COMPACT SUPPORT OF L^1 PENALIZED VARIATIONAL PROBLEMS

JONATHAN SIEGEL, OMER TEKIN

Abstract. We investigate the solutions to L^1 constrained variational problems. In particular, we are interested in the case where the L^1 term is weighted by some non-negative function. Extending previous results of Brezis et al., we prove that for a wide range of variational problems, the solutions have compact support. Additionally, we provide the results of some numerical experiments, where we computed the solutions to L^1 constrained elliptic problems using splitting and ADMM.

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1. Introduction. For finite dimensional optimization problems where the sparse structure of the solution is crucial, such a structure can be enforced using an L^1 penalization term. The advantage of the L^1 term is that it is convex, indeed it can be seen as a convex relaxation of the L^0 quasi-norm, which makes it much computationally efficient to solve the corresponding minimization problems. The resulting applications such as lasso regression, and compressed sensing has been proven to be successful in statistics and machine learning [12].

In this work, we consider infinite dimensional optimization problems, where the notion corresponding to sparsity is compact support. Indeed, spatial localization, which is provided by compact support, occurs naturally in many problems from physics and other disciplines.

In [2, 8] the authors pioneer the use of an L^1 penalty term to enforce compact support and obtain localized solutions to a class of PDE problems that can be recast as variational optimization problems. The approach taken is based upon the observation that the eigenvalues of a Hermitian operator T can be realized as the solutions to variational problems by the min-max theorem. To obtain compactly supported functions, an L^1 penalization term is added to the min-max variational problem. The resulting problem is

$$\operatorname*{arg\,min}_{\|x\|_2=1} \ \langle Tx, x\rangle + \frac{1}{\mu}\|x\|_1 \tag{1.1}$$

The resulting functions are called compressed modes (CMs), and have been studied in [11, 1].

We extend this idea and show that for a wide range of variational problems arising from eigenvalue, elliptic, and parabolic problems, adding a weighted L^1 term produces compactly supported solutions. In particular, we consider the three different types of problems described below.

In Section 2, we consider the solutions to the following problem on the whole $\mathbb{R}^n$

$$\operatorname*{arg\,min}_{\|u\|_2=1, u\in H^1(\mathbb{R}^n)} \|\nabla u\|_2^2 + \gamma\|u\|_1 \tag{1.2}$$

where $\gamma > 0$. This problem is motivated by the variational formulation of the first eigenvalue/eigenvector of the Laplace operator on a bounded domain. Note that the existence of a solutions to (1.2) is non-trivial. In fact, if we remove the L^1 term, this problem has no minimizer as the domain is the whole $\mathbb{R}^n$. The existence and compact support of solutions to (1.2) was studied in [1]. We provide a different approach to proving the existence of minimizers which is more flexible and allows us to deal with problems of the form

$$\underset{\|u\|_2=1, u \in H^1}{\arg\min} \|\nabla u\|_2^2 + \|w(x)u\|_1 \tag{1.3}$$

as long as the weight, $w(x)$ is a non-decreasing, non-zero, positive radial function.

In section 3, we consider the problem

$$f \in Lu + \beta(u) \tag{1.4}$$

on Ω with boundary data $u = \phi$ on $\partial\Omega$. Here Ω is an unbounded subset of $\mathbb{R}^n$, L is a uniformly elliptic operator, and β is a maximal monotone graph such that $\beta(0) = [\gamma_-, \gamma_+]$ where $\gamma_- < 0 < \gamma_+$. This problem was considered in [3], where it was shown, under certain assumptions, that solutions $u \in W^{2,p}$ exist, are unique, and have compact support. We extend this result to the problem

$$f \in Lu + \mu(x)\beta(u) \tag{1.5}$$

where μ is a non-negative weight function which is large outside of a compact set. Such class of weights are useful for the purpose of preserving the local information of the solution whereas still obtaining compactly supported solutions. We also show that if we remove the non-negativity assumption on μ, then uniqueness fails, but any solution still must have compact support. Note that if the non-negativity assumption on μ is removed, then the problem is no longer necessarily convex.

In section 4, we consider the variational inequality,

$$(u_t - \Delta u)(v - u) \geq f(v - u) \text{ a.e. } \quad \text{for } x \in \mathbb{R}^n, 0 < t < T, \tag{1.6}$$

for any non-negative measurable function v, and look for the solutions u with

$$u \geq 0 \text{ for } x \in \mathbb{R}^n, 0 < t < T,$$
$$u(x, 0) = u_0(x).$$

In [4], the existence and uniqueness of the above solution are verified and it is shown that if f is uniformly negative, then u has compact support in time. If, in addition, u_0 is compactly supported, then u has compact support in space and time. We extend this result by only requiring strict negativity outside of a compact set.

Finally, in section 5, we provide the results of some numerical experiments we performed in which we explicitly calculated solutions to the type of problem discussed in section 3. Namely, we numerically solve

$$\underset{u \in H^1}{\arg\min} \|\nabla u\|_2^2 - 2\langle f, u\rangle + \|w(x)u\|_1 \tag{1.7}$$

where w is the characteristic function of the complement of a ball.

Throughout the paper, we will use the following notation. Lebesgue spaces will be denoted by L^p and Sobolev spaces by $W^{k,p}$ and H^k if $p = 2$. L^p_{loc} will denote the space of functions which are in $L^p(\Omega)$ for every compact set Ω, note this is not a normed space. Additionally, we denote by H^k_{rad} the subspace of H^k consisting of radial functions. The symbol $\lesssim$ will be used when suppressing a uniform constant in an inequality. Unless otherwise stated the constant will only depend upon the dimension and the L^p norm which appears in the inequality and will never depend upon the functions which appear.

