

# Math 324 Summer 2018: Review Questions

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- Let  $f\vec{F}$  denote  $f(x, y, z)\vec{F}(x, y, z)$ . Show that  $\text{curl}(f\vec{F}) = f\text{curl}(\vec{F}) + \nabla f \times \vec{F}$ .
  - Assume  $f$  and  $g$  have continuous second-order partial derivatives in  $\mathbb{R}^3$ ,  $C$  is a simple closed loop with positive orientation that is the boundary of a parametric surface  $S$ . Note that  $S$  and  $C$  satisfy the conditions of Stokes' Theorem. Show that

$$\int_C f \nabla g \cdot d\vec{r} = 0.$$

- Recall that heat flow through an three-dimensional object can be described by the vector field  $\vec{F}(x, y, z) = -K\nabla u(x, y, z)$  where  $K$  is the conductivity of the object and  $u(x, y, z)$  is the temperature. Gelila, an engineer, designs cooling systems for concentrated photovoltaic panels. One of the middle layers of the panel has a critical design constraint. This critical layer of her solar panel can be described by the parametric equation  $\vec{r}(x, y) = x\hat{i} + y\hat{j} + (0.05x^2 + 0.05y^2)\hat{k}$  for  $(x, y)$  such that  $x^2 + y^2 \leq 1$ , where units are in meters. When she tests her cooling system, she notices the temperature settles at  $u(x, y, z) = x^2 - y^2 + z + 78$  degrees Celsius in an open region around the critical layer.

Compute the heat flux through the critical layer of Gelila's panel. You may assume the outward normal is the upward normal on the surface. You do not need to answer in the correct units, but note that your answer will depend on  $K$ .

- Let  $\vec{F}$  be an inverse square field, that is,  $\vec{F} = \frac{cx}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{cy}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{cz}{\sqrt{x^2+y^2+z^2}}\hat{k}$  for some constant  $c$ . Show that the flux of  $F$  across a sphere  $S$  with center the origin is independent of the radius of  $S$ . (Hint: You may use the fact that the outward normal of any sphere is  $\hat{n} = \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k})$ , where  $r$  is the radius.)
- Evaluate

$$\int_C (y + \sin(x))dx + (z^2 + \cos(y))dy + x^3dz$$

where  $C$  is the curve  $\vec{r}(t) = \sin(t)\hat{i} + \cos(t)\hat{j} + \sin(2t)\hat{k}$ , for  $0 \leq t \leq 2\pi$ . (Hint: Observe that  $C$  lies on the surface  $z = 2xy$ .)

- Let  $\vec{F}(x, y, z) = z\tan^{-1}(y^2)\hat{i} + z^3\ln(x^2 + 8)\hat{j} + z\hat{k}$ . Find the flux of  $\vec{F}$  across  $S$ , the part of the paraboloid  $x^2 + y^2 + z = 20$  that lies above the plane  $z = 4$  and is oriented upward.

6. Consider the electric field generated by a single electron located at the origin in  $\mathbb{R}^3$

$$\vec{E} = \frac{\epsilon Q x}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{i}} + \frac{\epsilon Q y}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{j}} + \frac{\epsilon Q z}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{k}}$$

where  $Q$  is the charge of the electron and  $\epsilon$  a constant dependent on the units used.

- (a) Show that  $\text{div} \vec{E} = 0$  for the electric field.
- (b) Show that the flux of  $E$  through any closed surface that contains the origin is  $4\pi\epsilon Q$ .