RESEARCH STATEMENT

JOSH SOUTHERLAND

1. Introduction

I work at the intersection of harmonic analysis, geometry, and dynamics, and I study two primary objects: translation surfaces and metric graphs. Both objects are forms of a singular manifold, and I hope to connect the Laplacian to *dynamical* and *algebraic* properties of these objects. This turns out to be quite hard - the existence of singularities, despite how mild, create difficulties.

2. Translation Surfaces: Laplacian and Dynamics

A *translation surface* is a polygon or set of polygons in the plane such that each side of the polyon(s) is identified to a parallel side by translation [35], [36]. Translation surfaces come with a natural action of $SL_2(\mathbb{R})$, where the action of a matrix is just the usual linear action. Since the linear action sends parallel lines to parallel lines, the action sends translation surfaces to translation surfaces. We call the stabilizer of the $SL_2(\mathbb{R})$ action the *Veech group* of the surface.

For an example of a stabilizing element, consider the unit square with opposite sides identified by translation (a torus) and let $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

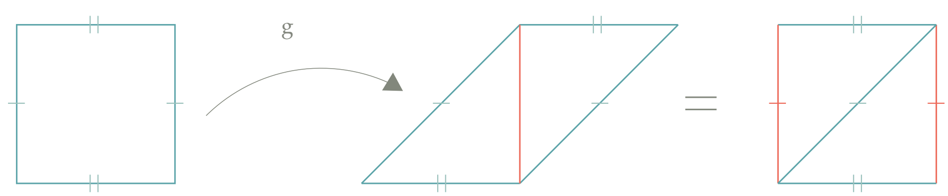

Figure 1. Application of a Veech group element and cut-paste equivalence

Using the cut-and-paste procedure pictured above we can reassemble the new polygon as the old, meaning the underlying topological space is the same. This means the Veech group is not always trivial. Visually, we see that the action of the matrix appears connected to linear maps on the surface. In fact, we can identify the Veech group with collection of derivatives of affine linear maps on the surface [33].

In 1989, Veech discovered a large class of surfaces that have "large" stabilizers, specifically, stabilizers that are lattices in $SL_2(\mathbb{R})$ [33]. Such lattices are necessarily non-cocompact, finite covolume discrete groups of $SL_2(\mathbb{R})$ [17]. We call these surfaces *lattice surfaces*.

Since Veech groups are non-cocompact, finite covolume discrete (Fuchsian) groups of $SL_2(\mathbb{R})$, they contain a hyperbolic element, which is a matrix with expanding and contracting eigenspaces. The corresponding affine linear map, after several applications, sufficiently "mixes" the points on the surface. In fact, the map will be *weakly mixing*. For example, consider Arnold's cat map (see Figure 2.

One can imagine that after several iterations, the cat will be quite blurred, illustrating that the points are moving a lot on the surface.

The fact that these hyperbolic elements are mixing tells us that the action of the Veech group of a lattice surface is *ergodic*, which means we can ask questions about the density of the orbits of the

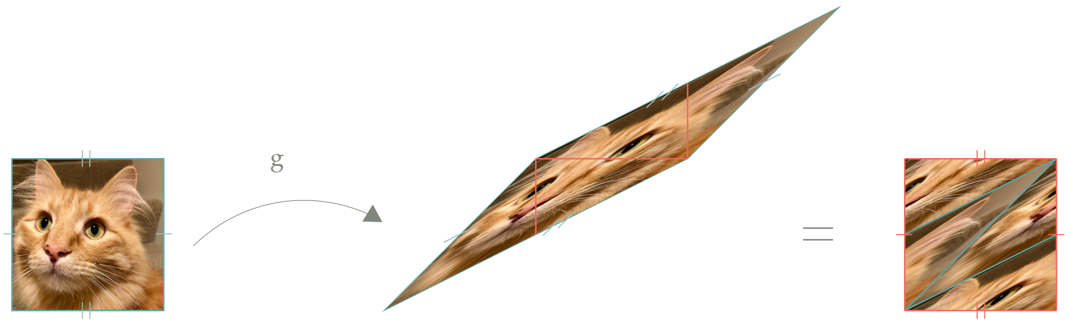

FIGURE 2. Arnold's Cat Map

action. One way to answer this is by proving that the action of the Veech group exhibits a *shrinking target property*. Fix a lattice surface S with Veech group Γ, and pick any $y \in S$. Let $B_g(y)$ denote the open ball of radius $\phi(||g||)$ (a decreasing function of the usual matrix norm). Does almost every $x \in S$ have the property that $g \cdot x \in B_g(y)$ for infinitely many $g \in \Gamma$? How fast can ϕ decrease (the target shrink) before this no longer holds?