2. L^1 **Constrained Eigenvalue Problems.** We first consider the problem

$$\underset{\|u\|_2=1, u\in H^1(\mathbb{R}^n)}{\arg\min} \|\nabla u\|_2^2 + \mu\|u\|_1 \tag{2.1}$$

There are two main ingredients to the existence and compact support proofs. The first is a rearrangement inequality and the second is a compactness result.

First we describe the rearrangement inequality. We define the symmetric decreasing rearrangement of a function f as follows.

Definition 2.1. *Let A be a borel measurable set in $\mathbb{R}^n$. The symmetric rearrangement of A is $A^* = \{x \in \mathbb{R}^d : |x| < r\}$ where r is chosen such that $|A| = |A^*|$.*

Now let $f : \mathbb{R}^d \to \mathbb{C}$ be a borel measurable function. The symmetric decreasing rearrangement of f is

$$f^*(x) = \int_0^\infty \chi_{\{|f|>\lambda\}^*}(x)d\lambda$$

Note that f^* has the same distribution function as f, i.e. $|\{|f| > \lambda\}| = |\{|f^*| > \lambda\}|$ for all λ. In particular, $\|f\|_p = \|f^*\|_p$ for all p.

We need the following theorem concerning the symmetric decreasing rearrangement, due to Polyá and Szego.

Theorem 2.2. *Let $f \in W^{1,p}$ for $1 \le p \le \infty$. Then $f^* \in W^{1,p}$ and*

$$\|\nabla f^*\|_p \le \|\nabla f\|_p$$

Note that the above theorem is related to the isoperimetric inequality. In fact, for $p = 1$ it implies the isoperimetric inequality. We will only need the case $p = 2$ of the above theorem, which can be found in [9].

Next we give the compactness result that we need. We need the following theorem.

Theorem 2.3. *Fix $d \ge 2$. Then $H^1_{rad} \cap L^1$ is compactly contained in L^p for $1 < p < \frac{2d}{d-2}$.*

In order to prove this we will need the following lemmas from harmonic analysis. (Note that $\lesssim$ means $\le$ up to a constant independent of the function showing up on both sides.)

Lemma 2.4. *Let $u \in H^1$, then for $2 \le p \le \frac{2d}{d-2}$ (for $d = 1, 2$, $2 \le p < \infty$) we have*

$$\|u\|_p \lesssim \|\nabla u\|_2^\theta \|u\|_2^{1-\theta}$$

where $\theta = \frac{2d-p(d-2)}{2p}$. The previous lemma is the well-known Galgiardo Nierenberg inequality [7]. It follows from the Sobolev embedding theorem in dimension ≥ 3. In dimensions 1 and 2 it is a generalization of Sobolev Embedding.

The next lemma is called the radial Sobolev inequality.

Lemma 2.5. *Let $d \ge 2$ and $1 \le q < \frac{2d}{d-2}$. Let $f \in L^q \cap \dot{H}^1$ be radial. Then*

$$r^{\frac{2(d-1)}{q+2}}|f(r)| \lesssim \|f\|_q^{\frac{q}{q+2}} \|\nabla f\|_2^{\frac{2}{q+2}}$$

a.e.

Proof. Notice that since $|\nabla|f|| \le |\nabla f|$ a.e., it suffices to consider the case where $f \ge 0$. We claim that it also suffices to consider the case where f is a Schwartz function. This is so because Schwartz functions are dense in $L^q \cap \dot{H}^1$ and if $f_n \to f$ in $L^q \cap \dot{H}^1$, then a subsequence converges to f a.e.

So assume that f is a non-negative, radial Schwartz function. We have

$$r^{d-1}|f(r)|^{1+\frac{q}{2}} = r^{d-1}\left(1+\frac{q}{2}\right)\int_r^\infty |f(t)|^{\frac{q}{2}} f'(t)dt$$

Since $t \geq r$ in the above integration we have that the above is bounded by

$$\left(1+\frac{q}{2}\right)\int_r^\infty |f(t)|^{\frac{q}{2}} |f'(t)| t^{d-1} dt$$

We now apply Cauchy-Schwartz to bound the above by

$$\left(1+\frac{q}{2}\right)\left(\int_r^\infty |f(t)|^q t^{d-1} dt\right)^{\frac{1}{2}}\left(\int_r^\infty |f'(t)|^2 t^{d-1} dt\right)^{\frac{1}{2}}$$

$$\leq \left(1+\frac{q}{2}\right)\|f\|_q^{\frac{q}{2}}\|\nabla f\|_2$$

Taking everything to the power $\left(1+\frac{q}{2}\right)^{-1}$, we obtain the lemma. □

The next lemma is a criterion for a collection of functions in L^p to be compact, essentially a version of the Arzela-Ascoli theorem for $p < \infty$, due to Kolmogorov and Riesz [6, 10].

Lemma 2.6. *Let $X \subset L^p$. Then X is precompact in L^p iff the following hold*

1. *X is uniformly bounded, i.e. there exists $M > 0$ s.t. $\|f\|_p < M$ for all $f \in X$.*
2. *X is uniformly equicontinuous, i.e. for all $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\|f(x) - f(x-y)\|_{L^p(dx)} < \epsilon$$

whenever $|y| < \delta$, for all $f \in X$.

3. *X is uniformly tight, i.e. for every $\epsilon > 0$, there exists an $R > 0$ such that*

$$\|f\|_{L^p(B(0,R)^c)} < \epsilon$$

for all $f \in X$.

We are now in a position to prove Theorem 2.

Proof. We prove this by verifying each of the conditions given in lemma 3, when X is the unit ball B_1 in $H^1_{rad} \cap L^1$. First of all, B_1 is uniformly bounded in L^p by the Gagliardo-Nierenberg inequality and interpolation of L^p norms.

