Math 122
Introduction to Statistics
Confidence Intervals

ONE POINT APPROXIMATIONS

Mean: The best one point approximation of a population mean μ based on a sample is the sample mean $\bar{x}$.

Proportion: The best one point approximation of a population proportion p based on a sample is the sample is the sample proportion $\hat{p}$.

One point approximations are usually close to population parameters but not equal to them. To give approximations which quantify how close we are to the actual population parameters, we turn to confidence intervals.

INTERPRETATION

Confidence interval approximations will give us statements such as the ones below (which all say the same thing).

- We are 95% sure that the population mean is between 10 and 14.
- We are 95% sure that the population mean is 12 with a margin of error of 2.
- We are 95% sure that the population mean is $\mu = 10 \pm 2$.
- The 95% confidence interval of the population mean is $10 < \mu < 14$.
- The 95% confidence interval of the population mean is $(10, 14)$.

What does this mean? If the process that was use to create this interval is repeated many times to create many intervals, then we would expect about 95% of the intervals to contain the actual mean.

Note: We are confident in the *process* and not necessarily the actual *interval*.

INGREDIENTS

Confidence Level: This is a level of confidence in the process which we will use to find intervals. The most common value we will use for the confidence level is 95%. We will not frequently need notation for the confidence level. For now, we can denote the confidence level as CL.

Significance Level: Our calculations will not refer directly to the confidence level CL but to $1 - CL$. We call this the significance level and denote it α. So $\alpha = 1 - CL$.

Point Approximation: Our confidence intervals will be built around a one point approximation. This will be for us either the sample mean or the sample proportion. This is the center of the confidence interval.

Margin of Error: Our confidence intervals are centered around the one point approximation and extend the same distance above and below this center. This distance above and below is the margin of error. We will denote it as E.

Interval: If the one point approximation $\bar{x}$ or $\hat{p}$ is known, then the confidence interval is

$$\bar{x} - E < \mu < \bar{x} + E$$

or

$$\hat{p} - E < p < \hat{p} + E$$

DERIVATION FOR MEANS KNOWING σ

The ingredients described for a confidence interval should indicate that to find an interval, we just need the one point approximation $\bar{x}$ and the margin of error E. We get $\bar{x}$ directly from the sample. We derive a formula for E here (which we will not actually use).

We want to find E so that

$$CL = P(\bar{x} - E < \mu < \bar{x} + E).$$

Subtracting $\bar{x}$ gives

$$CL = P(-E < \mu - \bar{x} < E).$$

The center expression here is similar to the z score in the Central Limit Theorem, so we will manipulate things until we see a z score here. First, negating and changing the order gives

$$CL = P(E > \bar{x} - \mu > -E)$$

or

$$CL = P(-E < \bar{x} - \mu < E).$$

Now, according to the Central Limit Theorem, the standard deviation for $\bar{x}$ is $\sigma/\sqrt{n}$, so we divide by this

$$CL = P\left(-\frac{E}{\sigma/\sqrt{n}} < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < \frac{E}{\sigma/\sqrt{n}}\right).$$

The expression in the middle should have a standard normal distribution (by the Central Limit Theorem). So we want:

$$CL = P\left(-\frac{E}{\sigma/\sqrt{n}} < z < \frac{E}{\sigma/\sqrt{n}}\right).$$

The area under the standard normal density curve between $-\frac{E}{\sigma/\sqrt{n}}$ and $\frac{E}{\sigma/\sqrt{n}}$ should be CL. By symmetry, the area to the left of $-\frac{E}{\sigma/\sqrt{n}}$ and the area to the right of $\frac{E}{\sigma/\sqrt{n}}$ should both be $\frac{1-CL}{2}$ or $\alpha/2$. This means that $\frac{E}{\sigma/\sqrt{n}}$ should be the number $z_{\alpha/2} = -invnorm(\alpha/2)$. We will call this the *critical value*.

From $\frac{E}{\sigma/\sqrt{n}} = z_{\alpha/2}$ we find

$$E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

We now *could* find a confidence interval for a mean μ given the standard deviation σ and a simple random sample of size n in this manner:

1. Calculate the sample average $\bar{x}$.
2. Find $z_{\alpha/2} = -invnorm(\alpha/2)$ (where $\alpha = 1 - CL$).
3. Let $E = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.
4. The interval is $\bar{x} - E < \mu < \bar{x} + E$.

Nonsense: Of course, this is not too realistic. If we do not know μ, then how are we to know σ? We can use the normal distribution as an approximation to the binomial distribution and follow the above process to find a usable formula for margin of error for estimating a proportion. For means, we can use Student's t distribution to find a formula for E. This leads to:
Confidence Level=CL
Significance Level=$\alpha = 1 - CL$

	Critical Value	Margin of Error	Interval
Means	$t_{\alpha/2} = -invt(\alpha/2, n-1)$	$t_{\alpha/2}\frac{s}{\sqrt{n}}$	$\bar{x} - E < \mu < \bar{x} + E$
Proportions	$z_{\alpha/2} = -invnorm(\alpha/2)$	$z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$	$\hat{p} - E < p < \hat{p} + E$

Width: Notice in these formulas that the larger n is, the smaller the confidence interval will be. Also, the larger CL is, the larger $z_{\alpha/2}$ will be – making the interval wider. Larger samples correspond to narrower intervals. Higher confidence corresponds to wider intervals.

