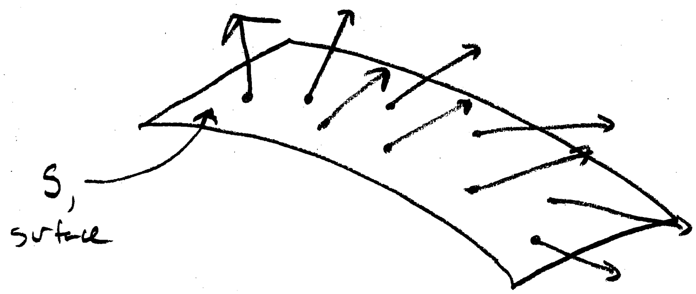
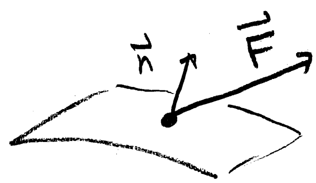
16.7 (cont.)

Next, we want to define a surface integral over a vector field.

Big Picture:



At each point on our surface, there is a vector from a vector field $\vec{F}$. What we would like to do is dot each of these vectors with a unit! normal vector,

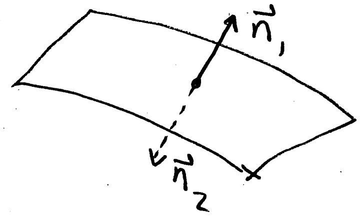


which will tell us how much of $\vec{F}$ is "leaving" or "passing through" the surface. In other words, we want something like:

$$\iint_S \vec{F} \cdot \vec{n} \, dS ,$$

where $dS = |\vec{r}_u \times \vec{r}_v| \, dA$, for $\vec{r}(u,v)$ be a parametric equation describing the surface.

But aren't there two normal vectors? Yes!



By convention, if we think of (u,v) as the parameter domain, we will always cross $\vec{r}_u \times \vec{r}_v$ in that order. Thus, our choice of a right-handed coordinate system gives us a way to consistently pick one of the normal vectors. (The "outward" one.)

<u>Exercise</u> $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$. Let $\vec{r}(u,v)$ be a parametric equation of a graph of a function, i.e., let $u=x, v=y$, and

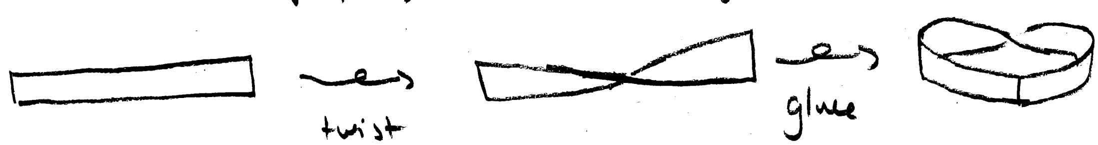
$$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle.$$

What is the outward normal in this situation?

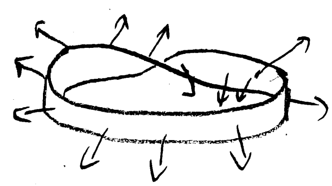
Show this! $\Longrightarrow$ $$\left(\vec{n} = \frac{-\frac{\partial g}{\partial x}\hat{\imath} + -\frac{\partial g}{\partial y}\hat{\jmath} + \hat{k}}{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \right)$$

<u>REMARK</u> Notice that the $\hat{k}$ term is <u>positive</u>. This means for a graph of a function, we always pick the $\vec{n}$ pointing <u>up</u>. We call this "choosing a positive orientation", or the "outward normal."

CAUTION Choosing an outward normal on a surface is <u>not</u> always possible. We need our surface to be "orientable" and some are not. For example, consider a Möbius Strip: take a strip of paper, twist, and glue the ends.



Now, pick an "outward" normal at a point, and travel along the surface. When you get back to the point you started at, the vector now points in the opposite direction ... !



Ah...

Something has gone terribly wrong with this surface ... it is <u>not</u> orientable!

Do not worry though ... we won't see any of these surfaces.

Now, we can make a definition.

Why?

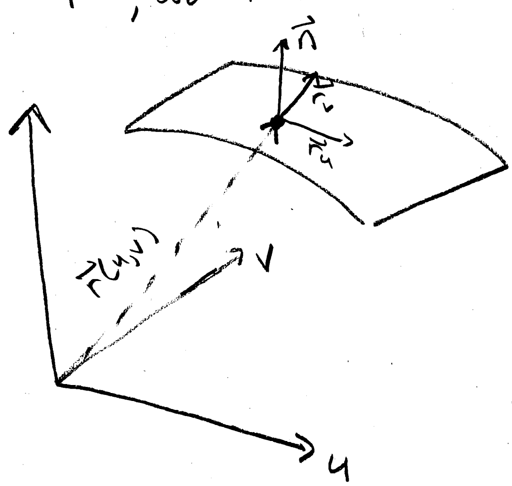
- If $\vec{F} = \rho(x,y,z) \cdot \vec{v}(x,y,z)$, where ρ is the density of a fluid and $\vec{v}$ the velocity, then this integral computes the rate of flow through S.

- If $\vec{F} = \vec{E}$ is an electric field, then the surface integral is giving us the electric flux of $\vec{E}$ through the surface.

- If $\vec{F}$ is a heat flow, then this integral would compute the rate of heat flow across the surface S.

Great. Give me an $\vec{F}$, and I can evaluate this. Except...

what's $\vec{n}$?



- $\vec{r}_u$ and $\vec{r}_v$ live in the tangent plane

- How do we get something normal?

$\vec{r}_u \times \vec{r}_v$!