Next we must verify equicontinuity. If $p \geq 2$ we see that by Gagliardo-Nierenberg,

$$\|f(x) - f(x-y)\|_{L^p(dx)} \lesssim \|\nabla f(x) - \nabla f(x-y)\|^{\theta}_{L^2(dx)}\|f(x) - f(x-y)\|^{1-\theta}_{L^2(dx)}$$

$$\leq 2\|\nabla f\|_2^{\theta}\|\nabla f\|_2^{1-\theta}|y|^{1-\theta} \leq \|f\|_{H^1}|y|^{1-\theta}$$

Now since $1-\theta > 0$ we get equicontinuity. For $1 < p < 2$ we use interpolation of L^p norms in combination with the result for $p \geq 2$. In particular, we write

$$\|f(x) - f(x-y)\|_{L^p(dx)} \lesssim \|f(x) - f(x-y)\|^{\theta}_{L^1(dx)}\|f(x) - f(x-y)\|^{1-\theta}_{L^2(dx)}$$

Now the first term above is bounded by $2\|f\|_1$ and the second term can be bounded as before in terms of a power of $|y|$. Since $p > 1$, $\theta < 1$ and we get the desired equicontinuity.

Finally we verify the tightness. To do this we write

$$\int_{|x|>R} |f(x)|^p dx = \int_{|x|>R} |f(x)|^\delta |f(x)|^{p-\delta} dx$$

Now using Lemma 2 with $q = 2$ we see that $|f(x)| \lesssim |x|^{-\frac{d-1}{2}}$. Thus the above integral is

$$\lesssim R^{-\delta \frac{d-1}{2}} \int_{|x|>R} |f(x)|^{p-\delta} dx$$

Setting $\delta = p - 1$ we get

$$\int_{|x|>R} |f(x)|^p dx \lesssim R^{-(p-1)\frac{d-1}{2}} \|f\|_1$$

which completes the proof.
□

In order to show that existence of compactly supported minimizers to the original problem, we proceed as follows.

Let x_n be a minimizing sequence, i.e. $\|x_n\|_2 = 1$ and $\|\nabla x_n\|_2^2 + \|x_n\|_1$ converges to the optimal value. By the Polyá-Szego theorem and the trivial properties of the symmetric decreasing rearrangement, we see that taking the symmetric rearrangement of the x_n results in another minimizing sequence. Hence we may assume that the x_n are radial, non-negative, and decreasing. Note that x_n is bounded in $H^1_{rad} \cap L^1$, so by the compactness result we can take a subsequence which converges in L^2. We can also take a further subsequence which converges weakly in H^1_{rad} and in L^1 as well (by the Banach–Alaoglu theorem). Call u the limit (in L^2) of this sequence. Then we have $\|u\|_2 = 1$ since our sequence converges strongly in L^2. We also have, from the properties of weak convergence, that $\|\nabla u\|_2^2 \leq \lim \|\nabla x_n\|_2^2$ and $\|u\|_1 \leq \lim \|x_n\|_1$. But since x_n is a minimizing sequence we can't have strict inequality in the preceeding two inequalities. Hence $\|\nabla u\|_2^2 + \|u\|_1$ is optimal and we have found a minimizer.

We can use the above compactness result to show the existence of radial, non-negative, decreasing minimizers to problems of the form

$$\operatorname*{arg\,min}_{\|u\|_2=1, u\in H^1} \|\nabla u\|_2^2 + \|w(x)u\|_1 \tag{2.2}$$

as long as the weight, $w(x)$ is a non-decreasing, non-zero, positive radial function.

Theorem 2.7. *There exist radial, non-negative, decreasing minimizers to*

$$\operatorname*{arg\,min}_{\|u\|_2=1, u\in H^1} \|\nabla u\|_2^2 + \|w(x)u\|_1$$

where $w(x)$ is a non-decreasing, non-zero, positive radial function.

Proof. First we will show that

$$\|w(x)u^*\|_1 \leq \|w(x)u\|_1$$

where u^* is the symmetric decreasing rearrangement of u. To show this we note that

$$\|w(x)u\|_1 = \int_{\mathbb{R}^d} w(x)|u(x)| dx$$

$$= \int_{\mathbb{R}^d} \int_0^\infty \chi_{\{w>\lambda\}}(x) d\lambda \int_0^\infty \chi_{\{|u|>\mu\}}(x) d\mu dx$$

Here $\chi_{\{w>\lambda\}}(x)$ is the characteristic function of the set $\{w(x) > \lambda\}$ and $\chi_{\{|u|>\mu\}}(x)$ is the characteristic function of the set $\{w(x) > \mu\}$. This equality follows since

$$|f(x)| = \int_0^\infty \chi_{\{|f|>\lambda\}}(x) d\lambda$$

for all measurable f.

Now we switch the order of integration in the above to obtain

$$\|w(x)u\|_1 = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \chi_{\{w>\lambda\}}(x)\chi_{\{|u|>\mu\}}(x) dx d\lambda d\mu$$

$$= \int_0^\infty \int_0^\infty |\{w > \lambda\} \cap \{|u| > \mu\}| d\lambda d\mu$$

Now I claim that $|\{w > \lambda\} \cap \{|u| > \mu\}| \geq |\{w > \lambda\} \cap \{u^* > \mu\}|$ for all λ and μ. This follows since by assumption, $\{w > \lambda\}$ is the complement of a ball centered at the origin and $\{u^* > \mu\}$ is a ball centered at the origin. Thus if $|\{w > \lambda\} \cap \{u^* > \mu\}| > 0$ then $\{u^* > \mu\}$ covers the entire complement of $\{w > \lambda\}$. Since $|\{u^* > \mu\}| = |\{|u| > \mu\}|$, we have that $|\{w > \lambda\} \cap \{|u| > \mu\}| \geq |\{w > \lambda\} \cap \{u^* > \mu\}|$.