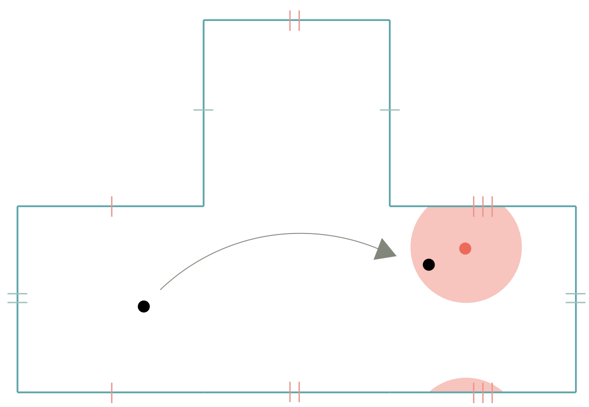

FIGURE 3. Hitting the target

2.1. **The torus.** As an example, consider the action of $SL_2(\mathbb{Z})$ on the unit torus. The torus is an example of a *translation surface*, and $SL_2(\mathbb{Z})$ is its *Veech group*. In fact, $SL_2(\mathbb{Z})$ is a lattice in $SL_2(\mathbb{R})$, so the torus is an example of a lattice surface.

Our interest is in connecting this dynamical system to the Laplacian on the torus. The action of the Veech group on the surface induces a group representation on the bounded linear functionals of $L^2(\mathbb{T}^2)$, the *Koopman representation*, $\pi : SL_2(\mathbb{Z}) \to \mathcal{B}(L^2(\mathbb{T}^2))$, where $\pi(g)f(x) = f(g^{-1}x)$. Now, recall the eigenfunctions of the Laplacian, $\Delta = -(\partial_x^2 + \partial_y^2)$, are solutions to $\Delta f = \lambda f$. We can compute the eigenfunctions: $e^{2\pi imx}e^{2\pi iny}$, where $(m, n) \in \mathbb{Z}^2$.

Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. Then

$$\pi(g)e^{2\pi imx}e^{2\pi iny} = e^{2\pi i(dm-cn)x}e^{2\pi i(an-bm)y}$$

This is significant: the Koopman representation sends eigenspaces of the Laplacian to eigenspaces. In other words, *the action of the Veech group plays nicely with the spectral properties of the Laplacian.* In fact, we can say precisely how the eigenspaces are permuted by noting how (m, n) is permuted: by the inverse transpose of g.

$$(m, n) \to (dm - cn, an - bm) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}$$

Now take the subspace of functions orthogonal to constant functions, $L^2_0(\mathbb{T}^2)$. On this subspace, the Fourier transform $\mathscr{F}$ intertwines the Koopman representation with a representation on the bounded linear functionals on $l^2(\mathbb{Z}^2 \setminus 0)$, $\pi : SL_2(\mathbb{Z}) \to \mathcal{B}(l^2(\mathbb{Z}^2 \setminus 0))$ given by $\pi(g)f(\vec{n}) = g^t\vec{n}$.

$$\begin{array}{ccc} L^2_0(\mathbb{T}^2) & \xrightarrow{\mathscr{F}} & l^2_0(\mathbb{Z}^2 \setminus 0) \\ \big\downarrow \pi_0(g) & & \big\downarrow \widehat{\pi_0}(g) \\ L^2_0(\mathbb{T}^2) & \xrightarrow{\mathscr{F}} & l^2_0(\mathbb{Z}^2 \setminus 0) \end{array}$$

We can use this to decompose the transpose representation. In 2016, Finkelshtein did this to show that the L^2 norm of an averaged Koopman representation (averaged over a measure on the group) is equivalent to the L^2-norm of a Markov operator associated to a random walk on the group $SL_2(\mathbb{Z})$. He then used this, along with spectral estimates of an induced representation on the visual boundary of $SL_2(\mathbb{Z})$, to prove a shrinking target property on tori.

