MATH 324 A
Exam II
July 30, 2018

Name ______________________________

Student ID #______________ Section ______________

HONOR STATEMENT

"I affirm that my work upholds the highest standards of honesty and academic integrity at the University of Washington, and that I have neither given nor received any unauthorized assistance on this exam."

SIGNATURE: ______________________________

SOLUTIONS!

1	16	
2	6	
3	6	
4	10	
5	12	
6	10	
Total	60	

- Your exam should consist of this cover sheet, followed by 6 problems. Check that you have a complete exam.
- Pace yourself. You have 60 minutes to complete the exam and there are 6 problems. Try not to spend more than 10 minutes on each page.
- Show all your work and justify your answers.
- Your answers should be exact values rather than decimal approximations. (For example, $\frac{\pi}{4}$ is an exact answer and is preferable to its decimal approximation 0.7854.)
- You may use TI-30X IIS calculator and one 8.5×11-inch sheet of handwritten notes. All other electronic devices (including graphing calculators) are forbidden.
- The use of headphones or earbuds during the exam is not permitted.
- There are multiple versions of the exam, you have signed an honor statement, and cheating is a hassle for everyone involved. DO NOT CHEAT.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!

1. (16 points) Clearly indicate whether each statement is true or false. You may assume, unless otherwise noted, that vector fields are smooth on an open, connected domain D.

   (a) **TRUE / FALSE** A line integral over a vector field $\vec{F}$ on the domain D is independent of path if and only if the vector field on D is conservative.

   (b) **TRUE / FALSE** Suppose $\vec{F} = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$ and $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ on D, which is simply-connected. Then line integrals over curves in D are independent of path.

   (c) **TRUE / FALSE** A line integral over a conservative vector field on D will only depend on the initial point and terminal point of the curve one is integrating over.

   (d) **TRUE / FALSE** Suppose $\int_C \vec{F} \cdot d\vec{r} = 0$ on every closed loop C in the domain $D = \{(x,y) : 1 < x^2 + y^2 < 4\}$. Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

   (e) **TRUE / FALSE** Let $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ be a smooth vector field defined on $D = \{(x,y) : x^2 + y^2 < 1\}$. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then for every closed curve C, $\int_C P(x,y)dx + Q(x,y)dy = 0$.

   (f) **TRUE / FALSE** A line integral with respect to arc length is dependent on the orientation of the curve.

   (g) **TRUE / FALSE** Let $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ be a smooth vector field defined on $D = \{(x,y) : x^2 + y^2 < 1\}$. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then for every closed curve C, $\int_C \vec{F} \cdot d\vec{r} = 0$.

   (h) **TRUE / FALSE** Consider the vector field $\vec{F}(x,y) = \langle y^3 cos(x), -3y^2 sin(x)\rangle$. F is conservative on $\mathbb{R}^2$.

2. (6 points) Suppose f is a differentiable function of x and y, and $g(u,v) = f(e^u + \sin(v), e^u + \cos(v))$. Use the table of values to calculate $\frac{\partial g}{\partial u}(0,0)$ and $\frac{\partial g}{\partial v}(0,0)$. Show your all of your work.

	f	g	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial y}$
$(0,0)$	2	0	8	4
$(1,2)$	0	2	7	5

$x(0,0) = (e^0 + \sin(0)) = 1$

$y(0,0) = e^0 + \cos(0) = 2$

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} \Rightarrow \frac{\partial g}{\partial u}(0,0) = \frac{\partial f}{\partial x}\Big|_{(x(0,0),y(0,0))}\cdot\frac{\partial x}{\partial u}\Big|_{(0,0)} + \frac{\partial f}{\partial x}\Big|_{(x(0,0),y(0,0))}\cdot\frac{\partial y}{\partial u}\Big|_{(0,0)}$$

$$= 7\cdot(e^u)\big|_{(0,0)} + 5(e^u)\big|_{(0,0)}$$

$$= 7\cdot 1 + 5\cdot 1 = \boxed{12}$$

Similarly,

$$\frac{\partial g}{\partial v}(0,0) = \frac{\partial f}{\partial x}\Big|_{(1,2)}\cdot\frac{\partial x}{\partial v}\Big|_{(0,0)} + \frac{\partial f}{\partial y}\Big|_{(1,2)}\cdot\frac{\partial y}{\partial v}\Big|_{(0,0)} = 7\cdot(\cos(v))\big|_{(0,0)} + 5\cdot(-\sin(v))\big|_{(0,0)}$$

$$= 7\cdot 1 + 5\cdot 0 = \boxed{7}$$

3. (6 points) Let $f(x,y) = e^x \sin(y) + 2xy$.