Integrating this with respect to λ and μ we get that

$$\|w(x)u^*\|_1 \leq \|w(x)u\|_1$$

Thus, by taking symmetric decreasing rearrangements we may assume that any minimizing sequence consists of radial functions. Now the compactness result implies the existence of a minimizer if we can uniformly bound the $|\cdot|_1$ norm of the minimizing sequence.

This follows since under the assumptions on w, there is a radius R and a constant $C > 0$ such that $w(x) > C$ if $|x| > R$. Consequently we see that $\|u\|_{L^1(\{|x|>R\})} < C\|w(x)u\|_1$, which implies that $\|u_n\|_{L^1(\{|x|>R\})}$ is uniformly bounded ($\|w(x)u_n\|_1$ is uniformly bounded as it is a minimizing sequence). Now any minimizing sequence satisfies $\|u_n\|_2 = 1$, and thus $\|u_n\|_{L^1(\{|x|<R\})} \leq R^{1/2}\|u_n\|_2$. So, since R only depends on w, we have a uniform bound on $\|u_n\|_1$ for any minimizing sequence.

The above compactness result finishes the proof. □

Next we consider the compact support of the solution. We prove the following

THEOREM 2.8. *The radial, non-negative, decreasing solutions to* (2.1) *and* (2.2) *have compact support.*

Proof. Note that the result will follow if we can show that the measure of the support is finite. This is true since a radial, non-negative, decreasing function will have support which is a ball. To this end we note that the solution u satisfies

$$\lambda u \in -\Delta u + w(x)\beta(u)$$

where β is the subdifferential of $|\cdot|$. Multiplying this by the sign of u and integrating we see that

$$\lambda\|u\|_1 = \int_{\mathbb{R}^n} -\Delta u \chi_{\{u>0\}} dx + \int_{\{u>0\}} w(x) dx$$

This is true since $u \cdot \operatorname{sgn}(u) = |u|$ and $\beta(u) \cdot \operatorname{sgn}(u) = \chi_{\{u>0\}}$. Now we consider the term

$$\int_{\mathbb{R}^n} -\Delta u \chi_{\{u>0\}} dx$$

the divergence theorem yields that this is equal to

$$\int_{\partial\{u>0\}} -\nabla u \cdot \nu dS$$

where ν is the outward normal of $\{u > 0\}$. Since $u > 0$ on the interior of $\{u > 0\}$ we have that $-\nabla u \cdot \nu \geq 0$. Thus the above integral is positive. Hence we obtain

$$\int_{\{u>0\}} w(x)dx \leq \lambda \|u\|_1 < \infty$$

Now by the assumptions on w, we have that $w(x) > C$ for $|x| > R$ for some $R > 0$ and $C > 0$. Hence

$$C|\{u > 0\} \cap \{|x| > R\}| \leq \int_{\{u>0\}} w(x)dx < \infty$$

Thus u has finite measure support and thus compact support as desired. □

3. L^1 Constrained Elliptic Problems. Let $\Omega \subset \mathbb{R}^n$ be an unbounded subset with smooth boundary and L a second order elliptic operator satisfying the same assumptions as in [3]. Let β be a maximal monotone graph in $\mathbb{R}^2$ such that $\beta(0) = [\gamma^-, \gamma^+]$ with $\gamma^- < 0$ and $\gamma^+ > 0$.

We wish to extend the results in [3] by determining when the problem

$$f \in Lu + \mu(x)\beta(u) \tag{3.1}$$

on Ω with boundary data $u = \phi$ on $\partial\Omega$ has solutions with compact support.

Note that we may attempt to divide the entire problem by μ to obtain

$$(f/\mu(x)) \in (L/\mu(x))u + \beta(u)$$

Now if μ is bounded away from 0 and positive, then $L/\mu(x)$ will still be an elliptic operator and we are in a position to apply the result from [3]. We wish to extend this to the case where μ can be taken to vanish and be negative. However, we need μ to be large outside of a compact set and we lose uniqueness if μ can be negative.

Precisely, we will prove the following

THEOREM 3.1. *Assume that*

$$\phi \in C_c^2(\partial\Omega) \text{ and } \beta(\phi) \in L^\infty(\partial\Omega)$$

$$f \in L^\infty_{loc} \text{ and } \gamma^- < \liminf_{|x|\to\infty} f \leq \limsup_{|x|\to\infty} f < \gamma^+$$

$$\mu \in L^\infty_{loc} \text{ and } \mu(x) \geq 1 \text{ for } x \geq R_0$$

Then all solutions $u \in H^2(\Omega)$ to the above variational problem have compact support. Moreover, if $\mu \geq 0$, then the solution exists and is unique. A key lemma in the proof will be the following maximal principle

LEMMA 3.2. *Let $u, v \in H^2(\Omega)$, assume that $\mu \geq 0$, and let $f \in Lu + \mu(x)\beta(u)$ and $g \in Lv + \mu(x)\beta(v)$ with $f \geq g$. Then if $u \geq v$ on $\partial\Omega$, $u \geq v$ a.e. on Ω.*

Proof. Consider the function $w = (v-u)_+ \in H^1(\Omega)$ (note that we can only guarantee that this function will be in $H^1(\Omega)$, not necessarily in $H^2(\Omega)$). We wish to show that $w = 0$. First we define $w^* \in H^1(\mathbb{R}^n)$ such that $w^* = w$ on Ω and $w^* = 0$ elsewhere. This function w^* will be in H^1 since w vanishes at the boundary of Ω and Ω has a smooth boundary. Additionally, extend L to all of $\mathbb{R}^n$ by setting it to be the negative Laplacian outside of Ω. Now we will show that $w^* = 0$. To do so we will show that w^* is a weak subsolution of L, i.e. $Lw^* \leq 0$ in a weak sense. Then the weak Harnack inequality implies that $w^* \leq 0$ (see [5] p.194).