2.2. **Spectral properties in dynamics.** My goal is to leverage a similar technique to prove a shrinking target property for square-tiled surfaces [29]. Here, a *square-tiled surface* is a translation surface that is a finitely branched cover of the unit square torus.

This problem is challenging for the following reason: the action of the Veech group on a square-tiled surface does not, in general, respect the eigenspaces of the Laplacian. In fact, the problem is worse. On square-tiled surfaces, neighborhoods of singularities cause problems. If we define the domain of the Laplacian to be the set of smooth functions with compact support away from the singularities, the domain of the adjoint of the Laplacian will include a set of functions with poles at the singularities [15]. The Koopman representation sends these functions to functions outside the domain of the adjoint [?SS21].

To address this for primitive square-tiled surfaces, I have attempted to push the dynamics on a square-tiled surface to a measurably conjugate dynamical system on a discrete set of labeled tori to study the induced action [29]. To push through the argument, one needs to show that the L^2-norm of an averaged Koopman operator is equal to the L^2-norm of an operator associated to a random walk on the acting group. However, only a bound between the L^2-norms can be achieved with the current machinery, not equality. This is currently being updated.

Question 2.1. Can the same technique be used to solve a shrinking target problem for lattice surfaces? Translation surfaces?

If the technique works for square-tiled surfaces, the answer to this question is almost surely yes. The technique will work for other surfaces that can be decomposed into rectangles, but with one technical caveat concerning the critical exponent. The critical exponent δ_Γ is the exponent required for convergence in the Poincaré series of the discrete group Γ [4] [24], which is equivalent to the exponential growth rate of the number of points in the orbit of Γ acting on the upper half-plane [31]. Patterson [24] showed that for a finitely generated Fuchsian group Γ, the critical exponent is precisely the Hausdorff dimension of the limit set, $\Lambda = \overline{\Gamma x} \cap S^1$, where S^1 is the circle at infinity. Sullivan [31] showed that in the general case of a Fuchsian group, the critical exponent is the Hausdorff dimension of the *radial* limit set, $\Lambda_r \subset \Lambda$ consisting of all points in the limit set such that there exists a sequence $\lambda_n x \to y$ remaining within a bounded distance of a geodesic ray ending at y.

Question 2.2. For arbitrary non-cocompact lattice subgroups G of $SL_2(\mathbb{R})$, is it true that for any subgroup $\Gamma \subset G$ and any $\varepsilon > 0$ we can find a convex, cocompact subgroup $\Gamma' \subset \Gamma$ such that $\delta_{\Gamma'} > \delta_\Gamma - \epsilon$?

2.3. **Spectral properties related to counting.** In 1989, Veech discovered an amazing fact about lattice surfaces: he packaged the lengths of simple closed geodesics on the surface (taking only one simple closed geodesic in a homotopy class) into a zeta function, and showed that the zeta function has very close ties to the cusps of the $SL_2(\mathbb{R})$ orbit - namely the zeta function can be expressed in terms of the Eisenstein series arising from the cusps on the $SL_2(\mathbb{R})$ orbit [33]. He then used this expression to count these geodesics, showing that the number of geodesics with length less than R grows quadratically.

Consider our example again: the torus. Once we fix a point on the torus, lines with rational slope will close up. Rational numbers are in bijection with elements in $(m,n) \in \mathbb{Z}^2$ such that m and n are coprime. We will refer to this subset of $\mathbb{Z}^2$ as $\mathbb{Z}^2_{\text{prim}}$.

