

Geometry Theorem Sequence

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1 Incidence Geometry

1.1 Incidence

Primitives. The undefined terms for incidence geometry are *point* and *line*.

Axiom. Lines are sets of points.

Definition. If P is a point and ℓ is a line then each of these means that $P \in \ell$:

- P is on ℓ .
- P lies on ℓ .
- ℓ is on P .
- ℓ lies on P .
- ℓ passes through P .
- P is incident with ℓ .
- ℓ is incident with P .
- ℓ contains P .

Definition. We call the set of all points under consideration *the plane* and denote it \mathbb{P} .

Incidence Axiom 1 (IA1). If P and Q are distinct points, then there is at least one line ℓ so that $P \in \ell$ and $Q \in \ell$.

Incidence Axiom 2 (IA2). If P and Q are distinct points, then there is at most one line ℓ so that $P \in \ell$ and $Q \in \ell$.

Incidence Axiom 3 (IA3). If ℓ is a line, then there are distinct points P and Q so that $P \in \ell$ and $Q \in \ell$.

Incidence Axiom 4 (IA4). There exist three distinct points P , Q , and R so that there is no line ℓ with $P \in \ell$ and $Q \in \ell$ and $R \in \ell$.

Incidence Axioms 1 and 2 guarantee that any two distinct points are contained in exactly one line. We give a name to this line.

Definition. If P and Q are distinct points then the unique line that contains both P and Q is denoted \overleftrightarrow{PQ} .

Suppose that P and Q are distinct points. It follows immediately from this definition that:

- If ℓ is a line that contains P and Q , then $\ell = \overleftrightarrow{PQ}$.
- $\overleftrightarrow{QP} = \overleftrightarrow{PQ}$.
- If ℓ is a line and $P \in \ell$ and $Q \in \ell$, then $\ell = \overleftrightarrow{PQ}$.

Exercise 1.1. Find a model which satisfies Incidence Axioms 1, 2, and 3 but not 4.

Exercise 1.2. Find a model which satisfies Incidence Axioms 1, 2, and 4 but not 3.

Exercise 1.3. Find a model which satisfies Incidence Axioms 1, 3, and 4 but not 2.

Exercise 1.4. Find a model which satisfies Incidence Axioms 2, 3, and 4 but not 1.

These three exercises establish that the incidence axioms are *independent* of each other. No one of them can be proven from the other three. If one of them were provable from the other three, then any model which satisfies the three would also have to satisfy the one.

Exercise 1.5. Find a model of IA1, IA2, IA3, and IA4 which has three or fewer points.

It follows from this exercise that you cannot prove the existence of four points. If you could, the model you found could not exist.

Proposition 1.6. *The following are true.*

1. *There is a point.*
2. *There is a line.*

Proposition 1.7. *If ℓ is a line, then there exist points P and Q so that $\ell = \overleftrightarrow{PQ}$.*

Proposition 1.8. *If ℓ and m are distinct lines, then there is at most one point P which is on both ℓ and m .*

Proposition 1.9 (IA4'). *There is a line ℓ and a point P so that $P \notin \ell$.*

Exercise 1.10. Assume IA1, IA2, IA3, and IA4' and use these assumptions to prove IA4.

This last exercise demonstrates that in the presence of the other Incidence Axioms, IA4 and IA4' are *equivalent*.

Definition. Points P_1, P_2, \dots, P_n are *collinear* if there is a line ℓ so that $P_i \in \ell$ for all i . Points which are not collinear are *noncollinear*. Lines $\ell_1, \ell_2, \dots, \ell_n$ are *concurrent* if there is a point P so that $P \in \ell_i$ for all i . Lines which are not concurrent are *nonconcurrent*.

Proposition 1.11. *There are three noncollinear points.*

Proposition 1.12. *There are three nonconcurrent lines.*

Proposition 1.13. *If ℓ is any line, then there exists at least one point P so that $P \notin \ell$.*

Proposition 1.14. *If P is any point, then there exist at least two lines ℓ and m so that $P \in \ell$ and $P \in m$.*

Definition. If ℓ and m are lines, and if P is a point so that $P \in \ell$ and $P \in m$, then ℓ and m *intersect* at P .

Proposition 1.15. *If ℓ is any line, then there exist lines m and n so that ℓ , m , and n are distinct and both m and n intersect ℓ .*

Proposition 1.16. *Suppose that A , B , and C are distinct points and that P is a point so that $P \in \overleftrightarrow{AB}$, $P \in \overleftrightarrow{AC}$, and $P \in \overleftrightarrow{BC}$. Then A , B , and C are collinear.*

Proposition 1.17. *If P is any point, then there exists at least one line ℓ such that $P \notin \ell$.*

Proposition 1.18. *If P is any point, then there exist points Q and R so that P , Q , and R are noncollinear.*

Proposition 1.19. *If P and Q are two distinct points, then there exists a point R so that P , Q , and R are noncollinear.*

2 Incidence and Betweenness Geometry

2.1 Betweenness

Primitives. We add a new undefined term, *betweenness*.

Axiom. Betweenness is a ternary (three-place) relation on the set of all points. To denote that points P , Q , and R are in the betweenness relation, we use the notation $P * Q * R$.

Definition. If P , Q , and R are points and the relation $P * Q * R$ holds, we say that Q is *between* P and R .

Betweenness Axiom 1 (BA1). If A , B , and C are points so that $A * B * C$, then A , B , and C are distinct collinear points and $C * B * A$.

Betweenness Axiom 2 (BA2). If B and D are distinct points, then there are points A , C , and E so that $A * B * D$, $B * C * D$, and $B * D * E$.

Betweenness Axiom 3 (BA3). If A , B , and C are distinct collinear points, then exactly one of these holds: $A * B * C$ or $A * C * B$ or $B * A * C$.

Definition. Suppose that A and B are distinct points. The *line segment* \overline{AB} is defined as

$$\overline{AB} = \{P | A * P * B\} \cup \{A, B\}.$$

Definition. If $P = Q$ is any point, then $\overleftrightarrow{PQ} = \{P\} = \{Q\}$. Note that this IS NOT a line. Similarly, if $P = Q$ is any point, then $\overline{PQ} = \{P\} = \{Q\}$. Note that this IS NOT a line segment.

Definition. Suppose that A and B are distinct points. The *ray* \overrightarrow{AB} is defined as

$$\overrightarrow{AB} = \overline{AB} \cup \{P | A * B * P\}.$$

Proposition 2.1. If A , B , and P are points so that $P * A * B$ or $A * B * P$ then P is not in \overline{AB} .

Proposition 2.2. If A , B , and P are points so that $P * A * B$ then P is not in \overrightarrow{AB} .

Proposition 2.3. If A and B are distinct points, then $\overline{AB} = \overline{BA}$.

Proposition 2.4. If A and B are distinct points, then $\overrightarrow{AB} \neq \overrightarrow{BA}$.

Proposition 2.5. If A and B are distinct points, then $\overline{AB} \subseteq \overleftrightarrow{AB}$.

Proposition 2.6. If A and B are distinct points, then $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.

Proposition 2.7. If A and B are distinct points, then $\overleftrightarrow{AB} \subseteq \overleftrightarrow{AB}$.

Proposition 2.8. If A and B are distinct points then $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.

Proposition 2.9. If A and B are distinct points then $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

2.2 Plane Separation

Primitives. We add a new undefined term, *half-plane*.

Definition. A set of points S is *convex* if whenever A and C are in S and $A * B * C$ then B is in S also.

Plane Separation Axiom 1. Suppose ℓ is a line. There are two sets H_1 and H_2 called the half-planes bounded by ℓ so that

1. $\mathbb{P} = H_1 \cup \ell \cup H_2$.
2. $H_1 \neq \emptyset$ and $H_2 \neq \emptyset$.
3. $H_1 \cap H_2 = H_1 \cap \ell = H_2 \cap \ell = \emptyset$.
4. H_1 and H_2 are convex.
5. If $P \in H_1$ and $Q \in H_2$ then $\overline{PQ} \cap \ell \neq \emptyset$.

Proposition 2.10. If ℓ is a line and A is a point not on ℓ , then there is exactly one half-plane bounded by ℓ which contains A .

Definition. If ℓ is a line and A is a point not on ℓ , then the half-plane bounded by ℓ containing A is denoted $H_\ell(A)$. The two half-planes bounded by ℓ are called the *sides* of ℓ . If H is a half-plane bounded by ℓ , we will sometimes express $A \in H$ as “ A is *in* H ” and sometimes as “ A is *on* H .” We will call the two distinct sides of ℓ the *opposite sides* of ℓ . If A and B are two points not on ℓ and $H_\ell(A) = H_\ell(B)$, then we say that A and B are *on the same side* of ℓ . If $H_\ell(A) \neq H_\ell(B)$, then we say that A and B are *on opposite sides* of ℓ .

Proposition 2.11. If A and B are distinct points not on a line ℓ , then A and B are on the same side of ℓ if and only if $\overline{AB} \cap \ell = \emptyset$. A and B are on opposite sides of ℓ if and only if $\overline{AB} \cap \ell \neq \emptyset$.

Theorem 2.12 (Pasch’s Theorem). Suppose that A , B , and C are distinct noncollinear points and that ℓ is a line intersecting \overline{AB} .

- ℓ intersects \overline{AC} or \overline{BC} .
- If ℓ does not contain A , B , or C , then ℓ intersects exactly one of \overline{AC} and \overline{BC} .

This theorem is proven through the following claimable lemmas.

Lemma 2.13 (Part of the proof of 2.12). Suppose that A , B , and C are distinct noncollinear points and that ℓ is a line intersecting \overline{AB} . Then ℓ intersects \overline{AC} or \overline{BC} .

Hint: A and B are on opposite sides of ℓ .

Lemma 2.14 (Part of the proof of 2.12). Suppose that A , B , and C are distinct noncollinear points and that ℓ is a line intersecting \overline{AB} . If ℓ does not contain A , B , or C , then ℓ intersects exactly one of \overline{AC} and \overline{BC} .

Hint: Suppose that ℓ intersects both \overline{AB} and \overline{AC} . Argue that B and C are on the same side of ℓ .

Proposition 2.15. Suppose A , B , and C are points not on a line ℓ . If A and B are on opposite sides of ℓ but B and C are on the same side of ℓ , then A and C are on opposite sides of ℓ .

Proposition 2.16 (Ray Theorem). Suppose that A is a point on a line ℓ and that B is a point not on ℓ . Every point on \overrightarrow{AB} other than A is on the same side of ℓ as B .

2.3 Betweenness of Four Points

Definition. Let A, B, C , and D be points. The relation $A - B - C - D$ holds if $A * B * C$, $A * B * D$, $A * C * D$, and $B * C * D$.

Proposition 2.17. *Let A, B, C , and D be points. If $A - B - C - D$, then A, B, C , and D are distinct collinear points.*

Proposition 2.18. *Let A, B, C , and D be points. If $A - B - C - D$, then $D - C - B - A$.*

Proposition 2.19 (Four Point Squeezing 1). *Let A, B, C , and D be points. If $A * B * C$ and $A * C * D$ then $A - B - C - D$.*

Hint:

1. Show that all four points lie on the line \overleftrightarrow{AC} .
2. Find some point E not on \overleftrightarrow{AC} , and let $\ell = \overleftrightarrow{EC}$.
3. What is the point of intersection of ℓ and \overleftrightarrow{AC} ?
4. Are A and D on the same side of ℓ or different sides?
5. Are A and B on the same side of ℓ or different sides?
6. Are B and D on the same side of ℓ or different sides?

Proposition 2.20 (Four Point Squeezing 2). *Let A, B, C , and D be points. If $A * B * D$ and $B * C * D$ then $A - B - C - D$.*

Proposition 2.21 (Four Point Overlap). *Let A, B, C , and D be points. If $A * B * C$ and $B * C * D$ then $A - B - C - D$.*

Proposition 2.22. *Let A, B, C , and D be distinct collinear points with $A * B * C$. Then exactly one of these is true: $D - A - B - C$ or $A - D - B - C$ or $A - B - D - C$ or $A - B - C - D$.*

Hint:

1. Consider all the betweenness relations for A, B , and D . All but one should solve themselves.
2. Assume you are in the one case from (1) which is different.
3. Consider all the betweenness relations for A, C, D . All but one should solve themselves.
4. Assume you are also in the one case from (3) which is different.
5. You should now be done with *at least one*.
6. To get exactly *exactly one*, assume that two hold and establish a contradiction.

Proposition 2.23. *Let A, B, C , and D be distinct collinear points. There is a permutation X, Y, Z, W of A, B, C, D so that $X - Y - Z - W$.*

Proposition 2.24. *If A, B , and C are points so that $A * B * C$ then $\overline{AB} \subseteq \overline{AC}$.*

Proposition 2.25. *If A, B , and C are points so that $A * B * C$ then $\overline{AB} \cup \overline{BC} = \overline{AC}$.*

Proposition 2.26. If A , B , and C are points so that $A * B * C$ then $\overline{AB} \cap \overline{BC} = \{B\}$.

Definition. If A , B , and C are points so that $B * A * C$ then \overrightarrow{AB} and \overrightarrow{AC} are *opposite rays*.

Proposition 2.27. If A , B , and C are points so that $B * A * C$ then $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$.

Proposition 2.28. If A , B , and C are points so that $B * A * C$ then $\overrightarrow{AB} \cup \overrightarrow{AC} = \overleftarrow{AB}$.

Proposition 2.29. If A , B , and C are points so that $A * B * C$ then $\overrightarrow{AB} = \overrightarrow{AC}$.

Proposition 2.30. If A , B , and C are distinct points so that $C \in \overrightarrow{AB}$ then $\overrightarrow{AB} = \overrightarrow{AC}$.

Proposition 2.31. If X and Y are distinct points on the ray \overrightarrow{AB} then it is not the case that $X * A * Y$.

Proposition 2.32. If C is any point on the ray \overrightarrow{AB} and $C \neq A$, then there is a point D on \overrightarrow{AB} so that $A * C * D$.

Proposition 2.33. If \overrightarrow{AB} and \overrightarrow{CD} are rays so that $\overrightarrow{AB} = \overrightarrow{CD}$ then $A = C$.

Definition. If \overrightarrow{AB} is a ray, then A is the *vertex* of the ray.

Proposition 2.34. If X and Y are distinct points on the segment \overline{AB} then it is not the case that $X * A * Y$.

Hint: Suppose $X * A * Y$ and get a contradiction. There are cases: $X = B$, $Y = B$, or A , B , X , and Y are distinct.

Proposition 2.35. If X and Y are distinct points on the segment \overline{AB} then it is not the case that $X * B * Y$.

Proposition 2.36. If \overline{AB} and \overline{CD} are line segments so that $\overline{AB} = \overline{CD}$ then $\{A, B\} = \{C, D\}$.

Definition. If \overline{AB} is a line segment, then A and B are the *endpoints* of the segment.

Proposition 2.37. Every line contains infinitely many points.

Hint: Let A be any point on a line ℓ . You should construct a sequence $\langle X_0, X_1, \dots \rangle$ recursively so that for each n , $A * X_{n+1} * X_n$. After you have A and X_0 , Axiom BA2 will help to find points. Four point betweenness may help to show they are distinct.

Proposition 2.38. Every line segment contains infinitely many points.

2.4 Angles

Definition. Suppose that A , B , and C are noncollinear points. The *angle* $\angle BAC$ is defined to be $\overrightarrow{AB} \cup \overrightarrow{AC}$.

Proposition 2.39. Suppose that A , B , and C are noncollinear points. Then $\angle BAC = \angle CAB$.

Proposition 2.40. Suppose that A , B , and C are noncollinear points. If $B' \in \overrightarrow{AB}$, $C' \in \overrightarrow{AC}$, $B' \neq A$, and $C' \neq A$, then A , B' , and C' are noncollinear and $\angle BAC = \angle B'AC'$.

Proposition 2.41. Suppose that A , B , and C are noncollinear points. Then A is the only point on $\angle BAC$ which is not between any two other points of $\angle BAC$.

Proposition 2.42. If $\angle BAC$ and $\angle EDF$ are angles and $\angle BAC = \angle EDF$, then $A = D$.

Proposition 2.43. If $\angle BAC$ and $\angle DAE$ are angles and $\angle BAC = \angle DAE$ then either $\overrightarrow{AB} = \overrightarrow{AD}$ and $\overrightarrow{AC} = \overrightarrow{AE}$ or $\overrightarrow{AB} = \overrightarrow{AE}$ and $\overrightarrow{AC} = \overrightarrow{AD}$.

Definition. Suppose that A , B , and C are noncollinear points. Then A is the *vertex* of $\angle BAC$, and \overrightarrow{AB} and \overrightarrow{AC} are the *sides* of $\angle BAC$.

Proposition 2.44. Suppose that $\angle BAC$ and $\angle B'AC$ are angles and $\angle BAC = \angle B'AC$. Then $H_{\overrightarrow{AC}}(B) = H_{\overrightarrow{AC}}(B')$.

Definition. Suppose that A , B , and C are noncollinear points. The *interior* of $\angle BAC$ is $H_{\overrightarrow{AC}}(B) \cap H_{\overrightarrow{AB}}(C)$. The *exterior* of $\angle BAC$ is the set of all points which are not on $\angle BAC$ and are not in the interior of $\angle BAC$.

Proposition 2.45. Suppose that A , B , and C are noncollinear points. A point D is in the interior of $\angle BAC$ if and only if D is on the same side of \overleftrightarrow{AC} as B and D is on the same side of \overleftrightarrow{AB} as C .

Proposition 2.46 (Crossbar Betweenness). Suppose that A , B , and C are noncollinear points, and suppose D is a point on \overleftrightarrow{BC} . Then D is in the interior of $\angle BAC$ if and only if $B * D * C$.

Hint: Can \overline{BD} intersect \overleftrightarrow{AC} ?

Proposition 2.47. Suppose that A , B , and C are noncollinear points, and suppose D is a point in the interior of $\angle BAC$. Then every point on \overrightarrow{AD} other than A is in the interior of $\angle BAC$.

Hint: Ray Theorem 2.16.

Definition. Suppose that A , B , C , and D are points. The ray \overrightarrow{AD} is *between* the rays \overrightarrow{AB} and \overrightarrow{AC} if A , B , and C are noncollinear and D is in the interior of $\angle BAC$.

Proposition 2.48. Suppose that A , B , and C are noncollinear points and that D is a point different from A . If \overrightarrow{AD} intersects the interior of \overline{BC} then D is in the interior of $\angle BAC$.

Lemma 2.49 (Z-Lemma). Suppose that A and B are distinct points on a line ℓ and that C and D are points on opposite sides of ℓ . Then \overrightarrow{AC} and \overrightarrow{BD} are disjoint.

Theorem 2.50 (Crossbar Theorem). Suppose A , B , and C are noncollinear points and that D is in the interior of $\angle BAC$. Then \overrightarrow{AD} intersects \overline{BC} .

Hint:

1. Find points E and F so that $E * A * B$ and $F * A * D$ and let $\ell = \overleftrightarrow{ED}$.
2. Apply an earlier theorem to conclude that, since ℓ intersects \overline{EB} then ℓ must intersect exactly one of \overline{EC} and \overline{BC} .
3. We need to show that the portion of ℓ that intersects \overline{EC} or \overline{BC} is \overrightarrow{AD} and not \overrightarrow{AF} . We also have to show that \overrightarrow{AD} intersects \overline{BC} and not \overline{EC} .
4. Use the Z-Lemma to argue that $\overrightarrow{AF} \cap \overrightarrow{EC} = \emptyset$. (This requires some arguments about sides.) Conclude $\overrightarrow{AF} \cap \overline{EC} = \emptyset$.

