Homework #4 Answer Key

1. To show that $\emptyset$ is a subset of any set X, we need to show that if $x \in \emptyset$, then $x \in X$ (by definition of a subset). Since the statement $x \in \emptyset$ is always false, the conditional statement "if $x \in \emptyset$, then $x \in X$" is always true. Hence, $\emptyset$ is a subset of any set X, as desired. $\boxtimes$

2. proof: Assume $A \subset B \subset C$ and $A = C$. To show $A = B = C$, we will show $A = B$ and $B = C$. By THM 1, we know $A = B$ if and only if $A \subset B$ and $B \supset A$. By assumption, we have that $A \subset B$, so we need only prove $B \subset A$. Let $x \in B$. Since $B \subset C$, we see that $x \in C$, by definition. Similarly, since $A = C$, we have $C \subset A$ by THM 1, so we can conclude $x \in A$ by definition of subset. Thus, if $x \in B$, $x \in A$, which by definition means $B \subset A$. Hence, $A = B$.

Similarly, by THM 1, $B = C$ if and only if $B \subset C$ and $C \subset B$. By assumption, $B \subset C$, so we need only show $C \subset B$. Assume $x \in C$. Since $C = A$, we know by THM 1 $C \subset A$, thus $x \in A$ by definition.

By assumption, we have that $A \subset B$, so by definition of subset, we know $x \in B$. Hence, if $x \in C$, then $x \in B$, which means $C \subset B$ by defintion. By <u>THM 1</u>, since $B \subset C$ and $C \subset B$, we can conclude $B = C$ as desired. $\boxtimes$

3. <u>proof</u>: (2) Assume $A, B \subset X$ and let A^c denote the complement of A relative to X.
To show set equality, $(A \cap B)^c = A^c \cup B^c$, we need to show $x \in (A \cap B)^c$ if and only if $x \in A^c \cup B^c$. Let $x \in X$.

$x \in (A \cap B)^c \iff x \notin A \cap B$ by definition of complement,

$\iff x \notin A$ or $x \notin B$, by definition of intersection, and negation of "and",

$\iff x \in A^c$ or $x \in B^c$, by definition of complement,

$\iff x \in A^c \cup B^c$, by definition of union.

Thus, $(A \cap B)^c = A^c \cup B^c$, as desired.

$\boxtimes$

4. <u>proof:</u> Assume $A, B, C \subset X$ and let A^c denote the complement of A relative to X. We need to show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Observe the following:

$$A^c \cap (B^c \cup C^c) = (A^c \cap B^c) \cup (A^c \cap C^c)$$

by the left-most equation in THM 3 (3), i.e. the top equation.

Take $A^c \cap (B^c \cup C^c)$ and apply De Morgan's rules twice:

$$\begin{aligned} A^c \cap (B^c \cup C^c) &= A^c \cap (B \cap C)^c \\ &= (A \cup (B \cap C))^c. \end{aligned}$$

Now, if we complement both sides

$$(*) \quad \begin{aligned} (A^c \cap (B^c \cup C^c))^c &= ((A \cup (B \cap C))^c)^c \\ &= A \cup (B \cap C). \end{aligned}$$

by De Morgan's Rule (3). Notice how this set is the set in the equality we are trying to prove.

Now take $(A^c \cap B^c) \cup (A^c \cap C^c)$ and apply De Morgan's Rules twice:

$$(A^c \cap B^c) \cup (A^c \cap C^c) = (A \cup B)^c \cup (A \cup C)^c$$
$$= ((A \cup B) \cap (A \cup C))^c.$$

As before, if we complement both sides,

$$(**) \qquad ((A^c \cap B^c) \cup (A^c \cap C^c))^c = (((A \cup B) \cap (A \cup C))^c)^c$$
$$= (A \cup B) \cap (A \cup C)$$

by De Morgan's rule (3).

Combining equations (*) and (**) with the left most equation in THM3 (3), we see

$$A \cup (B \cap C) = (\underbrace{A^c \cap (B^c \cup C^c)}))^c \text{, by } (*)$$
$$= (\overbrace{(A^c \cap B^c) \cup (A^c \cap C^c)})^c \text{, by left-most equation in THM3 (3)}$$
$$= (A \cup B) \cap (A \cup C) \text{, by } (**).$$

Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, as desired. ☒

NOTE We had to assume $A, B, C \subset X$ ... but it's not really a problem! (Notice THM3 does not make this assumption.) It's not a problem because we can always take $X = A \cup B \cup C$.

