

3. Notation

To avoid confusion between logical symbols and lattice operations, we will use \vee and \wedge for logical “or” and “and.” We will use additive notation $(+, \Sigma)$ for lattice joins, and multiplicative notation $(\cdot, \Pi, \text{ or juxtaposition})$ for lattice meets. We will refer to the intersection of arbitrary binary relations with \cap , while we refer to the intersection of equivalence relations with product notation. When equivalence relations might be mixed with non-equivalence relations, we will use \cap .

A primitive positive formula is a formula of the form $\exists \wedge (\text{atomic})$. Suppose that ϕ is a primitive positive formula with two free variables v_1 and v_2 which employs binary relation symbols r_1, \dots, r_l . Suppose also that $\theta_1, \dots, \theta_l$ are binary relations on a set A . By interpreting each r_i as θ_i , ϕ can be used to define a new binary relation on A . We will denote this new binary relation as $\Phi(\theta_1, \dots, \theta_l)$. To denote that $\langle a, b \rangle$ satisfies this new relation, we will write $\langle a, b \rangle \in \Phi(\theta_1, \dots, \theta_l)$ rather than $\Phi(\theta_1, \dots, \theta_l)(a, b)$. On any set A , Φ induces an operation on binary relations of A which maps an l -tuple $\langle \theta_1, \dots, \theta_l \rangle$ of binary relations to the relation $\Phi(\theta_1, \dots, \theta_l)$. We will also call this operation Φ and usually denote it by $\Phi(r_1, \dots, r_l)$. When there is possible confusion about the underlying set A involved, we will denote the operation by $\Phi^A(r_1, \dots, r_l)$. We will also use $\Phi(r_1, \dots, r_l)$ to denote the primitive positive formula. Generally, it will be apparent from context whether $\Phi(r_1, \dots, r_l)$ refers to the formula or the operation.

By a compatible relation on an algebra \mathbf{A} we mean a subuniverse of a (finite) direct power of A . In [5], the author exploits the following lemma which follows from the fact that a set of relations on a finite set is the set of all relations compatible with an algebra on the set if and only if the relations are closed under primitive positive definitions [1, 2].

Lemma 3.1 (Corollary 2.2 of [5]). *Suppose \mathcal{L} is a 0–1 lattice of equivalence relations on a finite set A . There is an algebra \mathbf{A} on A with $\text{Con}\mathbf{A} = \mathcal{L}$ if and only if every equivalence relation on A which can be defined from \mathcal{L} by a primitive positive formula is already in \mathcal{L} .*

4. The variety generated by \mathbf{M}_3

In this section, we prove

Theorem 4.1. *Every finite lattice in the variety generated by \mathbf{M}_3 is representable.*

We do so through the following sequence of definitions, lemmas, and corollaries. Let $B = \{1, 2, 3\}$. Denote the lattice of equivalence relations on B by $\text{Eq}B$. Then $\text{Eq}B$ is isomorphic to \mathbf{M}_3 . Let $i \in B$. Denote the unique atom of $\text{Eq}B$ in which

$\{i\}$ is an equivalence class by μ_i . Denote the universal relation on B by 1_B and the smallest equivalence relation by 0_B .

Lemma 4.2. *Suppose that α , β , and γ are equivalence relations on the three element set B . The equivalence relation $(\alpha + \beta)\gamma$ is the smallest equivalence relation on B containing $(\alpha \circ \beta) \cap \gamma$.*

Proof. Suppose that σ is an equivalence relation on B containing $(\alpha \circ \beta) \cap \gamma$. We will prove that $(\alpha + \beta)\gamma \leq \sigma$. If α or β is in $\{0_B, 1_B\}$, then this is easy. Assume this is not the case. Then $\alpha = \mu_i$ and $\beta = \mu_j$ for some $i, j \in B$. If $i = j$ then

$$(\alpha + \beta)\gamma = \alpha \cap \gamma = (\alpha \circ \beta) \cap \gamma \leq \sigma.$$

