Homework #8 Answer Key

1. Let $S = \{ \varphi \mid \varphi : \mathbb{Z} \to \{0,1\} \}$ be a set of maps. We will show that this set is uncountable. Assume for the sake of contradiction that S is countable. Then, $S \sim \mathbb{N}$, and by symmetry, $\mathbb{N} \sim S$. Thus, there is a bijection $f : \mathbb{N} \to S$. We can use this bijection to label our maps in S:

"enumerate"

$$f(1) = \varphi_1 \leftarrow \text{label}$$
$$f(2) = \varphi_2$$
$$\vdots$$
$$f(k) = \varphi_k .$$
$$\vdots$$

Now, notice that $\mathbb{Z}$ is countable, by THM 3 from Week 7. Thus, $\mathbb{Z} \sim \mathbb{N}$, and by symmetry, $\mathbb{N} \sim \mathbb{Z}$. This means there exists a bijection $g : \mathbb{N} \to \mathbb{Z}$. We can use this bijection to label our integers:

$$g(1) = n_1$$
$$g(2) = n_2$$
$$\vdots$$
$$g(k) = n_k .$$
$$\vdots$$

Now, take a map in S, say φ_k, and notice that φ_k assigns a 0 or a 1 for every integer. This means

$$\begin{array}{ccccc}
\mathbb{Z}: & n_1 & n_2 & n_3 & n_4 \\
 & \downarrow \varphi_k & \downarrow \varphi_k & \downarrow \varphi_k & \downarrow \\
 & 0 \text{ or } 1 & 0 \text{ or } 1 & 0 \text{ or } 1 & 0 \text{ or } 1 .
\end{array}$$

We can list the maps in S and record the values for each integer:

$$\begin{array}{c|ccccc}
 & n_1 & n_2 & n_3 & n_4 & \cdots \\
\hline
\varphi_1 & b_{11} & b_{12} & b_{13} & b_{14} & \cdots \\
\varphi_2 & b_{21} & b_{22} & b_{23} & b_{24} & \cdots \\
\varphi_3 & b_{31} & b_{32} & b_{33} & b_{34} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\varphi_k & b_{k1} & b_{k2} & b_{k3} & b_{k4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Here, b_{ij} is the value of $\varphi_i(n_j)$. In other words:

$$\boxed{\varphi_i(n_j) = b_{ij}}$$

(Notice, we have identified each map with a one-sided string of 0's and 1's.)

Now, define a new map φ^* where for every $i \in \mathbb{N}$,

$$\varphi^*(n_i) \neq \varphi_i(n_i).$$

We can do this since $\varphi_i(n_i) \in \{0,1\}$, so we can always choose the number that $\varphi_i(n_i)$ is not.
(For example, if $\varphi_i(n_i) = 1$, then we pick $\varphi^*(n_i) = 0$.)

Now, notice that φ^* is a map from $\mathbb{Z}$ to $\{0,1\}$, so it should be in our list. In other words, $\varphi^* = \varphi_k$ for some $k \in \mathbb{N}$. However, $\varphi^*(n_k) \neq \varphi_k(n_k)$ for any k, so it cannot be in our list! Thus, our assertion that S is countable is wrong. We can conclude S is uncountable. (Note that there are infinitely many maps in S.)

2. a) $\mathcal{P} = \emptyset$

b) $\mathcal{P} = \{\{1\}\}$

c) $\mathcal{P}_1 = \{ \{1\}, \{2\}\}$

$\mathcal{P}_2 = \{ \{1,2\} \}$

d) $\mathcal{P}_1 = \{ \{1\}, \{2\}, \{3\}\}$

$\mathcal{P}_2 = \{ \{1,2\}, \{3\} \}$

$\mathcal{P}_3 = \{ \{1,3\}, \{2\} \}$

$\mathcal{P}_4 = \{ \{2,3\}, \{1\} \}$

$\mathcal{P}_5 = \{ \{1,2,3\} \}$.