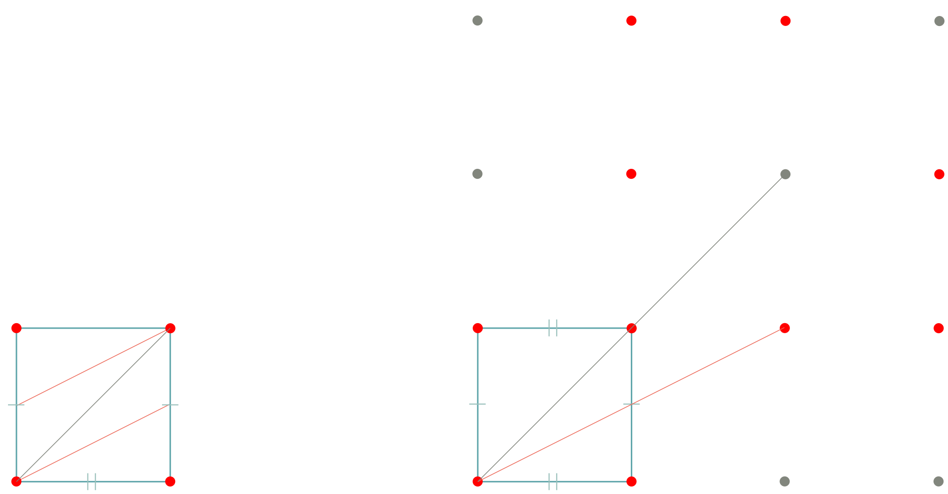

FIGURE 4. Primitive and non-primitive vectors

Note that the gray vector in Figure 4 does not correspond to a simple closed curve since it winds around the surface two times, whereas the red-orange vector does correspond to a simple closed geodesic.

We call $\mathbb{Z}^2_{\text{prim}}$ the collection of the *holonomy vectors* of the torus. They represent the simple closed geodesics we are counting on the torus.

We will show how Veech's theorem works on the torus, and then draw a connection to the spectrum of the Laplacian. First, we package the lengths in a zeta function. Let L denote the set of lengths.

$$\zeta_{\text{length}}(s) = \sum_{l \in L} \frac{1}{l^s}$$

Using Veech's machinery, we can deduce the following. Let

$$E(z,s) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{\operatorname{Im}(z)^s}{|m+nz|^{2s}}$$

be an Eisenstein series, and let $\zeta_R(s)$ be the Riemann zeta function. Then,

$$\zeta_{\text{length}}(s) = \frac{1}{\zeta_R(2s)} E(i,s).$$

Note that $\zeta_R(s)$ appears because the length spectrum is summed over $\mathbb{Z}^2_{\text{prim}}$ and $\mathbb{Z}^2 \setminus (0,0) = \cup_{n\in\mathbb{N}} nZ^2_{\text{prim}}$. And, of course, we do not need the full strength of Veech's result to deduce this formula. We can compute this directly.

Now, we can solve for eigenvalues of the Laplacian on the torus by solving the equation $\Delta f = \lambda f$. The spectrum of the torus is given by:

$$\sigma(\Delta) = \{4\pi^2(n^2 + m^2) : n, m \in \mathbb{Z}\}$$

Then $\zeta s := \sum_{\lambda \in \text{Spec}(\Delta)} \frac{1}{\lambda^s}$, and

$$\zeta_{\text{eig}}(s) = (2\pi)^{-2s} E(i, s).$$

Remark 2.1. Using a Mellin transform, we can express the zeta function as an integral function connected to spectrum to the trace of the heat kernel:

$$\zeta_{\text{eig}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\text{Tr}(e^{-\Delta t}) - 1\right) dt$$

This becomes important in Section 3 below.

Now, it is not hard to see that there is a relationship between the two zeta functions.

$$\zeta_{\text{eig}}(s) = (2\pi)^{-2s} \zeta_R(2s) \zeta_{\text{length}}(s)$$

Curiously, though likely happenstance, the zeta-function that packages eigenvalues on the circle $\mathbb{R}/\mathbb{Z}$ is given by $2(2\pi)^{-2s}\zeta_R(2s)$.

This kind of relationship between the spectrum and eigenvalues should be expected on the torus. After all, the Poisson summation formula makes a very similar connection (with the exception of summing over an entire lattice instead of the associated primitive lattice), meaning the Poisson summation formula counts with multiplicity.