- How do we make it a unit vector?

$$\boxed{\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

DEF If $\vec{F}$ is a continuous vector field defined on an <u>oriented</u> surface S with a unit normal vector $\vec{n}$, then the <u>surface integral of $\vec{F}$ over S</u> is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{n}\, dS.$$

We also call this integral the <u>flux</u> of $\vec{F}$ across S.

<u>How to compute</u>:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \left(\vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right) |\vec{r}_u \times \vec{r}_v|\, dA = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA}$$

parameter domain (u,v)

EXERCISE If our parametric surface is the graph of a function, $\vec{r}(x,y)$, how does $\iint_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_y)\, dA$ reduce? Let $F = P\hat{i} + Q\hat{j} + R\hat{k}$.

Show this! $\Rightarrow$ $\left(\iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA \right)$

EXAMPLE 1

Find the flux of the vector field $\vec{F}(x,y,z) = z\hat{\imath} + y\hat{\jmath} + x\hat{k}$ across the unit sphere.

STEP 1 Parametrize the surface: unit sphere. We've done this!

$$\left.\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta \\ z &= \cos\phi \end{aligned}\right] \Rightarrow D = \{(\phi,\theta) : 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

$$\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi \rangle$$

STEP 2 $\vec{r}_\phi \times \vec{r}_\theta = \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi \rangle$

STEP 3 Plug in to vector field:

$$\vec{F}(\vec{r}(\phi,\theta)) = \langle z(\phi,\theta),\ y(\phi,\theta),\ x(\phi,\theta) \rangle$$
$$= \langle \cos\phi,\ \sin\phi\sin\theta,\ \sin\phi\cos\theta \rangle$$

STEP 4 Then compute!

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^{\pi} F(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi\times\vec{r}_\theta)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \langle \cos\phi,\ \sin\phi\sin\theta,\ \sin\phi\cos\theta \rangle \cdot \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\theta \rangle\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta\right)d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta\right)d\phi\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^{\pi} \underbrace{\sin^2\phi\cos\phi}\cos\theta\, d\phi\, d\theta + \int_0^{2\pi}\int_0^{\pi}\sin^3\phi\sin^2\theta\, d\phi\, d\theta$$

$\left[\begin{array}{l} u = \sin\phi \\ du = \cos\phi\, d\phi \end{array}\right]$

$$= 2\int_0^{2\pi}\left[\frac{\sin^3\phi}{3}\right]_0^{\pi}\cos\theta\, d\theta + \int_0^{2\pi}\int_0^{\pi}(1-\cos^2\phi)\sin\phi\cdot\sin^2\theta\, d\phi\, d\theta$$

$$= 2\int_0^{2\pi} 0\cdot\cos\theta\, d\theta + \int_0^{2\pi}\sin^2\theta\left(\int_0^{\pi}\left(\sin\phi - \underbrace{\cos^2\phi\sin\phi}_{}\right)d\phi\right)d\theta$$

u sub: $u = \cos\phi$
$du = -\sin\phi\, d\phi$

$$= 0 + \int_0^{2\pi}\sin^2\theta\cdot\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi}\sin^2\theta\cdot\left(\left(1-\tfrac{1}{3}\right)-\left(-1+\tfrac{1}{3}\right)\right)d\theta$$

$$= \frac{4}{3}\int_0^{2\pi}\sin^2\theta\, d\theta$$

$$= \frac{4}{3}\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{4}{3}\cdot\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}$$

$$\boxed{= \frac{4}{3}\pi}$$

EXAMPLE 7

Let $\vec{F} = -K\,\nabla u$, where K is the conductivity of a substance and $u(x,y,z)$ is the temperature. ($\vec{F}$ is the heat flow!)

The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere of radius a with center at the center of the ball.

SOLUTION Impose a coordinate system: let the center of the ball be the origin. Then

square of the distance

$$u(x,y,z) = C\,(x^2+y^2+z^2),$$

where C is a constant. Then

$$\vec{F} = -K\nabla u = -K(\,C2x\,\hat{\imath} + C2y\,\hat{\jmath} + C2z\,k\,)$$

$$= -KC\,(\,2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}\,)$$

The sphere of radius a centered at the origin is $x^2+y^2+z^2=a^2$. Our usual parametrization tells us

$$\vec{n} = \frac{\vec{r}_\phi \times \vec{r}_\theta}{|\vec{r}_\phi \times \vec{r}_\theta|} = \frac{a\sin^2\phi\cos\theta\,\hat{\imath} + a\sin^2\phi\sin\theta\,\hat{\jmath} + a\sin\phi\cos\phi\,k)}{a^2\sin\phi}$$

$$= \frac{1}{a}\left(\sin\phi\cos\phi\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\phi\,\hat{k}\,\right)$$

$$\boxed{= \frac{1}{a}\,(\,x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\,)}$$

always true for spheres!

Then, notice

$$\vec{F} \cdot \vec{n} = -\frac{2KC}{a}\left(x^2+y^2+z^2\right).$$

On S, $x^2+y^2+z^2=a^2$, so

$$\vec{F} \cdot \vec{n} = -2aKC \text{ on } S!$$

Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = -2aKC \boxed{\iint_S dS}$$

$$= -2aKC \cdot A(S)$$

from before!

$$= -2aKC \cdot \boxed{4\pi a^2}$$

$$\boxed{= -8KC\pi a^3}$$

EXERCISE Distill the argument above that shows the outward normal vector on any sphere of radius r is

$$\frac{1}{r}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).$$

EXERCISE Work through example 5 in 16.7 (text).

EXERCISES [#4, #9, #17, #23, #27, #31, #43, #47, 48, 49.] in 16.7