5. Use the Z-Lemma to argue that $\overrightarrow{AF} \cap \overrightarrow{BC} = \emptyset$. (This requires some arguments about sides.)
Conclude $\overrightarrow{AF} \cap \overrightarrow{BC} = \emptyset$.
6. At this point, we know that \overrightarrow{AD} must intersect \overrightarrow{EC} or \overrightarrow{BC} .
7. Use the Z-Lemma to argue that $\overrightarrow{CE} \cap \overrightarrow{AD} = \emptyset$. (This requires some arguments about sides.)
Conclude $\overrightarrow{AD} \cap \overrightarrow{EC} = \emptyset$.

Proposition 2.51. *Suppose A, B, C, D , and E are distinct points so that \overrightarrow{AD} is between \overrightarrow{AB} and \overrightarrow{AC} and that \overrightarrow{AE} is between \overrightarrow{AB} and \overrightarrow{AD} . Then \overrightarrow{AE} is between \overrightarrow{AB} and \overrightarrow{AC} .*

Hint: Apply the Crossbar Theorem to get a point D' in $\overrightarrow{AD} \cap \overrightarrow{BC}$. Then apply the Crossbar Theorem to $\overrightarrow{BD'}$ and use four point betweenness.

Proposition 2.52. *Suppose A, B, C, D , and E are distinct points so that \overrightarrow{AD} is between \overrightarrow{AB} and \overrightarrow{AC} and that \overrightarrow{AE} is between \overrightarrow{AB} and \overrightarrow{AD} . Then \overrightarrow{AD} is between \overrightarrow{AE} and \overrightarrow{AC} .*

Proposition 2.53 (Interior Inclusion). *Suppose that A, B , and C are noncollinear points. If D is in the interior of $\angle BAC$ and E is in the interior of $\angle BAD$, then E is in the interior of $\angle BAC$.*

Proposition 2.54. *If three distinct rays have the same vertex, then at most one of the rays is between the other two.*

Proposition 2.55. *Suppose that A, B, C , and D are four points such that C and D are on the same side of \overleftrightarrow{AB} and D is not on \overleftrightarrow{AC} . Then either C is in the interior of $\angle BAD$ or D is in the interior of $\angle BAC$.*

Exercise 2.56. Draw three rays with a common endpoint so that no one of them is between the other two.

Definition. Suppose A, B , and C are points so that $B * A * C$. Let D be a point not on \overleftrightarrow{BC} . The angles $\angle BAD$ and $\angle DAC$ are a *linear pair*.

Proposition 2.57. *Every angle is part of a linear pair.*

Proposition 2.58. *If $C * A * B$ and D is in the interior of $\angle BAE$ then E is in the interior of $\angle DAC$.*

Proposition 2.59 (Supplementary Interiors 1). *The interiors of angles in a linear pair are disjoint.*

Hint: Suppose that E is in the interior of both angles in a linear pair with common vertex A . Consider sides of $\ell = \overleftrightarrow{AE}$.

Proposition 2.60 (Supplementary Interiors 2). *If $\angle BAD$ and $\angle DAC$ form a linear pair, and if E is on the same side of \overleftrightarrow{BC} as D , then exactly one of these must hold:*

1. E is on \overrightarrow{AD} .
2. E is in the interior of $\angle BAD$.
3. E is in the interior of $\angle DAC$.

Hint: Suppose (1) does not hold and consider sides of \overleftrightarrow{AD} .

Proposition 2.61. Suppose that A, B, C, D , and E are points so that $B * A * C$. If \overrightarrow{AD} is between \overrightarrow{AB} and \overrightarrow{AC} then \overrightarrow{AE} is between \overrightarrow{AD} and \overrightarrow{AC} .

Definition. Suppose that A, B , and C are noncollinear points. The triangle $\triangle ABC$ is defined to be

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

Proposition 2.62. If A, B , and C are noncollinear points then

$$\triangle ABC = \triangle ACB = \triangle BAC = \triangle BCA = \triangle CAB = \triangle CBA.$$

Proposition 2.63. Suppose that A, B , and C are noncollinear points. The points A, B , and C are the only points in $\triangle ABC$ which are not between other points on $\triangle ABC$.

Proposition 2.64. If $\triangle ABC$ and $\triangle DEF$ are triangles so that $\triangle ABC = \triangle DEF$ then $\{A, B, C\} = \{D, E, F\}$. It follows that

$$\{\overline{AB}, \overline{BC}, \overline{AC}\} = \{\overline{DE}, \overline{EF}, \overline{DF}\}$$

and that

$$\{\angle ABC, \angle BCA, \angle CAB\} = \{\angle DEF, \angle EFD, \angle FDE\}.$$

Definition. Suppose that A, B , and C are noncollinear points. Then A, B , and C are the *vertices* of $\triangle ABC$. \overline{AB} , \overline{BC} , and \overline{AC} are the *sides* of $\triangle ABC$, and the angles $\angle ABC$, $\angle BCA$, and $\angle CAB$ are the *angles* of $\triangle ABC$. The *interior* of $\triangle ABC$ is the intersection of the interiors of the angles of $\triangle ABC$. The *exterior* of $\triangle ABC$ is the set of all points not on $\triangle ABC$ and not in the interior of $\triangle ABC$.

Proposition 2.65. Suppose that A, B , and C are noncollinear points. The interior of $\triangle ABC$ is the set $H_{\overrightarrow{AB}}(C) \cap H_{\overrightarrow{BC}}(A) \cap H_{\overrightarrow{AC}}(B)$.

2.5 Convexity

Definition. A set of points S is *convex* if whenever A and C are in S and $A * B * C$ then B is in S also.

Proposition 2.66. The empty set is convex.

Proposition 2.67. If A is a point then $\{A\}$ is convex.

Proposition 2.68. Lines are convex.

Proposition 2.69. Rays are convex.

Hint: Let \overrightarrow{XY} be a ray and let $A, B \in \overrightarrow{XY}$. There are several cases to consider. For example, it could be that $X = A$ or $X = B$ or neither.

Proposition 2.70. Half planes are convex.

Proposition 2.71. If S and T are convex sets, then $S \cap T$ is a convex set.

Proposition 2.72. Line segments are convex.

Hint: Use rays and intersections.

Proposition 2.73. Interiors of angles and triangles are convex.

Exercise 2.74. Show by means of a picture that angles are not convex. (This might bother you.)

3 Incidence, Betweenness, and Congruence Geometry

3.1 Segment Congruence

Primitives. We add here a new primitive, *segment congruence*.

Axiom. Segment congruence is a binary relation on the set of all segments. If \overline{AB} and \overline{CD} are segments, then $\overline{AB} \cong \overline{CD}$ indicates that the segment congruence relation holds between \overline{AB} and \overline{CD} .

Segment Congruence Axiom 1 (SCA1). *Segment congruence is an equivalence relation on the set of all segments.*

Segment Congruence Axiom 2 (SCA2 Point Construction). *Suppose that \overline{AB} is a segment and \overrightarrow{CD} is a ray. There is a unique point E on \overrightarrow{CD} which is distinct from C so that $\overline{AB} \cong \overline{CE}$.*

Segment Congruence Axiom 3 (SCA3). *Suppose that \overline{AC} and $\overline{A'C'}$ are congruent segments. If B is a point so that $A * B * C$ then there is a point B' so that $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$.*

Proposition 3.1. *The point B' in SCA3 is unique.*

Proposition 3.2 (Segment Addition). *Suppose that $A * B * C$ and $A' * B' * C'$. If $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$ then $\overline{AC} \cong \overline{A'C'}$.*

Hint:

1. Use an axiom to find C'' on \overrightarrow{AC} with $\overline{AC''} \cong \overline{A'C'}$.
2. Use an axiom to find B'' in $\overline{AC''}$ so that $A * B'' * C''$, $\overline{AB''} \cong \overline{A'B'}$, and $\overline{B''C''} \cong \overline{B'C'}$. (We say that B'' corresponds to B' .)
3. Introduce some rays and use uniqueness from SCA2 to argue $B = B''$.
4. Introduce some rays and use uniqueness from SCA2 to argue $C = C''$.

Proposition 3.3 (Segment Subtraction). *Suppose that $A * B * C$ and $A' * B' * C'$. If $\overline{AC} \cong \overline{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$ then $\overline{BC} \cong \overline{B'C'}$.*

Hint:

1. Use an axiom to find B'' in $\overline{A'C'}$ corresponding to B .
2. Introduce some rays and use uniqueness from SCA2 to argue that $B'' = B'$.
3. You are almost there.

Definition. Suppose that \overline{AB} and \overline{CD} are segments. We write $\overline{AB} < \overline{CD}$ or $\overline{CD} > \overline{AB}$ if there is a point E so that $C * E * D$ and $\overline{AB} \cong \overline{CE}$.

Proposition 3.4. *Suppose that $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$. Then $\overline{AB} < \overline{EF}$.*

Proposition 3.5 (Segment Substitution). *If $\overline{AB} < \overline{CD}$, $\overline{CD} \cong \overline{C'D'}$, and $\overline{AB} \cong \overline{A'B'}$ then $\overline{A'B'} < \overline{C'D'}$.*

Hint: Use axiom SCA3.

Proposition 3.6. If $A * B * C$ then $\overline{AB} < \overline{AC}$.

Proposition 3.7. Suppose that $\overrightarrow{AB} = \overrightarrow{AC}$. If $\overline{AB} < \overline{AC}$ then $A * B * C$.

Lemma 3.8. If \overline{AB} is a line segment, then it is not the case that $\overline{AB} < \overline{AB}$.

Hint: Use the definition and uniqueness in SC2.

Proposition 3.9 (Segment Trichotomy). If \overline{AB} and \overline{CD} are segments, then exactly one of these holds: $\overline{AB} < \overline{CD}$ or $\overline{AB} \cong \overline{CD}$ or $\overline{AB} > \overline{CD}$.

Hint: Find E on \overline{CD} with $\overline{AB} \cong \overline{CE}$. Do cases now on betweenness with C , D , and E . This will give at least one. Now suppose two at a time hold and arrive at a contradiction.

3.2 Angle Congruence

Primitives. We add here a new primitive, *angle congruence*.

Axiom. Angle congruence is a binary relation on the set of all angles. If $\angle ABC$ and $\angle DEF$ are angles, then $\angle ABC \cong \angle DEF$ indicates that the angle congruence relation holds between $\angle ABC$ and $\angle DEF$.

Definition. By an *oriented triangle* we will mean a triangle with a specified ordering of its vertices. If A , B , and C are noncollinear points, then the triangle with these vertices in the order A , B , then C is denoted $\triangle ABC$.

Note. We are using the same notation for an oriented triangle as for a non-oriented triangle. Generally, if congruence is under consideration, we will mean an oriented triangle.

Definition. Two oriented triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *congruent* if

- $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{AC} \cong \overline{A'C'}$,
- $\angle ABC \cong \angle A'B'C'$, $\angle BCA \cong \angle B'C'A'$, and $\angle CAB \cong \angle C'A'B'$.

Angle Congruence Axiom 1 (ACA1). *Angle congruence is an equivalence relation on the set of all angles.*

Angle Congruence Axiom 2 (ACA2 Angle Construction or Ray Construction). *If $\angle BAC$ is any angle and \overrightarrow{DE} is any ray, then on either side of \overrightarrow{DE} there is a unique ray \overrightarrow{DF} so that $\angle BAC \cong \angle EDF$.*

Angle Congruence Axiom 3 (ACA3 Side-Angle-Side Congruence or SAS). *If $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$, $\angle CAB \cong \angle C'A'B'$, and $\overline{AC} \cong \overline{A'C'}$, then $\triangle ABC \cong \triangle A'B'C'$.*

Proposition 3.10 (Triangle Construction). *Suppose that A , B , and C are noncollinear points and that $\overline{A'B'}$ is a segment with $\overline{AB} \cong \overline{A'B'}$. Then on either side of $\overleftrightarrow{A'B'}$ is a point C' with $\triangle ABC \cong \triangle A'B'C'$.*

Hint: Apply Angle Construction followed by Point Construction followed by SAS.

Proposition 3.11. *The point C' in Triangle Construction is unique.*

Definition. An *isosceles triangle* is a triangle in which at least two sides are congruent. An *equilateral triangle* is a triangle in which all three sides are congruent.

Theorem 3.12 (Isosceles Triangle Theorem). *If $\triangle ABC$ is an isosceles triangle in which $\overline{AB} \cong \overline{AC}$ then $\angle ABC \cong \angle ACB$.*

Hint: Apply SAS to two orientations of the triangle.

Corollary 3.13. *If $\triangle ABC$ is an equilateral triangle, then all three angles in $\triangle ABC$ are congruent.*

Note. Note that this is a conditional. We are not declaring that an equilateral triangle actually exists.

Proposition 3.14. *If A , B , and C are noncollinear points, then $\triangle ABC \cong \triangle ABC$.*

Proposition 3.15. *If $\triangle ABC$ and $\triangle A'B'C'$ are triangles with $\triangle ABC \cong \triangle A'B'C'$ then $\triangle A'B'C' \cong \triangle ABC$.*

Proposition 3.16. *If $\triangle ABC$, $\triangle A'B'C'$, and $\triangle A''B''C''$ are triangles with $\triangle ABC \cong \triangle A'B'C'$ and $\triangle A'B'C' \cong \triangle A''B''C''$, then $\triangle ABC \cong \triangle A''B''C''$.*

Proposition 3.17. *Suppose that angles $\angle BAD$ and $\angle DAC$ are a linear pair and that angles $\angle B'A'D'$ and $\angle D'A'C'$ are a linear pair. If $\angle BAD \cong \angle B'A'D'$ then $\angle DAC \cong \angle D'A'C'$.*

Hint:

1. Use SCA2 to replace B' , C' , and D' so that the angles do not change but $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, and $\overline{AD} \cong \overline{A'D'}$.
2. Figure out how to apply SAS to $\triangle BAD$, then to $\triangle DBC$, and then to $\triangle ACD$. Segment addition may play a role.

Definition. Suppose ℓ and m are two lines intersecting at a point A . Suppose that B and D are on m and C and E are on ℓ so that $B * A * D$ and $C * A * E$. Then $\angle BAC$ and $\angle DAE$ are *vertical angles*.

Theorem 3.18 (Vertical Angles Theorem). *Vertical angles are congruent.*

Hint: Using the symbols from the definition of vertical angles, $\angle BAC$ and $\angle DAE$ both form linear pairs with $\angle BAE$.

Proposition 3.19 (Converse of Vertical Angles Theorem). *Suppose that $A * B * C$ and that D and E are on opposite sides of \overleftrightarrow{AB} . If $\angle DBC \cong \angle EBA$ then D , B , and E are collinear.*

Hint: Extend the line \overleftrightarrow{DB} across \overleftrightarrow{AC} and use the Vertical Angles Theorem. Then use uniqueness from ACA2.

Proposition 3.20. *Suppose that \overleftrightarrow{AD} is a line, and let B and C be on opposite sides of \overleftrightarrow{AD} . Suppose that α and β are a linear pair so that $\alpha \cong \angle BAD$ and $\beta \cong \angle DAC$. Then B , A , and C are collinear and $\angle BAD$ and $\angle DAC$ form a linear pair.*

Hint:

1. Find E so that $B * A * E$ and argue that C and E are on the same side of \overleftrightarrow{AD} .

2. Use 3.17 to argue that $\angle DAE \cong \beta$.
3. Use an axiom to argue that $\overrightarrow{AC} = \overrightarrow{AE}$.
4. Find a line that contains A , B , and C .
5. Use the fact that B and C are on opposite sides of \overleftrightarrow{AD} to argue that $B * A * C$.
6. Now argue that $\angle BAD$ and $\angle DAC$ are a linear pair.

Definition. An angle α is a *right angle* if α forms a linear pair with an angle congruent to α .

Corollary 3.21. Suppose that A , B , C , and D are points so that $\angle DAB$ and $\angle DAC$ are right angles. Then A , B , and C are collinear.

Proposition 3.22 (Angle Addition). Suppose that D is an interior point of the angle $\angle BAC$ and that D' is an interior point of the angle $\angle B'A'C'$. If $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$ then $\angle BAC \cong \angle B'A'C'$.

Hint:

1. Use SCA2 to replace B' and C' so that we can assume that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$.
2. Use the Crossbar Theorem to find E on \overrightarrow{AD} so that $B * E * C$. Argue that B and C are on opposite sides of \overleftrightarrow{AD} .
3. Argue that B' and C' are on opposite sides of $\overleftrightarrow{A'D'}$.
4. Use an axiom to find E' on $\overleftrightarrow{A'D'}$ with $\overline{AE} \cong \overline{A'E'}$. Note that we do not know that E' is on $\overleftrightarrow{B'C'}$, but we hope it is.
5. Apply SAS to triangles $\triangle BAE$ and $\triangle CAE$.
6. Use 3.20 to show that B' , E' , and C' are collinear.
7. Argue that $\overline{BE} \cong \overline{B'E'}$ and $\overline{EC} \cong \overline{E'C'}$.
8. Argue that $\overline{BC} \cong \overline{B'C'}$.
9. Figure out how to apply SAS to $\triangle ABC$.
10. Use triangle congruence to arrive at $\angle BAC \cong \angle B'A'C'$.

Proposition 3.23. Suppose $\angle BAC \cong \angle B'A'C'$ and that \overrightarrow{AD} is a ray between \overrightarrow{AB} and \overrightarrow{AC} . There is a ray $\overrightarrow{A'D'}$ between $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ so that $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$.

Hint:

1. Replace B' and C' so that we can assume $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$.
2. Apply SAS to $\triangle BAC$.
3. Use the Crossbar Theorem to find a point E on \overrightarrow{AD} so that $B * E * C$.
4. Somehow, find E' so that $B' * E' * C'$, $\overline{BE} \cong \overline{B'E'}$, and $\overline{EC} \cong \overline{E'C'}$.

5. Apply SAS to $\triangle ABE$.

6. Let $D' = E'$ and argue that D' satisfies the proposition.

Proposition 3.24 (Angle Subtraction). *Suppose that D is an interior point of the angle $\angle BAC$ and that D' is an interior point of the angle $\angle B'A'C'$. If $\angle BAC \cong \angle B'A'C'$ and $\angle BAD \cong \angle B'A'D'$ then $\angle DAC \cong \angle D'A'C'$.*

Hint:

1. Apply 3.23 to get a ray $\overrightarrow{A'D''}$ as in that theorem (D'' is playing the role of D' .)

2. Argue that D' and D'' are on the same side of $\ell = \overleftrightarrow{A'B'}$.

3. Apply uniqueness from axiom ACA2.

Definition. If $\angle BAC$ and $\angle EDF$ are angles then we write $\angle BAC < \angle EDF$ or $\angle EDF > \angle BAC$ if there is a ray \overrightarrow{DG} between \overrightarrow{DE} and \overrightarrow{DF} so that $\angle BAC \cong \angle EDG$.

Proposition 3.25. *Suppose that $\angle BAC < \angle EDF$ and $\angle EDF < \angle HGI$. Then $\angle BAC < \angle HGI$.*

Proposition 3.26 (Angle Substitution). *Let α , α' , β , and β' be angles. If $\alpha < \beta$ and $\alpha \cong \alpha'$ and $\beta \cong \beta'$ then $\alpha' < \beta'$.*

Proposition 3.27. *If \overrightarrow{AD} is between \overrightarrow{AB} and \overrightarrow{AC} then $\angle BAD < \angle BAC$.*

Proposition 3.28. *Suppose that α and β are a linear pair and that α' and β' are a linear pair. If $\alpha > \alpha'$ then $\beta < \beta'$.*

Proposition 3.29. *If α is any angle, then it is not the case that $\alpha < \alpha$.*

Proposition 3.30 (Angle Trichotomy). *If α and β are angles, then exactly one of these must hold: $\alpha < \beta$ or $\alpha \cong \beta$ or $\alpha > \beta$.*

Hint: Construct angles congruent to the given angles from three points $B*A*C$. Use Supplementary Interiors.