Assume that $i \neq j$. We proceed now by cases on γ . If $\gamma = 0_B$ then

$$(\alpha + \beta)\gamma = 0_B \leq \sigma.$$

If $\gamma = 1_B$, then $(\alpha \circ \beta) \cap \gamma = \alpha \circ \beta$. Since $(\alpha \circ \beta) \cap \gamma \leq \sigma$, and since $(\alpha \circ \beta) \cap \gamma = \alpha \circ \beta$, we know that $\alpha \leq \sigma$ and that $\beta \leq \sigma$. It follows then that

$$(\alpha + \beta)\gamma = \alpha + \beta \leq \sigma.$$

Next assume that $\gamma = \mu_k$ for some $k \in B$. If $k \in \{i, j\}$, then $(\alpha \circ \beta) \cap \gamma = \gamma$. If $k \notin \{i, j\}$, then $\langle j, i \rangle \in (\alpha \circ \beta) \cap \gamma$ while $\langle i, j \rangle \notin (\alpha \circ \beta) \cap \gamma$, so $0_B < (\alpha \circ \beta) \cap \gamma < \gamma$. In either case, $0_B < (\alpha \circ \beta) \cap \gamma \leq \gamma = \mu_k$, so the only equivalence relations above $(\alpha \circ \beta) \cap \gamma$ are $\mu_k = \gamma$ and 1_B . Therefore, either $\sigma = \gamma$ or $\sigma = 1_B$. In either case, $(\alpha + \beta)\gamma \leq \sigma$. We have proven that if σ is any equivalence relation on B containing $(\alpha \circ \beta) \cap \gamma$ then $(\alpha + \beta)\gamma \leq \sigma$. This establishes the lemma since clearly $(\alpha \circ \beta) \cap \gamma \subseteq (\alpha + \beta)\gamma$. \square

Definition 4.3. For any primitive positive formula $\Phi(r_1, \dots, r_l)$ of the form

$$\langle v_1, v_2 \rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists v_3 \bigwedge_{i=1}^l r_i(v_{j_i}, v_{k_i})$$

define a lattice term $T_\Phi(r_1, \dots, r_l)$ by

$$T_\Phi(r_1, \dots, r_l) = \left(\left[\prod \{r_i : \{j_i, k_i\} = \{1, 3\}\} \right] + \left[\prod \{r_i : \{j_i, k_i\} = \{3, 2\}\} \right] \right) \\ \cdot \left[\prod \{r_i : \{j_i, k_i\} = \{1, 2\}\} \right].$$

If any of the sets in this definition are empty, we follow the tradition that the meet over an empty set is the largest element of a lattice, so that this is actually a term in the language of lattices with a greatest element (the language with symbols $+$, \cdot , and 1).

Lemma 4.4. *If $\Phi(r_1, \dots, r_l)$ is a primitive positive formula of the form*

$$\langle v_1, v_2 \rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists v_3 \bigwedge_{i=1}^l r_i(v_{j_i}, v_{k_i})$$

and if $\theta_1, \dots, \theta_l \in \text{Eq}B$, then $T_\Phi(\theta_1, \dots, \theta_l)$ is the smallest equivalence relation on B containing $\Phi(\theta_1, \dots, \theta_l)$.

Proof. Define $\alpha = \prod\{\theta_i : \{j_i, k_i\} = \{1, 3\}\}$, $\beta = \prod\{\theta_i : \{j_i, k_i\} = \{3, 2\}\}$, and $\gamma = \prod\{\theta_i : \{j_i, k_i\} = \{1, 2\}\}$. Notice that each of these is an equivalence relation on B , and that these are the meets which occur within $T_\Phi(\theta_1, \dots, \theta_l)$ so that $T_\Phi(\theta_1, \dots, \theta_l) = (\alpha + \beta)\gamma$.

We claim that $\Phi(\theta_1, \dots, \theta_l) = (\alpha \circ \beta) \cap \gamma$. To see this, suppose first that $\langle x_1, x_2 \rangle \in \Phi(\theta_1, \dots, \theta_l)$. This means that there is an element $x_3 \in B$ so that $\theta_i(x_{j_i}, x_{k_i})$ is true for $i = 1, \dots, l$. Suppose that $\{j_i, k_i\} = \{1, 3\}$. Then from $\theta_i(x_{j_i}, x_{k_i})$, we know either $\theta_i(x_1, x_3)$ or $\theta_i(x_3, x_1)$. Since θ_i is symmetric, this means $\theta_i(x_1, x_3)$. This is true for all i for which $\{j_i, k_i\} = \{1, 3\}$, so $\langle x_1, x_3 \rangle \in \alpha$. Similar arguments when $\{j_i, k_i\} = \{3, 2\}$ and $\{j_i, k_i\} = \{1, 2\}$ will establish that $\langle x_3, x_2 \rangle \in \beta$ and $\langle x_1, x_2 \rangle \in \gamma$. Hence $\langle x_1, x_2 \rangle \in (\alpha \circ \beta) \cap \gamma$ and $\Phi(\theta_1, \dots, \theta_l) \subseteq (\alpha \circ \beta) \cap \gamma$.