So let $q \in C_0^1(\mathbb{R}^n)$, $v \geq 0$ be a test function and consider

$$\langle q, Lw^* \rangle = \int_{\mathbb{R}^n} a_{ij} D_i q D_j w^* + a_i D_i w^* q + aqw^* dx$$

where a_{ij}, a_i, a are as in [3] within Ω and $a_{ij} = \delta_{ij}, a_i = 0, a = 0$ outside of Ω. Notice further that the integral outside of Ω vanishes since w^* and Dw^* are 0 a.e. outside of Ω. So we have

$$\langle q, Lw^* \rangle = \int_{\Omega} a_{ij} D_i q D_j w^* + a_i D_i w^* q + aqw^* dx$$

Moreover, since $w^* = (v-u)_+$ within Ω, we have that Dw^* and w^* are 0 whenever $v \leq u$ (at least a.e.). Thus we have that

$$\langle q, Lw^* \rangle = \int_{\{v>u\}} a_{ij} D_i q D_j (v-u) + a_i D_i (v-u) q + aq(v-u) dx$$

We now integrate the first term by parts and use the definition of L to see that

$$\langle q, Lw^* \rangle = \int_{\{v>u\}} qL(v-u)dx + \int_{\partial\{v>u\}} q(\nu \cdot a_{ij} D_j (v-u))dS$$

where ν is the outward normal to $\partial\{v > u\}$. Note that since $\{v > u\}$ is the set $\{v - u > 0\}$, $D(v-u)$ is a non-negative multiple of the inward pointing normal. Hence, since a_{ij} is positive definite we see that the second integral above is non-positive. Thus we obtain

$$\langle q, Lw^* \rangle \leq \int_{\{v>u\}} qL(v-u)dx = \int_{\{v>u\}} q(g - f - w(x)(h-j))dx$$

where $h \in \beta(v)$ and $j \in \beta(u)$ (since $f \in Lu + \mu(x)\beta(u)$ and $g \in Lv + \mu(x)\beta(v)$). But on the set where we are integrating, $v > u$ which implies by the monotonicity of β, that $h \geq j$. Thus since $q \geq 0$ and $w \geq 0$, we get that

$$\langle q, Lw^* \rangle \leq \int_{\{v>u\}} q(g-f)dx \leq 0$$

Hence w^* is a weak subsolution of L and thus as remarked above, $w^* \leq 0$. Since we have by definition that $w^* \geq 0$, we see that $w^* = 0$ as desired. □

We proceed with the proof of the main theorem.

Proof. The argument presented in [3] applies to the present situation using the above maximum principle, provided that $\mu \geq 0$. The only difference is that the r_0 in [3] must be chosen larger than the R_0 in our statement of the theorem.

Thus it is only left to consider the case where w is not necessarily positive. First, choose $\epsilon > 0$ let $R > R_0$ so large that $\phi(x) = 0$ and $\gamma^- + \epsilon < f(x) < \gamma^+ - \epsilon$ for $|x| > R$ (this can be done for small enough epsilon by assumption) and consider the domain $\Omega^* = \Omega \cap \{|x| > R\}$. Let $u \in H^2$ be a solution to the given variational problem. We first show that $u \in L^\infty_{loc}(\Omega^*)$.

To this end, we first extend u to u^* on the entire set $\{|x| > R\}$ by setting $u^* = 0$ outside of Ω. Then again we will have $u^* \in H^1$ since u vanishes on $(\partial\Omega) \cap \{|x| > R\}$. It will suffice to show that $u^* \in L^\infty_{loc}(\{|x| > R\})$.

A computation which is essentially the same as the one performed in the above lemma implies that $Lu^*_+ \leq 0$ on $\{|x| > R\}$ (this requires that $f - \mu(x)\beta(u) \leq 0$ wherever $u > 0$ as $\mu(x) \geq 1$ and $f < \gamma^+$ for $|x| > R$).

For each point x with $|x| > R$ we choose a ball $B_\rho(x)$ about x which is still contained in $\{|x| > R\}$. We can now use again the weak harnack inequality (see [5] p.194) (as $u \in L^2$ since $u \in H^1$) and the analogous argument applied to u_- to conclude that $u \in L^\infty(B_\rho(x))$.

Now consider the domain $\Omega^* = \Omega \cap \{|x| > R'\}$ where $R' > R$. First we note that u is bounded on $\partial\Omega^*$. This follows since outside of a radius R, $u = 0$ on $\partial\Omega$ and on $\partial\{|x| > R'\}$, u is locally bounded and thus bounded since $\{|x| = R'\}$ is a compact set.

Now we proceed to construct a function $v \in C^2_c(\Omega^*)$ such that $g \in Lv + \mu(x)\beta(v)$ with $g \geq f$. Thus by the above maximum principle applied to Ω^*, $u \leq v$. Analogously we can construct $v \in C^2_c(\Omega^*)$ such that $u \geq v$. This will imply that u has compact support.

In particular, we construct v of the form

$$v(x) = \begin{cases} \frac{\lambda}{2}(|x| - R)^2 & \text{for } R' \leq |x| < R \\ 0 & \text{for } R \leq |x| \end{cases}$$

where λ and R are to be determined. Simple computations which are given in [3] imply that

$$Lv \geq -\lambda K' - \lambda K(R - |x|) + \frac{1}{2}\delta\lambda(R - |x|)^2$$

where $K' = \sup_\Omega \sum_i a_{ii}$, $K^2 = \sum_i \|a_i\|^2_{L^\infty(\Omega)}$ and $\delta > 0$ is such that $a \geq \delta$ (this is one of the assumptions in [3]).