3.3 Right Angles and Perpendicular Lines

Definition. An angle α is a *right angle* if α forms a linear pair with an angle congruent to α . If α is a right angle and β is an angle so that $\beta < \alpha$, then β is an *acute* angle. If $\alpha < \beta$ then β is an *obtuse* angle.

Proposition 3.31. *If α is a right angle and $\gamma \cong \alpha$, then γ is a right angle.*

Proposition 3.32. *If α is a right angle and γ forms a linear pair with α , then γ is a right angle.*

Proposition 3.33. *Any two right angles are congruent.*

Hint: Use Angle Trichotomy.

Definition. Two lines ℓ and m are *perpendicular* if ℓ and m intersect at a point A and there exist $B \neq A$ on ℓ and $C \neq A$ on m so that $\angle BAC$ is a right angle. In this case, we write $\ell \perp m$.

Proposition 3.34. Suppose that ℓ and m are lines that intersect at a point A and that there exist $B \neq A$ on ℓ and $C \neq A$ on m so that $\angle BAC$ is a right angle. Then for all $B' \neq A$ on ℓ and $C' \neq A$ on m $\angle B'AC'$ is a right angle.

Hint: There are several cases as to whether B' and C' are on the rays \overrightarrow{AB} and \overrightarrow{AC} or their opposites.

Proposition 3.35. If ℓ is a line and P is a point not on ℓ then there is a line m passing through P which is perpendicular to ℓ .

Hint:

1. Let $A \neq B$ be two points on ℓ .
2. Find P' on the side of ℓ opposite from P so that $\angle PAB \cong \angle P'AB$ and $\overline{AP} \cong \overline{AP'}$.
3. Let Q be a point on ℓ so that $P * Q * P'$. (Why does this exist?)
4. Argue that $\angle PQB$ is a right angle if $A = Q$.
5. Assume $A \neq Q$.
6. Figure out how to apply SAS to $\triangle PAQ$.
7. Show that $\angle PQA$ is a right angle.

Corollary 3.36. Right angles exist.

Corollary 3.37. Perpendicular lines exist.

Proposition 3.38. If ℓ is a line and P is a point on ℓ then there is a unique line m passing through P which is perpendicular to ℓ .

Hint: Use Angle Construction.

3.4 Parallel Lines

Definition. Two lines ℓ and m are *parallel* if they do not intersect. In this case, we write $\ell \parallel m$.

Theorem 3.39 (Alternate Interior Angles Theorem). Suppose that C and D are on opposite sides of \overleftrightarrow{AB} . If $\angle CAB \cong \angle DBA$ then $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$.

Hint:

1. Suppose that \overleftrightarrow{AC} and \overleftrightarrow{BD} intersect in some point X . Without loss of generality, assume that X is on the same side of \overleftrightarrow{AB} as C so that $D * B * X$.
2. Find Y somehow on \overleftrightarrow{BD} so that $\overline{BY} \cong \overline{BX}$.
3. Note that $\angle ABY$ and $\angle ABX$ are a linear pair.
4. Apply SAS to $\triangle BAX$ and $\triangle BAY$ to get that $\angle BAY \cong \angle ABX$.
5. Since $\angle ABY$ and $\angle ABX$ are a linear pair, and since $\angle BAX \cong \angle ABY$ and $\angle ABX \cong \angle BAY$ then $\angle BAX$ and $\angle BAY$ are a linear pair. Why?

6. This implies that $Y * A * X$ so $Y \in \overleftrightarrow{AX} = \overleftrightarrow{AC}$.

7. You are a breath away from a contradiction. How many points of intersection do \overleftrightarrow{BD} and \overleftrightarrow{AC} have?

Definition. Suppose that ℓ and ℓ' are two distinct lines. A *transversal* of ℓ and ℓ' is a third line which intersects ℓ and ℓ' . Suppose that t is a transversal of ℓ and ℓ' . Let B be the intersection of t with ℓ , and let B' be the intersection of t with ℓ' . Let A and C be points on ℓ on opposite sides of t , and let A' and C' be points on ℓ' on opposite sides of t so that A and A' are on the same side of t . The four angles $\angle ABB'$, $\angle CBB'$, $\angle A'B'B$, and $\angle C'B'B$ are *interior angles*. The angles $\angle ABB'$ and $\angle C'B'B$ are *alternate interior angles* as are $\angle CBB'$ and $\angle A'B'B$.

Note. An alternative statement of the Alternate Interior Angles Theorem is: If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles are congruent, then $\ell \parallel \ell'$.

Definition (Continuing Definition 3.4). Let D and D' be points on t so that $D * B * B' * D'$. These pairs of angles are *corresponding angles*:

$$\angle ABD \text{ and } \angle A'B'D'; \angle ABB' \text{ and } \angle A'B'D'$$

$$\angle CBD \text{ and } \angle C'B'D'; \angle CBB' \text{ and } \angle C'B'D'.$$

Proposition 3.40 (Corresponding Angles Theorem). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of corresponding angles are congruent, then $\ell \parallel \ell'$.*

Proposition 3.41. *Suppose that ℓ and ℓ' are distinct lines. If there is one line m so that $\ell \perp m$ and $\ell' \perp m$ then $\ell \parallel \ell'$.*

Hint: Draw a picture. Label some points. Use the AIAT.

Proposition 3.42. *If ℓ is a line and P is a point not on ℓ then there is a unique line m passing through P which is perpendicular to ℓ .*

Hint: The new thing here is the word *unique* (why?). Assume there are two such lines and find a contradiction.

Definition. If P is a point not on a line ℓ , and if m is the unique line passing through P perpendicular to ℓ , then the point of intersection of ℓ and m is the *foot* of the perpendicular.

Proposition 3.43 (Existence of Parallels). *Suppose that P is a point not on a line ℓ . There is at least one line ℓ' through P parallel to ℓ .*

Hint: Use 3.41.

Note. The method of proof you (probably) used in 3.43 is called the *Double Perpendicular Construction*.

3.5 Inequalities for Triangles

Definition. Suppose that A , B , and C are noncollinear points. The angles $\angle ABC$, $\angle BCD$, $\angle CAB$ are called *interior* angles of $\triangle ABC$. Any angle forming a linear pair with an interior angle of a triangle is an *exterior* angle. If an exterior angle forms a linear pair with one of the interior angles, the other two interior angles are *remote* interior angles (remote to the given exterior angle). In other words if $B * C * D$, then the angle $\angle ACD$ is an exterior angle with remote interior angles $\angle CAB$ and $\angle CBA$.

Lemma 3.44. *Suppose that A , B , and C are noncollinear points. If $B * C * D$ then the exterior angle $\angle ACD$ is greater than the remote interior angle $\angle CAB$ (whose vertex is not on \overleftrightarrow{CD}).*

Hint:

1. Suppose by way of contradiction that $\angle ACD > \angle CAB$ fails. Apply Angle Trichotomy to get cases.
2. The Alternate Interior Angles Theorem will handle the case when $\angle ACD = \angle CAB$.
3. Suppose $\angle ACD < \angle CAB$. Find E so that $B * E * C$ and $\angle CAE \cong \angle ACD$. How?
4. Apply the Alternate Interior Angles Theorem and get a contradiction and conclude $\angle ACD > \angle CAB$.

Theorem 3.45 (Exterior Angle Theorem). *Suppose that A , B , and C are noncollinear points. If $B * C * D$ then the exterior angle $\angle ACD$ is greater than both remote interior angles $\angle CAB$ and $\angle CBA$.*

Hint: One angle is handled by the lemma. For the other, find a point F with $A * C * F$ and apply the lemma and the Vertical Angles Theorem.

Definition. Suppose that $\triangle ABC$ is a triangle. The side \overline{BC} is *opposite* to the angle $\angle CAB$. The side \overline{AB} is opposite to the angle $\angle BCA$, and the side \overline{CA} is opposite to the angle $\angle ABC$.

Definition. A *right triangle* is a triangle in which at least one angle is a right angle.

Proposition 3.46. *If $\triangle ABC$ is a triangle and $\angle ABC$ is a right angle, then $\angle CAB$ and $\angle BCA$ are acute angles.*

Hint: Draw a picture. Use the Exterior Angles Theorem and the definition of right angle.

Corollary 3.47. *A right triangle has exactly one right angle.*

Definition. In any right triangle, the side opposite the right angle is the *hypotenuse*. The other two sides are *legs*.

Proposition 3.48. *If α and β are a linear pair, then α is acute if and only if β is obtuse.*

Hint: Name some points to make the discussion easier. If β is obtuse, then use the definition of obtuse and angle ordering to find a right angle “inside” β . Then use the definition of angle ordering with α . For the other direction, use the contrapositive, but notice there are cases.

Proposition 3.49. *A triangle can have at most one obtuse angle.*

Proposition 3.50. Suppose that $\triangle ABC$ is a triangle and let F be the foot of the perpendicular from C to \overleftrightarrow{AB} . If the angles $\angle CAB$ and $\angle CBA$ are acute, then $A * F * B$.

Hint: This involves contradiction and the Exterior Angle Theorem along with a strange diagram that would happen if the foot were not where we want it to be.

Theorem 3.51 (Scalene Inequality). If A , B , and C are noncollinear points then $\overline{BC} > \overline{AC}$ if and only if $\angle BAC > \angle ABC$.

This theorem is proven by the following claimable lemmas.

Lemma 3.52 (Part of the proof of 3.51). Suppose that A , B , and C are noncollinear points. If $\overline{BC} > \overline{AC}$ then $\angle BAC > \angle ABC$.

Hint:

1. Find D so that $C * D * B$ and $\overline{CD} \cong \overline{AC}$.
2. Apply the Isosceles Triangle Theorem to $\triangle ACD$.
3. Apply the Exterior Angles Theorem to $\angle ADC$.
4. Do some transitivity and substitution.

Lemma 3.53 (Part of the proof of 3.51). Suppose that A , B , and C are noncollinear points. If $\angle BAC > \angle ABC$ then $\overline{BC} > \overline{AC}$.

Hint:

1. Use Angle Trichotomy.
2. The Isosceles Triangle Theorem rules out one possibility.
3. The previous lemma rules out the other possibility.
4. That leaves you with one option, which happens to be the one we want.

Proposition 3.54. Suppose that P is a point not on a line ℓ . Let F be the foot of the perpendicular from P to ℓ . If Q is any point on ℓ other than F , then $\overline{PF} < \overline{PQ}$.

Lemma 3.55. Suppose that A , B , and C are noncollinear points and $B * D * C$. If $\overline{AC} > \overline{AB}$ or $\overline{AC} = \overline{AB}$ then $\overline{AC} > \overline{AD}$.

Hint:

1. Apply the Exterior Angle Theorem to $\angle ADC$.
2. If $\overline{AC} > \overline{AB}$ show that that $\angle ADC > \angle ACD$.
3. If $\overline{AC} = \overline{AB}$ use the Isosceles Triangle Theorem and substitution to conclude $\angle ADC > \angle ACD$.
4. Now use the Scalene Inequality.

Theorem 3.56 (Hinge Theorem). Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$. Then $\angle BAC > \angle B'A'C'$ if and only if $\overline{BC} > \overline{B'C'}$.

This theorem is proven by the following claimable lemmas.

Lemma 3.57 (Part of the proof of 3.56). *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$. If $\angle BAC > \angle B'A'C'$ then $\overline{BC} > \overline{B'C'}$.*

Hint:

1. One of \overline{AC} or \overline{AB} must be longer. Without loss of generality, assume that $\overline{AC} > \overline{AB}$ or $\overline{AC} \cong \overline{AB}$.
2. Find D so that $B * D * C$ and $\angle BAD \cong \angle B'A'C'$.
3. Argue that $\overline{AC} > \overline{AD}$.
4. Find E on \overrightarrow{AD} so that $\overline{AE} \cong \overline{A'C'} \cong \overline{AC}$.
5. Now $\triangle BAE \cong \triangle B'A'C'$ by SAS. We just need to show that $\overline{BC} > \overline{BE}$.
6. Establish this string of comparisons:

$$\angle BCE = \angle DCE < \angle ACE \cong \angle CED < \angle CEB$$

7. Apply 3.51

Lemma 3.58 (Part of the proof of 3.56). *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$. If $\overline{BC} > \overline{B'C'}$ then $\angle BAC > \angle B'A'C'$.*

Hint: Suppose that it is not the case that $\angle BAC > \angle B'A'C'$. Angle Trichotomy gives two other options. One of them can be ruled out by the use of SAS. The other can be ruled out by the previous lemma.

Theorem 3.59 (Triangle Inequality Form 1). *Let $\triangle ABC$ be a triangle and let D be a point such that $A * B * D$ and $\overline{BD} \cong \overline{BC}$. Then $\overline{AC} < \overline{AD}$.*

Hint: Establish these comparisons and apply 3.51:

$$\angle ADC = \angle BDC \cong \angle BCD < \angle ACD.$$

3.6 Triangle Congruence

Theorem 3.60 (Angle-Angle-Side Congruence or AAS). *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\angle CAB \cong \angle C'A'B'$ and $\angle ABC \cong \angle A'B'C'$ and $\overline{BC} \cong \overline{B'C'}$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Hint:

1. Use Triangle Construction to find a point D on the same side of \overleftrightarrow{BC} as A so that $\triangle DBC \cong \triangle A'B'C'$.
2. Use ACA2 to argue that $\overrightarrow{BD} = \overrightarrow{BA}$.
3. Now do cases as to whether $B * A * D$ or $B * D * A$ or $D = A$ and use exterior angles.

Theorem 3.61 (Angle-Side-Angle Congruence or ASA). *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\angle ABC \cong \angle A'B'C'$ and $\overline{BC} \cong \overline{B'C'}$ and $\angle BCA \cong \angle B'C'A'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Hint: Start off like you did for AAS. Argue that $\overrightarrow{BD} = \overrightarrow{BA}$ and that $\overrightarrow{CD} = \overrightarrow{CA}$. How many points of intersection do \overrightarrow{BA} and \overrightarrow{CA} have if $D \neq A$?

Proposition 3.62 (Converse of Isosceles Triangle Theorem). *If $\triangle ABC$ is a triangle in which $\angle ABC \cong \angle ACB$ then $\overline{AB} \cong \overline{AC}$.*

Hint: Apply ASA to two orientations of the triangle.

Corollary 3.63. *If all three angles in a triangle are congruent, then the triangle is equilateral.*

Theorem 3.64 (Side-Side-Side Congruence or SSS). *Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are triangles so that $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$ and $\overline{BC} \cong \overline{B'C'}$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Hint: Use Angle Trichotomy and the Hinge Theorem to argue that $\angle BAC = \angle B'A'C'$. Then use SAS.

Exercise 3.65. Draw a picture indicating that Angle-Side-Side congruence is not a thing.

Theorem 3.66 (Hypotenuse-Leg Congruence or HL). *Let $\triangle ABC$ and $\triangle A'B'C'$ be right triangles with $\angle CAB$ and $\angle C'A'B'$ right angles. If $\overline{BC} \cong \overline{B'C'}$ and $\overline{AB} \cong \overline{A'B'}$ then $\triangle ABC \cong \triangle A'B'C'$.*

Hint:

1. Find D on the opposite side of \overleftrightarrow{AB} so that $\triangle ABD \cong \triangle A'B'C'$. We just need to show that $\triangle ABD \cong \triangle ABC$.
2. Use linear pairs to argue that A , C , and D are collinear.
3. Argue that $\triangle CDB$ exists and is isosceles.
4. Use AAS to show that $\triangle DAB \cong \triangle CAB$.

3.7 Midpoints and Bisectors

Definition. A *midpoint* of a segment \overline{AB} is a point M so that $A * M * B$ and $\overline{AM} \cong \overline{MB}$.

Lemma 3.67. *If a segment has a midpoint, then it has exactly one.*

Hint: Suppose that there are two midpoints and use the definition of segment order and segment trichotomy.

Lemma 3.68. *Suppose that A and B are distinct points and that C and D are points on opposite sides of \overleftrightarrow{AB} so that $\angle BAD$ and $\angle ABC$ are right angles. If M is the point of intersection of \overline{CD} and \overleftrightarrow{AB} then $M \neq A$ and $M \neq B$.*

Hint: Use proof by contradiction and the Exterior Angle Theorem.

Lemma 3.69. *Suppose that A and B are distinct points and that C and D are points on opposite sides of \overleftrightarrow{AB} so that $\angle BAD$ and $\angle ABC$ are right angles. If M is the point of intersection of \overline{CD} and \overleftrightarrow{AB} then $A * M * B$.*

Hint: Use proof by contradiction, vertical angles, the Exterior Angle Theorem, and acute and obtuse angles.

Note. The proof here that the midpoint of \overline{AB} exists uses points off the line \overleftrightarrow{AB} . That may seem odd. If we had already assumed Dedekind's Axiom or if we had assumed a Ruler Postulate we could prove existence of the midpoint from within \overleftrightarrow{AB} .

Theorem 3.70 (Midpoint Existence). *Every segment has a unique midpoint.*

Hint: Suppose that \overline{AB} is a segment. Find points C and D on opposite sides of \overleftrightarrow{AB} so that $\angle ABC$ and $\angle BAD$ are right angles and so that $\overline{BC} \cong \overline{AD}$. Then use the previous lemmas and triangle congruence.

Definition. A *perpendicular bisector* of a segment \overline{AB} is a line ℓ perpendicular to \overleftrightarrow{AB} which intersects \overline{AB} at its midpoint.

Theorem 3.71 (Existence of Perpendicular Bisectors). *Every segment has a unique perpendicular bisector.*

Theorem 3.72 (Pointwise Characterization of Perpendicular Bisectors). *Suppose that A and B are distinct points and that P is any point. P lies on the perpendicular bisector of \overline{AB} if and only if $\overline{PA} \cong \overline{PB}$.*

This theorem is proven by the following claimable lemmas.

Lemma 3.73 (Part of the proof of 3.72). *Suppose that A and B are distinct points. If P lies on the perpendicular bisector of \overline{AB} then $\overline{PA} \cong \overline{PB}$.*

Hint: Similar triangles.

Lemma 3.74 (Part of the proof of 3.72). *Suppose that A and B are distinct points. If P is a point so that $\overline{PA} \cong \overline{PB}$ then P lies on the perpendicular bisector of \overline{AB} .*

Hint: Similar triangles.

Definition. Suppose that $\angle BAC$ is an angle. A ray \overrightarrow{AD} is a *bisector* of $\angle BAC$ if \overrightarrow{AD} is between \overrightarrow{AB} and \overrightarrow{AC} and if $\angle DAB \cong \angle DAC$.

Lemma 3.75. *If an angle has a bisector, then it has exactly one.*

Hint: Suppose that $\angle BAC$ is an angle and that \overrightarrow{AD} and $\overrightarrow{AD'}$ are bisectors of $\angle BAC$. Find B' on \overrightarrow{AB} and C' on \overrightarrow{AC} so that $\overline{AB'} \cong \overline{AC'}$. Let M be the intersection of \overrightarrow{AD} with $\overline{B'C'}$, and let M' be the intersection of $\overrightarrow{AD'}$ with $\overline{B'C'}$ (why do these exist?). Prove that M and M' are midpoints of $\overline{B'C'}$.

Theorem 3.76. *Every angle has a unique bisector.*

Hint: Let $\angle BAC$ be an angle. Find B' on \overrightarrow{AB} and C' on \overrightarrow{AC} so that $\overline{AB'} \cong \overline{AC'}$. Let M be the midpoint of $\overline{B'C'}$. Use triangle congruence to show that \overrightarrow{AM} is a bisector of $\angle BAC$.

