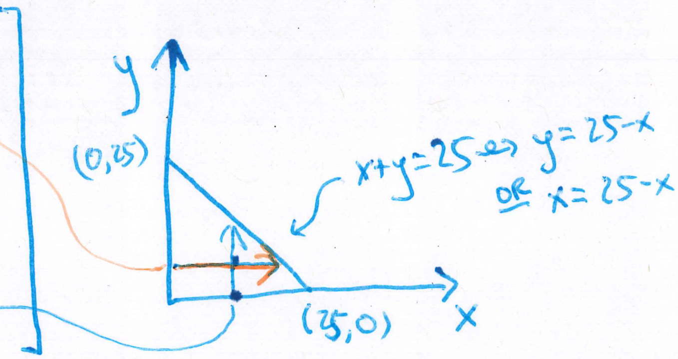
$I = \sqrt{I^2} = \sqrt{\pi}$. We choose the positive root since $e^{-(x^2)} > 0$ for all x, so $\int_{-\infty}^{\infty} e^{-(x^2)} dx > 0$.

3. (14 points) Let E be the set of points bounded by $x = 0$, $y = 0$, $z = 0$, $x + y = 25$, and $z = \sqrt{x+y}$. Write $\iiint_E f(x, y)\, dV$ as an iterated integral in three different ways. Do not solve.



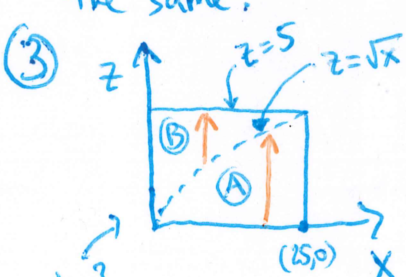
① $$\int_{y=0}^{y=25}\int_{x=0}^{x=25-y}\int_{z=0}^{z=\sqrt{x+y}} f(x,y)\,dz\,dx\,dy$$

② $$\int_{x=0}^{x=25}\int_{y=0}^{y=25-x}\int_{z=0}^{z=\sqrt{x+y}} f(x,y)\,dz\,dy\,dx$$



Whichever way you choose to set-up the integral the third way, it will become 2 integrals. Whether you do x or y first, it looks (basically) the same.

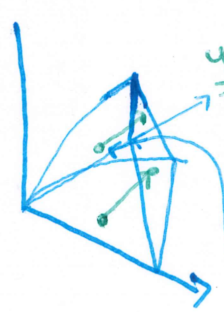
③



$$\overbrace{\int_{x=0}^{x=25}\int_{z=0}^{z=\sqrt{x}}\int_{y=0}^{y=25-x} f(x,y)\,dy\,dz\,dx}^{A} + \overbrace{\int_{x=0}^{x=25}\int_{z=\sqrt{x}}^{z=5}\int_{y=z^2-x}^{y=25-x} f(x,y)\,dy\,dz\,dx}^{B}$$

why?

because of this:



makes things y splits into two regions, A and B.

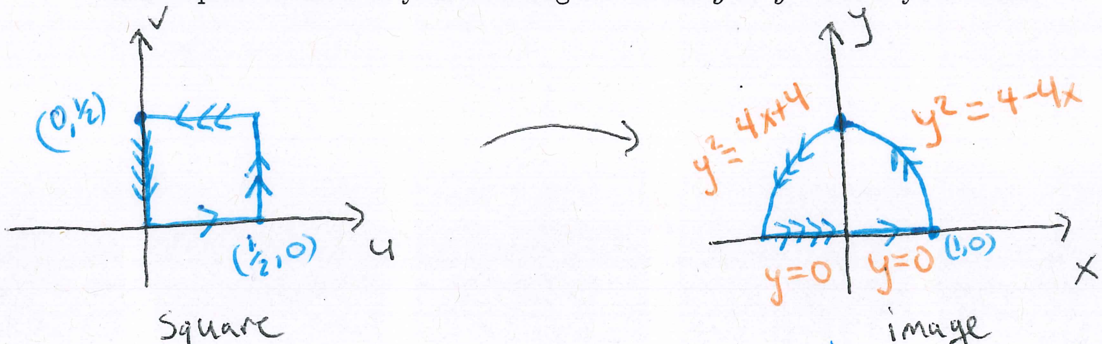
For B, $z=\sqrt{x+y}$

$z^2 = x+y$

$y = z^2 - x$

4. In this problem, we will use the change of variables $x = 4u^2 - 4v^2$, $y = 8uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4-4x$ and $y^2 = 4+4x$.