Question 2.3. Is there a connection between the cusps of the Veech group and eigenvalues of the flat Laplacian on lattice surfaces? If so, does this provide a pathway forwards for thinking about Veech's question: extending the prime geodesic theorem to a larger class of translation surfaces?

In 1998 Veech published a paper in which he defines the Siegel-Veech transform and uses it to give an answer to his question about a prime geodesic theorem [34]. In some sense, the Seigel-Veech transform is replacing the Eisenstein series above.

3. Metric Graphs and Tropical Curves

The following is joint work with Junaid Hasan and Farbod Shokrieh.

A *metric graph* is a compact, connected metric space such that any point has a neighborhood isometric to a star shaped set. In other words, it is a one-dimensional manifold, except at finitely many points which are isomorphic to stars with more than two branches.

Mikhalkin and Zharkov proved that there is a one-to-one correspondence between metric graphs and compact tropical curves, where *tropical curves* are connected topological spaces homeomorphic to a locally finite 1-dimensional simplicial complex equipped with a tropical structure, and the tropical structure is an atlas of maps into the semifield $\mathbb{R} \cup \{-\infty\}$ satisfying certain transition conditions [22]. The semifield here is the tropical semifield, where "addition" is $\max\{x, y\}$ and "multiplication" is $x + y$. Tropical curves are a special case of a tropical variety, the higher dimensional analogue.

The field of tropical geometry is analogous to complex geometry in some surprising ways. For example, there is a rather mysterious Riemann-Roch theorem for graphs that was proven by Baker and Norine [3] [10] [22]. The only known proof of this theorem relies on Dhar's burning algorithm,

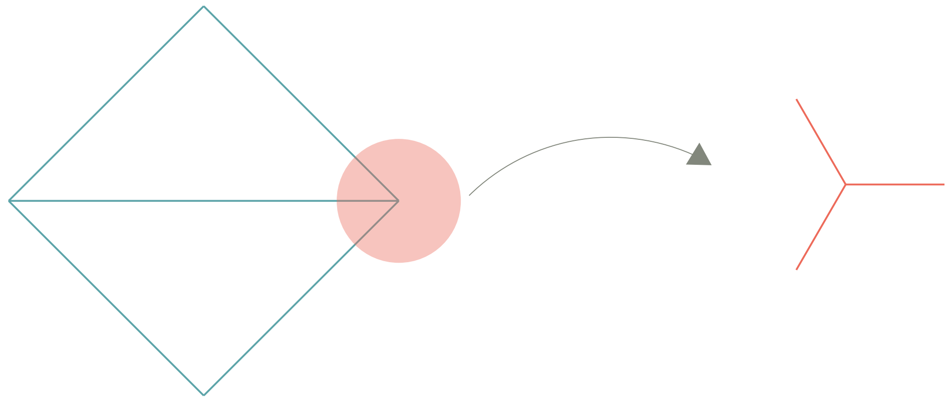

FIGURE 5. Chart to a star-shaped set

which is purely combinatorial. We ask whether or not an analytic proof of Riemann-Roch could be constructed. Classically, such proofs in complex geometry leverage properties of the Laplacian. However, in the case of the tropical Riemann-Roch theorem, it is unclear *what functions* the Laplacian would be acting on, leading to a deeper question about what sheaf we should be studying.

My focus is the role that harmonic analysis can play in helping answer these kinds of questions, starting with metric graphs.

3.1. **Regularized determinants, heights, and analytic torsion on metric graphs.** In 1973, Ray and Singer computed analytic torsion on complex manifolds [26]. Given the similarities between complex geometry and tropical geometry, it is a natural question to ask if analytic torsion can be computed on metric graphs, and whether or not this gives us any insight into other open questions in tropical geometry.

Analytic torsion is an invariant on a Riemannian manifold that depends on the underlying topology, a chain complex of forms on the surface, and a representation of the fundamental group of the surface. It is computed by taking alternating sums (up to a constant) of the logarithm of the determinant of the Hodge Laplacian. The alternating sum is over the determinant of the Laplacian on each piece of a twisted (by the representation) chain complex of smooth k-forms over the surface [25]. The representation must be picked in a way that guarantees that the cohomology associated to the twisted chain complex is acyclic (each homology group collapses).