Theorem 3.77 (Pointwise Characterization of Angle Bisector). *Suppose that P is a point in the interior of $\angle BAC$. Let F be the foot of the perpendicular from P to \overline{AB} , and let G be the foot of the perpendicular from P to \overline{AC} . Then \overrightarrow{AP} is the bisector of $\angle BAC$ if and only if $\overline{PF} \cong \overline{PG}$.*

This theorem is proven by the following claimable lemmas.

Lemma 3.78 (Part of the proof of 3.77). Suppose that P is a point in the interior of $\angle BAC$. Let F be the foot of the perpendicular from P to \overline{AB} , and let G be the foot of the perpendicular from P to \overline{AC} . If \overrightarrow{AP} is the bisector of $\angle BAC$ then $\overline{PF} \cong \overline{PG}$.

Lemma 3.79 (Part of the proof of 3.77). Suppose that P is a point in the interior of $\angle BAC$. Let F be the foot of the perpendicular from P to \overline{AB} , and let G be the foot of the perpendicular from P to \overline{AC} . If $\overline{PF} \cong \overline{PG}$ then \overrightarrow{AP} is the bisector of $\angle BAC$.

Proposition 3.80. If \overrightarrow{AD} is a bisector of an angle $\angle BAC$ then the angle $\angle BAD$ is acute.

3.8 Quadrilaterals

Definition. Suppose that A, B, C , and D are four distinct points. If \overline{AB} and \overline{CD} do not intersect and \overline{AD} and \overline{CB} do not intersect then

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$$

is a *quadrilateral*. A, B, C , and D are the *vertices* of $\square ABCD$. The line segments \overline{AB} , \overline{BC} , \overline{CD} , and \overline{DA} are the *sides* of $\square ABCD$. \overline{AB} and \overline{CD} are *opposite sides*, as are \overline{AD} and \overline{CB} .

Definition. Two quadrilaterals $\square ABCD$ and $\square XYZW$ are *congruent* if

$$\overline{AB} \cong \overline{XY} \text{ and } \overline{BC} \cong \overline{YZ} \text{ and } \overline{CD} \cong \overline{ZW} \text{ and } \overline{DA} \cong \overline{WX} \text{ and}$$

$$\angle ABC \cong \angle XYZ \text{ and } \angle BCD \cong \angle YZW \text{ and } \angle CDA \cong \angle ZWX \text{ and } \angle DAB \cong \angle WXY.$$

This is expressed as $\square ABCD \cong \square XYZW$.

Definition. A *convex quadrilateral* is a quadrilateral in which each vertex is in the interior of the angle formed by the other three vertices.

Note. This is standard but rather inconvenient terminology. A quadrilateral is not convex in the sense that we have been using that word. The interior of a convex quadrilateral is convex. However, defining the interior is awkward for general quadrilaterals.

Exercise 3.81. Draw a quadrilateral which is not a convex quadrilateral and explain.

Proposition 3.82. If $\square ABCD$ is a convex quadrilateral, then C and D are on the same side of \overline{AB} . The same holds for all other pairs of points.

Hint: D is in the interior of $\angle CAB$.

Definition. The *interior* of a convex quadrilateral is the intersection of the interiors of the angles determined by its sides

Proposition 3.83. The interior of a convex quadrilateral is convex.

Hint: Use intersections.

Definition. A quadrilateral $\square ABCD$ is a *parallelogram* if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$.

Proposition 3.84. Parallelograms are convex quadrilaterals.

Hint: Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, D and C are on the same side of \overleftrightarrow{AB} . Look up the definition of the interior of an angle.

Proposition 3.85. Suppose that $\triangle ABC$ is a triangle. If $A * D * B$ and $A * E * C$ then $\square DBCE$ is a convex quadrilateral.

Definition. The *diagonals* of $\square ABCD$ are the line segments \overline{AC} and \overline{BD}

Proposition 3.86. A quadrilateral is a convex quadrilateral if and only if its diagonals have an interior point in common.

Hint: Consider the crossbar theorem.

Corollary 3.87. If $\square ABCD$ and $\square ACBD$ are both quadrilaterals then $\square ABCD$ is not a convex quadrilateral.

Corollary 3.88. If $\square ABCD$ is a quadrilateral but not a convex quadrilateral, then $\square ABCD$ and $\square ACBD$ are both quadrilaterals.

3.9 Circles

Definition. Suppose that A and B are distinct points. The *circle* γ with center A and radius \overline{AB} is the set of all points X so that $\overline{AX} \cong \overline{AB}$. We denote the circle as $\gamma = \mathcal{C}(A, B)$. The point A is the center of γ . If $\overline{AX} \cong \overline{AB}$, then \overline{AX} is a *radius* of γ . We may also call $\mathcal{C}(A, B)$ the *circle centered at A passing through B or containing B* .

Definition. If γ is the circle with center A and radius \overline{AB} , then the *interior* of γ is the set of all points X so that $X = A$ or $\overline{AX} < \overline{AB}$. The *exterior* of γ is the set of all points X so that $\overline{AX} > \overline{AB}$.

Note. When $P \in \gamma = \mathcal{C}(A, B)$ we will usually say that P is *on* γ rather than P is *in* γ to avoid confusion with points inside the interior of γ , which we may say are *inside* γ .

Proposition 3.89 (Circle Trichotomy). Suppose that γ is any circle and X is any point. The exactly one of these holds: X is in the interior of γ or X is on γ or X is in the exterior of γ .

Definition. If γ is a circle, then the union of γ with its interior is called a *closed disk*. The interior of γ is called an *open disk*.

Lemma 3.90. Let X , Y , and C be noncollinear points so that $\overline{CX} < \overline{CY}$ or $\overline{CX} \cong \overline{CY}$. If $X * B * Y$ then $\overline{CB} < \overline{CY}$

Hint: Consider 3.55

Proposition 3.91. The interior of a circle is convex.

Proposition 3.92. Closed disks are convex.

Proposition 3.93. Suppose that γ is a circle, that A is in the interior of γ , and that B is in the exterior of γ . The intersection of \overline{AB} with the interior of γ is convex.

Proposition 3.94. Suppose that γ is a circle, that A is in the interior of γ , and that B is in the exterior of γ . The intersection of \overline{AB} with the exterior of γ is convex.

Hint: Use the assumptions and the standard proof outline for convexity to find points arranged like so: $A - X - Y - Z$ with A in the interior of the circle and X and Z in the exterior. Let C be the center of the circle, and let D be a point on the circle. Argue that

$$\angle CXY > \angle CAX > \angle CXA.$$

This forces $\angle CXY$ to be obtuse. The other angles in $\triangle CXY$ are then acute. Use the Scalene Inequality.

Lemma 3.95. *Suppose that C is a point in the interior of a circle γ . There is a circle centered at C contained entirely in the interior of γ .*

Hint: Let $\gamma = \mathcal{C}(A, B)$. Explain why there is a point D on \overrightarrow{AC} so that $D \in \gamma$. The midpoint of \overline{CD} may be helpful along with the triangle inequality.

Lemma 3.96. *Suppose that C is a point in the exterior of a circle γ . There is a circle centered at C contained entirely in the exterior of γ .*

Proposition 3.97. *Suppose that ℓ is a line and γ is a circle. The number of points in $\gamma \cap \ell$ is 0, 1, or 2.*

Hint:

1. Suppose that $\gamma = \mathcal{C}(C, R)$.
2. Suppose by way of contradiction that X, Y, Z are distinct points in $\gamma \cap \ell$ and that $X * Y * Z$.
3. Use the Isosceles Triangle Theorem to find a contradiction.

Definition. A *tangent* to a circle γ is a line that intersects γ exactly once. A *secant* to a circle γ is a line that intersects γ exactly twice. If C and D are points on a circle γ , then \overline{CD} is a *chord* of γ . A chord that contains the center of γ is a *diameter* of γ . If \overline{CD} is a diameter of γ , then C and D are *antipodal points* on γ .

Theorem 3.98 (Tangent Line Theorem). *Let t be a line and let $\gamma = \mathcal{C}(C, B)$ be a circle. Let $P \in \gamma \cap t$. Then t is a tangent to γ at P if and only if $\overrightarrow{CP} \perp t$.*

This theorem is proven by the following claimable lemmas.

Lemma 3.99 (Part of the proof of 3.98). *Let t be a line and let $\gamma = \mathcal{C}(C, B)$ be a circle. Let $P \in \gamma \cap t$. If t is a tangent to γ at P then $\overrightarrow{CP} \perp t$.*

Hint:

1. Let Q be the foot of the perpendicular from C to t , and suppose by way of contradiction that $Q \neq P$.
2. Find $R \in t$ so that $P * Q * R$ and $\overline{PQ} \cong \overline{QR}$.
3. Apply SAS and find a contradiction.

Lemma 3.100 (Part of the proof of 3.98). *Let t be a line and let $\gamma = \mathcal{C}(C, B)$ be a circle. Let $P \in \gamma \cap t$. If $\overrightarrow{CP} \perp t$ then t is a tangent to γ at P .*

Hint: Look at 3.54.

Proposition 3.101. *If P is a point on a circle γ then there is a line t that is tangent to γ at P .*

Proposition 3.102. *If t is a line tangent to a circle γ at a point P , then every point on t except P is in the exterior of γ .*

Proposition 3.103 (Secant Line Theorem). *If \overline{AB} is a chord of the circle $\gamma = \mathcal{C}(C, R)$, then C lies on the perpendicular bisector of \overline{AB} .*

Proposition 3.104. *If \overline{AB} is a chord of a circle γ and $A * B * D$, then D is in the exterior of γ .*

Definition. Two circles α and β are *tangent* to each other at a point P if $\alpha \cap \beta = \{P\}$.

Proposition 3.105 (Tangent Circles Theorem). *If $\alpha = \mathcal{C}(A, R)$ and $\beta = \mathcal{C}(B, S)$ are tangent at a point P , then $A \neq B$, A , B , and P are collinear, and the circles have a common tangent at P .*

Definition. If γ is a circle that contains three noncollinear points A , B , and C , then γ is a *circumcircle* of $\triangle ABC$ and is said to *circumscribe* the triangle $\triangle ABC$. The center of γ is the *circumcenter* of $\triangle ABC$.

Theorem 3.106 (Circumscribed Triangle Theorem). *A triangle can be circumscribed if and only if the perpendicular bisectors of the sides of the triangle are concurrent. In this case, the center of the circumscribing circle is the point of intersection of the perpendicular bisectors of the sides.*

This theorem is proven through the following claimable lemmas.

Lemma 3.107 (Part of the proof of Theorem 3.106). *If the perpendicular bisectors of the sides of a triangle are concurrent, then the triangle can be circumscribed.*

Lemma 3.108 (Part of the proof of Theorem 3.106). *If a triangle can be circumscribed, then the perpendicular bisectors of the sides of the triangle intersect at the center of the circumscribing circle.*

Corollary 3.109. *Circumscribing circles are unique.*

Definition. A circle γ is *inscribed* in a triangle if the lines containing each side of the triangle are all tangent to the circle. In this case, the center of γ is an *incenter* of the triangle.

Proposition 3.110 (Inscribed Circle Theorem). *Every triangle has an inscribed circle whose center is the intersection of the angle bisectors of the triangle.*

Corollary 3.111. *Inscribed circles are unique.*

4 Continuity and Measurement

In this section we add a new axiom, Dedekind's Axiom, which essentially declares that lines have no holes in them. The point of this section is to establish the following theorems. The first two theorems on intersections were assumed by Euclid without explanation. Their proofs require an intimate knowledge of topological notions related to continuity. The second two theorems on measurement are often treated as axioms. However, doing so hides the fact that we can accomplish much in geometry without the notion of measurement. Treating the ability to measure as an axiom also hides subtle geometric properties within (unstated) assumptions about the real numbers. Some approaches to geometry will prove the measurement theorems first and then use them to prove the intersection theorems. This approach hides the fact that the intersection theorems are a product of the topology of the plane and are not dependent on the measurement theorems.

Theorem. 4.27 (Circle-Segment Intersection or Elementary Circular Continuity.) *Suppose that γ is a circle, that A is in the interior of γ , and that B is in the exterior of γ . Then \overline{AB} intersects the circle γ .*

Theorem. 4.31 (Circle-Circle Intersection or Circular Continuity.) *Suppose that γ and δ are two circles. If γ contains a point in the interior of δ and if γ contains a point in the exterior of δ , then γ and δ intersect in two points which are on opposite sides of the line connecting the centers of the circles.*

Theorem. 4.40 (Ruler Theorem.) *For any points A and B there is a unique real number called the length of \overline{AB} denoted as $|\overline{AB}|$. Suppose that A , B , C , and D are points.*

1. If $A \neq B$ $|\overline{AB}| > 0$.
2. $|\overline{AB}| = 0$ if and only if $A = B$.
3. $\overline{AB} \cong \overline{CD}$ if and only if $|\overline{AB}| = |\overline{CD}|$.
4. $\overline{AB} < \overline{CD}$ if and only if $|\overline{AB}| < |\overline{CD}|$.
5. If $A * B * C$ then $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$.
6. For every positive real number x , there is a segment \overline{AB} so that $|\overline{AB}| = x$.

Theorem. 4.48 (Protractor Theorem.) *For any angle α there is a unique real number called the measure of α denoted as $|\alpha|$. Suppose that $\angle CAB$ and $\angle ZXY$ are angles.*

1. $0 < |\angle CAB| < 180$.
2. If $\angle CAB$ is a right angle, then $|\angle CAB| = 90^\circ$.
3. $\angle CAB \cong \angle ZXY$ if and only if $|\angle CAB| = |\angle ZXY|$.
4. $\angle CAB < \angle ZXY$ if and only if $|\angle CAB| < |\angle ZXY|$.
5. If D is interior to $\angle CAB$ then $|\angle CAB| = |\angle CAD| + |\angle DAB|$.
6. For every real number x with $0 < x < 180$, there is an angle α so that $|\alpha| = x^\circ$.

4.1 Continuous Functions on the Plane

Definition. Suppose that S and T are subsets of \mathbb{P} , that $f : S \rightarrow T$ is a function, and that A is in S . The function f is *continuous* at A if for every circle ϵ centered at $f(A)$, there is a circle δ centered at A so that if P is in the intersection of S and the interior of ϵ then $f(P)$ is in the interior of δ . If f is continuous at every $P \in S$ then f is continuous on S .

Proposition 4.1. Suppose that \overline{AB} is a chord of the circle $\gamma = \mathcal{C}(C, R)$ not passing through C . Define a function $f : \overline{AB} \rightarrow \gamma$ in the following way. Suppose that $X \in \overline{AB}$. Since $C \notin \overline{AB}$, $C \neq X$, and we can consider the ray \overrightarrow{CX} . There is a unique point Y on \overrightarrow{CX} so that $\overline{CY} \cong \overline{CR}$. Let $f(X) = Y$. The function f is continuous on \overline{AB} .

Hint:

1. Let $X \in \overline{AB}$ and let $\epsilon = \mathcal{C}(f(X), S)$ be a circle centered at $f(X)$.
2. Suppose for now that $X \neq A$ and $X \neq B$.
3. Let ℓ be the line perpendicular to $\overleftrightarrow{Cf(X)}$ at $f(X)$.
4. Note that ℓ is tangent to γ at $f(X)$.
5. Let A' be the foot of the perpendicular from $f(X)$ to \overleftrightarrow{CA} , and let B' be the foot of the perpendicular from $f(X)$ to \overleftrightarrow{CB} . How do we know that neither of these is equal to $f(X)$?
6. Find a point $E \in \ell$ so that
 - $\overline{f(X)E} < \overline{f(X)S}$
 - $\overline{f(X)E} < \overline{f(X)A'}$
 - $\overline{f(X)E} < \overline{f(X)B'}$

How do we know such a E exists?

7. Let E' be the point on the ray opposite $\overrightarrow{f(X)E}$ so that $\overline{f(X)E} \cong \overline{f(X)E'}$.
8. Argue that E and E' are in the interior of $\angle ACB$.
9. The rays \overrightarrow{CE} and $\overrightarrow{CE'}$ must intersect \overline{AB} (why?). Call the points of intersection D and D' , respectively. If $\overline{XD'} < \overline{XD}$, let $\delta = \mathcal{C}(X, D')$. Otherwise, let $\delta = \mathcal{C}(X, D)$.
10. Let P be a point in the intersection of the interior of δ and \overline{AB} . Then \overrightarrow{CP} is between the rays \overrightarrow{CE} and $\overrightarrow{CE'}$.
11. Let P' be the intersection of \overrightarrow{CP} with ℓ . If $P' = f(X)$, there is nothing to prove (why?). Without loss of generalization, suppose that $f(X) * P' * E$.
12. The triangle $\triangle Cf(X)f(P)$ is an isosceles triangle with base angles at $f(X)$ and $f(P)$ (why?) so $\angle Cf(P)f(X)$ is acute.
13. The angle $\angle f(X)f(P)P'$ is obtuse (why?).
14. Support these inequalities: $\overline{f(X)f(P)} < \overline{f(X)P'} < \overline{f(X)E} < \overline{f(X)S}$.
15. Now, what if $X = A$ or $X = B$?

16. It follows that $f(P) \in \epsilon$ and that f is continuous at P .

Definition. We will call the function f in 4.1 the *projection* of \overrightarrow{AB} to γ .

Proposition 4.2. Suppose that A and R are distinct points. We define a function f mapping the entire plane \mathbb{P} to \overrightarrow{AR} . First, let $f(A) = A$. Now, if $X \neq A$ there is a unique point Y on \overrightarrow{AR} so that $\overrightarrow{AX} \cong \overrightarrow{AY}$. Let $f(X) = Y$. This function f is continuous on \mathbb{P} .

Hint:

1. Let X be a point.
2. Let ϵ be a circle centered at $f(X)$. Find antipodal points E and E' on ϵ so that $A * E' * E$ and $\epsilon = \mathcal{C}(f(X), E)$ and $\overrightarrow{f(X)E} \cong \overrightarrow{f(X)E'}$.
3. Find $D \in \overrightarrow{AX}$ so that $A * X * D$ and $\overrightarrow{AD} \cong \overrightarrow{f(X)E}$. Let $\delta = \mathcal{C}(X, D)$, and let D' be the point on δ antipodal to D .
4. Argue that $f(D) = E$ and $f(D') = E'$.
5. Suppose that P is in the interior of δ . If $P \in \overrightarrow{AX}$ then $D' * P * D$ and $E' * f(P) * E$, so $f(P)$ is in the interior of ϵ .
6. Assume P is not on \overrightarrow{AX} .
7. Let F be the foot of the perpendicular from P to \overrightarrow{AX} .
8. Argue that F is in the interior of ϵ .
9. Support these inequalities: $\overrightarrow{AD'} < \overrightarrow{AF} < \overrightarrow{AP}$.
10. Let G be the point on \overrightarrow{XD} so that $\overrightarrow{XG} \cong \overrightarrow{XP}$.
11. Support these inequalities: $\overrightarrow{AP} < \overrightarrow{AG} < \overrightarrow{AD}$
12. We have $\overrightarrow{AD'} < \overrightarrow{AP} < \overrightarrow{AD}$.
13. Argue that $\overrightarrow{AE'} < \overrightarrow{Af(P)} < \overrightarrow{AE}$.
14. Argue that $f(P)$ is in the interior of ϵ , and finish the proof.

Definition. We will call the function f in 4.2 the *distance function ruled by* \overrightarrow{AR} .

Proposition 4.3. Suppose that $f : R \rightarrow S$ is continuous on R and that $g : S \rightarrow T$ is continuous on S . The function $g \circ f : R \rightarrow T$ is continuous on R .