Next, suppose that $\langle x_1, x_2 \rangle \in (\alpha \circ \beta) \cap \gamma$. Since $\langle x_1, x_2 \rangle \in \alpha \circ \beta$, there is an $x_3 \in B$ with $x_1 \alpha x_3 \beta x_2$. Suppose $i \in \{1, \dots, l\}$. If $\{j_i, k_i\} = \{1, 3\}$ then $\theta_i(x_1, x_3)$ (since $\langle x_1, x_3 \rangle \in \alpha$). Since θ_i is symmetric, it follows that $\theta_i(x_{j_i}, x_{k_i})$. Similar arguments establish $\theta_i(x_{j_i}, x_{k_i})$ when $\{j_i, k_i\} = \{3, 2\}$ or $\{j_i, k_i\} = \{1, 2\}$. Thus we have $\theta_i(x_{j_i}, x_{k_i})$ for all i , so x_3 witnesses that $\langle x_1, x_2 \rangle \in \Phi(\theta_1, \dots, \theta_l)$. This provides the reverse inclusion to conclude that $\Phi(\theta_1, \dots, \theta_l) = (\alpha \circ \beta) \cap \gamma$. The lemma now follows from Lemma 4.2. \square

Definition 4.5. Suppose that $n \geq 2$. Define F_n to be the set of all functions $f : \{1, \dots, n\} \rightarrow \{1, 2, 3\}$ with $f(1) = 1$ and $f(2) = 2$. Suppose that $\Phi(r_1, \dots, r_l)$ is a primitive positive formula of the form

$$\langle u_1, u_2 \rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists u_3, \dots, u_n \bigwedge_{i=1}^l r_i(u_{g_i}, u_{h_i}).$$

Let $f \in F_n$. Define $\Phi_f(r_1, \dots, r_l)$ to be the primitive positive formula

$$\langle v_1, v_2 \rangle \in \Phi_f(r_1, \dots, r_l) \leftrightarrow \exists v_3 \bigwedge_{i=1}^l r_i(v_{f(g_i)}, v_{f(h_i)}).$$

Lemma 4.6. *Suppose that $\Phi(r_1, \dots, r_l)$ is the primitive positive formula*

$$\langle u_1, u_2 \rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists u_3, \dots, u_n \bigwedge_{i=1}^l r_i(u_{g_i}, u_{h_i}).$$

Let $x_1, x_2 \in B$ and $\theta_1, \dots, \theta_l \in \text{Eq}B$. Then $\langle x_1, x_2 \rangle \in \Phi(\theta_1, \dots, \theta_l)$ if and only if there is a function $f \in F_n$ so that $\langle x_1, x_2 \rangle \in \Phi_f(\theta_1, \dots, \theta_l)$.

Proof. Suppose that $\langle x_1, x_2 \rangle \in \Phi(\theta_1, \dots, \theta_l)$. If $x_1 = x_2$, then $\langle x_1, x_2 \rangle$ is in any $\Phi_f(\theta_1, \dots, \theta_l)$. Suppose then that $x_1 \neq x_2$. There is exactly one element in B and not in $\{x_1, x_2\}$. Call this element z . There are $x_3, \dots, x_n \in B$ so that $\theta_i(x_{g_i}, x_{h_i})$ for each $i = 1, \dots, l$. Define $f : \{1, \dots, n\} \rightarrow \{1, 2, 3\}$ by

$$f(i) = \begin{cases} 1 & x_i = x_1 \\ 2 & x_i = x_2 \\ 3 & x_i = z. \end{cases}$$

Let $y_1 = x_1$, $y_2 = x_2$, and $y_3 = z$. It follows that $y_{f(g_i)} = x_{g_i}$ and $y_{f(h_i)} = x_{h_i}$ for all i . Since $\theta_i(x_{g_i}, x_{h_i})$ for all i , this means $\theta_i(y_{f(g_i)}, y_{f(h_i)})$. Hence, $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \in \Phi_f(\theta_1, \dots, \theta_l)$.

