Math 122
Introduction to Statistics
One Proportion Z Test

NOTATION

In these sections, we will use variables such as H_0 and H_1 to represent statements (sentences). This may at first seem odd, but it is really just short-hand. Actually, the first historic use of variables was by Aristotle to represent sentences rather than the traditional use you may be familiar with to represent numbers.

HYPOTHESIS TESTING IDEA

The basis for hypothesis testing is the rare event rule:

Rare Event Rule: If an assumption H_0 implies that the probability of an observed event is exceptionally small, then that assumption is probably false.

Suppose that an assumption H_0 implies that the probability of a certain event is .00001. If we actually observe that event, then one of two things has happened. Either H_0 is true and we have observed something very, very unlikely, or H_0 is false. It is more likely that H_0 is false.

In hypothesis testing, we begin by defining two statements H_0 (called the null hypothesis) and its opposite H_1 (called the alternative hypothesis) based on the claim being tested. In some sense, at the end of the test we will believe either H_0 or H_1. We make observations or collect data relevant to the claim being tested. We then assume H_0 and use this assumption to calculate the probability of seeing data as extreme as we have observed. This probability is the P-value P. If P is exceptionally small, then the Rare Event Rule says that it is unlikely that H_0 is true. In this case, we will reject H_0 in favor of H_1. On the other hand, if P is large, then H_0 might be true so we do not reject it.

NORMAL APPROXIMATION TO A BINOMIAL DISTRIBUTION

Recall that we have a version of the Central Limit Theorem which relates to proportions.

Central Limit Theorem Variant: Suppose that x is the number of successes in simple random samples of size n selected from a population in which the proportion of success is p. If $np \geq 5$ and $n(1-p) \geq 5$, then the sample successes x are approximately normally distributed with mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$.

Example: Suppose that x is a binomial random variable with $n = 100$ and $p = 0.25$. Find $P(x \leq 30)$ using a binomial distribution and using a normal approximation.

Solution: Using a binomial distribution:

$$P(x \leq 30) = binomcdf(100, 0.25, 30) = 0.8962$$

To use the normal approximation, we must convert to a standard normal using

$$\mu = np = 100 \cdot 0.25 = 25 \text{ and } \sigma = \sqrt{np(1-p)} = \sqrt{100 \cdot 0.25 \cdot 0.75} = 4.3301.$$

Then

$$\begin{aligned} P(x \leq 30) &= P(z \leq \frac{30-25}{4.3301}) \\ &= P(z \leq 1.1547) \\ &= normalcdf(-9, 1.1547) \qquad &= 0.8759 \end{aligned}$$

Notice that the two solutions are not identical, but are close. The larger n is, the closer the approximation should be.

Example: Suppose that x is a binomial random variable with $n = 1000$ and $p = 0.25$. Find $P(x \leq 260)$ using a binomial distribution and using a normal approximation.

Solution: Using a binomial distribution:

$$P(x \leq 300) = binomcdf(1000, 0.25, 260) = 0.7791$$

To use the normal approximation, we must convert to a standard normal using

$$\mu = np = 1000 \cdot 0.25 = 250 \text{ and } \sigma = \sqrt{np(1-p)} = \sqrt{1000 \cdot 0.25 \cdot 0.75} = 13.6931.$$

Then

$$\begin{aligned} P(x \leq 260) &= P(z \leq \frac{260-250}{13.6931}) \\ &= P(z \leq 0.7303) \\ &= normalcdf(-9, 0.7303) \qquad &= 0.7674 \end{aligned}$$

Notice that the two solutions are not identical, but are close. The larger n is, the closer the approximation should be.

HYPOTHESIS TEST WITH NORMAL APPROXIMATION

We can perfom hypothesis tests for proportions using a normal approximation rather than a binomial distribution. This will give only approximate answers, but it has the advantage that it is much simpler *when doing things by hand*. For that reason, hypothesis testing about proportions has traditionally been done using the normal approximation to the binomial distribution.

Binomial Tests: We can use the normal approximation to the binomial distribution to perform hypothesis tests. In the days before calculators, this was important because computations with the normal distribution are much easier by hand than computations with the binomial. We demonstrate this with an example below from the Binomial Test section.

Example: Bob believes that a coin he has lands on T more often than it lands on H. He flips the coin 103 times and sees an H 49 times. Use this information to test the claim that the probability that this coin lands on H is less than one half.

1. **Parameters:** Let p be the probability that this coin lands on H. Let $n = 103$, and let x be the number of H's that appear when this coin is flipped 103 times.

2. **Symbolic Claim:** The claim is $p < 0.5$.

3. **Opposite of Claim:** The opposite of the claim is $p \geq 0.5$.

4. **H_0 and H_1:** H_0 is $p \geq 0.5$ and H_1 is $p < 0.5$.

5. **Significance Level:** We will default to a significance level of $\alpha = 0.05$.

6. **Distribution and Test Statistic:** Each flip of the coin is an independent trial that ends in either H or T. We are assuming that the probability of an H on each trial is $p = 0.5$. The variable x is the number of H's that appear in 103 flips. Thus, x has a binomial distribution. Our observed value of x is $O = 49$.

   We will treat x as a normal random variable with mean and standard deviation
   $$\mu = np = 103 \cdot 0.5 = 51.5 \text{ and } \sigma = \sqrt{np(1-p)} = \sqrt{103 \cdot 0.5 \cdot 0.5} = 5.0744.$$
   Our *test statistic* is the z-score of our observed value of x. This is
   $$z = \frac{x - \mu}{\sigma} = \frac{49 - 51.5}{5.0744} = -0.4927.$$

7. **P-value:** Since H_1 involves $<$, this is a left-tailed test. Then
   $$P = P(z < -0.4927) = \mathit{normalcdf}(-9, -0.4927) = 0.3111.$$

8. **Formal Conclusion:** Since $P > \alpha$, we do not reject H_0 (and we do not support H_1).

9. **Final Conclusion:** Since our claim is the same as H_1, we first rephrase our formal conclusion to refer to H_1. Since we are not rejecting H_0, we are not supporting H_1. This means that we are not supporting our claim. Our conclusion is:

   > There is not enough sample evidence to support the claim that the probability that this coin lands on H is less than one half.

Interpretation: The observation of 49 out of 103 flips ending in H is not extreme enough to conclude that the coin is not fair. The coin is probably fair.

Performing a hypothesis test here with a binomial distribution (instead of a normal distribution) gives $P = 0.3468$. Notice that the P-values are close but not equal.