The definition of analytic torsion relies on the following definition of the determinant of the Laplacian, which we call a *regularized determinant*.

$$\det(\Delta) = e^{-\zeta'(0)}$$

where $\zeta(s) = \sum_{\lambda \in \sigma} \frac{1}{\lambda^s}$ is the zeta function packaging the spectrum of the Laplacian, σ. Here, we should note that we are using an analytic continuation of the zeta function to a meromorphic function on $\mathbb{C}$, and the function takes a value at 0, hence is holomorphic in a neighborhood of zero, so $\zeta'(0)$ makes sense. One can show this using the integral representation of the zeta function given in Remark 2.1 coupled with the small time asymptotics of the heat kernel.

It is not hard to check that this definition generalizes the usual determinant of a linear transformation on a finite dimensional vector space. The usual determinant is the product of the (finitely many) eigenvalues λ_i and

$$\prod_{i=1}^{n} \lambda_i = e^{-\frac{d}{ds}\left(\sum_{i=1}^{n} \frac{1}{\lambda_i^s}\right)\Big|_{s=0}}.$$

The derivative of the zeta function, or the logarithm of the determinant, is of independent interest since it connects to the *theory of heights* [27].

3.2. **The circle.** Consider the circle $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, which we think of as a simple example of a graph. We can compute eigenvalues of the operator directly by solving $\Delta f = \lambda f$. The non-zero spectrum σ is $\{n^2 : n \in \mathbb{Z}\}$.

We package the non-zero spectrum in a zeta-function, $\zeta(s)$, which has a nice connection to the Riemann zeta function $\zeta_R(s)$:

$$\begin{aligned}\zeta(s) &= \sum_{\lambda\in\sigma} \frac{1}{\lambda^s} \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{n^{2s}} \\ &= 2\sum_{n\in\mathbb{N}} \frac{1}{n^{2s}} \\ &= 2\zeta_R(2s)\end{aligned}$$

This means $\zeta'(s) = 4\zeta_R'(2s)$, hence $\zeta'(0) = 4(-\frac{1}{2}\log 2\pi) = -2\log 2\pi$. This gives the following value for the regularized determinant:

$$\det(\Delta) = e^{-\zeta'(0)} = (2\pi)^2 = 4\pi^2$$

To compute the analytic torsion, we pick a representation of the fundamental group $\mathbb{Z}$ and use the deRham cohomology on S^1. However, this is where our curiosities diverge from the classical set-up: what forms should we be using on the graph?

3.3. **General metric graphs.** For the case of a general graph, there is an additional complication in the spectral theory. Similar to translation surfaces, a generic metric graph may have more than one self-adjoint Laplacian. The choice of self-adjoint Laplacian is intimately tied to the sheaf we are interested in. For example, Kurasov and Sarnak have recently computed trace formulae for metric graphs. They define the Laplacian on a set of sufficiently differentiable functions which are continuous at vertices and the sum of the incoming slopes at vertices is zero [20].

Our interest is in a similar set of functions (or rather forms): we plan to use the theory of Chamber-Loir and Ducros [7] on tropical forms and currents. Our objective is to compute the analytic torsion on a general metric graph associated with the chain of superforms. This may not be possible, in full generality, so we are starting with simpler computations, for instance, on chains of loops where torsion computation is more easily linked to the Riemann Zeta function.

There is another question lingering for general metric graphs. Ray and Singer first defined analytic torsion with the suspicion that it was an analytic version of R-Torsion, an invariant defined on topological spaces, the difference being that R-Torsion is an invariant defined through the homology whereas analytic torsion is defined through the de Rham cohomology [25]. Several years later, Cheeger and Müller independently proved that this is true [8], [23].

Question 3.1. Is R-Torsion equivalent to analytic torsion on metric graphs? For what sheaf? Does this give any indication as to the kinds of duality to be expected between homology and cohomology in tropical geometry?

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UNIVERSITY OF WASHINGTON, SEATTLE, WA 98105

Email address: `jsouther@uw.edu`