

Lecture #7

15.10 (Webassign) / 15.9 (Book) Change of Variables in Multiple Integrals

REM U-sub

$$\int_0^{\pi} \sin(x^2) \cdot 2x \, dx = \int_{u(0)=0}^{u(\pi)=\pi^2} \sin(u) \, du$$

u = x²

$$du = 2x \, dx$$

Now, let's reverse the role of x and u :

$$\int_0^{\pi^2} \sin(x) \, dx = \int_0^{\pi} \sin(u^2) \cdot 2u \, du$$

~~u = x²~~

$$x = u^2 \leftarrow g(u) \text{ or } x(u)$$

$$dx = 2u \, du$$

~~g'(u) or x'(u)~~

NOTICE In this form, we can write a formula that says precisely how to change coordinates from x -coordinates to u -coordinates.

$$\int_a^b f(x) \, dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u)) \cdot g'(u) \, du$$

~~g^{-1}(a) = c~~

If you do not like the "g" notation (where $g(u)$ is u^2 in the example) and you would rather think of $x(u) = \underline{x=u^2}$, i.e. $x(u)=u^2$, then we could write the following:

$$\int_{a=x(a)}^{b=x(b)} f(x) dx = \int_{c=x^{-1}(a)}^{d=x^{-1}(b)} f(x(u)) x'(u) du$$

Both formulas are identical! ~~One might say the other~~

REMARK This equation, coming from the idea of u-substitution, gives us a way to change from x-coordinates to u-coordinates in the one dimensional case.

Our goal is to extend this idea into n-dimensions!
We'll start with 2 though.

REM Polar coordinates! We have already done this!

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \text{--- --- ---}$$

(but which one? ... lots of derivatives)

And integration:

$$\iint_R f(x,y) dx dy = \iint_S f(r \cos \theta, r \sin \theta) \cdot r dr d\theta$$

① S is written in polar coordinates.

② has something to do with a derivative!

We will focus on (1) for a moment, and hold off on the derivative issue (2). If we change coordinates to something other than polar, we need to know how to re-write the region R in the new coordinates.

EXAMPLE 1

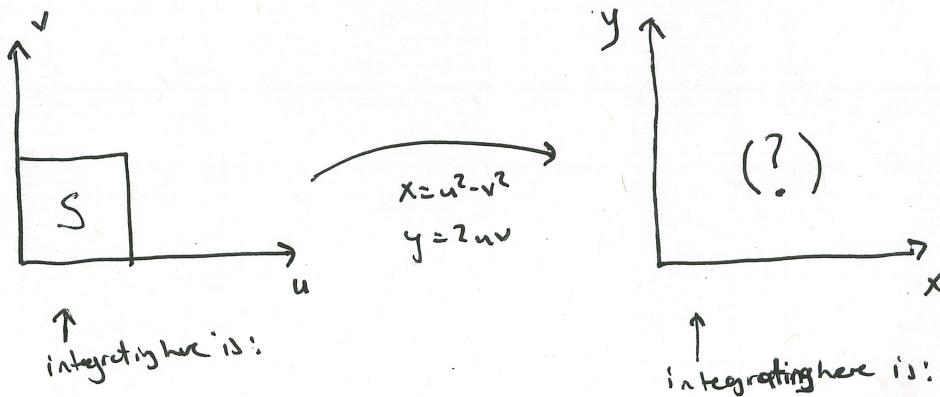
Consider the change of coordinates defined by

$$x = u^2 - v^2$$

$$y = 2uv.$$

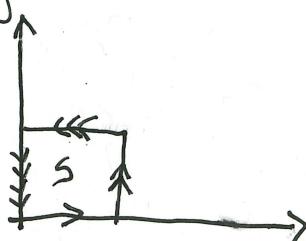
Find the image of the square $S = \{(u,v) | 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

NOTICE We are going backwards! S is given in the u - v plane:



$$\iint_S f(x(u,v), y(u,v)) \cdot \text{some derivative} \cdot du dv = \iint_{(?)} f(x,y) dx dy$$

To find the image of S under $x = u^2 - v^2$ and $y = 2uv$, we need to figure out where the edges of S go, so let's label them:



Start with \rightarrow , where $v=0$, $0 \leq u \leq 1$. Plug it into our equation!

$$x = u^2 - 0 = u^2$$

$$y = 0$$

So, along this first edge, it gets mapped to $x=u^2$ from 0 to 1, when $y=0$.