(a) (6 points) What is the image of the square $S = \{(u,v)\,|\,0 \le u \le \frac{1}{2},\ 0 \le v \le \frac{1}{2}\}$ under the transformation given? To receive full credit, clearly draw and label each edge of the square, draw the image, and indicate which edge in the image corresponds to which edge in the square. Additionally, label each edge in the image with equations. Solve each equation used in your labeling for either y or y^2. Show your work.



$u=0$, $0 \le v \le \frac{1}{2}$
$\Rightarrow x = -4v^2$, $y = 0$
$\Rightarrow y=0$ between $x=-1$ and $x=0$.

$0 \le u \le \frac{1}{2}$, $v = 0$
$\Rightarrow x = 4u^2$, $y = 0$ $\Rightarrow$ $y=0$ between $x=0$, $x=1$

$u = 1/2$, $0 \le v \le 1/2$
$\Rightarrow x = 1-4v^2 \Rightarrow x = 1 - \frac{1}{4}y^2$, $y^2 = 4-4x$
& $y = 4v$, $v = \frac{1}{4}y$ plug in $(0 \le y \le 2)$

$0 \le u \le \frac{1}{2}$, $v = \frac{1}{2}$
$\Rightarrow x = 4u^2 - 1 \Rightarrow x = \frac{1}{4}y^2 - 1$, $y^2 = 4x+4$
& $y = 4u$, $u = \frac{1}{4}y$ plug in $(0 \le y \le 2)$

(b) (4 points) Compute the Jacobian of the coordinate transformation. What is the absolute value of the Jacobian?

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 8u & -8v \\ 8v & 8u \end{vmatrix} = 64u^2 + 64v^2 > 0 \Rightarrow |J| = |64u^2+64v^2| = 64u^2+64v^2.$$

(c) (6 points) Compute the integral $\iint_R y\,dA$. (If needed, complete the computation on the back of this page.)

$$\iint_R y\,dA = \underbrace{\int_0^{1/2}\int_0^{1/2}}_{T^{-1}(R) = \text{Square}} \overbrace{8uv}^{y}\,\overbrace{(64(u^2+v^2))}^{|J|}\,du\,dv = 512\int_0^{1/2}\int_0^{1/2}(u^3v + uv^3)\,du\,dv$$

$$= 512\int_0^{1/2}\left[\frac{1}{4}u^4v + \frac{1}{2}u^2v^3\right]_0^{1/2} dv = 512\int_0^{1/2}\left(\frac{1}{64}v + \frac{1}{8}v^3\right)dv$$

$$= 512\left[\frac{1}{64}\cdot\frac{1}{2}v^2 + \frac{1}{8}\cdot\frac{1}{4}v^4\right]_0^{1/2} = 512\left[\frac{1}{512} + \frac{1}{512}\right] = \boxed{2}.$$

5. (a) (2 points) Convert the following equation in rectangular coordinates to an equation in spherical coordinates and reduce the answer to the most simplified form.

$$x^2 + y^2 + z^2 = \sqrt{x^2 + y^2}$$

$$\rho^2 = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$$

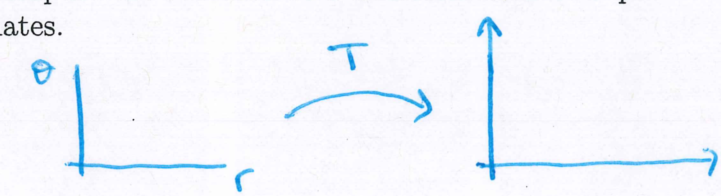
$$\rho^2 = \sqrt{\rho^2 \sin^2\phi \,(\cos^2\theta + \sin^2\theta)}$$

$$\rho^2 = \rho \sin\phi$$

$$\boxed{\rho = \sin\phi}$$

(b) (4 points) Compute the Jacobian of a transformation from polar coordinates to Cartesian coordinates.

REM :



from polar ———→ to cartesian

$$T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$* \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

(c) (4 points) Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $A(D)\sqrt{1 + a^2 + b^2}$, where a, b, and c are constants.

$$SA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \qquad \frac{\partial z}{\partial x} = a \qquad \frac{\partial z}{\partial y} = b$$

$$= \iint_D \sqrt{1 + a^2 + b^2}\, dA$$

$$= \sqrt{1 + a^2 + b^2} \cdot \iint_D dA = \left(\sqrt{1 + a^2 + b^2}\right) \cdot A(D).$$