Next, suppose that $f \in F_n$ and that $\langle x_1, x_2 \rangle \in \Phi_f(\theta_1, \dots, \theta_l)$. This means there is an $x_3 \in B$ with $\theta_i(x_{f(g_i)}, x_{f(h_i)})$ for all i . For each $i = 1, \dots, l$, let $y_i = x_{f(i)}$. Then $y_1 = x_1$, $y_2 = x_2$, and for all $i = 1, \dots, l$, $y_{g_i} = x_{f(g_i)}$ and $y_{h_i} = x_{f(h_i)}$. Suppose that $i \in \{1, \dots, l\}$. Since $\theta_i(x_{f(g_i)}, x_{f(h_i)})$, it follows that $\theta_i(y_{g_i}, y_{h_i})$. Thus $\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \in \Phi(\theta_1, \dots, \theta_l)$. \square

Lemma 4.6 immediately gives us

Corollary 4.7. Suppose that $\Phi(r_1, \dots, r_l)$ is the primitive positive formula

$$\langle u_1, u_2 \rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists u_3, \dots, u_n \bigwedge_{i=1}^l r_i(u_{g_i}, u_{h_i}).$$

Let $\theta_1, \dots, \theta_l \in \text{Eq}B$. Then $\Phi(\theta_1, \dots, \theta_l) = \bigcup_{f \in F_n} \Phi_f(\theta_1, \dots, \theta_l)$. \square

Definition 4.8. Suppose that $\Phi(r_1, \dots, r_l)$ is the primitive positive formula

$$\langle u_1, u_2 \rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists u_3, \dots, u_n \bigwedge_{i=1}^l r_i(u_{g_i}, u_{h_i}).$$

Define the following term in the language of lattices with a greatest element

$$Q_\Phi(r_1, \dots, r_l) = \sum_{f \in F_n} T_{\Phi_f}(r_1, \dots, r_l).$$

Corollary 4.9. Suppose that $\Phi(r_1, \dots, r_l)$ is the primitive positive formula

$$\langle u_1, u_2 \rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists u_3, \dots, u_n \bigwedge_{i=1}^l r_i(u_{g_i}, u_{h_i}).$$

Let $\theta_1, \dots, \theta_l \in \text{Eq}B$. If $\Phi(\theta_1, \dots, \theta_l)$ is an equivalence relation on B , then

$$\Phi(\theta_1, \dots, \theta_l) = Q_\Phi(\theta_1, \dots, \theta_l).$$

Proof. Suppose that $\Phi(\theta_1, \dots, \theta_l)$ is an equivalence relation. By Corollary 4.7 and Lemma 4.4

$$\Phi(\theta_1, \dots, \theta_l) = \bigcup_{f \in F_n} \Phi_f(\theta_1, \dots, \theta_l) \subseteq \sum_{f \in F_n} T_{\Phi_f}(\theta_1, \dots, \theta_l) = Q_\Phi(\theta_1, \dots, \theta_l).$$

Thus $\Phi(\theta_1, \dots, \theta_l) \subseteq Q_\Phi(\theta_1, \dots, \theta_l)$. Now note that $\Phi_f(\theta_1, \dots, \theta_l) \subseteq \Phi(\theta_1, \dots, \theta_l)$ for each $f \in F_n$ by Corollary 4.7. Therefore, since $\Phi(\theta_1, \dots, \theta_l)$ is an equivalence relation, Lemma 4.4 tells us that $T_{\Phi_f}(\theta_1, \dots, \theta_l) \subseteq \Phi(\theta_1, \dots, \theta_l)$. This now gives that

$$Q_\Phi(\theta_1, \dots, \theta_l) = \sum_{f \in F_n} T_{\Phi_f}(\theta_1, \dots, \theta_l) \subseteq \Phi(\theta_1, \dots, \theta_l).$$

We then have $Q_\Phi(\theta_1, \dots, \theta_l) = \Phi(\theta_1, \dots, \theta_l)$ as desired. \square

Definition 4.10. Let $\mathcal{M} = \text{Eq}B$. This lattice is isomorphic to \mathbf{M}_3 . Let m be a positive integer. Let $\theta_1, \dots, \theta_m$ be equivalence relations on B . By $\langle \theta_1, \dots, \theta_m \rangle$ we will mean the equivalence relation on B^m defined so that $\langle x_1, \dots, x_m \rangle$ is related to $\langle y_1, \dots, y_m \rangle$ precisely when $x_i \theta_i y_i$ for all $i = 1, \dots, m$. We will denote the lattice of all equivalence relations on B^m of the form $\langle \theta_1, \dots, \theta_m \rangle$ where each $\theta_i \in \mathcal{M}$ as \mathcal{M}^m . Note that this lattice is isomorphic to \mathbf{M}_3^m .