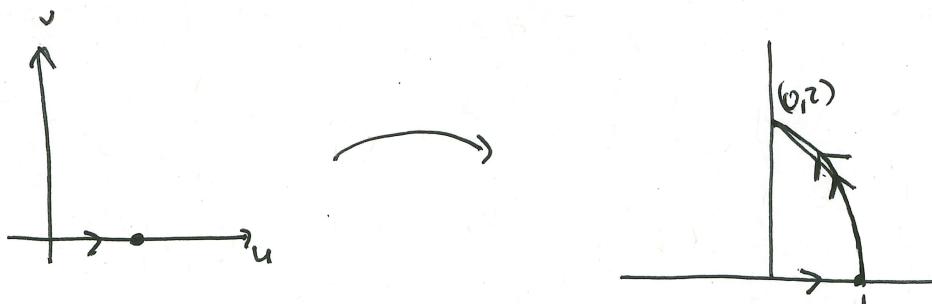
Next, let's do \uparrow , where $u=1$, $0 \leq v \leq 1$. Plug in:

$$\begin{aligned} x &= 1 - v^2 \\ y &= 2v \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{System of equations! solve } x(y) \text{ or } y(x).$$

$$\left(\begin{array}{l} \therefore (\frac{1}{2}y = v) \\ \therefore v \\ \therefore \end{array} \right)$$

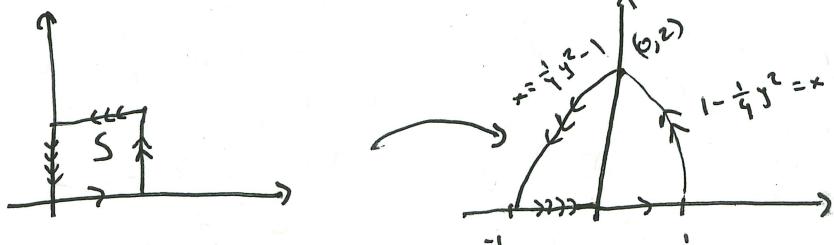
$$x = 1 - \left(\frac{1}{2}y\right)^2$$

$$x = 1 - \frac{y^2}{4}$$



Exercise do this for the other two sides!

Solution:



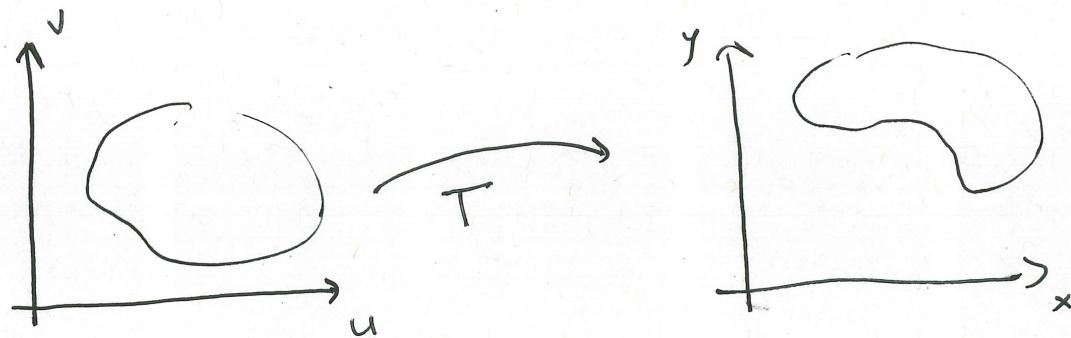
REMARK

We typically think of

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

as a transformation from \mathbb{R}^2 to \mathbb{R}^2 , i.e.

$$T(u,v) = (x(u,v), y(u,v)), \text{ where}$$



T maps from the u,v -plane to the xy -plane.

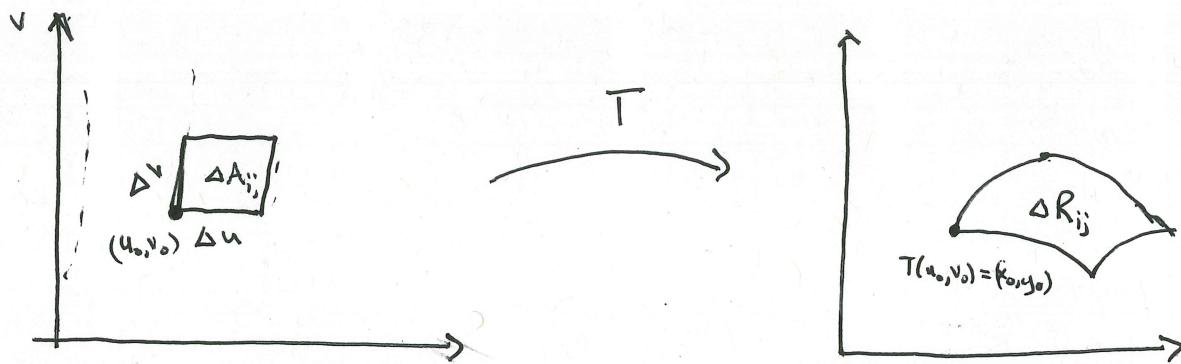
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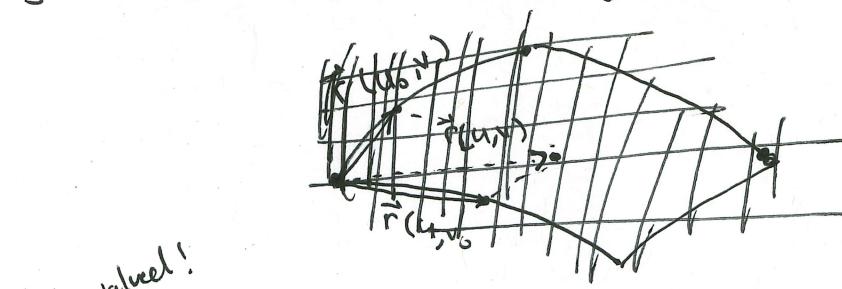
For this to work, we need T to be a C^1 -transformation, in other words, it needs to have ^{all} partial derivatives of order 1, and the map must be one-to-one onto the image. This property means there is an inverse transformation, which we want! (Why?) More to the point, we really want the map to be one-to-one ... otherwise two 'coordinates' in the uv -plane get sent to the same 'coordinate' in the xy -plane. (What value of the function do you assign to this point when you integrate?)