We can now choose λ small enough, so that the above expression is greater than $-\epsilon$ uniformly in R. This is because the expression is a quadratic in $(|x| - R)$ with positive leading coefficient. Thus there is a minimal value that can be attained which is independent of R.

We then simply choose R large enough, so that $v \geq u$ on $\{|x| = R'\}$. This can be done since $u(x) = \frac{\lambda}{2}(R - R')^2$ on $\{|x| = R'\}$.

Now choose g such that $g \in Lv + \mu(x)\beta(v)$ where $v > 0$ and $g = Lv + \mu(x)\gamma^+$ where $v = 0$ ($v \geq 0$ so this covers all cases). Then by definition of γ^+, $g \in Lv + \mu(x)\beta(v)$. Note that since v is a monotone graph, we have $g \geq Lv + \mu(x)\gamma^+$ everywhere. Now $Lv \geq -\epsilon$, $f < \gamma^+ - \epsilon$, and $\mu(x) \geq 1$ on Ω^* imply that $g \geq f$ on Ω^*. Combined with $v \geq u$ on $\{|x| = R'\}$, this implies that $v \geq u$ as desired.

The analogous argument with a subsolution concludes the proof that u must be compactly supported. $\square$

Unfortunately, we don't obtain a bound on the support which is independent of u. In particular, the size of the support depends upon $\|u\|_{L^\infty(\partial\Omega^*)}$ which in turn can be controlled by the L^p norm of u ($p > 1$, by the weak Harnack inequality). Thus we cannot, in general, reduce the existence to a bounded domain. However, if the variational problem arises in the context of a minimization problem which allows one to control the L^p norm of the solution, then existence can be reduced to a bounded domain.

Finally, note that uniqueness fails if μ is allowed to be negative. Indeed, take any $u \in C_c^\infty$, $u \geq 0$ and define $\mu(x) = \Delta u$ if x is in the support of u and $\mu(x) = 1$ otherwise. Additionally, let β be the subdifferential of $|\cdot|$. Then it is easy to see that $0 \in -\Delta u + \mu(x)\beta(u)$. However, we also clearly have $0 \in -\Delta 0 + \mu(x)\beta(0)$. Hence the solution isn't unique in this case.

4. L^1 Constrained Parabolic Problems. We now consider the applications of L^1-constrained problems to the class of parabolic PDEs. In particular, we consider the following variational obstacle problem for the heat equation given in [4]:

$$(u_t - \Delta u)(v - u) \geq f(v - u) \text{ a.e. } \text{ for } x \in \mathbb{R}^n, 0 < t < T, \tag{4.1}$$

for any non-negative measurable function v, and look for the solutions u with

$$\begin{aligned} u \geq 0 \text{ for } x \in \mathbb{R}^n, 0 < t < T, \\ u(x,0) = u_0(x). \end{aligned}$$

The existence and uniqueness for this problem is given in the above mentioned article. Furthermore, they prove theorems regarding the compact support of the solution (Theorem 3.1. and Theorem 3.2. in [4]) under the uniform negativity constraint on f, namely that there exist a positive real number ν, such that

$$f \leq -\nu. \tag{4.2}$$

For sufficient regularity assumptions, they also require

$$\begin{aligned} f \in L^\infty(\mathbb{R}^n \times (0,T)), \\ f_t \in L^\infty(\mathbb{R}^n \times (0,T)). \end{aligned} \tag{4.3}$$

Theorem 4.1 (Theorem 3.1. in [4]). *Suppose (4.2) and (4.3) are satisfied. Then, there is a positive number T_0 such that $u(x,t) \equiv 0$ for $t \geq T_0$.*

Theorem 4.2 (Theorem 3.2. in [4]). *Suppose (4.2) and (4.3) are satisfied. Suppose further that u_0 has compact support. Then, there is a positive number R_0 such that $u(x,t) = 0$ if $|x| > R_0$.* In this section, we show that we can relax the condition (4.2) so that it only holds away from a ball centered at the origin. Namely, we only require

$$f(x,t) \leq -\nu \text{ for } |x| > K, \tag{4.4}$$

along with non-strict negativity condition on f:

$$f(x,t) \leq 0. \tag{4.5}$$

THEOREM 4.3. *Suppose (4.3), (4.4), and (4.5) are satisfied. Then, there is a positive number T_0 such that $u(x,t) \equiv 0$ for $t \geq T_0$.*

THEOREM 4.4. *Suppose (4.3) and (4.4) are satisfied. Suppose further that u_0 has compact support. Then, there is a positive number R_0 such that $u(x,t) = 0$ if $|x| > R_0$.*

COROLLARY 1. *Let u satisfy the following PDE with a compactly supported initial condition u_0,*

$$u_t - \Delta u + \chi_{|x|>K}(x)\mu p(u) = f,$$

where p is the sub-differential of the absolute value function. Suppose further that $f(x,t) \to 0$, as $(x,t) \to \infty$. Then, u is compactly supported on the (x,t)-space.