Lemma 4.11. *For any positive integer m , every 0–1 sublattice of \mathcal{M}^m is the congruence lattice of an algebra on B^m .*

Proof. Let \mathcal{L} be a 0–1 sublattice of \mathcal{M}^m . We will show that \mathcal{L} is closed under primitive positive definitions. Let $\Phi(r_1, \dots, r_l)$ be the primitive positive formula

$$\langle u_1, u_2 \rangle \in \Phi(r_1, \dots, r_l) \leftrightarrow \exists u_3, \dots, u_n \bigwedge_{i=1}^l r_i(u_{g_i}, u_{h_i}).$$

We know from Corollary 4.9 that there is term $Q_\Phi(r_1, \dots, r_l)$ in the language of lattices with a greatest element so that if $\theta_1, \dots, \theta_l \in \text{Eq}B$ and if $\Phi(\theta_1, \dots, \theta_l)$ is an equivalence relation, then $\Phi(\theta_1, \dots, \theta_l) = Q_\Phi^\mathcal{M}(\theta_1, \dots, \theta_l)$. Since Q_Φ is a term in the language of lattices with a greatest element, there is a pure lattice term $R_\Phi(r_1, \dots, r_l, y)$ so that \mathcal{M} satisfies $Q_\Phi^\mathcal{M}(r_1, \dots, r_l) = R_\Phi^\mathcal{M}(r_1, \dots, r_l, 1)$.

Suppose that $\theta_1, \dots, \theta_l \in \mathcal{L}$ and that $\Phi^{B^m}(\theta_1, \dots, \theta_l)$ is an equivalence relation. For each i , there are equivalence relations $\theta_i^1, \dots, \theta_i^m$ so that $\theta_i = \langle \theta_i^1, \dots, \theta_i^m \rangle$. By virtue of Φ being a primitive positive formula

$$\Phi^{B^m}(\theta_1, \dots, \theta_l) = \langle \Phi^B(\theta_1^1, \dots, \theta_l^1), \dots, \Phi^B(\theta_1^m, \dots, \theta_l^m) \rangle.$$

Since $\Phi^{B^m}(\theta_1, \dots, \theta_l)$ is an equivalence relation, each $\Phi^B(\theta_1^i, \dots, \theta_l^i)$ is an equivalence relation on B . Therefore,

$$\begin{aligned} \Phi^{B^m}(\theta_1, \dots, \theta_l) &= \langle \Phi^B(\theta_1^1, \dots, \theta_l^1), \dots, \Phi^B(\theta_1^m, \dots, \theta_l^m) \rangle \\ &= \langle Q_\Phi^{\mathcal{M}}(\theta_1^1, \dots, \theta_l^1), \dots, Q_\Phi^{\mathcal{M}}(\theta_1^m, \dots, \theta_l^m) \rangle \\ &= \langle R_\Phi^{\mathcal{M}}(\theta_1^1, \dots, \theta_l^1, 1), \dots, R_\Phi^{\mathcal{M}}(\theta_1^m, \dots, \theta_l^m, 1) \rangle \\ &= R_\Phi^{\mathcal{M}^m}(\theta_1, \dots, \theta_l, \langle 1, \dots, 1 \rangle). \end{aligned}$$

Since \mathcal{L} is a 0–1 sublattice of \mathcal{M}^m , and since R_Φ is a lattice term, this is an element of \mathcal{L} . The lattice \mathcal{L} is thus closed under primitive positive definitions which yield equivalence relations, so by 3.1 it is the congruence lattice of an algebra on B^m . \square

Every finite lattice in the variety generated by \mathbf{M}_3 is isomorphic to a 0–1 sublattice of a finite direct power of \mathbf{M}_3 . Therefore, Theorem 4.1 now follows directly from Corollary 4.11.

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