Homework 3: Answer Key

1. (a) Let $\mathbb{Z}$ be the set of integers. Then the statement in set-theoretic form is

$$\{n \in \mathbb{Z} \mid |n| < n^2\} = \mathbb{Z}. \quad \text{(false!)}$$

The conditional form:

"If n is an integer, then $|n| < n^2$."

(b) Let $\mathcal{O}$ be the set of odd integers. The statement in set-theoretic form is

$$\{n \in \mathcal{O} \mid n^{15} \in \mathcal{O}\} = \mathcal{O}.$$

The conditional form:

"If n is an odd integer, then n^{15} is odd."

(c) Let E be the set of even integers. The statement in set-theoretic form is

$$\{n \in E \mid n-1 \text{ odd}\} = E.$$

The conditional form:

"If n is an even integer, $n-1$ is odd."

2. (a) Counterexample: Let $n = -1$. Then $-1 < |-1| = 1$,

so $-1 \not> 1$ (trichotomy axiom).

(b) Counter example: Let $n = 1$. Then $2 \cdot 1$ is even

since 2 is a divisor. Thus,

$2 \cdot 1$ is not odd.

(c) Counter example: Let $n = 0$. Since $2 \cdot 0 = 0$, we

see that 2 is a divisor of zero,

thus 0 is even. Notice

$$0^2 = 0.$$

so

$$0^2 \not> 0. \text{ (trichotomy axiom)}$$

3. (a) "For all" form:

For all odd integers a, b, and c, $ax^2 + bx + c = 0$

does not have a rational solution.

"If, then" form:

If a, b, and c are odd integers, $ax^2 + bx + c = 0$

does not have a rational solution.

(b). • For all integers p, q, and r, $p+q+r$ is not odd <u>or</u> the number of odd elements in $\{p, q, r\}$ is not even.

• If p, q, and r are integers, then $p+q+r$ is not odd <u>or</u> the number of odd elements in $\{p, q, r\}$ is not even.

(*)

(c). • For all prime numbers p, $p \le q$ for some prime number q.

4. Negation:

"There exists an $x \in \mathbb{R}$ and an $\varepsilon > 0$ such that for all $\delta > 0$, there exists a y such that $|x-y| < \delta$, but $|f(x) - f(y)| \ge \varepsilon$."

5. <u>proof</u>: Assume n is a negative, odd integer. Then $-n > 0$ by Elementary Proposition 11, thus $-n$ is a positive, odd integer. Since we have already proven that positive,

odd integers are of the form $2k+1$ for some integer k, we know $-n = 2k+1$ for some integer k. Then,

$$-n = 2k+1$$
$$(-1)(-n) = (-1)(2k+1)$$
$$n = -2k-1$$
$$n = -2k-2+1$$
$$n = 2(-k-1)+1.$$

Since $-k-1$ is an integer, we see that n is of the desired form.

$\blacksquare$

6. (a) proof: (We still need to prove, "If n is an integer of the form $2k+1$ for some integer k, then n is odd.")

Assume $n = 2k+1$ for some k and n is even for the sake of contradiction.

Then $n = 2m$ for some integer m, by definition of even. Since $n = 2k+1$ and $n = 2m$, we can write

$$2m = 2k+1$$

$$2m-2k = 1$$

$$2(m-k) = 1$$

Since $m-k$ is an integer (integers are closed under addition), we see that this means 2 is a divisor of 1. In other words, 1 is even.

Notice, however, that $2\cdot 0 = 0$ and $2\cdot 1 = 2$:

$$2\cdot 0 = 0 < 1 < 2 = 2\cdot 1$$

Now, for integers $k<0$, $2k<0$ by elementary property 11. For integers $\ell > 1$, $2\ell > 2$ by elementary property 10. Thus:

$$2k < 0 = 2\cdot 0 < 1 < 2\cdot 1 = 2 < 2\ell,$$

so we see that 1 cannot be even, a contradiction.

$\boxtimes$

<u>REMARK</u> You can also do this directly by considering $n-2k=1$ and showing n is not even.