Hint:

1. Suppose that $X \in S$ and that ϵ is a circle centered at $g \circ f(X) = g(f(X))$.
2. Since g is continuous on S , there is a circle δ_g centered at $f(X)$ so that if P is any point in the intersection of the interior of δ_g and S , then $g(P) \in \epsilon$.
3. Since f is continuous on R , there is a circle δ centered at X so that if P is any point in the intersection of the interior of δ and R , then $f(P) \in \delta_g$.
4. Now finish the proof.

4.2 Dedekind Cuts

Definition. A *Dedekind cut* of a line ℓ is a partition $\{\Sigma_1, \Sigma_2\}$ of ℓ into two nonempty convex subsets. The sets Σ_1 and Σ_2 are called the *slices* of the Dedekind cut.

Proposition 4.4. *If ℓ is any line, then there is at least one Dedekind cut of ℓ .*

Definition. A *cut point* of a Dedekind cut of a line ℓ is a point C on ℓ so that for all $X, Y \in \ell$, if $C * X * Y$ then X and Y are in the same slice.

Proposition 4.5. *Suppose that C is a cut point of a Dedekind cut of a line ℓ . If X and Y are points on ℓ so that $X * C * Y$ then X and Y are in different slices.*

Hint: Suppose by way of contradiction that X and Y are in the same slice. Apply convexity to conclude something about C . Select a point Z in the other slice and do cases with four point betweenness.

Proposition 4.6. *Suppose that $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of a line ℓ and that C is a cut point. If $X \in \Sigma_1$ and $Y \in \Sigma_2$, then $X * C * Y$.*

Hint: Do cases on betweenness.

Proposition 4.7. *Suppose that $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of a line ℓ and that C is a cut point. If $X \in \Sigma_1$ and $X \neq C$, then there is some $Y \in \Sigma_1$ so that $X * Y * C$.*

Hint: There has to be something between X and C . The definition of cut point then helps.

Proposition 4.8. *If a Dedekind cut of a line has a cut point, then it has exactly one cut point.*

Hint: Suppose that a Dedekind cut $\{\Sigma_1, \Sigma_2\}$ has two distinct cut points C and D . Consider points X, Y , and Z so that $X * C * Y$, $C * Y * D$, and $Y * D * Z$. Without loss of generality, assume that $X \in \Sigma_1$. Apply the assumption that C is a cut point and figure out which slice the other points are in. Now apply the assumption that D is a cut point and figure out which slice the other points are in.

Completeness Axiom 1. (*Dedekind's Axiom*) *Every Dedekind cut of a line has a cut point.*

Exercise 4.9. Use Dedekind cuts to prove that midpoints of line segments exist.

Note. We can also define Dedekind cuts and cut points for segments and rays. In each case, a Dedekind cut is a a partition $\{\Sigma_1, \Sigma_2\}$ of the segment or ray into two nonempty convex subsets. If we do so, all of the theorems above hold for segments and rays also. The reason for this is the next proposition for line segments and its corresponding proposition for rays.

Proposition 4.10. *Every Dedekind cut of a line segment has a cut point.*

Hint: Let $\{\Sigma_1, \Sigma_2\}$ be a Dedekind cut of a line segment \overline{AB} with $A \in \Sigma_1$ and $B \in \Sigma_2$. Let $\Gamma_1 = \{P \in \overline{AB} : P * A * B \text{ or } P \in \Sigma_1\}$ and $\Gamma_2 = \{P \in \overline{AB} : A * B * P \text{ or } P \in \Sigma_2\}$. Show that $\{\Gamma_1, \Gamma_2\}$ is a Dedekind cut of \overline{AB} and that its cut point is in \overline{AB} .

4.3 The Intermediate Value Theorem

Theorem 4.11 (Intermediate Value Theorem). *Suppose that \overline{AC} is a segment and \overleftrightarrow{XZ} is a line, and let $f : \overline{AC} \rightarrow \overleftrightarrow{XZ}$ be a function that is continuous on \overline{AC} . If $f(A) = X$ and $f(B) = Z$ and $X * Y * Z$, then there is a B so that $A * B * C$ and $f(B) = Y$.*

This theorem is proven through the following claimable lemmas.

Setup for the proof of Theorem 4.11: Let Σ_1 be the set of all $P \in \overline{AC}$ for which there is a $Q \in \overline{AC}$ so that either $A * P * Q$ or $P * Q * C$ and $f(Q) * Y * Z$, and let Σ_2 be $\overline{AC} - \Sigma_1$.

Note. Informally, think of A as being to the left of C and X as being to the left of Z . Σ_1 is the set of all points which are to the left of a point Q so that $f(Q)$ is to the left of Y .

Lemma 4.12. $A \in \Sigma_1$.

Hint:

1. Let M be the midpoint of \overline{XY} . Note that $X = f(A)$.
2. Let $\epsilon = \mathcal{C}(X, M)$.
3. There is a circle δ centered at A so that if P is in \overline{AC} and if P is in the interior of δ , then $f(P) \in \epsilon$.
4. Select R so that $\delta = \mathcal{C}(A, R)$. Note that $A * R * C$ (why?).
5. Let Q be the midpoint of \overline{AR} .
6. Then $A * Q * C$ and $f(Q) * Y * Z$.

Lemma 4.13. $C \in \Sigma_2$.

Hint: There is no $Q \in \overline{AC}$ with $A * C * Q$ or with $C * Q * C$.

Lemma 4.14. Σ_1 is convex.

Hint:

1. Suppose that $D \neq F$ in Σ_1 and that $D * E * F$.
2. There is some $G \in \overline{AC}$ with either $A * F * G$ or $F * G * C$ and $f(G) * Y * Z$.
3. Then $A * E * G$ and $f(G) * Y * Z$, so $G \in \Sigma_1$.

Lemma 4.15. Σ_2 is convex.

Hint:

1. Suppose $D \neq F$ in Σ_2 and that $D * E * F$. To show $E \in \Sigma_2$, we just need to show that $E \notin \Sigma_1$.
2. If there were some $G \in \overline{AC}$ with either $A * E * G$ or $E * G * C$ and $f(G) * Y * Z$, then we would also have $A * D * G$. This would put $D \in \Sigma_1$, a contradiction.

Lemma 4.16. $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut.

More setup for the proof of 4.11: Let B be the cut point of $\{\Sigma_1, \Sigma_2\}$.

Lemma 4.17. $B \neq A$

Hint:

1. Suppose $B = A$
2. Note that $X = f(A) = f(B)$.
3. Let M be the midpoint of \overline{XY} .
4. Let $\epsilon = \mathcal{C}(X, M)$.
5. There is a circle δ centered at B so that if P is in \overline{AC} and if P is in the interior of δ , then $f(P) \in \epsilon$.
6. Select R so that $\delta = \mathcal{C}(B, R)$. Note that $B * R * C$ (why?).
7. Let Q be the midpoint of \overline{BR} .
8. Then $B * Q * C$ and $f(Q) * Y * Z$.

Lemma 4.18. If $A \neq U$ and $f(U) * Y * Z$ then there is some $V \in \Sigma_1$ so that $A * U * V$

Hint:

1. Let M be the midpoint of $\overline{f(U)Y}$.
2. Let $\epsilon = \mathcal{C}(f(U), M)$.
3. There is a circle δ centered at U so that if P is in the interior of δ and if P is in \overline{AC} then $f(P) \in \epsilon$.
4. Select V in \overline{AC} so that V is in the interior of δ and $A * U * V$. This is possible because $A \neq U$.
5. Since $V \in \delta$, $f(V) \in \epsilon$. This implies $f(U) * f(V) * M * Y$, so $f(V) * Y * Z$.
6. Thus, $V \in \Sigma_1$ and $A * U * V$.

Lemma 4.19. If $A \neq U$ and $f(U) * Y * Z$, then $U \in \Sigma_1$.

Lemma 4.20. It cannot be that $f(B) = Z$.

Hint:

1. Suppose that $f(B) = Z$.
2. Let M be the midpoint of \overline{YZ} , and let $\epsilon = \mathcal{C}(Z, M)$.
3. There is a circle δ centered at B so that if P is in the interior of δ and if P is in \overline{AC} then $f(P) \in \epsilon$.
4. There is an R with $A * R * B$ so that $\delta = \mathcal{C}(B, R)$ (why?).
5. Because B is the cutpoint of $\{\Sigma_1, \Sigma_2\}$, $A * R * B$ and $A \in \Sigma_1$, we have $R \in \Sigma_1$.
6. There is some $Q \in \overline{AC}$ so that either $A * R * Q$ or $R * Q * C$ and $f(Q) * Y * Z$.

7. Since $A * R * B$, $A * R * Q$.
8. Since $A \neq Q$ and $f(Q) * Y * Z$, there is some $V \in \Sigma_1$ so that $A * Q * V$. This means that $Q \in \Sigma_1$.
9. It cannot be that $Q \in \delta$ because then $f(Q) \in \epsilon$ and it would not be the case that $f(Q) * Y * Z$.
10. Since $A * R * Q$ and Q is not in $\delta = \mathcal{C}(B, R)$, it follows that $A * B * Q$. This is a contradiction since $A, Q \in \Sigma_1$.

Lemma 4.21. $B \neq C$.

Lemma 4.22. If $C \neq U$ and $U \in \Sigma_2$ and $Y * f(U) * Z$ then there is some $V \in \Sigma_2$ so that $V * U * C$.

Hint:

1. Let M be the midpoint of $\overline{Yf(U)}$.
2. Let ϵ be the circle $\mathcal{C}(f(U), M)$.
3. By continuity, there is a circle δ centered at U so that if P is in the interior of δ and $P \in \overline{AC}$ then $f(P) \in \epsilon$.
4. Select R on \overrightarrow{UC} so that $\delta = \mathcal{C}(U, R)$.
5. Let $V \in \delta$ so that $V * U * C$ and $V \neq A$.
6. Since $V \in \delta$, we know that $f(V) \in \epsilon$, so $Y * M * f(V)$.
7. It has to be that $V \in \Sigma_2$.
 - (a) Suppose $V \in \Sigma_1$.
 - (b) Suppose that $A * V * Q$ and $f(Q) * Y * Z$.
 - (c) Note that $Q \in \Sigma_1$.
 - (d) It cannot be that $Q \in \delta$ because then we would have $X * Y * M * f(Q)$ and could not have $f(Q) * Y * Z$.
 - (e) Since $A * V * Q$ and Q is not in $\delta = \mathcal{C}(U, R)$ but $V \in \delta$, then it must be that $A * U * Q$.
 - (f) But then $A \in \Sigma_1$ and $Q \in \Sigma_1$ but $U \in \Sigma_2$, a contradiction.

More setup for the proof of 4.11: Either $f(B) = Y$ or $f(B) * Y * Z$ or $Y * f(B) * Z$.

Lemma 4.23. It is not the case that $f(B) * Y * Z$.

Hint: If $f(B) * Y * Z$, then there is a $V \in \Sigma_1$ so that $A * B * V$. However, since $A * B * V$ and $B \in \Sigma_1$, then $V \in \Sigma_2$, which is a contradiction.

Lemma 4.24. $B \in \Sigma_2$.

Hint:

1. If $B \in \Sigma_1$, then there is a Q so that $A * B * Q$ and $f(Q) * Y * Z$.
2. This Q is in Σ_1 , but then A and Q are on opposite sides of B but in the same slice, a contradiction.

Lemma 4.25. *It is not the case that $Y * f(B) * Z$.*

Hint:

1. We know $B \in \Sigma_2$. Suppose $Y * f(B) * Z$. This implies that there is a $V \in \Sigma_2$ so that $V * B * C$. Since $C \in \Sigma_2$, this forces $V \in \Sigma_1$, a contradiction.

Lemma 4.26. $f(B) = Y$.

Hint: The only option left is that $B \in \Sigma_2$ and $f(B) = Y$.

4.4 Intersections

Theorem 4.27. *(Circle-Segment Intersection or Elementary Circular Continuity) Suppose that γ is a circle, that A is in the interior of γ , and that B is in the exterior of γ . Then \overline{AB} intersects the circle γ .*

Hint:

1. Let $\gamma = \mathcal{C}(C, R)$.
2. Let $f : \overline{AB} \rightarrow \overrightarrow{CR}$ be the distance function ruled by \overrightarrow{CR} .
3. Argue that $f(A) * R * f(B)$.
4. Apply the Intermediate Value Theorem.

Lemma 4.28. *Suppose that γ is a circle centered at C with radius \overline{CR} and that D is any point other than C . Let A be the unique point on \overrightarrow{CD} with $\overline{CA} \cong \overline{CR}$. If X is any point on γ other than A , then $\overline{DA} < \overline{DX}$.*

Hint:

1. What if X is on \overleftrightarrow{CA} ?
2. Suppose X is not on \overleftrightarrow{CA}
3. Suppose first that $C * A * D$.
 - (a) Let F be the foot of the line through X perpendicular to \overleftrightarrow{CD} .
 - (b) Use the Exterior Angle Theorem to argue that it is not the case that $C * A * F$.
 - (c) Argue that $A \neq F$ and $D \neq F$.
 - (d) So $F * A * D$.
 - (e) Consider the right triangle $\triangle DFX$.
 - (f) Support these comparisons:

$$\overline{DA} < \overline{DF} < \overline{DX}.$$

4. Now suppose that $C * D * A$.

- (a) Support these comparisons:

$$\angle DXA < \angle CXA \cong \angle CAX.$$

(b) Apply the Scalene Inequality.

Lemma 4.29. *Suppose that γ is a circle centered at C with radius \overline{CR} and that D is any point other than C . Let A be the unique point on the ray opposite \overrightarrow{CD} with $\overline{CA} \cong \overline{CR}$. If X is any point on γ other than A , then $\overline{DA} > \overline{DX}$.*

Hint:

1. What if X is on \overleftrightarrow{CA} ?
2. Suppose X is not on \overleftrightarrow{CA} .
3. Apply the Triangle Inequality Form 1 to the triangle $\triangle DCX$ with $D * C * A$ and $\overline{CA} \cong \overline{CX}$.

Lemma 4.30. *Suppose that γ is the circle centered at C with radius \overline{CR} and that δ is any circle centered at a point $D \neq C$. If γ contains a point in the interior of δ and a point in the exterior of δ , then there is a point A on \overleftrightarrow{CD} and points B and B' on either side of \overleftrightarrow{CD} so that*

- A , B , and B' are on γ .
- Either A is in the interior of δ and both B and B' are in the exterior of δ or A is in the exterior of δ and both B and B' are in the interior.
- Angles $\angle ACB$ and $\angle ACB'$ are right angles.

Hint:

1. Let X be the unique point on \overleftrightarrow{CD} with $\overline{CX} \cong \overline{CR}$, and let Y be the unique point on the ray opposite \overleftrightarrow{CD} with $\overline{CY} \cong \overline{CR}$.
2. Use 4.28 and 4.29 to argue that X is in the interior of δ and Y is in the exterior of δ .
3. Let ℓ be the line perpendicular to \overleftrightarrow{CD} through point C , and let Z and W be the two points where ℓ intersects γ . (Why do these exist?)
4. Argue that either both of Z and W are in the interior of δ or both are in the exterior.
5. If both of Z and W are in the interior of δ , then let $A = Y$ and $B = Z$, and $B' = W$.
6. If both of Z and W are in the exterior of δ , then let $A = X$ and $B = Z$, and $B' = W$.

Theorem 4.31 (Circle-Circle Intersection or Circular Continuity). *Suppose that γ and δ are two circles. If γ contains a point in the interior of δ and if γ contains a point in the exterior of δ , then γ and δ intersect in two points which are on opposite sides of the line connecting the centers of the circles.*

Hint:

1. Let $\gamma = \mathcal{C}(C, R)$ and let $\delta = \mathcal{C}(D, S)$.
2. Find A , B and B' as in 4.30.
3. Let $f : \overline{AB} \rightarrow \gamma$ be the projection of \overline{AB} to γ .
4. Let $g : \gamma \rightarrow \overleftrightarrow{DS}$ be the distance function ruled by \overleftrightarrow{DS} .

5. Let $h : \overrightarrow{AB} \rightarrow \overrightarrow{DS}$ be $g \circ f$.
6. The function $h = g \circ f$ is continuous.
7. Argue that $h(A) * S * h(B)$.
8. Apply the Intermediate Value Theorem.
9. This (eventually) gives a point on $\gamma \cap \delta$ on the same side of \overrightarrow{CD} as B . A similar argument gives another point on the same side as B' .

4.5 Segment Measurement

Definition. Let O and I be any two distinct points. For any $A \neq O$ in \overrightarrow{OI} , let $-A$ be the point in the ray opposite \overrightarrow{OA} so that $\overrightarrow{OA} \cong \overrightarrow{O(-A)}$.

Definition. The *dyadic rationals* are all real numbers of the form $\frac{a}{2^b}$ where a is any integer and b is a nonnegative integer. Denote the set of dyadic rationals by \mathbb{D} .

Definition. For any nonnegative integer n and any point P on \overrightarrow{OI} , we define a point $n \cdot P$. First $n \cdot O = O$ for all nonnegative integers n . Let $P \neq O$ be a point on \overrightarrow{OI} . We use recursion to define $n \cdot P$.

- $0 \cdot P = O$
- $1 \cdot P = P$.
- If $n \geq 1$ and $n \cdot P$ is defined, then $(n+1) \cdot P$ is the unique point on the ray opposite $\overrightarrow{(n \cdot P)O}$ so that $\overrightarrow{(n \cdot P)((n+1) \cdot P)} \cong \overrightarrow{OP}$.

Note. Think of $n \cdot P$ as the point you arrive at by starting off at O and marking off n segments congruent to P in the direction of \overrightarrow{OP} .

Proposition 4.32 (Archimedean Principle). *Suppose that P is any point in \overrightarrow{OI} and that Q is any point in \overrightarrow{OI} different from O . There is a positive integer n so that $-I * P * n \cdot Q$.*

Note. We use $-I$ here only because we are allowing P to be O . In that case, we cannot write $O * O * n \cdot Q$.

Hint:

1. Let Σ_1 be the set of all P in \overrightarrow{OI} for which there is such an n .
2. Note that $n \cdot Q \in \Sigma_1$ for all positive integers n .
3. Σ_1 is not empty.
4. Let $\Sigma_2 = \overrightarrow{OI} - \Sigma_1$.
5. We want Σ_2 to be empty. Suppose by way of contradiction that it is not.
6. Show that $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of \overrightarrow{OI} .
7. Let C be the cut point of $\{\Sigma_1, \Sigma_2\}$.

8. If $C = n \cdot Q$ for some positive integer n , then argue that $(n-1) \cdot Q * C * (n+1) \cdot Q$ but $(n-1) \cdot Q$ and $(n+1) \cdot Q$ are both in Σ_1 . This would be a contradiction. So C cannot be $n \cdot Q$ for any positive integer n .
9. Suppose by way of contradiction that $C \in \Sigma_1$.
10. There is some positive integer n so that $-I * C * n \cdot Q$. Select the smallest such n . Let $m = n-1$ so that $-I * m \cdot Q * C * (m+1) \cdot Q$. Why is this a contradiction?
11. This means that $C \in \Sigma_2$.
12. Find $R \in \overrightarrow{CO}$ so that $\overline{CR} \cong \overline{OQ}$. Use R to construct a point S with $R * C * S$ and $S \in \Sigma_1$.
13. You have a contradiction.

Definition. For any nonnegative integer n and any point P on \overrightarrow{OI} we define a point $\frac{1}{2^n} \cdot P$ on \overrightarrow{OI} . First $\frac{1}{2^0} \cdot O = O$ for all nonnegative integers n . Let $P \neq O$ be a point on \overrightarrow{OI} . We use recursion to define $\frac{1}{2^n} \cdot P$.

