Math 122
Introduction to Statistics
Poisson Distributions

Poisson Random Variable: A random variable x has a Poisson distribution if x is the number of occurrences of some rare event over some interval of time, distance, area, or volume.

Examples:

1. The number of major earthquakes during a year.
2. The number of births at a hospital in a year.
3. The number of emails received in an hour.
4. The number of automobile accidents on a given mile of road.
5. The number of bug pieces in a tablespoon of peanut butter.
6. The number of dandelions on a square foot of land.

NOTATION AND FORMULAS

- x =number of events.
- μ =average number of events.
- σ =standard deviation= $\sqrt{\mu}$.
- $P(x)$ =probability of exactly x successes= $\frac{\mu^x e^{-\mu}}{x!}$.

WE WILL USE TECHNOLOGY TO DO ALL OF THESE CALCULATIONS. WE WILL NOT USE THIS FORMULA!

To find	Use
$P(x = a)$	$poissonpdf(\mu, a)$
$P(x \leq a)$	$poissoncdf(\mu, a)$
$P(x < a)$	$poissoncdf(\mu, a - 1)$
$P(x > a)$	$1 - poissoncdf(\mu, a)$
$P(x \geq a)$	$1 - poissoncdf(\mu, a - 1)$

Example: There were 93 major earthquakes in the world over the past 100 years.

1. What is the probability that in the next year there will be EXACTLY 2 major global earthquakes?
2. What is the probability that in the next year there will be NO MORE THAN 2 major global earthquakes?

3. What is the probability that in the next year there will be AT LEAST 2 major global earthquakes?
4. What are the usual numbers of earthquakes in a year according to the Range Rule of Thumb?

Solution: If we let x be the number of major earthquakes in a year, then x should have a Poisson distribution. The mean for this random variable is

$$\mu = \frac{93}{100} = 0.93.$$

1. $P(x = 2) = \mathit{poissonpdf}(0.93, 2) = 0.1706.$
2. $P(x \leq 2) = \mathit{poissoncdf}(0.93, 2) = 0.9321$
3. $P(x \geq 2) = 1 - P(x \leq 1) = 1 - \mathit{poissoncdf}(0.93, 1) = 0.2385.$
4. To apply the Range Rule of Thumb, we need the standard deviation:

   $$\sigma = \sqrt{\mu} = \sqrt{0.93} = 0.9644.$$

   The minimum usual value is

   $$\mu - 2\sigma = 0.93 - 2 \cdot 0.9644 = -0.9988.$$

   The maximum usual value is

   $$\mu + 2\sigma = 0.93 - 2 \cdot 0.9644 = 2.8588.$$

   Numbers of earthquakes between -0.9988 and 2.8588 are not unusual. The usual numbers of earthquakes in a year are 0, 1, and 2. More than this is unusual.

Example: 120 children are born each year at Seward Memorial Hospital.

1. What is the probability on a given day that there are no children born at Seward Memorial?
2. What is the probability on a given day that there are more than 2 children born at Seward Memorial. Is this unusual (by the 5% rule)?
3. Seward Memorial has two birthing rooms. Is this adequate?

Solution: Let x be the number of children born on any day at Seward Memorial. Then x has a Poisson distribution with

$$\mu = \frac{120}{365} = 0.3288.$$

1. $P(x = 0) = \mathit{poissonpdf}(0.3288, 0) = 0.7198$

2. $P(x > 2) = 1 - P(x \leq 2) = 1 - \mathit{poissoncdf}(0.3288, 2) = 0.0046$. This is unusual.

3. Two rooms is adequate 99.54% of the time.

Example: 250 children are born each year at a certain hospital. How many birthing rooms should they have so that there are enough birthing rooms 99.5% of the time?

Solution: The number of births per day has a Poisson distribution with $\mu = \frac{250}{360} = 0.6849$. We simply use trial and error to find a number of births so that the probability of having that number or less is at least 99.5%.

$$P(x \leq 0) = \mathit{poissoncdf}(0.6849, 0) = 0.5041$$
$$P(x \leq 1) = \mathit{poissoncdf}(0.6849, 1) = 0.8494$$
$$P(x \leq 2) = \mathit{poissoncdf}(0.6849, 2) = 0.9677$$
$$P(x \leq 3) = \mathit{poissoncdf}(0.6849, 3) = 0.9946$$
$$P(x \leq 4) = \mathit{poissoncdf}(0.6849, 4) = 0.9993$$

Four rooms will be enough.

BINOMIAL VS. POISSON

A Binomial Distribution with $n \geq 100$ and $np \leq 10$ is close to the Poisson Distribution with $\mu = np$.

Example: In the Illinois Pick3 Lottery, you select a sequence of 3 digits (0-9). There is one winning sequence. Suppose that you play the game every day for one year. What is the probability that you win at least twice?

Solution: Each time you play, there are 1000 possible choices and one winning choice, so the probability of winning each time you play is $p = \frac{1}{1000} = 0.001$.

Binomial: Let x be the number of times that you win when you play this game 365 times. Then x has a binomial distribution with $n = 365$ and $p = 0.001$.

$$P(x \geq 2) = 1 - P(x \leq 1) = 1 - \mathit{binomcdf}(365, 0.001, 1) = 0.052341.$$

Poisson: Since $n \geq 100$ and $np = 0.365 \leq 10$, we can assume that x has a Poisson distribution with $\mu = np = 0.365$.

$$P(x \geq 2) = 1 - P(x \leq 1) = 1 - \mathit{poissoncdf}(0.365, 1) = 0.052421.$$

Notice that the two solutions agree to three decimal places.