(b) proof: Assume m and n are odd integers. By our theorem, we know $m = 2k+1$ for some integer k and $n = 2\ell+1$ for some integer ℓ. Then

$$\begin{aligned} m+n &= 2k+1+2\ell+1 \\ &= 2k+2\ell+2 \\ &= 2(k+\ell+1). \end{aligned}$$

Since $k+\ell+1$ is an integer, we see that 2 is a divisor of $m+n$. Thus $m+n$ is even, as desired. $\boxtimes$

7. proof: First, assume n is odd. Then $n = 2k+1$ for some k, by our theorem. Computing, we see

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)+1. \end{aligned}$$

Since integers are closed under

multiplication and addition, we know that $2k^2+2k$ is an integer. In other words, n^2 is of the form $2\ell+1$ for the integer $\ell = 2k^2+2k$.

By our theorem (specifically, the statement we proved in Question 6a!), we see that this means n^2 is odd.

Conversely, assume n is even.
(We are proving the other implication, but by using the contrapositive! We are showing that n^2 odd implies n is odd by assuming n is not odd, i.e. even, and exhibiting that n^2 is not odd, i.e. even.)
Since n is even, we know 2 is a divisor, hence $n = 2m$ for some integer m. Then

$$n^2 = (2m)^2 = 2(2m^2),$$

hence 2 is a divisor of n^2. Thus, we see that n^2 is even, as desired.

$\blacksquare$

8. We want to prove: For all integers p, q, and r $p+q+r$ is not odd or the number of odd elements in $\{p, q, r\}$ is not even. In other words, if p, q, and r are integers, then $p+q+r$ is even or the number of odd elements in $\{p, q, r\}$ is odd.

proof: Assume p, q, and r are integers. Consider $p+q+r$. If $p+q+r$ is even, we are done. If $p+q+r$ is odd, we need to show that the number of odd elements in $\{p, q, r\}$ is odd. Assume $p+q+r$ is odd. We will need the following three facts to complete the direct proof.

Can conclude $m+n$ is odd, as desired.

Now, consider the following. $p+q+r = (p+q)+r$, so since we are assuming $p+q+r$ is odd, we know by Facts 1, 2, and 3 (and because integers are either even nor odd, not both), that either r is even and $p+q$ is odd or that r is odd and $p+q$ is even.

Case 1: (r is even and $p+q$ is odd)

By applying Facts 1, 2, and 3 again, we see that either p is odd and q is even or p is even and q is odd. In either case, the number of odd elements in $\{p, q, r\}$ is 1, an odd integer since $1 = 2 \cdot 0 + 1$ (using our theorem)!

Case 2: (r is odd and $p+q$ is even)

By applying Facts 1, 2, 3 again, either both p and q are odd or both p and q are even. Thus, there are either 3 or 1 odd integers in $\{p, q, r\}$. Since $1 = 2 \cdot 0 + 1$ and $3 = 2 \cdot 1 + 1$, both 1 and 3 are odd, so we can conclude that an odd number of elements in $\{p, q, r\}$ is odd, as desired.

$\boxtimes$

1. The sum of two even integers is even.

2. The sum of two odd integers is even.

3. The sum of an odd integer and an even integer is odd.

To see 1, let m and n be even integers. Then $m=2k$ for some integer k and $n=2\ell$ for some integer ℓ. Computing, we see $m+n = 2k+2\ell = 2(k+\ell)$. Since $k+\ell$ is an integer (integers are closed under addition), we see that $m+n$ is divisible by 2, hence even.

or cite Question 6b!

To see 2, let m and n be odd integers. Then, by our theorem, we know $m=2k+1$ and $n=2\ell+1$. Computing, we see $m+n = 2k+1+2\ell+1 = 2(k+\ell+1)$. Since $k+\ell+1$ is an integer, we see that $m+n$ is divisible by 2, hence even.

To see 3, let m be an odd integer and let n be an even integer. Then, by the definition of even and our theorem about odd numbers, $n=2\ell$ for some integer ℓ and $m=2k+1$ for some integer k. Notice, $m+n = 2k+1+2\ell = 2(k+\ell)+1$. By our theorem about odd integers (Question 6a), we