- $\frac{1}{2^0} \cdot P = P$
- If $n \geq 0$ and $\frac{1}{2^n} \cdot P$ is defined, then $\frac{1}{2^{(n+1)}} \cdot P$ is the midpoint of $\overline{O(\frac{1}{2^n} \cdot P)}$.

Note. To find $\frac{1}{2^n} \cdot P$ we first find the midpoint $\frac{1}{2} \cdot P$ of \overline{OP} . Then we find the midpoint $\frac{1}{2^2} \cdot P$ of $\overline{O(\frac{1}{2} \cdot P)}$. We continue until we have found n midpoints.

Proposition 4.33 (Archimedean Principle for Halving). *Suppose that P and Q are points \overrightarrow{OI} other than O . There is a nonnegative integer n so that $0 * [\frac{1}{2^n} \cdot Q] * P$.*

Hint:

1. Let Σ_1 be the points P in \overrightarrow{OI} for which such an n exists. And let $\Sigma_2 = \overrightarrow{OI} - \Sigma_1$.
2. Note that $\frac{1}{2^n} \cdot Q \in \Sigma_1$ for all nonnegative integers n .
3. Prove that $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of \overrightarrow{OI} and let C be its cut point.
4. We want to show that $C = O$ so that $\Sigma_2 = \{O\}$.
5. Suppose that $C \neq O$.
6. Argue that $O * C * Q$.
7. Suppose that $C \in \Sigma_1$. Use the definition of this to argue that the midpoint of \overline{OC} is in Σ_1 to get a contradiction.
8. So $C \in \Sigma_2$. Find the unique point A on \overrightarrow{CQ} so that $\overline{CA} \cong \overline{OC}$. Argue that $A \in \Sigma_1$ and use this to argue that $C \in \Sigma_1$. You have another contradiction.

Definition. If a and b are nonnegative integers, then we use $\frac{a}{2^b} \cdot P$ to represent $a \cdot (\frac{1}{2^b} \cdot P)$.

Proposition 4.34. *Suppose that $P \neq Q$ are points on \overrightarrow{OI} . There are positive integers a and b so that $P * [\frac{a}{2^b} \cdot I] * Q$.*

Hint: Find R on \overrightarrow{OI} so that $\overline{OR} \cong \overline{PQ}$. Find b so that $0 * \left[\left(\frac{1}{2^b}\right) \cdot I\right] * R$. Then find an appropriate a .

Proposition 4.35. *Suppose that a, b, c , and d are nonnegative integers. Then $\frac{a}{2^b} < \frac{c}{2^d}$ if and only if $-I * \frac{a}{2^b} \cdot I * \frac{c}{2^d} \cdot I$.*

Definition. For each point P on \overrightarrow{OI} and each nonnegative integer k , let $m_k(P)$ be the greatest nonnegative integer so that either $P = \left(\frac{m_k(P)}{2^k} \cdot I\right)$ or $-I * \left(\frac{m_k(P)}{2^k} \cdot I\right) * P$.

Lemma 4.36. *For any P , the sequence $\left\langle \frac{m_k(P)}{2^k} \right\rangle$ is bounded and increasing (nondecreasing) and therefore converges.*

Definition. Define a function $\# : \overrightarrow{OI} \rightarrow \mathbb{R}$ by $\#(P) = \lim_{k \rightarrow \infty} \frac{m_k(P)}{2^k}$.

Theorem 4.37. *The following hold for all $P, Q \in \overrightarrow{OI}$.*

- $\#(P)$ is a nonnegative real number.
- $\# \left(\frac{a}{2^b} \cdot I\right) = \frac{a}{2^b}$.
- $\#(O) = 0$
- $\#(I) = 1$
- $\#(P) < \#(Q)$ if and only if $-I * P * Q$.

Proposition 4.38. *For every positive real number x there is a point $P \in \overrightarrow{OI}$ so that $\#(P) = x$.*

Definition. If $A \neq B$ are points, let $P_{\overline{AB}}$ be the unique point on \overline{OI} so that $\overline{OP_{\overline{AB}}} \cong \overline{AB}$. If $A = B$, let $P_{\overline{AB}} = O$. The distance between A and B is $d(A, B) = \#P_{\overline{AB}}$. For any line segment \overline{AB} , the length of \overline{AB} is $|\overline{AB}| = d(A, B)$.

Proposition 4.39. *The following hold for any points P and Q*

- $d(P, Q) \geq 0$.
- $d(P, Q) = 0$ if and only if $P = Q$.
- $d(P, Q) = d(Q, P)$.

Theorem 4.40 (Ruler Theorem). *Suppose that A, B, C , and D are points.*

1. $|\overline{AB}| > 0$.
2. $|\overline{AB}| = 0$ if and only if $A = B$.
3. $\overline{AB} \cong \overline{CD}$ if and only if $|\overline{AB}| = |\overline{CD}|$.
4. $\overline{AB} < \overline{CD}$ if and only if $|\overline{AB}| < |\overline{CD}|$.
5. If $A * B * C$ then $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$.
6. For every positive real number x , there is a segment \overline{AB} so that $|\overline{AB}| = x$.

This theorem is proven by the following claimable lemmas.

Lemma 4.41. $|\overline{OI}| = 1$.

Lemma 4.42. If \overline{AB} and \overline{CD} are segments, then $|\overline{AB}| = |\overline{CD}|$ if and only if $\overline{AB} \cong \overline{CD}$.

Lemma 4.43. If \overline{AB} and \overline{CD} are segments, then $|\overline{AB}| < |\overline{CD}|$ if and only if $\overline{AB} < \overline{CD}$.

Lemma 4.44. If \overline{AB} is a segment, then $|\overline{AB}| = |\overline{BA}|$.

Lemma 4.45. If A , B , and C are three distinct points, then $A * B * C$ if and only if $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$.

Lemma 4.46. For every positive real number x , there is a segment \overline{AB} so that $|\overline{AB}| = x$.

4.6 Angle Measurement

For segment measurement, we basically chose arbitrary points O and I as a fundamental unit along with an (almost) arbitrary number 1, assigned the length of \overline{OI} to be 1 and derived all other lengths from this assignment using dyadic rationals. We can do the same thing with angles. We can select an arbitrary right angle, assign it the arbitrary measure of 90° (read as *90 degrees*), and then use dyadic rationals to assign a measure between 0 and 180 to every other angle. The one thing we are missing is the notion of Dedekind cut for angles.

Definition. Suppose that \overline{AB} is any line and D is any point not on \overline{AB} . Let \mathbb{F} be the set of all angles $\angle CAB$ where C is a point on the same side of \overline{AB} as D . (In case you are wondering, \mathbb{F} is for *fan*.) A Dedekind cut of \mathbb{F} is a partition $\{\Sigma_1, \Sigma_2\}$ of \mathbb{F} into two nonempty disjoint sets called *slices* so that if α and γ are angles in one slice and β is an angle between α and γ , then β is in the same slice as α and γ . A cut point of a Dedekind cut is an angle α in \mathbb{F} so that if β and γ are angles so that β is between α and γ , then β and γ are in the same slice.

Theorem 4.47 (Completeness for Angles). *Every Dedekind cut of \mathbb{F} has a cut point.*

Armed with this completeness theorem, we can define a measure of angles (denoted $|\angle ABC|$) so that the following theorem is true.

Theorem 4.48 (Protractor Theorem). *Suppose that $\angle CAB$ and $\angle ZXY$ are angles.*

1. $0 < |\angle CAB| < 180$.
2. If $\angle CAB$ is a right angle, then $|\angle CAB| = 90^\circ$.
3. $\angle CAB \cong \angle ZXY$ if and only if $|\angle CAB| = |\angle ZXY|$.
4. $\angle CAB < \angle ZXY$ if and only if $|\angle CAB| < |\angle ZXY|$.
5. If D is interior to $\angle CAB$ then $|\angle CAB| = |\angle CAD| + |\angle DAB|$.
6. For every real number x with $0 < x < 180$, there is an angle α so that $|\alpha| = x^\circ$.

5 Neutral Geometry

5.1 Adding Measurements

Proposition 5.1 (Triangle Inequality). *If A , B , and C are three noncollinear points, then $|\overline{AC}| < |\overline{AB}| + |\overline{BC}|$.*

Hint: Use 3.59 and 4.40.

Proposition 5.2. *If A and B are distinct points and x is a positive real number, then there exists a unique point $C \in \overleftrightarrow{AB}$ so that $|\overline{AC}| = x$.*

Hint: Use 4.40 and Point Construction.

Proposition 5.3. *If A and B are distinct points and x is a real number between 0 and 180, then on either side of \overleftrightarrow{AB} there exists a unique point C with $|\angle CAB| = x^\circ$.*

Hint: Use 4.48 and Angle Construction.

Proposition 5.4. *Suppose that A , B , C , and D are four distinct points and that C and D are on the same side of \overleftrightarrow{AB} . Then $|\angle DAB| < |\angle CAB|$ if and only if D is in the interior of $\angle CAB$.*

Hint: Use the definition of betweenness for angles and the definition of order for angles and 4.48.

Theorem 5.5 (Linear Pair Theorem). *If $\angle DAC$ and $\angle DAB$ are a linear pair then $|\angle DAC| + |\angle DAB| = 180^\circ$.*

Hint: Find E on the same side of \overleftrightarrow{BC} as D so that $\angle EAB$ is a right angle. The ray \overrightarrow{AE} “splits” one of the angles involved. Now use 4.48 twice.

5.2 The Saccheri-Legendre Theorem

Definition. The *angle sum* of a triangle $\triangle ABC$ is the sum of the measures of the three interior angles of $\triangle ABC$:

$$\sigma(\triangle ABC) = |\angle ABC| + |\angle BCA| + |\angle CAB|.$$

Proposition 5.6. *Suppose that $\triangle ABC$ is any triangle. Then $|\angle CAB| + |\angle ABC| < 180^\circ$.*

Hint: Use the Exterior Angle Theorem to get an inequality and then add the non-remote interior angle to each side.

Proposition 5.7. *If $\triangle ABC$ is any triangle and E is a point in the interior of \overline{BC} then*

$$\sigma(\triangle AEB) + \sigma(\triangle AEC) = \sigma(\triangle ABC) + 180^\circ.$$

Proposition 5.8. *If $\triangle ABC$ is a triangle, then there is a point D not on \overleftrightarrow{AB} so that $\sigma(\triangle ABD) = \sigma(\triangle ABC)$ and so that the measure of one of the interior angles of $\triangle ABD$ is less than or equal to $\frac{1}{2}|\angle CAB|$.*

Hint: Let E be the midpoint of \overline{BC} , and let D be the point on \overrightarrow{AE} so that $A * E * D$ and $\overline{ED} \cong \overline{AE}$. Use SAS to get a triangle congruence. Apply 5.7 to find $\sigma(\triangle ABC)$ and $\sigma(\triangle ABD)$. This will show that the angle sums are the same. For the portion involving $\frac{1}{2}$, write $|\angle BAC|$ as a sum using 4.48.

Theorem 5.9 (Saccheri-Legendre Theorem). *If $\triangle ABC$ is any triangle then $\sigma(\triangle ABC) \leq 180^\circ$.*

Hint: Suppose by way of contradiction that $\sigma(\triangle ABC) > 180^\circ$ and let $\epsilon = \sigma(\triangle ABC) - 180$. Apply 5.8 repeatedly to get a triangle with the same angle sum and one angle less than ϵ . Explain why this is a contradiction.

Corollary 5.10. *The sum of the measures of two interior angles of a triangle is less than or equal to the measure of their remote exterior angle.*

Hint: Use the Saccheri-Legendre Theorem and the Linear Pair Theorem.

Corollary 5.11 (Converse to Euclid's Fifth Postulate). *Let ℓ and ℓ' be two lines cut by a transversal t . If ℓ and ℓ' meet on one side of t , then the sum of the measures of the two interior angles on that side of t is strictly less than 180° .*

Hint: Use 5.6.

5.3 More on Quadrilaterals

Definition. If $\square ABCD$ is a convex quadrilateral, then the *angle sum* of $\square ABCD$ is the sum of the measures of the four interior angles of the quadrilateral:

$$\sigma(\square ABCD) = |\angle ABC| + |\angle BCD| + |\angle CDA| + |\angle DAB|.$$

Proposition 5.12. *If $\square ABCD$ is a convex quadrilateral then $\sigma(\square ABCD) \leq 360^\circ$.*

Definition. The *defect* of a triangle $\triangle ABC$ is

$$\delta(\triangle ABC) = 180 - \sigma(\triangle ABC).$$

The *defect* of a convex quadrilateral $\square ABCD$ is

$$\delta(\square ABCD) = 360 - \sigma(\square ABCD).$$

Proposition 5.13 (Additivity of Defect for Triangles). *If $\triangle ABC$ is a triangle and $B * E * C$ then $\delta(\triangle ABC) = \delta(\triangle AEB) + \delta(\triangle AEC)$.*

Proposition 5.14 (Additivity of Defect for Quadrilaterals). *If $\square ABCD$ is a convex quadrilateral, then $\delta(\square ABCD) = \delta(\triangle ACB) + \delta(\triangle ACD)$.*

Definition. A quadrilateral with four right angles is a *rectangle*.

Proposition 5.15. *Every rectangle is a parallelogram.*

Theorem 5.16. *The following are equivalent.*

1. *There is a triangle with defect 0° .*
2. *There is a right triangle with defect 0° .*
3. *There is a rectangle.*
4. *There exist rectangles with arbitrarily long sides.*
5. *The defect of every right triangle is 0° .*
6. *The defect of every triangle is 0° .*

This theorem is proven through the following sequence of claimable lemmas.

Lemma 5.17 (Part of proof of 5.16). *If there is a triangle with defect 0° then there is a right triangle with defect 0° .*

Hint: Draw a triangle and drop a perpendicular to the side between two acute angles. Then calculate some defects. Note that all defects are nonnegative.

Lemma 5.18 (Part of proof of 5.16). *If there is a right triangle with defect 0° then there is a rectangle.*

Hint: Draw a triangle $\triangle ABC$ with right angle at B . Find a point D on the opposite side from \overleftrightarrow{AC} to form a triangle congruent to $\triangle ABC$ in such a way that a rectangle is formed.

Lemma 5.19 (Part of proof of 5.16). *If there is a rectangle then there exist rectangles with arbitrarily long sides.*

Hint: Extend two parallel sides of the rectangle and select points on these two sides to construct a rectangle which is twice as long. Be careful not to use results that were ingrained in you when you did Euclidean geometry in high school. You should encounter many triangle congruences.

Lemma 5.20 (Part of proof of 5.16). *If there exist rectangles with arbitrarily long sides then the defect of every right triangle is 0° .*

Hint: Suppose that $\triangle ABC$ is a right triangle with right angle at A . Find a rectangle with one angle $\angle CAB$ whose sides are longer than the legs of $\triangle ABC$. Divide this rectangle into triangles and calculate some defects.

Lemma 5.21 (Part of proof of 5.16). *If the defect of every right triangle is 0° then the defect of every triangle is 0° .*

Hint: Take a triangle and drop a perpendicular to one of the sides.

Lemma 5.22 (Part of proof of 5.16). *If the defect of every triangle is 0° then there is a triangle with defect 0° .*

Definition. A *Saccheri quadrilateral* is a quadrilateral $\square ABCD$ so that $\angle ABC$ and $\angle DAB$ are right angles and $\overline{AD} \cong \overline{CB}$. The segment \overline{AB} is the *base* and the segment \overline{CD} is the *summit*. The angles at A and B are the *base angles* while the angles at C and D are the *summit angles*.

Proposition 5.23. *There exists a Saccheri quadrilateral.*

Proposition 5.24 (Properties of Saccheri Quadrilaterals). *If $\square ABCD$ is a Saccheri quadrilateral with base \overline{AB} then*

1. *The diagonals \overline{AC} and \overline{BD} are congruent.*
2. *The summit angles $\angle ADC$ and $\angle BCD$ are congruent.*
3. *The segment joining the midpoint of the base \overline{AB} and the midpoint of the summit \overline{CD} is perpendicular to both the base and the summit.*
4. *$\square ABCD$ is a parallelogram.*
5. *$\square ABCD$ is a convex quadrilateral.*

6. The summit angles $\angle ADC$ and $\angle BCD$ are either right or acute.

This theorem is proven through the following sequence of claimable lemmas.

Lemma 5.25 (Part of proof of 5.24). *The diagonals \overline{AC} and \overline{BD} are congruent.*

Hint: Use a triangle congruence.

Lemma 5.26 (Part of proof of 5.24). *The summit angles $\angle ADC$ and $\angle BCD$ are congruent.*

Hint: Use a triangle congruence and the previous result.

Lemma 5.27 (Part of proof of 5.24). *The segment joining the midpoint of the base \overline{AB} and the midpoint of the summit \overline{CD} is perpendicular to both the base and the summit.*

Hint: Let M be the midpoint of the base. Use multiple triangle congruences, starting with $\triangle DAM$.

Lemma 5.28 (Part of proof of 5.24). $\square ABCD$ is a parallelogram.

Hint: Use the Alternate Interior Angles Theorem.

Lemma 5.29 (Part of proof of 5.24). $\square ABCD$ is a convex quadrilateral.

Lemma 5.30 (Part of proof of 5.24). *The summit angles $\angle ADC$ and $\angle BCD$ are either right or acute.*

Definition. A *Lambert quadrilateral* is a quadrilateral with at least three right angles.

Proposition 5.31. *There exists a Lambert quadrilateral.*

Theorem 5.32 (Properties of Lambert Quadrilaterals). *If $\square ABCD$ is a Lambert quadrilateral with right angles at vertices A , B , and C then*

1. $\square ABCD$ is a parallelogram.
2. $\square ABCD$ is a convex quadrilateral.
3. $\angle ADC$ is either right or acute.
4. $\overline{BC} < \overline{AD}$ or $\overline{BC} \cong \overline{AD}$.

This theorem is proven through the following sequence of claimable lemmas.

Lemma 5.33 (Part of proof of 5.32). $\square ABCD$ is a parallelogram.

Hint: Use 3.41

Lemma 5.34 (Part of proof of 5.32). $\square ABCD$ is a convex quadrilateral.

Lemma 5.35 (Part of proof of 5.32). $\angle ADC$ is either right or acute.

Lemma 5.36 (Part of proof of 5.32). $\overline{BC} < \overline{AD}$ or $\overline{BC} \cong \overline{AD}$.

Hint: Suppose that $\overline{BC} > \overline{AD}$. Find C' in \overline{BC} so that $\overline{BC'} \cong \overline{AD}$. Find a Saccheri quadrilateral in your diagram and use the Exterior Angle Theorem.

5.4 Neutral Area

Recall that the *interior* of $\triangle ABC$ is the intersection of the interiors of the angles of $\triangle ABC$.

Definition. Suppose that $\triangle ABC$ is a triangle. Denote the interior of $\triangle ABC$ as $\text{Int}\triangle ABC$. The *triangular region associated with $\triangle ABC$* is the union of the triangle and its interior. We denote this by $\blacktriangle ABC$, so $\blacktriangle ABC = \triangle ABC \cup \text{Int}\triangle ABC$.

Definition. A set $\{T_1, T_2, \dots, T_n\}$ of triangular regions are *nonoverlapping* if for all $i \neq j$ either $T_i \cap T_j = \emptyset$ or $T_i \cap T_j$ is a subset of an edge of T_i and an edge of T_j .