Now, we have to figure out (2) ... understand how the derivative plays a role.

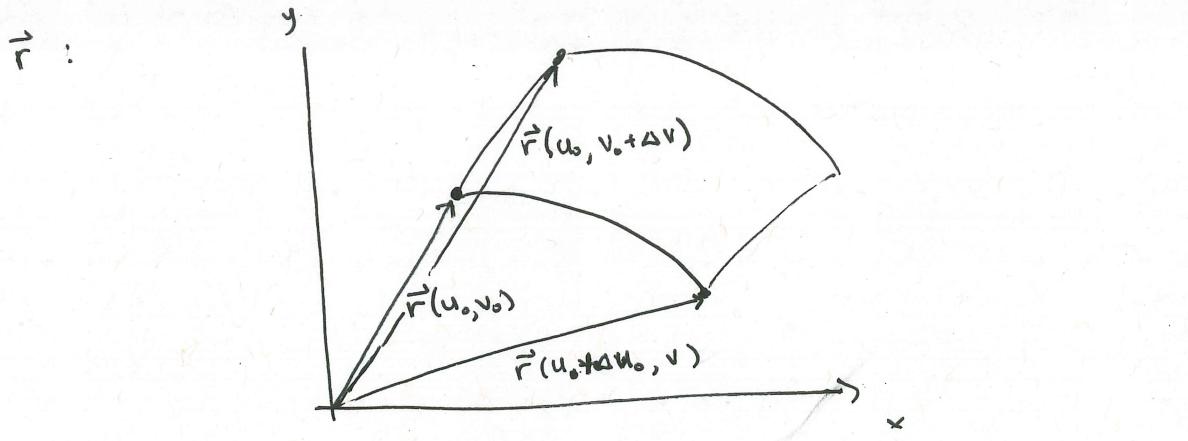
To understand this, we need to understand how a small piece of area $\Delta A = \Delta u \cdot \Delta v$ in the uv -plane produces a small piece of area in the xy -plane under the change of coordinates (or transformation).



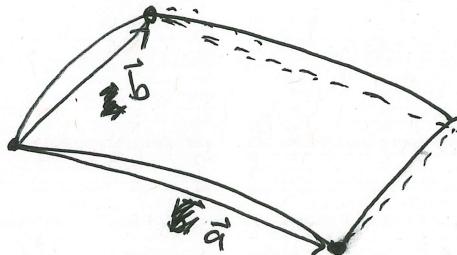
Now, look closely at ΔR_{ij} . Can we define a function along boundary? Yes! By defining a vector function on the region



$$T(u, v) \rightsquigarrow \vec{r}(u, v) = \underbrace{g(u, v)}_{\text{or } \rightarrow x(u, v)} \hat{i} + \underbrace{h(u, v)}_{\text{or } \rightarrow y(u, v)} \hat{j} \quad (\text{this is the coordinate transformation!})$$



Notice, as ΔR_{ij} gets small (when $\Delta u \cdot \Delta v$ gets small), we can approximate the area of ΔR_{ij} by computing the area of the parallelogram: $|\vec{a} \times \vec{b}|$, where



$$\vec{a} = \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)$$

$$\vec{b} = \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)$$

Notice, as $\Delta u \rightarrow 0$ and $\Delta v \rightarrow 0$, we can actually say what \vec{a} and \vec{b} will be:

$$\lim_{\Delta u \rightarrow 0} \frac{\vec{a}}{\Delta u} = \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)}{\Delta u} = \vec{r}_u$$

$$\lim_{\Delta v \rightarrow 0} \frac{\vec{b}}{\Delta v} = \frac{\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)}{\Delta v} = \vec{r}_v$$