The proofs of Theorems 4.3 and 4.4 rely on the maximum principle applied to the family of functions β_ϵ, $u_{R,\epsilon}$ defined in [4]. We merely repeat the definitions of these functions here. β_ϵ is a $C^\infty(\mathbb{R})$ function satisfying:

$$\begin{aligned} \beta_\epsilon(x) &= 0 \text{ for } x > 0, \\ \lim_{\epsilon\to 0} \beta_\epsilon(x) &= -\infty \text{ for } x < 0, \\ \beta_\epsilon'(x) &> 0 \text{ for } x > 0. \end{aligned}$$

Then, for a given initial data u_0 and a source term f, the functions $u_{R,\epsilon}$ are the solution to the following problem:

$$\begin{aligned} u_t - \Delta u + \beta_\epsilon(u) &= f, \text{ for } |x| < R, 0 < t < T, \\ u(x,0) &= u_0(x), \text{ for } |x| < R, \\ u(x,t) &= 0, \text{ for } |x| = R, t > 0. \end{aligned}$$

Proof. [Proof of Theorem 4.3] We follow a similar construction as in [4]. From Theorem 2.1 in [4], there exists $M > 0$ such that $u_{R,\epsilon}(x,1) \leq M$. Let

$$v(x,t) = \begin{cases} M - \nu(t-1) \text{ for } |x| > K, \\ M - \nu(t-1) + \nu(K^2 - |x|^2)/2d \text{ for } |x| \leq K \end{cases}$$

and let $w = \max(0, v)$. Then, $w(x,t) = 0$ for $t > T_0 := 1 + M/\nu + K^2/2d$. Furthermore,

$$w_t - \Delta w = \begin{cases} 0 \text{ if } |x| < K \\ -\nu \text{ if } |x| > K, 1 \leq t \leq T_0 \\ 0 \text{ if } |x| > K, t > T_0 \end{cases}$$

In particular, w satisfies

$$w_t - \Delta w + \beta_\epsilon(w) \geq f.$$

Therefore, by the maximum principle applied on $w - u_{R,\epsilon}$, we conclude that $u_{R,\epsilon}(x,T_0) \leq 0$. Letting, $R \to \infty$, and $\epsilon \to 0$, we obtain $u(x,T_0) = 0$. Hence, $u \equiv 0$ for $t \geq T_0$. ☐

Proof. [Proof of Theorem 4.4] Let ρ denote the radius of the support of u_0, as in the original proof. The only difference is that, we proceed with $\tilde{\rho}$ such that $\tilde{\rho} >$

$\max(\rho, K)$, and construct the comparison function in maximum principle argument in a slightly different way.

From Theorem 2.1 in [4], we know the existence of $N > 0$ such that

$$|u_{R,\epsilon}(t,x)| \leq N \text{ for } x \in \mathbb{R}^n, \tilde{\rho} \leq |x| \leq R, 0 < t < T_0.$$

For arbitrary positive constants μ, R_0 and for $r = |x|$, let w solve the heat equation for $|x| < \tilde{\rho}$ with source term f:

$$\begin{aligned} w_t - \Delta w &= f \text{ for } |x| < \tilde{\rho} \\ w(x,t) &= \mu(R_0 - \tilde{\rho})^2 \text{ for } |x| = \tilde{\rho}, t > 0 \\ w(x,0) &= \mu(R_0 - r)^2 \text{ for } |x| < \tilde{\rho}. \end{aligned}$$

We choose parameters μ, R_0 such that $2\mu \leq \nu$, $\mu(R_0 - \tilde{\rho})^2 \geq N$, so that

$$\begin{aligned} w_t - \Delta w + \beta_\epsilon(w) &\geq -\nu \text{ if } |x| > \tilde{\rho} \\ w &\geq N \text{ if } |x| = \tilde{\rho}. \end{aligned}$$

Now, applying the maximum principle to $w - u_{R,\epsilon}$, we conclude that $w - u_{R,\epsilon} \geq 0$ if $\tilde{\rho} < |x| < R, 0 < t < T_0$. Therefore,

$$u_{R,\epsilon}(x,t) = 0 \text{ if } R_0 \leq |x| \leq R, 0 < t < T_0.$$

Letting $R \to \infty$, we obtain the spatial compactness of u, as desired. ☐

Proof. [Proof of Corollary 1] Observe that u_+ is a solution to the variational inequality (4.1) when f is replaced by $f - \chi_{|x|>K}(x)\mu$, so that the RHS of the variational inequality is strictly negative for large values of x and t. Now, by Theorem 4.4, u_+ is compactly supported in the space variable x.

Suppose $|f| < \frac{\mu}{2}$ for $t > T$. Then, Theorem 4.3 is applicable for u_+ provided that we replace the initial time with $t = T$ instead of $t = 0$, so that u_+ has compact support in t.

Repeating the above arguments for u_-, we conclude that u is compactly supported, as desired. ☐

5. Numerical Results for L^1 Constrained Elliptic Problems. We numerically investigated solutions to the L^1 constrained elliptic problem

$$\arg\min_{u \in H^1} \|\nabla u\|_2^2 - 2\langle f, u\rangle + \|w(x)u\|_1 \tag{5.1}$$

Specifically, we solved

$$\arg\min_{u \in H_0^1(\Omega)} \|\nabla u\|_2^2 - 2\langle f, u\rangle + \|w(x)u\|_1 \tag{5.2}$$

where Ω is the unit cube $[0,1]^2$. By making $w(x)$ large enough in relation to f, we can, by the above arguments force the support of the solution to lie in Ω, thus we obtain a solution to the first problem by solving the second.

To solve the above problem we used a splitting scheme in combination with ADMM. Specifically, we rewrote the problem as

$$\arg\min_{u,v \in H_0^1(\Omega)} \|\nabla v\|_2^2 - 2\langle f, u\rangle + \|w(x)u\|_1 \tag{5.3}$$

subject to the constraint $u = v$. which we then solved using ADMM. We ended up with the following iteration

$$\begin{aligned}
v_{n+1} &= \underset{v \in H_0^1(\Omega)}{\arg\min} \|\nabla v\|_2^2 + \frac{\mu}{2}\|v - u_n - \lambda_n\|_2^2 \\
u_{n+1} &= \underset{u \in H_0^1(\Omega)}{\arg\min} \|w(x)u\|_1 - 2\langle f, u\rangle + \frac{\mu}{2}\|v_{n+1} - u - \lambda_n\|_2^2 \\
\lambda_{n+1} &= \lambda_n + (u_{n+1} - v_{n+1})
\end{aligned}$$

The first of these problems can be solved by solving the Poisson equation. The second minimizer is given in closed form by a pointwise shrink operator.