Proposition 5.37. If $\triangle ABC$ is a triangle and $E \in \overline{AB}$ then $\blacktriangle ABC = \blacktriangle AEC \cup \blacktriangle BEC$

Proposition 5.38. If $\square ABCD$ is a convex quadrilateral, then $\blacktriangle ACB \cup \blacktriangle ACD = \blacktriangle BDA \cup \blacktriangle BDC$

Proposition 5.39. The quadrilateral region defined by a convex quadrilateral $\square ABCD$ is

$$\blacksquare ABCD = \blacktriangle ACB \cup \blacktriangle ACD = \blacktriangle BDA \cup \blacktriangle BDC.$$

Definition. Suppose that R is a subset of the plane. A *triangulation* of R is a finite set $\{T_1, T_2, \dots, T_n\}$ of nonoverlapping triangular regions so that $R = T_1 \cup T_2 \cup \dots \cup T_n$.

Exercise 5.40. Draw two different triangulations of a triangular region.

Exercise 5.41. Draw two different triangulations of a quadrilateral region.

Exercise 5.42. Give an example of a subset of the plane which has no triangulation.

Definition. A *polygonal region* is a subset of the plane that has a triangulation.

Exercise 5.43. Draw a somewhat complex polygonal region and two triangulations of it.

Proposition 5.44. The intersection of two polygonal regions is a polygonal region.

Proposition 5.45. The union of two polygonal regions is a polygonal region.

Axiom (Neutral Area Postulate). Every polygonal region R is associated with a nonnegative real number $\alpha(R)$ called the *area* of R so that:

- Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles. If $\triangle ABC \cong \triangle DEF$ then $\alpha(\blacktriangle ABC) = \alpha(\blacktriangle DEF)$.
- If R_1 and R_2 are nonoverlapping polygonal regions, then $\alpha(R_1 \cup R_2) = \alpha(R_1) + \alpha(R_2)$.

Proposition 5.46. Suppose that k is any nonnegative real number. For any polygonal region R , let $\beta(R) = k \cdot \alpha(R)$. Then β satisfies the two conditions in the Neutral Area Postulate for α .

Note. This proposition says that if there is an area function α , then any nonnegative multiple of α is also a legitimate area function.

5.5 Statements Equivalent to the Euclidean Parallel Postulate

Here are three options for a parallel postulate:

- **Euclidean Parallel Posulate:** For every line ℓ and for every point P not on ℓ , there is exactly one line m so that P lies on m and $m \parallel \ell$.
- **Elliptic Parallel Posulate:** For every line ℓ and for every point P not on ℓ , there is no line m so that P lies on m and $m \parallel \ell$.
- **Hyperbolic Parallel Posulate:** For every line ℓ and for every point P not on ℓ , there are at least two lines m and n so that P lies on m and n and $m \parallel \ell$ and $n \parallel \ell$.

Note. *Elliptic* means *too few*, and *hyperbolic* means *too many*.

Proposition 5.47. *The Elliptic Parallel Postulate does not hold in Neutral Geometry.*

Note. Below is a list of statements which are equivalent to the Euclidean Parallel Postulate, assuming the axioms of Neutral Geometry. What this means is that if we assume all of the axioms of Neutral Geometry and we assume the Euclidean Parallel Postulate, then we can prove each of these, and if we assume all of the axioms of Neutral Geometry and we assume any one of these, then we can prove the Euclidean Parallel Postulate. It also means that if we have any model of Neutral Geometry, then that model satisfies the Euclidean Parallel Postulate if and only if it satisfies these statements.

Definition. The triangles $\triangle ABC$ and $\triangle DEF$ are *similar* if $\angle ABC \cong \angle DEF$ and $\angle BCA \cong \angle EFD$ and $\angle CAB \cong \angle FDE$. We express this as $\triangle ABC \sim \triangle DEF$.

- **Euclid's Fifth Postulate:** If ℓ and ℓ' are two lines cut by a transversal t in such a way that the sum of the measures of the two interior angles one one side of t is less than 180° , then ℓ and ℓ' intersect on that side of t .
- **Hilbert's Parallel Posulate:** For every line ℓ and for every point P not on ℓ , there is at most one line m so that P lies on m and $m \parallel \ell$.
- **Converse to the Alternating Interior Angles Theorem:** If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.
- **Angle Sum Posulate:** The sum of the angles in any triangle is 180° .
- **Clairaut's Axiom:** There is a rectangle.
- **Transitivity of Parallelism:** If $\ell \parallel m$ and $m \parallel n$ then either $\ell = n$ or $\ell \parallel n$.
- **Proclus's Axiom:** Suppose ℓ and ℓ' are parallel lines and that t is a line other than ℓ . If t intersects ℓ , then t intersects ℓ' .
- **Perpendicular Transversal Condition:** Suppose that ℓ and ℓ' are parallel lines cut by a transversal t . If $t \perp \ell$, then $t \perp \ell'$.
- **$\perp \circ \parallel \circ \perp$ Condition:** If ℓ , m , n , and k are lines so that $m \perp k$, $k \parallel \ell$, and $\ell \perp n$, then either $m = n$ or $m \parallel n$.
- **Wallis's Postulate:** If $\triangle ABC$ is a triangle and \overline{DE} is a segment, then there exists a point F so that $\triangle ABC \sim \triangle DEF$.

- **The Pythagorean Theorem:** Suppose that $\triangle ABC$ is a right triangle with right angle at C . Then $|\overline{AB}|^2 = |\overline{AC}|^2 + |\overline{BC}|^2$.
- **Circumscribed Triangle Condition:** Every triangle can be circumscribed.

Theorem 5.48. *The Euclidean Parallel Postulate is equivalent to Euclid's Fifth Postulate.*

This is proven by the following claimable lemmas.

Lemma 5.49 (Part of the proof of 5.48). *The Euclidean Parallel Postulate implies Euclid's Fifth Postulate.*

Hint: Suppose ℓ and ℓ' are cut by a transversal t so that the interior angles on one side of t add to less than 180° . Call the points of intersection of t with ℓ and ℓ' B and B' , respectively. Find the unique line ℓ'' through B' so that the nonalternating interior angles made by ℓ , ℓ'' and t add to 180° (why does this exist?). Argue that:

- $\ell'' \neq \ell'$
- $\ell'' \parallel \ell$
- ℓ' is not parallel to ℓ .
- Argue that the point of intersection must be on the correct side of t .

Lemma 5.50 (Part of the proof of 5.48). *Euclid's Fifth Postulate implies the Euclidean Parallel Postulate.*

Hint: Suppose that P is a point not on a line ℓ . Drop a perpendicular t from P to ℓ and call the foot Q . Let m be a perpendicular to t at P . Argue that $m \parallel \ell$. You just need to show that no other line through P is parallel to ℓ . Suppose that m' is a line through P other than m . Argue that m' cannot be perpendicular to t . Use Euclid's Fifth Postulate now to argue that m' must intersect ℓ .

Theorem 5.51. *The following statements are equivalent to the Euclidean Parallel Postulate.*

- *The Converse to the Interior Angles Theorem*
- *The Angle Sum Postulate*
- *Clairaut's Axiom*

This is proven by the following claimable lemmas.

Lemma 5.52 (Part of the proof of 5.51). *The Euclidean Parallel Postulate implies the Converse to the Interior Angles Theorem.*

Hint: Suppose that ℓ and ℓ' are cut by a transversal t which intersects ℓ and ℓ' at B and B' respectively. Let ℓ'' be a line through B' which makes alternate interior angles congruent. Argue that $\ell'' \parallel \ell$ and then that $\ell' = \ell''$.

Lemma 5.53 (Part of the proof of 5.51). *The Converse to the Alternate Interior Angles Theorem implies the Angle Sum Postulate.*

Hint: Let $\triangle ABC$ be a triangle. Draw a line ℓ through A parallel to \overleftrightarrow{BC} . Identify two transversals, and use the Converse to the Interior Angles Theorem to find angles with vertex A congruent to $\angle ABC$ and $\angle ACB$. Now use the Linear Pair Theorem.

Lemma 5.54 (Part of the proof of 5.51). *Suppose that \overline{PQ} is a segment and that Q' is a point so that $\angle PQQ'$ is a right angle. For every $\epsilon > 0$ there is a point T on $\overrightarrow{QQ'}$ so that $|\angle PTQ| < \epsilon$.*

Lemma 5.55 (Part of the proof of 5.51). *The Angle Sum Postulate implies the Euclidean Parallel Postulate.*

Hint: Let P be a point not on a line ℓ through which there are two lines parallel to ℓ , and let Q be the foot of the perpendicular from P to ℓ . Let m be a line perpendicular to \overleftrightarrow{PQ} at P . Then $m \parallel \ell$ (why?). Suppose that m' is another line through P parallel to ℓ . Let S be a point on m' on the same side of m as Q , and let R be a point on m on the same side of \overleftrightarrow{PQ} as S . Use the previous lemma to find a point T on ℓ on the same side of \overleftrightarrow{PQ} as S so that $|\angle QTP| < |\angle SPR|$. Argue that $\sigma(\triangle QTP) < 180^\circ$.

Lemma 5.56 (Part of the proof of 5.51). *The Angle Sum Postulate is Equivalent to Clairaut's Axiom.*

Theorem 5.57. *The Euclidean Parallel Postulate is equivalent to Hilbert's Parallel Postulate.*

Theorem 5.58. *The Euclidean Parallel Postulate is equivalent to the Transitivity of Parallelism.*

This is proven by the following claimable lemmas.

Lemma 5.59 (Part of the proof of 5.58). *The Euclidean Parallel Postulate implies the Transitivity of Parallelism.*

Hint: Suppose that $m \parallel \ell$ and $\ell \parallel n$ but that m is not parallel to n . Show that $m = n$.

Lemma 5.60 (Part of the proof of 5.58). *Transitivity of Parallelism implies the Euclidean Parallel Postulate.*

Hint: Suppose that P is a point not on a line ℓ and that m and n are lines through P parallel to ℓ . Apply Transitivity of Parallelism.

Theorem 5.61. *The Euclidean Parallel Postulate is equivalent to Proclus's Axiom.*

This is proven by the following claimable lemmas.

Lemma 5.62 (Part of the proof of 5.61). *The Euclidean Parallel Postulate implies Proclus's Axiom.*

Hint: Suppose that $\ell \parallel \ell'$ and that t intersects ℓ . If t does not intersect ℓ' , then t is parallel to ℓ .

Lemma 5.63 (Part of the proof of 5.61). *Proclus's Axiom implies the Euclidean Parallel Postulate.*

Hint: Suppose that P is a point not on a line ℓ and that m and t are lines through P parallel to ℓ . Suppose that $t \neq m$ and apply Proclus's Axiom.

Theorem 5.64. *The Euclidean Parallel Postulate is equivalent to the Perpendicular Transversal Condition.*

This is proven by the following claimable lemmas.

Lemma 5.65 (Part of the proof of 5.64). *The Converse to the Alternate Interior Angle Theorem implies the Perpendicular Transversal Condition.*

Hint: Suppose that ℓ and ℓ' are parallel lines cut by a transversal t and that $t \perp \ell$. Apply the Converse to the Alternate Interior Angle Theorem.

Lemma 5.66 (Part of the proof of 5.64). *The Perpendicular Transversal Condition implies the Euclidean Parallel Postulate.*

Hint: Suppose that P is a point not on a line ℓ . Let Q be the foot of the perpendicular from P to ℓ , and let m be the line perpendicular to \overrightarrow{PQ} at P . Note that $m \parallel \ell$. Suppose that m' is any line through P parallel to ℓ . Apply the Perpendicular Transversal Condition using ℓ and m' . Use the result to argue that $m = m'$.

Theorem 5.67. *The Euclidean Parallel Postulate is equivalent to the $\perp \circ \parallel \circ \perp$ Condition.*

This is proven by the following claimable lemmas.

Lemma 5.68 (Part of the proof of 5.67). *Proclus's Axiom and the Perpendicular Transversal Condition together imply the $\perp \circ \parallel \circ \perp$ Condition.*

Hint: Argue that n intersects k and m intersects l . Now apply the Perpendicular Transversal Condition.

Lemma 5.69 (Part of the proof of 5.67). *The $\perp \circ \parallel \circ \perp$ Condition implies the Euclidean Parallel Postulate.*

Hint: Suppose that P is a point not on a line ℓ . Let Q be the foot of the perpendicular from P to ℓ , and let m be the line perpendicular to \overrightarrow{PQ} at P . Note that $m \parallel \ell$. Suppose that m' is a line through P parallel to ℓ . Let R be the foot of the perpendicular from Q to m' and apply the $\perp \circ \parallel \circ \perp$ Condition.

Theorem 5.70. *The Euclidean Parallel Postulate is equivalent to Wallis's Postulate.*

This is proven by the following claimable lemmas.

Lemma 5.71 (Part of the proof of 5.70). *Euclid's Fifth Postulate and the Angle Sum Postulate together imply Wallis's Postulate.*

Hint: Find rays \overrightarrow{DG} and \overrightarrow{EH} so that $\angle CAB \cong \angle GDE$ and $\angle ABC \cong \angle DEH$. Argue that Euclid's Fifth Postulate applies so that these two rays intersect at a point F . Then use the Angle Sum Postulate to argue that $\angle ACB \cong \angle DEF$.

Lemma 5.72 (Part of the proof of 5.70). *Wallis's Postulate implies the Euclidean Parallel Postulate.*

Hint: Suppose that P is a point not on a line ℓ . Let Q be the foot of the perpendicular from P to ℓ , and let m be the line perpendicular to \overrightarrow{PQ} at P . Note that $m \parallel \ell$. Let m' be a line through P parallel to ℓ . Suppose by way of contradiction that $m \neq m'$. Choose a point S on m' so that S and Q are on the same side of m , and let R be the foot of the perpendicular from S to \overrightarrow{PQ} . Find a point T so that $\triangle PRS \sim \triangle PQT$. Argue that T is on ℓ and that T is on m' . The Protractor Theorem may be helpful.

5.6 The Euclidean Parallel Postulate vs. The Hyperbolic Parallel Postulate

Proposition 5.73. *In any model of Neutral Geometry, if the Hyperbolic Parallel Postulate holds, then the Euclidean Parallel Postulate fails.*

Proposition 5.74. *In any model of Neutral Geometry, if the Euclidean Parallel Postulate fails, then the Hyperbolic Parallel Postulate holds.*

Hint: Suppose that P is a point not on a line ℓ . Construct a parallel through P using the usual construction. Let Q be the foot of the perpendicular to ℓ from P , and let m be the line through P perpendicular to \overleftrightarrow{PQ} . Construct a Lambert quadrilateral with two right angles on ℓ and one side \overline{PQ} so that the angle at P is not a right angle. Use the quadrilateral to find another parallel to ℓ through P .

Note. At first glance 5.74 seems not to say much - if there are more than one parallel, then there are at least two. However, what the result says is that if the Euclidean Parallel Postulate fails anywhere in the geometry, then it fails everywhere. There are not some exterior points with just one parallel and others with two or more.

Corollary 5.75. *In any model of Neutral Geometry, either the Euclidean Parallel Postulate holds or the Hyperbolic Parallel Postulate holds.*

6 A Little Euclidean Geometry

In this section, we assume the Euclidean Parallel Postulate:

Axiom (Euclidean Parallel Postulate). For every line ℓ and for every point P not on ℓ , there is exactly one line m so that P lies on m and $m \parallel \ell$.

Note. Since we are assuming the Euclidean Parallel Postulate, we can make use of all of the statements proven to be equivalent to the Euclidean Parallel Postulate in the last section.

6.1 Basic Parallelograms and Triangles

Proposition 6.1. *If $\square ABCD$ is a parallelogram, then $\triangle ABC \cong \triangle CDA$ and $\triangle ABD \cong \triangle CDB$.*

Proposition 6.2. *If $\square ABCD$ is a parallelogram, then $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$.*

Proposition 6.3. *If $\square ABCD$ is a parallelogram, then $\angle ABC \cong \angle CDA$ and $\angle DAB \cong \angle BCD$.*

Proposition 6.4. *If $\square ABCD$ is a parallelogram, then \overline{AC} and \overline{BD} intersect at a point M which is the midpoint of both \overline{AC} and \overline{BD} .*

Proposition 6.5. *The sum of the angles in any convex quadrilateral is 360° .*

Proposition 6.6. *Every Saccheri quadrilateral is a rectangle.*

Proposition 6.7. *Every Lambert quadrilateral is a rectangle.*

Proposition 6.8. *In any equilateral triangle, all of the angles are 60° .*

Proposition 6.9. *There is an equilateral triangle.*

Proposition 6.10. *There is a triangle whose angles measure 30° , 60° , and 90° .*

Proposition 6.11. *A right triangle is isosceles if and only if the angles of the triangle measure 45° , 45° , and 90° .*

Proposition 6.12. *Suppose that ℓ and m are parallel lines, that $P, Q \in \ell$, and that R and S are, respectively, the feet of the perpendiculars from O and Q to m . Then $\overline{PR} \cong \overline{QS}$.*

Proposition 6.13. *Suppose that ℓ and m are different lines, that $P, Q \in \ell$ are on the same side of m , and that R and S are, respectively, the feet of the perpendiculars from O and Q to m . If $\overline{PR} \cong \overline{QS}$ then $\ell \parallel m$.*

Proposition 6.14. *If there are three positive numbers that add to 180, then there is a triangle in which the measures of the angles are those three positive numbers.*

6.2 The Parallel Projection Theorem

Proposition 6.15 (Equal Projections). *Let ℓ , m , and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A , B , and C , respectively, and let t' be a transversal that cuts these lines at points A' , B' , and C' , respectively. Assume that $A * B * C$. If $\overline{AB} \cong \overline{BC}$ then $\overline{A'B'} \cong \overline{B'C'}$.*

Hint: Let t'' be a line through A' either equal to or parallel to t , and let A'' , B'' , and C'' be the appropriate points of intersection of t'' with the given parallels. (Note, $A'' = A'$.) Let t''' be the line through B' either equal to or parallel to t , and let A''' , B''' , and C''' be the appropriate points of intersection of t''' with the given parallels. (Note, $B''' = B'$.) Now use parallelograms and congruent triangles to establish the desired equality. There are a number of cases here. For classroom presentation, you can assume that the points are all distinct other than the ones noted.

Proposition 6.16 (Parallel Projection for Integer Ratios). *Let ℓ , m , and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A , B , and C , respectively, and let t' be a transversal that cuts these lines at points A' , B' , and C' , respectively. Assume that $A * B * C$. If $\frac{|AC|}{|AB|} = n$ for some positive integer n then $\frac{|A'C'|}{|A'B'|} = n$.*

Hint: Note that $|AC| = n \cdot |AB|$. Let $A_0 = A$, and for $k \geq 0$, let A_{k+1} be the point on t on the same side of ℓ as C so that $A_k A_{k+1} \cong AB$. What is A_1 ? What is A_n ? For each k , let ℓ_k be the line through A_k parallel to ℓ . The lines m and n are two of the ℓ_k . Which ones? For each k , let A'_k be the intersection of ℓ_k with t' . Now apply 6.15 repeatedly.

Note. Note that 6.15 is the special case of 6.16 when $n = 2$.

Proposition 6.17 (Parallel Projection for Rational Ratios). *Let ℓ , m , and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A , B , and C , respectively, and let t' be a transversal that cuts these lines at points A' , B' , and C' , respectively. Assume that $A * B * C$. If $\frac{|AC|}{|AB|} = \frac{n}{m}$ for some positive integers n and m then $\frac{|A'C'|}{|A'B'|} = \frac{n}{m}$.*

Hint: Note that $|AC| = n \cdot \frac{|AB|}{m}$. Find a point D so that $A * D * B$ and $|AD| = \frac{|AB|}{m}$. Define D' appropriately and use 6.16.

