(1) (Geometry Question) (Note: This problem is repeated in Chapter 5 with more parts.) Suppose we are given the unit square A in the plane with corners $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

(a) Find a linear transformation T that sends A to the parallelogram B with corners $(0,0)$, $(1,2)$, $(2,2)$ and $(1,0)$.
(b) Where does T send the point $(1/2, 1/2)$, which was in A?
(c) Is the linear transformation T unique? Why or why not?
(d) What linear transformation T' would send A to itself?
(e) Suppose we want to not only send A to B but also push B in the horizontal direction by one unit. What map can do this?
(f) Let L be the linear span of the side of B with corners $(0,0)$ and $(1,2)$. Write L in parametric form: $\mathbf{p} + \mathbf{q}t$ where t varies in some range and $\mathbf{p}$, $\mathbf{q}$ are vectors. What is the range of t and what are $\mathbf{p}$ and $\mathbf{q}$?
(g) Find the point in A that maps under T to the point $(1/2, 1)$ on L. In your parametric representation of L, what is the representation of $(1/2, 1)$?
(h) How can you map A to a parallelogram C of area 4 while still keeping $(0,0)$ and $(1,0)$ as two of its corners?
(i) What is the general formula for the linear transformation that sends A to a parallelogram of area k while still keeping $(0,0)$ and $(1,0)$ as two of its corners?

(2) (Geometry Question) How can you map the triangle $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ to the plane so that its area is preserved and one of its corners is $(0,0)$?

(3) (after 3.2) Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}.$$
Find a 3×2 matrix B with $AB = I_2$. Is there more than one matrix B with this property? Justify your answer.

(4) (after 3.2) Find a 2×3 matrix A and a 3×2 matrix B such that $AB = I$ but $BA \neq I$.

(5) (after 3.2) Find a 2×2 matrix A, which is not the zero or identity matrix, satisfying each of the following equations.
  a) $A^2 = 0$
  b) $A^2 = A$
  c) $A^2 = I_2$

(6) (after 3.2) Let
$$B = \begin{bmatrix} 1 & z \\ 4 & 3 \end{bmatrix}.$$
Find all values of z such that the linear transformation T induced by B fixes no line in $\mathbb{R}^2$. (By "fixing a line" we mean that $T(\mathbf{v}) = \mathbf{v}$ for every point $\mathbf{v}$ on the line.)

(7) (after 3.3) Find a 3×2 matrix A and a 2×3 matrix B such that AB is invertible or explain why such matrices cannot exist. Answer the same question with the requirement that BA be invertible.

(8) An ellipse whose axes of symmetry are parallel to the x- and y-axes has equation
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$
You want to find the equation of such an ellipse that passes through the points $(5, 9)$, $(13, 7)$, $(-1, 0)$, and $(11, 8)$. Using those 4 points, you can write down 4 equations in the 4 unknowns a, b, h, k. The trouble is, however, that the resulting equations won't look anything like a linear system. You'll have unknowns inside squares and in denominators. The following steps will allow you to convert this problem to a linear algebra problem and solve it without lengthy computations. The method suggested in the hints is certainly not the only way to do it, but its advantage is that it avoids tedious computations.

(a) Rewrite the equations with variables $\alpha, \beta, \gamma, \delta$ that are defined to make the equations look more linear (though not completely linear). Hint: set $\delta = 1/a^2$ and $\gamma = \delta h = h/a^2$, and similarly in the $(y - k)/b^2$ term.

(b) To get rid of the non-linear terms, change from 4 equations with 4 unknowns to 3 homogeneous equations with 4 unknowns by subtracting successive equations (the 2nd from the 1st, etc.).

(c) Set up the 3×4 coefficient matrix and solve for $\alpha, \beta, \gamma, \delta$. Let s denote the free variable.

(d) Convert back to the notation of the original equation.

(e) To solve for s, plug in the coordinates of one of the 4 points. Then write down the equation of the ellipse.

(9) Sinusoidal functions are a basic building block for modeling periodic phenomena (things that repeat at regular intervals). For background please see the "sinusoidal functions supplement" linked to from Week 4 of the week-by-week syllabus on the Math 124 webpage (which has the same url as Math 308 except with 308 replaced by 124). Suppose we want to use a sinusoidal function to give a (rough) formula for the way the temperature T in degrees Celsius varies with time t (hour of the day on the 24-hour clock) at a certain time of the year. That function will have the form
$$T = f(t) = A \sin\left(\frac{2\pi}{24}(t - C)\right) + D,$$
where A is the amplitude, D is the vertical shift (the horizontal line around which the temperature oscillates), and C is the phase shift. Suppose we know the average temperature at three different times of day, meaning that the sinusoid curve in the tT-plane passes through the three points $(8, 4)$, $(12, 10)$, $(16, 13)$. (Note: On the 24-hour clock, 12 is noon and 16 is 4 pm.)

(a) The three points give three equations with three unknowns A, C, D, but they are not linear equations. Change to the new variables defined by: $\alpha = D$, $\beta = A \cos\left(\frac{2\pi}{24}C\right)$, $\gamma = A \sin\left(\frac{2\pi}{24}C\right)$. Note that after you find α, β, γ you can quickly find A, C, D, because $D = \alpha$, $A = \sqrt{\beta^2 + \gamma^2}$, and $\frac{2\pi}{24}C = \text{Arctan}(\gamma/\beta)$. Using the trig identity for $\sin(X - Y)$, write the three equations in terms of the unknowns α, β, γ.

(b) Write the augmented matrix for this system of equations, transform it to reduced echelon form, and find α, β, γ.

(c) Write the formula for $T = f(t)$. Be careful (by checking the value at one of the points) about deciding which of two possible values for C is correct.

(10) A more realistic scenario is that you have a large number of data points (t_i, T_i) and want to fit a sinusoid curve to them. You almost certainly won't be able to fit the curve exactly if there are more than 3 data points. So you use the "least squares" method, described below. In order to keep the computations from getting tedious to do by hand, we'll suppose there are just 4 data points: (2:27:29, 5.0), (3:32:31, 4.0), (6:00:00, 5.0), and (8:27:29, 7.0) (where the first one means that at 2 hours 27 minutes 29 seconds past midnight the temperature was 5.0 degrees Celsius).

(a) Write 4 equations of the form $\alpha + s_i\beta - c_i\gamma = T_i$, where the s_i are certain sine values and the c_i are certain cosine values.

(b) We want to find values of the three unknowns α, β, γ that make the sum of the squares of the difference between the right and left hand sides as small as possible. To do this, set the three partial derivatives of this sum equal to zero and simplify, getting three linear equations with three unknowns.

(c) Write the augmented matrix for this system, and solve for α, β, γ.

(d) Write the formula $T = f(t)$ for the best sinusoidal fit.