The numerical results we obtained were as follows. In the first example, we let f and w be as below

(a) f (b) w

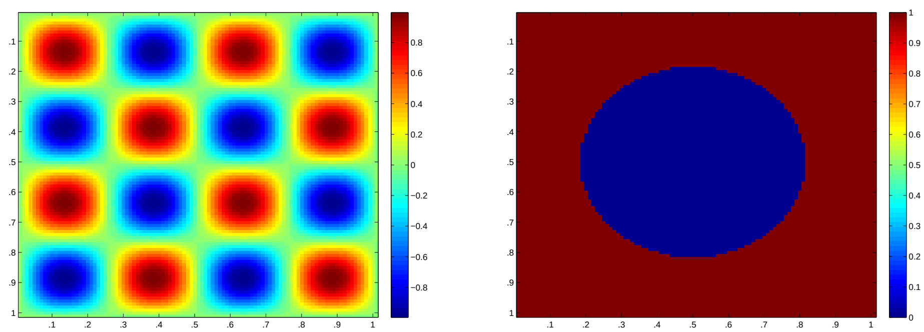

Figure 5.1: Plots of f and the weight w

(a) Weight is $\frac{w}{2}$ (b) Weight is w

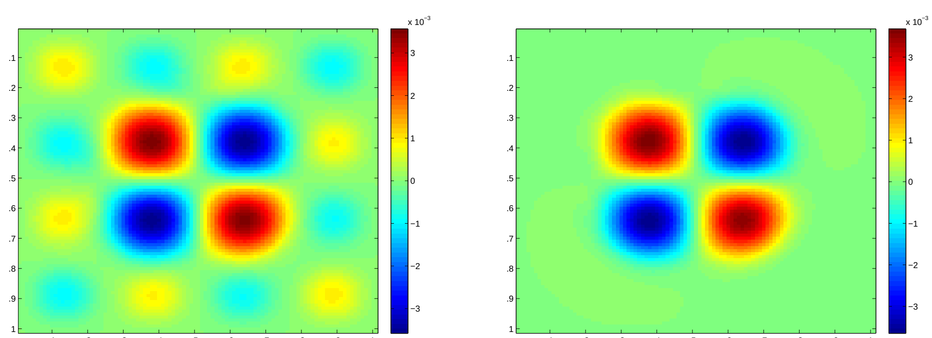

Figure 5.2: Plots of the solution with weights $\frac{w}{2}$ and w

Figure 5.2 shows the results we obtained with two different scalings of the L^1 weight. Notice that $f = \sin(4\pi x)\sin(4\pi y)$ is an eigenfunction for the Dirichlet laplacian, so that the solution to the elliptic problem without the L^1 term is a scaled version of f (namely $\frac{1}{32\pi^2} f$).

We see that although we obtain compact support, the solution is very close to the solution of the laplacian within the circle, where there is no L^1 term.

Now we let f be a function which is not an eigenfunction and use the same w as before. The function f and the solution to $\Delta u = f$ are given below.

(a) f (b) Solution to $\Delta u = f$

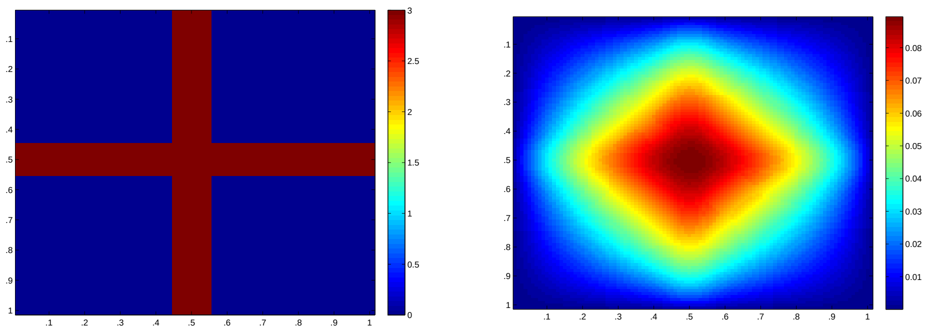

Figure 5.3: Plots of new function f and the solution to $\Delta u = f$

In this case, we obtain for two different scalings of the L1 term:

(a) Weight is w (b) Weight is $3w$

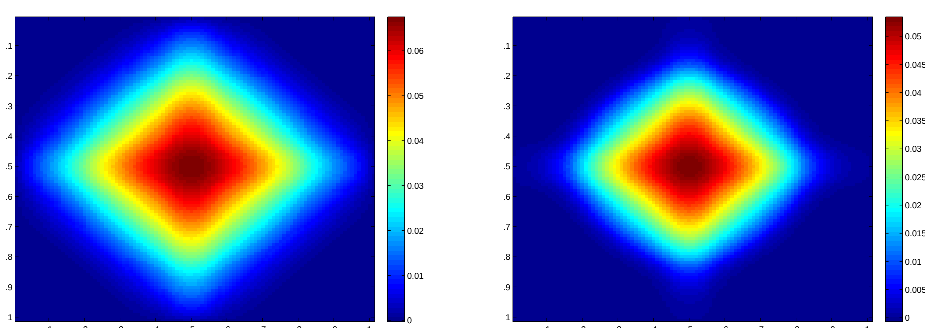

Figure 5.4: Plots of the solution with weights w and $3w$

Again, we see that although we obtain compact support, the solution is close to the solution of the laplacian within the circle, where there is no L^1 term.

Thus we propose that the solutions to such L^1 constrained elliptic problems could be used as C_0^1 local approximations to the unconstrained elliptic problem.

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