Theorem 6.18 (Parallel Projection Theorem). *Let ℓ , m , and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A , B , and C , respectively, and let t' be a transversal that cuts these lines at points A' , B' , and C' , respectively. Assume that $A * B * C$. Then $\frac{|AC|}{|AB|} = \frac{|A'C'|}{|A'B'|}$.*

Hint: The theorem holds if $\frac{|AC|}{|AB|}$ is rational by 6.17. Suppose that r is a rational number and that $0 < r < \frac{|AC|}{|AB|}$. Find D so that $A * D * C$ and $\frac{|AC|}{|AD|} = r$. Use 6.17 to show that $0 < r < \frac{|A'C'|}{|A'B'|}$. Use this idea to show that $\frac{|AC|}{|AD|}$ and $\frac{|A'C'|}{|A'B'|}$ are greater than the same rational numbers. It follows that they are equal.

Corollary 6.19. *Let ℓ , m , and n be distinct parallel lines. Let t be a transversal that cuts these lines at points A , B , and C , respectively, and let t' be a transversal that cuts these lines at points A' , B' , and C' , respectively. Assume that $A * B * C$. Then $\frac{|AC|}{|A'C'|} = \frac{|AB|}{|A'B'|}$.*

6.3 Similar Triangles

Theorem 6.20 (Similar Triangles Theorem). *Suppose that $\triangle ABC \sim \triangle DEF$. Then $\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}$.*

Hint: Without loss of generality, assume that $\overline{DE} < \overline{AB}$. Find G so that $A * G * B$ and $\overline{AG} \cong \overline{DE}$. Draw lines through A and G parallel to \overrightarrow{BC} . Construct a triangle congruent to $\triangle DEF$. Use 6.18.

Corollary 6.21. Suppose that $\triangle ABC \sim \triangle DEF$. There is a real number r so that $|\overline{DE}| = r \cdot |\overline{AB}|$, $|\overline{EF}| = r \cdot |\overline{BC}|$, and $|\overline{FD}| = r \cdot |\overline{CA}|$.

Proposition 6.22 (SAS Similarity). If $\triangle ABC$ and $\triangle DEF$ are triangles so that $\angle CAB \cong \angle FDE$ and $\frac{|\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{DF}|}{|\overline{DE}|}$ then $\triangle ABC \sim \triangle DEF$

Proposition 6.23 (Converse to Similar Triangles Theorem). Suppose that $\triangle ABC$ and $\triangle DEF$ are triangles. If $\frac{|\overline{AB}|}{|\overline{DE}|} = \frac{|\overline{AC}|}{|\overline{DF}|} = \frac{|\overline{BC}|}{|\overline{EF}|}$ then $\triangle ABC \sim \triangle DEF$.

6.4 Euclidean Area

Note. It is cumbersome always to refer to the area of a triangular region. Therefore, when we are working with a triangular region or a quadrilateral region, we will refer to the area of the triangle or the area of the quadrilateral when we really mean the area of the associated region.

Proposition 6.24. Suppose that $\triangle ABC$ is a triangle. Let D be the foot of the perpendicular from C to \overleftrightarrow{AB} , and let E be the foot of the perpendicular from B to \overleftrightarrow{AC} . Then $|\overline{AB}| \cdot |\overline{CD}| = |\overline{AC}| \cdot |\overline{EB}|$.

Hint: Do cases on whether $\angle BAC$ is right, acute, or obtuse. The right case is easy. In the other two cases, show that $\triangle ADC \sim \triangle AEB$. Then use the Similar Triangles Theorem.

Definition. Let $\triangle ABC$ be an oriented triangle, and let D be the foot of the perpendicular from C to A . The *base* of $\triangle ABC$ is \overline{AB} and the *height* is $|\overline{CD}|$.

Note. Proposition 6.24 tells us that the product (*length of base*) \cdot *height* is the same for every orientation of $\triangle ABC$.

Definition. A *square* is a rectangle all of whose sides have the same length.

Definition. Suppose that $\square ABCD$ is a square with side length 1. For the rest of this section, let K be $\alpha(\square ABCD)$, the area of a square with side length 1.

Lemma 6.25. Suppose that d is a positive integer. The area of a square with side length $\frac{1}{d}$ is $K \cdot \frac{1}{d^2}$.

Hint: Partition the square into d rows of d congruent squares and use additivity of area.

Lemma 6.26. Suppose that a , c , and d are positive integers. A rectangle with a side of length $\frac{a}{d}$ and a side of length $\frac{c}{d}$ has area $K \cdot \frac{a}{d} \cdot \frac{c}{d}$.

Hint: Partition the square into rows of squares with side length $\frac{1}{d}$.

Lemma 6.27. Suppose that $\square ABCD$ is a rectangle so that $|\overline{AB}|$ and $|\overline{CD}|$ are rational. The area of $\square ABCD$ is $K \cdot |\overline{AB}| \cdot |\overline{CD}|$.

Theorem 6.28. The area of a rectangle $\square ABCD$ is $K \cdot |\overline{AB}| \cdot |\overline{CD}|$.

Hint: $\alpha(\square ABCD) = \sup\{\sup(\alpha(\square EFGH)) : \square EFGH \text{ is a rectangle and } |\overline{EF}| \text{ and } |\overline{GH}| \text{ are rational and } |\overline{EF}| < |\overline{AB}| \text{ and } |\overline{GH}| < |\overline{CD}|\}$

Note. This theorem guarantees that the area of a rectangle $\square ABCD$ is a constant times $|\overline{AB}| \cdot |\overline{CD}|$. The constant in question is the area of a square with edge length 1. We know from 5.46 that any multiple of an area function is a legitimate area function, so we can basically choose the area of a square of edge length 1 to be anything we want. We choose 1.

Axiom (Euclidean Area Postulate). The area of a square with edge length 1 is 1.

Theorem 6.29 (Euclidean Area). *The area of a rectangle $\square ABCD$ is $\alpha(\square BCD) = |\overline{AB}| \cdot |\overline{CD}|$.*

Proposition 6.30. *The area of a right triangle is one half the product of the lengths of its legs.*

Proposition 6.31. *The area of a triangle is one half the product of the length of its base and its height.*

Proposition 6.32. *Suppose that $\triangle ABC$ and $\triangle DEF$ are similar triangles and let $r = \frac{|\overline{DE}|}{|\overline{AB}|}$. Then $\alpha(\triangle DEF) = r^2 \alpha(\triangle ABC)$.*

6.5 The Pythagorean Theorem

Theorem 6.33 (The Pythagorean Theorem). *Suppose that $\triangle ABC$ is a right triangle with right angle at C . Then $|\overline{AB}|^2 = |\overline{AC}|^2 + |\overline{BC}|^2$.*

Hint: Drop a perpendicular from C to \overline{AB} and call the foot D . It may be helpful to make these designations: $a = |\overline{BC}|$, $b = |\overline{AC}|$, $c = |\overline{AB}|$, $h = |\overline{CD}|$, $x = |\overline{AD}|$, $y = |\overline{DB}|$. Show that three triangles involved are similar. Invoke the Similar Triangles Theorem and express a and b in terms of x , y , and c . Then do some substituting.

Note. It appears that the original proofs of the Pythagorean Theorem used similarity. However, the Pythagoreans discovered that not all real numbers are rational, so there was some skepticism attached to similarity proofs. By the time of Euclid, proofs of the Pythagorean Theorem relied on area arguments.

Proposition 6.34. *In Neutral Geometry (not assuming the Euclidean Parallel Postulate) the Pythagorean Theorem implies the Euclidean Parallel Postulate.*

Hint: Consider an isosceles right triangle $\triangle ABC$ with right angle at C . Drop a perpendicular from C to \overline{AB} and call the foot D . Use similar triangles to show that $\sigma(\triangle ABC) = 180^\circ$.

Proposition 6.35 (Converse to the Pythagorean Theorem). *Suppose that $\triangle ABC$ is a triangle in which $|\overline{AB}|^2 = |\overline{AC}|^2 + |\overline{BC}|^2$. The $\angle BCA$ is a right angle.*

Hint: Construct a right triangle with legs congruent to \overline{AC} and \overline{BC} . Use the Pythagorean Theorem to find the length of the third side. Now show that this triangle is congruent to $\triangle ABC$.

6.6 Circles

Proposition 6.36. *In Euclidean Geometry, every triangle can be circumscribed.*

Hint: Use 3.106.

Proposition 6.37. *If the Euclidean Parallel Postulate fails, then there is a triangle that cannot be circumscribed.*

Hint:

1. Since the Euclidean Parallel Postulate fails, there are no rectangles.
2. Let $\square ABCD$ be a Saccheri quadrilateral with base \overline{AB} .

3. Let M be the midpoint of \overline{AB} and let N be the midpoint of \overline{CD} .
4. Show that $\overline{MN} < \overline{AD}$.
5. Find E on \overrightarrow{MN} so that $\overline{ME} \cong \overline{AD} \cong \overline{BC}$.
6. Show that $\square AMED$ and $\square MBCE$ are Saccheri quadrilaterals.
7. Use this to show that the perpendicular bisectors of \overline{DE} and \overline{CE} are parallel.
8. Use this to show that $\triangle CDE$ cannot be circumscribed.

Theorem 6.38. *The Euclidean Parallel Postulate is equivalent to the statement that every triangle can be circumscribed.*

Proposition 6.39 (Thales's Theorem).

Suppose that M is the midpoint of \overline{AB} in a triangle $\triangle ABC$. If $\overline{AM} \cong \overline{MC}$ then the angle $\angle ACB$ is a right angle.

Hint: Isosceles Triangle Theorem.

Proposition 6.40. *Suppose that M is the midpoint of \overline{AB} in a triangle $\triangle ABC$. If $\angle ACB$ is a right triangle, then $\overline{AM} \cong \overline{MC}$.*

Hint: Let D be a point on \overrightarrow{MC} so that $\overline{MD} \cong \overline{AM}$. Use contradiction to show that $D = C$.

Proposition 6.41. *If the angles in a triangle are 30° , 60° , and 90° . Then the length of the side opposite the 30° angle is half the length of the side opposite the 90° angle.*

Proposition 6.42. *If a right triangle is such that the length of one leg is half the length of the hypotenuse, then the angles in the triangle are 30° , 60° , and 90° .*

Definition. Suppose γ is a circle with center C . An *inscribed angle* of γ is an angle $\angle PQR$ where P , Q , and R are on γ . The *arc intercepted* by an inscribed angle $\angle PQR$ is the set of all points on γ which are in the interior of $\angle PQR$. A *central angle* of γ is an angle of the form $\angle PCR$ where P and R are on γ .

Definition. Suppose $\angle PQR$ is an inscribed angle of a circle γ with center C . If P and Q are on opposite sides of \overrightarrow{RC} or if R and Q are on opposite sides of \overrightarrow{PC} then $\angle PCR$ is the *corresponding angle* of $\angle PQR$.

Exercise 6.43. Draw an inscribed angle which does not have a corresponding angle.

Theorem 6.44 (Central Angle Theorem). *The measure of an angle corresponding to an inscribed angle is twice the measure of the inscribed angle.*

Hint: Suppose that $\angle PQR$ is an inscribed angle of a circle γ with center C and that $\angle PCR$ is a corresponding angle for $\angle PQR$. There are three cases:

1. $Q * C * R$ – In this case use the Isosceles Triangle Theorem.
2. C is in the interior of $\angle PQR$. – Find S antipodal to Q and use the angles $\angle PQS$ and $\angle RQS$ and the first case.
3. Neither of the first two cases holds. – Find the antipodal point S again. Either P is in the interior of $\angle RQS$ or R is in the interior of $\angle PQS$. Each case gives you two angles to which you can apply the first case.

6.7 Dissection Theory

Definition. Two polygonal regions R and R' are *equivalent by dissection* if there are triangulations $R = T_1 \cup T_2 \cup \cdots \cup T_n$ and $R' = T'_1 \cup T'_2 \cup \cdots \cup T'_n$ containing the same number of triangles so that for each i , $T_i \cong T'_i$.

Lemma 6.45. *If R is any polygonal region, then $R \equiv R$.*

Lemma 6.46. *If R_1 and R_2 are polygonal regions and $R_1 \equiv R_2$, then $R_2 \equiv R_1$.*

Lemma 6.47. *If R_1 , R_2 , and R_3 are polygonal regions so that $R_1 \equiv R_2$ and $R_2 \equiv R_3$ then $R_1 \equiv R_3$.*

Lemma 6.48. *Every triangle is equivalent by dissection to a rectangle.*

Hint: Assume that the angles of $\triangle ABC$ at A and B are acute. Draw a line ℓ connecting the midpoints of \overline{AC} and \overline{BC} . Drop perpendiculars from A and B to ℓ .

Lemma 6.49. *If $\square ABCD$ is a rectangle and $1 \leq |\overline{BC}| < 2$ then $\square ABCD$ is equivalent by dissection to a rectangle with an edge of length 1.*

Hint:

1. Draw $\square ABCD$. Let $a = |\overline{AB}|$ and $b = |\overline{AD}|$. Note that $1 \leq b < 2$. If $1 = b$, then we are done (why?), so assume $1 < b < 2$.
2. Let E be a point so that $A * E * D$ and $|\overline{AE}| = 1$.
3. Let F be a point so that $A * B * F$ and $|\overline{AF}| = ab$.
4. Somehow find G so that $\square AFGE$ is a rectangle.
5. Let H be the point of intersection of \overrightarrow{BC} and \overrightarrow{EG} .
6. Let I be the point of intersection of \overrightarrow{DF} and \overrightarrow{EG} .
7. Let J be the point of intersection of \overrightarrow{BC} and \overrightarrow{DF} .
8. Argue that the triangles $\triangle AFD$, $\triangle BFJ$, $\triangle EID$, $\triangle CDJ$, and $\triangle GIF$ are all similar.
9. Use similar triangles to find the lengths of the sides of all of those triangles.
10. Argue that $E * I * H * G$ and $C * H * J * B$.
11. Some of those triangles are congruent. Which ones?
12. Finish the proof.

Lemma 6.50. *Any rectangle is equivalent by dissection to a rectangle with a side of length greater than 2.*

Hint: Cut the rectangle in half and stack the halves.

Lemma 6.51. *Any rectangle with a side of length greater than two is equivalent by dissection to the union of a rectangle with a side of length 1 and a rectangle with a side of length less than 2 but greater than or equal to 1.*

Hint: After applying the last theorem, slice rectangles of height 1 off of the rectangle.

Lemma 6.52. *Any rectangle R is equivalent by dissection to a rectangle with edge lengths 1 and $\alpha(R)$.*

Lemma 6.53. *Every polygonal region R is equivalent by dissection to a rectangle with edge lengths 1 and $\alpha(R)$.*

Theorem 6.54 (Fundamental Theorem of Dissection Theory). *Two polygonal regions are equivalent by dissection if and only if they have the same area.*

7 Even Less Hyperbolic Geometry

In this section, we assume the Hyperbolic Parallel Postulate:

Axiom (Hyperbolic Parallel Postulate). For every line ℓ and for every point P not on ℓ , there are at least two lines m and n so that P lies on m and n and $m \parallel \ell$ and $n \parallel \ell$.

7.1 Triangles and Quadrilaterals

Proposition 7.1. *If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) < 180^\circ$ and $0^\circ < \delta(\triangle ABC) < 180^\circ$.*

Proposition 7.2. *For every convex quadrilateral $\square ABCD$, $\sigma(\square ABCD) < 360^\circ$.*

Proposition 7.3. *There is no rectangle.*

Proposition 7.4. *The fourth angle in a Lambert quadrilateral is acute.*

Proposition 7.5. *The summit angles in a Saccheri quadrilateral are acute.*

Hint: Draw two Lambert quadrilaterals.

Proposition 7.6. *In a Lambert quadrilateral, the length of a side between two right angles is less than the length of the opposite side.*

Hint: We know that the side between two right angles is less than or equal its opposite by 5.32. Assume equality and use a fact about Saccheri quadrilaterals.

Definition. The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is called the *altitude* of the quadrilateral. Its length is called the *height* of the quadrilateral.

Proposition 7.7. *In a Saccheri quadrilateral, the altitude is shorter than either side.*

Hint: Draw the altitude and look for Lambert quadrilaterals.

Proposition 7.8. *In any Saccheri quadrilateral, the summit is longer than the base.*

Hint: Draw the altitude and look for Lambert quadrilaterals.

Proposition 7.9 (AAA Congruence). *Suppose $\triangle ABC$ and $\triangle DEF$ are triangles. If $\triangle ABC \sim \triangle DEF$ then $\triangle ABC \cong \triangle DEF$.*

Hint: Suppose by way of contradiction the triangles are not congruent. To avoid SAS, it has to be that one triangle has two sides longer than the corresponding sides of the other. Without loss of generality, assume $\overline{AB} > \overline{DE}$ and $\overline{AC} > \overline{DF}$. Construct a copy of $\triangle DEF$ inside $\triangle ABC$. Within the diagram is a quadrilateral with defect 0° .

Proposition 7.10. *Any two Saccheri quadrilaterals with congruent summits and equal defects are congruent.*

Hint: Suppose that the quadrilaterals are $\square ABCD$ and $\square A'B'C'D'$. Argue that the quadrilaterals share the same angles. Find $E \in \overrightarrow{DA}$ and $F \in \overrightarrow{CB}$ so that $\overline{DE} \cong \overline{D'A'}$ and $\overline{CF} \cong \overline{C'B'}$. Use SAS a couple of times to argue that $\square A'B'C'D' \cong \square EFCD$. Now use Alternate Interior Angles to argue that if $A \neq E$ then $B \neq F$ and that there is a rectangle in your diagram.

7.2 Common Perpendiculars

Definition. Suppose that P is a point not on a line ℓ . The *distance* from P to ℓ is $|\overline{PF}|$ where F is the foot of the perpendicular from P to ℓ . We denote this as $d(P, \ell)$.

Proposition 7.11. *Suppose that ℓ and m are distinct lines. There are at most two points P and Q on m with $d(P, \ell) = d(Q, \ell)$.*

Hint: Suppose that there are three such points, P , Q , and R and assume $P * Q * R$. Draw the diagram. Identify two Saccheri triangles and use the Exterior Angles Theorem along with what we know about summit angles to arrive at a contradiction.

Definition. Two lines ℓ and m *admit a common perpendicular* if there is a line n so that $n \perp \ell$ and $n \perp m$.

Proposition 7.12. *If $\ell \parallel m$ and if there are two points on m that are the same distance from ℓ , then ℓ and m admit a common perpendicular.*

Hint: Saccheri quadrilateral.

Proposition 7.13. *If lines ℓ and m admit a common perpendicular, then they admit exactly one common perpendicular.*

Hint: Rectangles?

Proposition 7.14. *Let ℓ and m be parallel lines cut by a transversal t . Alternate interior angles are congruent if and only if ℓ and m admit a common perpendicular and t intersects the perpendicular segment at its midpoint.*

This proposition is proven by the following claimable lemmas.

Lemma 7.15 (Part of the proof of 7.14). *Let ℓ and m be parallel lines cut by a transversal t . If both pairs of alternate interior angles formed by t are congruent then ℓ and m admit a common perpendicular, and t intersects the perpendicular segment at its midpoint.*

Hint: Suppose ℓ and m be parallel lines cut by a transversal t and that both pairs of alternate interior angles formed by t are congruent. Let R be the point where t crosses ℓ and S the point where t crosses m . What if the alternate interior angles are right angles? Assume they are not. Let M be the midpoint of \overline{ST} . Let P be the foot of the perpendicular from M to ℓ , and let Q be the foot of the perpendicular from M to m . Argue that P and Q are on opposite sides of t . Argue that the two triangles in your diagram are congruent. Argue that M , Q , and P are collinear.

Lemma 7.16 (Part of the proof of 7.14). *Let ℓ and m be parallel lines cut by a transversal t . If ℓ and m admit a common perpendicular and t intersects the perpendicular segment at its midpoint then the alternate interior angles formed by t are congruent.*

Hint: Congruent triangles.