(a) At the point $(0,\pi)$, in which direction is the slope of the surface $f(x,y)$ the largest?

$$\nabla f = (e^x \sin(y) + 2y)\,\hat{\imath} + (e^x \cos(y) + 2x)\,\hat{\jmath}$$

$$\nabla f(0,\pi) = 2\pi\,\hat{\imath} + {-1}\,\hat{\jmath} = \langle 2\pi, -1\rangle$$

(b) What is the slope of the surface at the point $(0,\pi)$ in the direction of the unit vector $\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$?

$$D_{\vec{u}} f(0,\pi) = \nabla f(0,\pi)\cdot\vec{u} = \langle 2\pi, 1\rangle\cdot\langle \tfrac{\sqrt{3}}{2}, \tfrac{-1}{2}\rangle$$

$$= \pi\sqrt{3} + {-\tfrac{1}{2}}$$

4. (10 points) Compute

$$\vec{F} = (\quad)\hat{\imath} + (\quad)\hat{\jmath}$$

$$\int_C \sin(y)e^{x\sin(y)}dx + x\cos(y)e^{x\sin(y)}dy = \int_C \vec{F}\cdot d\vec{r}$$

where C is the line segment from $(1,\pi)$ to $(2,\pi)$ followed by the line segment from $(2,\pi)$ to $(2,2\pi)$. If you use a Theorem, state why you can use the Theorem.

Not necessary to write but nice to notice!

Notice: $\dfrac{\partial P}{\partial y} = \sin(y)\cdot e^{x\sin(y)}\cdot x\cos(y) + \cos(y)e^{x\sin(y)}$

$\dfrac{\partial Q}{\partial x} = x\cos(y)\cdot e^{x\sin(y)}\cdot \sin(y) +$

$\Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, smooth on $\underbrace{\text{all of } \mathbb{R}^2}_{\text{Simply connected!}}$ $\Rightarrow \vec{F} = P\hat{\imath} + Q\hat{\jmath}$ is conservative!

Find f such that $\vec{F} = \nabla f$: $\dfrac{\partial f}{\partial x} = \sin(y)e^{x\sin(y)}$, $\boxed{\dfrac{\partial f}{\partial y} = x\cos(y)e^{x\sin(y)}}$

$f(x,y,z) = \int \sin(y)e^{x\sin(y)}dx + h(y)$ ← "constant" is possibly a function of y.

$f(x,y,z) = e^{x\sin(y)} + h(y)$

$\boxed{\dfrac{\partial f}{\partial y} = e^{x\sin(y)}\cdot x\cos(y)} + h'(y)$

$\Rightarrow h'(y) = 0 \quad \Rightarrow h(y) = C$, constant.

We can pick any constant, and the resulting $f(x,y,z)$ will work:

Let $C=0 \Rightarrow \boxed{f(x,y,z) = e^{x\sin(y)}}$ ← If you found this directly, it shows $\vec{F}$ is conservative! (*)

Then, by the Fundamental Theorem of Line Integrals:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(2,2\pi) - f(1,\pi)$$

← starting + ending points!

$$= 1 - 1 = 0$$

4. (10 points) Compute

Alternative Solution..

$$\int_C \sin(y)e^{x\sin(y)}\,dx + x\cos(y)e^{x\sin(y)}\,dy$$

where C is the line segment from $(1,\pi)$ to $(2,\pi)$ followed by the line segment from $(2,\pi)$ to $(2,2\pi)$. If you use a Theorem, state why you can use the Theorem.

$C_1: \begin{cases} x(t) = t \\ y(t) = \pi \end{cases}, \quad 1 \le t \le 2$ ⇐ first line segment.

$$\int_1^2 \underbrace{\sin(\pi)\cdot e^{t\sin(\pi)}\cdot \underbrace{1}_{x'(t)}\,dt}_{\downarrow} + \underbrace{t\cdot\cos(\pi)\cdot e^{t\sin(\pi)}\cdot \underbrace{0}_{y'(t)}\,dt}_{0}$$

$$= \int_1^2 0\cdot e^0\,dt = \int_1^2 0\,dt = 0$$

$C_2: \begin{cases} x(t) = 2 \\ y(t) = t\pi \end{cases}, \quad 1 \le t \le 2$ ⇐ second line segment.

$$\int_1^2 \sin(\pi t)\,e^{2\sin(\pi t)}\cdot\underbrace{0}_{x'(t)=0}\,dt + 2\cos(\pi t)\,e^{2\sin(\pi t)}\cdot\pi\,dt$$

$$= \int_1^2 2\pi\cos(\pi t)\cdot e^{2\sin(\pi t)}\,dt \qquad u = 2\sin(\pi t), \quad du = 2\pi\cos(\pi t)\,dt$$

$$= \int_{t=1}^{t=2} e^u\,du = \left[e^{2\sin(\pi t)}\right]_1^2 = \left[e^0 - e^0\right] = \underline{0}.$$

$$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} = \boxed{0}$$

5. (12 points)

(a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1 y_2 - x_2 y_1.$$

Parametrize:

$$\begin{cases} x(t) = at+b \\ y(t) = ct+d \end{cases}, \quad 0 \le t \le 1 \qquad \begin{matrix} x(0) = b = x_1 \\ y(0) = d = y_1 \end{matrix} \Rightarrow \begin{cases} x(t) = at + x_1 \\ y(t) = bt + y_1 \end{cases} \Rightarrow \boxed{\begin{cases} x(t) = (x_2 - x_1)t + x_1 \\ y(t) = (y_2 - y_1)t + y_1 \end{cases} \text{ for } 0 \le t \le 1}$$