5. <u>proof:</u> ① Let $\mathcal{A}$ be a collection of subsets of a set X.

We will show that $x \in \left(\bigcup_{A \in \mathcal{A}} A\right)^c$ if and only if $x \in \bigcap_{A \in \mathcal{A}} (A^c)$, which by definition proves set equality. Here, ^{c} denotes the complement relative to X.

$$x \in \left(\bigcup_{A \in \mathcal{A}} A\right)^c \iff x \notin \bigcup_{A \in \mathcal{A}} A$$, by definition of the complement,

$$\iff x \notin \{y : y \in A \text{ for some } A \in \mathcal{A}\},$$

by definition of a union of a collection,

negating the condition on the set! "There exists some $A \in \mathcal{A}$ such that $y \in A$," becomes, "For all $A \in \mathcal{A}$, $y \notin A$."

$$\iff x \in \{y : y \notin A \text{ for all } A \in \mathcal{A}\},$$

by negation,

$$\iff x \in \{y : y \in A^c \text{ for all } A \in \mathcal{A}\},$$

by definition of the complement,

$$\iff x \in \bigcap_{A \in \mathcal{A}} (A^c)$$ by definition of an arbitrary intersection.

Hence, $\left(\bigcup_{A \in \mathcal{A}} A\right)^c = \bigcap_{A \in \mathcal{A}} A^c$, as desired.

(2) We prove (2) similarly to (1). Let $\mathcal{a}$ be a collection of subsets of a set X. We show $x \in \left(\bigcap_{A \in \mathcal{a}} A\right)^c$ if and only if $x \in \bigcup_{A \in \mathcal{a}} (A^c)$, which by definition proves set equality.

$$x \in \left(\bigcap_{A \in \mathcal{a}} A\right)^c \iff x \notin \bigcap_{A \in \mathcal{a}} A$$ by definition of the complement,

$$\iff x \notin \{y : y \in A \text{ for all } A \in \mathcal{a}\},$$ by definition of an arbitrary intersection,

$$\iff x \in \{y : y \notin A \text{ for some } A \in \mathcal{a}\},$$ by negation (universal to existential),

$$\iff x \in \{y : y \in A^c \text{ for some } A \in \mathcal{a}\}$$ by definition of the complement,

$$\iff x \in \bigcup_{A \in \mathcal{a}} (A^c),$$ by definition of an arbitrary union.

Hence, $\left(\bigcap_{A \in \mathcal{a}} A\right)^c = \bigcup_{A \in \mathcal{a}} (A^c)$, as desired. ∎

6. proof: We prove THM 7 by induction.

Base Case: Let $n=0$. Then $X=\emptyset$ (and we see that $|X|=|\emptyset|=0$).

$\mathcal{P}(\emptyset)=\{\emptyset\}$, so $|\mathcal{P}(\emptyset)|=1$.

Now, observe

$$|\emptyset|=0,$$

$$|\mathcal{P}(\emptyset)|=2^0=1,$$

which completes the base case.

(Starting at a one-element set is also okay.)

Inductive Step: Assume any n-element set has 2^n elements in its power set. Following the hint, let A be an $n+1$-element set. We want to show that $|\mathcal{P}(A)|=2^{n+1}$. Pick an element $a\in A$. Then $\{a\}$ is a one-element set, and $A\setminus\{a\}$ is an n-element set. Define $B=A\setminus\{a\}$, the n-element set. Then since $A\setminus\{a\}\cap\{a\}=\emptyset$, we see that

$$A=B\sqcup\{a\}.$$

Now, let $X \subset A$ be an arbitrary subset of A.

We have two cases:

① $a \notin X$, in which case $X \subset B$.

or

② $a \in X$, in which case $X = C \sqcup \{a\}$, where $C \subset B$.

In case ①, the number of subsets X of A with this condition is number of subsets of the n-element set B. By the <u>inductive hypothesis</u>, we know this is 2^n.

In case ②, the number of subsets X of the form $C \sqcup \{a\}$ where $C \subset B$ is the number of subsets of B. By the inductive hypothesis, we know this is 2^n.

In total, $|\mathcal{P}(A)| = 2^n + 2^n = 2(2^n) = 2^{n+1}$, as desired.

$\blacksquare$