WE WILL NOT USE THESE FORMULAS TO CALCULATE CONFIDENCE INTERVALS. WE WILL USE TECHNOLOGY.

A note about E: Some calculators do not explicitly report the margin of error E. If your calculator just gives you an interval (a, b), then $E = \frac{b-a}{2}$. The sample average in this case is $\bar{x} = \frac{a+b}{2}$.

Example: A newscast reports that the President's approval rating based on a sample is 37% with a margin of error of the confidence interval estimate of the approval rating?
Solution: The sample proportion is $\hat{p} = 37\%$. The margin of error is $E = 4\%$. Our confidence interval is $\hat{p} - E < p < \hat{p} + E$, which is $33\% < p < 41\%$.

Example: In one study of heights of adult males, a 95% confidence interval estimate of the average height μ of an adult male in inches was $68.9 < \mu < 70.3$. What were the margin of error and sample mean for this study?

Solution: The error and mean were:

$$E = \frac{70.3 - 68.9}{2} = 0.70$$

$$\bar{x} = \frac{68.9 + 70.3}{2} = 69.6$$

Example: In a pre-election poll, 217 of 500 voters polled (about 43.4%) preferred candidate A and 241 of 500 (about 48.2%) preferred candidate B. Use this information to construct 95% confidence interval estimates of the proportions of the population who prefer candidate A and B. Is candidate B ahead of candidate A in the polls?

Solution: Let p_A be the proportion who prefer A. Let p_B be the proportion who prefer B. Using technology, our 95% estimates are:

$$39.1\% < p_A < 47.7\% \text{ and } 43.8\% < p_B < 52.6\%.$$

These intervals would look like:

$$0.391 < p_A < 0.477 \text{ and } 0.438 < p_B < 0.526$$

if directions requested decimals rather than percentages. Since the intervals overlap each other, there is no statistical difference in the two proportions. The candidates are in a statistical tie.

Example: A sample of 12 pine needles had these lengths (in mm):

107 123 99 102 106 119 115 103 111 109 102 100

Use this data to find the 95% confidence interval estimate of the average length of one of these pine needles.

Solution: Using technology, we get:

$$\bar{x} = 108$$

$$E = 4.89$$

$$103.11 < \mu < 112.89$$

Example: A sample of 100 of the pine needles in the last problem had an average length of 106.2mm with a standard deviation of 4.3mm. Use this information to construct a 95% confidence interval.

Solution: Using technology again:

$$E = 0.85$$

$$105.35 < \mu < 107.05$$

SAMPLE SIZE

For proportions, we can solve for n in the equation

$$E = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

to try to determine what size sample we should use to obtain a particular margin of error. When we do so, we get:

$$n = \hat{p}(1-\hat{p})\left(\frac{z_{\alpha/2}}{E}\right)^2$$

Unfortunately, this expression includes a $\hat{p}$. Fortunately, the value $\hat{p}(1-\hat{p})$ can be no larger than $\frac{1}{4}$ (this can be verified by considering the graph of $y = x(1-x)$). So, it will be good enough to use

$$n = \left(\frac{1}{4}\right)\left(\frac{z_{\alpha/2}}{E}\right)^2 = \left(\frac{1}{2}\right)^2\left(\frac{z_{\alpha/2}}{E}\right)^2 = \left(\frac{z_{\alpha/2}}{2E}\right)^2.$$

Sufficient Sample Size: To approximate a proportion with margin of error no larger than E, use a sample size of

$$n = \left(\frac{z_{\alpha/2}}{2E}\right)^2.$$

NOTE: This sample size is *sufficient* (good enough) but not *necessary* (sometimes a smaller sample would be just as good).

Example: A poll is to be done to estimate what proportion of the population of voters prefers candidate A. We want a margin of error of no more than 3%. What sample size should be used for a 90% interval? For a 99% interval?

Solution: For the 90% interval, $\alpha = 0.1$, so

$$z_{\alpha/2} = -invnorm(0.05) = 1.6448$$

Then we can use

$$n = \left(\frac{1.6448}{2\cdot(.03)}\right)^2 = 751.49.$$

We round up to $n = 752$ since we can not poll 0.49 people. (Note the use of .03 for 3%.)

For the 99% interval, $\alpha = 0.01$, so

$$z_{\alpha/2} = -invnorm(0.005) = 2.5758$$

Then we can use

$$n = \left(\frac{2.5758}{2 \cdot (.03)}\right)^2 = 1842.98.$$

We round up to $n = 1843$ since we can not poll 0.98 people.