$t=0 \Rightarrow (x_1, y_1)$

$t=1 \Rightarrow (x_2, y_2)$

$x(1) = a + x_1 = x_2$

$y(1) = bt + y_1 = y_2$

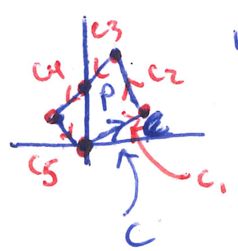
Compute

cancel

$$\int_0^1 (\underbrace{(x_2-x_1)t + x_1}_{x(t)}) \cdot \underbrace{(y_2 - y_1)}_{y'(t)}\,dt - (\underbrace{(y_2-y_1)t + y_1}_{y(t)}) \cdot \underbrace{(x_2-x_1)}_{x'(t)}\,dt = \int_0^1 \left(x_1(y_2-y_1) - (y_1)(x_2-x_1)\right)dt$$

$$= \int_0^1 \left(x_1y_2 - \cancel{x_1y_1} - y_1x_2 + \cancel{y_1x_1}\right)dt = \boxed{x_1y_2 - x_2y_1}$$

(b) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$. If you use a Theorem, state why you can use the Theorem.



We may use Green's theorem, since $\frac{\partial P}{\partial y} = -1$, $\frac{\partial Q}{\partial x} = 1$, they are continuous on $\mathbb{R}^2$, provided C is positively oriented and closed!

$$\iint_P dA = \frac{1}{2}\oint_C x\,dy - y\,dx$$

break C into line segments

$$= \frac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \int_{C_3} x\,dy - y\,dx + \int_{C_4} x\,dy - y\,dx + \int_{C_5} x\,dy - y\,dx\right)$$

apply part a

$$= \frac{1}{2}\left((0\cdot 1 - 2\cdot 0) + (2\cdot 3 - 1\cdot 1) + (1\cdot 2 - 3\cdot 0) + (0\cdot 1 - 2(-1)) + (-1\cdot 0 + 0\cdot 1)\right)$$

$$= \frac{1}{2}(9) = \boxed{9/2}$$

C positively oriented and closed!

(c) (Extra Credit! 2 points) Write an equation to compute the area of any polygon with n vertices (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), where the vertices are listed in counter-clockwise order. (You may work on the back of this page if you need more space.)

(apply part a)

$$A = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + \underbrace{(x_ny_1 - x_1y_n)}_{\text{don't forget the last line segment!}}\right]$$

6. (10 points) In this problem, you will compute the following integral $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ over two (possibly three!) curves C. If you decide to use a Theorem, state why you can use the Theorem.

(a) First, let C follow the unit circle $x^2+y^2=1$ with a positive orientation.

Notice:
$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)\cdot(-1) - (-y)\cdot 2y}{(x^2+y^2)^2} = \frac{-x^2 + -y^2 + 2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

But $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are not defined at $(0,0)$, which is inside C! ~~Cannot~~ Cannot use Green's theorem.

SO: parametrize: $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$, $0 \le t \le 2\pi$

compute:
$$\int_0^{2\pi} \frac{-\sin(t)}{\underbrace{(\cos^2 t+\sin^2 t)}_{=1}} \cdot \underbrace{(-\sin(t))}_{x'(t)} dt + \frac{\cos t}{\underbrace{(\cos^2 t+\sin^2 t)}_{=1}} \cdot (\cos t)\,dt = \int_0^{2\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{=1} dt = \boxed{2\pi}$$

(b) Second, let C follow the circle $x^2+y^2=25$ with a positive orientation.

Cannot apply Green's Theorem! (same as above: partials not defined at $(0,0)$.)

parametrize: $\begin{cases} x(t) = 5\cos t \\ y(t) = 5\sin t \end{cases}$, $0 \le t \le 2\pi$

$$\int_0^{2\pi} \frac{-(5\sin t)}{25\cos^2 t + 25\sin^2 t} \cdot (-5\sin t)\,dt + \frac{5\cos t}{25\cos^2 t + 25\sin^2 t}(5\cos t)\,dt$$
$$= \int_0^{2\pi} \frac{25\cos^2 t + 25\sin^2 t}{25(\cos^2 t + \sin^2 t)}dt = \int_0^{2\pi} dt = \boxed{2\pi}$$

(c) (Extra Credit! 3 points) Third, let C follow the curve $(x-5)^2+(y-5)^2=1$ with a positive orientation. (You may work on the back of this page if you need more space.)

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, and both are continuous and well-defined inside C.

(notice, $(0,0)$ is not inside the circle centred at $(5,5)$ with radius 1!)

Then, since C is positively oriented & closed,
$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \iint_{D} 0 \cdot dA = 0.$$
D ← interior