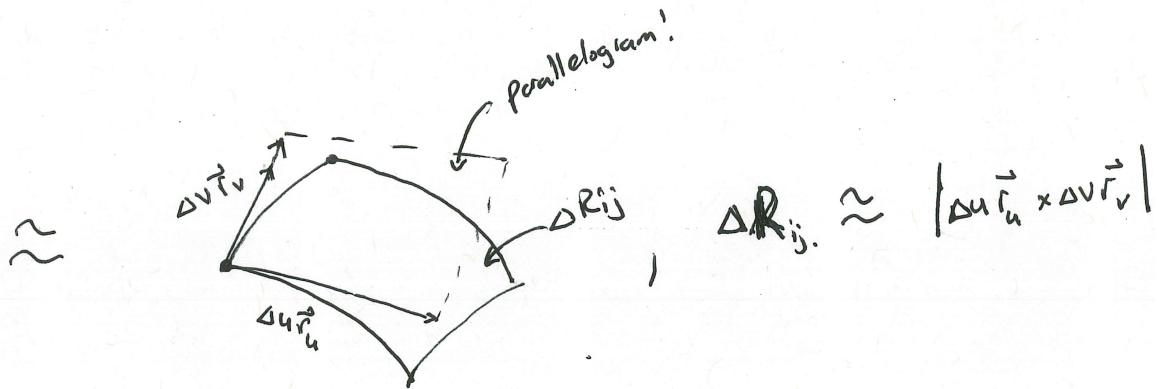
↑
partial w.r.t. u !
↓
partial w.r.t. v !

So, as Δu and Δv are getting very small ...

$$\vec{a} = \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \approx \Delta u \cdot \vec{r}_u$$

$$\vec{b} = \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0) \approx \Delta v \cdot \vec{r}_v$$

i.e.



Then, to approximate ΔR_{ij} , we want to compute:

$$|(\Delta u \cdot \vec{r}_u) \times (\Delta v \cdot \vec{r}_v)| = |\vec{r}_u \times \vec{r}_v| \cdot \underbrace{\Delta u \Delta v}_{\text{scalars}}^{(*)}$$

Then

$$\vec{r}_u = \frac{\partial x}{\partial u} \hat{i} + \frac{\partial y}{\partial u} \hat{j} \quad \leftarrow \text{depending on notation you like.}$$

$$\vec{r}_v = \frac{\partial x}{\partial v} \hat{i} + \frac{\partial y}{\partial v} \hat{j},$$

so,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k}$$

$$\text{and } |\vec{r}_u \times \vec{r}_v| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right| \quad \text{magnitude is just the scalar here!}$$

determined
of a 2×2
matrix.

DEF We say the Jacobian of a transformation is

$$|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}.$$

NOTE Typically, you call the 2×2 matrix J the Jacobian, and above, we are computing the determinant of this matrix J .

Then, we see that

$$\Delta A \approx |J| \cdot du \cdot dv, \text{ i.e.}$$

$$dx dy = |J| \cdot du \cdot dv.$$

Exercise Compute the Jacobian for the polar coordinate transformation.

Now, we can finally say what we mean by Change of variables in higher dimension.
DEF Change of Variable in a Double Integral

$$\iint_R f(x,y) dA = \iint_S f(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

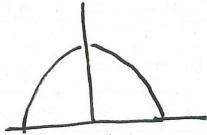
Notice We need the Jacobian to be non-zero, i.e. $T(u,v) = (x(u,v), y(u,v))$ must be an invertible transformation!

EXAMPLE 2

Use the change of variables $x=u^2-v^2$, $y=2uv$ to evaluate the integral $\iint_R y dA$, where R is the region bounded by the x -axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

①

R'



← same as image of square
in example 1!
(Show this!)

② Compute Jacobian:

$$|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 4u^2 + 4v^2 \geq 0$$

③ Compute integral:

$$\begin{aligned} \iint_R y dA &= \iint_S (2uv) \cdot |J| du dv = \iint_0^1 2uv (4u^2 + 4v^2) du dv \\ &= 8 \int_0^1 \int_0^1 (u^3 v + u v^3) du dv \\ &= 8 \int_0^1 \left[\frac{1}{4} u v + \frac{1}{2} u^2 v^3 \right]_0^1 dv \\ &= 8 \int_0^1 \left(\frac{1}{4} v + \frac{1}{2} v^3 \right) dv \\ &= 8 \left(\left[\frac{1}{8} v^2 + \frac{1}{8} v^4 \right]_0^1 \right) \\ &= 2 \end{aligned}$$