Math 122
Introduction to Statistics
Probability

EVENTS

Event: An event is an outcome of a procedure, experiment, or observation.

Example: If I flip a coin three times, I could see the progression HHT. That is an event. I could also see the event of flipping at least two H's. This event is composed of the four simpler events HHH, HHT, HTH, and THH.

Simple Event: A simple event is an event which cannot be decomposed into simpler components.

Example: In the three-flips example, the simple events are HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT.

Compound Event: A compound event is an event that is composed of multiple simple events.

Example: In the three-flips example, flipping at least two H's is a compound event. So is the event that the first flip is a T.

Sample Space: The sample space of a procedure is the set of all simple events.

Example: In the three-flips example, the sample space is

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Compound Events as Sets: A compound event is really just a set of simple events.

Example: The event that the first flip is a T is the set $\{THH, THT, TTH, TTT\}$.

And: If A and B are events, then the event (A and B) is the event that A and B both occur.

Example: In the three-flips example, if $A = \{THH, THT, TTH, TTT\}$ (the event that the first flip is a T) and $B = \{HHT, HTT, THT, TTT\}$ (the event that the last flip is a T), then (A and B) is the set of all events which are both in A and in B. That is (A and B) $= \{THT, TTT\}$. This is the event that A and B both happen – that the first flip is a T and the third flip is a T.

Or: If A and B are events, then the event (A or B) is the event that A or B or both occur.

Example: In the three-flips example, if $A = \{THH, THT, TTH, TTT\}$ (the event that the first flip is a T) and $B = \{HHT, HTT, THT, TTT\}$ (the event that the last flip is a T), then (A or B) is the set of all events which are in A or in B or both. That is (A or B) $= \{THH, THT, TTH, TTT, HHT, HTT\}$. This is the event that A or B or both happen – that the first flip is a T or the third flip is a T.

Complement of an Event (Not): The complement of a event A is the event that A does not occur. This is denoted $\bar{A}$.

Example: In the three-flips example, if $A = \{THH, THT, TTH, TTT\}$ (the event that the first flip is a T), then $\bar{A}$ is the set of all events which are not in A. That is $\bar{A} = \{HHH, HHT, HTH, HTT\}$. This is the event that A does not happen – that the first flip is not a T.

PROBABILITY

Each event is assigned a number called its probability. The probability of an event A is denoted $P(A)$. $P(A)$ is a number between 0 and 1 (inclusive).

- If $P(A) = 1$, then A is (essentially) certain to happen.
- If $P(A)$ is close to 1, then A is likely to happen.
- If $P(A)$ is close to 0, then A is unlikely.
- If $P(A) = 0$, then it is (essentially) certain that A will not happen.

RELATIVE FREQUENCY APPROXIMATION

To approximate the probability of an event A we can:

1. Repeat the procedure a number of times.
2. Count how many times A occurs.
3. $P(A) \approx \dfrac{\text{number of times } A \text{ occured}}{\text{number of times procedure repeated}}$.

Example: When a tack is flipped, it can land either point up or point down. If I flip the tack 200 times and it lands up 143 times, then an approximation to the probability that the tack will land up is $\frac{143}{200} = 0.715$. This may not be the real probability.

Law of Large Numbers: As a procedure is repeated a growing number of times, the relative frequency approximation of the probability of an event tends to approach the actual probability of the event.

Interpretation: The more times we repeat an experiment, the closer we expect the relative frequency approximation to be to the actual probability.

CLASSICAL APPROACH TO PROBABILITY

If all of the simple events in a procedure are equally likely then

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{number of simple events}}.$$

Example: In the three-flips example, there are 8 simple events which are all equally likely. If $A = \{THH, THT, TTH, TTT\}$ (the event that the first flip is a T), then $P(A) = \frac{4}{8} = 0.5$.

Example: Suppose that a bowl contains 3 red marbles and 2 blue marbles. A marble is selected from the bowl. We want to know the probability that the marble is red. Since there are 3 red marbles and 5 marbles total, then the probability that the marble is red is $\frac{3}{5}$. The probability that the marble is blue is $\frac{2}{5}$.

COMBINATIONS OF EVENTS

Complements: $P(\bar{A}) = 1 - P(A)$.
Example: In the three-flips example, let A be the probability that at least one H is flipped. The $\bar{A}$ is the probability that no H's are flipped. There is only one what in which $\bar{A}$ can occur – TTT – so $P(\bar{A}) = \frac{1}{8}$. Then $P(A) = 1 - P(\bar{A}) = \frac{7}{8}$.

At Least One: In the previous example, we used that the complement of "at least one" is "none." This is a trick we will use repeatedly.

$$\overline{\text{at least one}} = \text{none}.$$

Independent Events: Two events A and B are independent if the occurrence of one of the events does not affect the probability of the occurence of the other event. Events which are not independent are dependent.
Example: In the three-flips example, if A is the event that the first flip is a T and B is the event that the last flip is a T, then A and B are independent because the first and last flips do not affect each other. If C is the event that there is at least on T, then A and C are dependent because if A happens, then C must also happen.

Example: Suppose that a bowl contains 3 red marbles and 2 blue marbles. A marble is selected from the bowl. Its color is noted, and it is replaced. Then another marble is selected from the bowl. Its color is noted, and it is replaced. Since the distribution of colors of marbles in the bowl are the same for both selections, the two colors are independent. This is **selection with replacement**.

Example: Suppose that a bowl contains 3 red marbles and 2 blue marbles. A marble is selected from the bowl. Its color is noted, and it is NOT replaced. Then another marble is selected from the bowl. Its color is noted, and it is NOT replaced. Since the color of the first marble changes the distribution of colors of marbles in the bowl, the probabilities for the second selection are affected by the first selection. The two colors are dependent. This is **selection withOUT replacement**.

With and Without Replacement: Selections with replacement are independent. Selections without replacement are dependent.

And – Multiplication Rule, Independent Events: If A and B are independent, then $P(A \text{ and } B) = P(A) \times P(B)$.

Example: In the three-flips example, if A is the event that the first flip is a T and B is the event that the last flip is a T, then A and B are independent so $P(A \text{ and } B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Note that there are 4 ways in which A can occur, 4 ways in which B can occur, and 2 ways in which A and B can occur together. Since there are a total of 8 outcomes, this agrees with the classical approach.

Example: Suppose that a bowl contains 3 red marbles and 2 blue marbles. Two marbles are selected from the bowl with replacement. What is the probability that both marbles are blue?

Solution: The probability that the first marble is blue is $\frac{2}{5}$. Since this is selection with replacement, the probability that the second is blue is also $\frac{2}{5}$. So

$$P(\text{both blue}) = P(\text{first blue and second blue}) = P(\text{first blue}) \times P(\text{second blue}) = \frac{4}{25}.$$

Multiplication Rule – Dependent Events: If A and B are dependent, then we will informally say that $P(A \text{ and } B) = P(A) \times P(B)$, but when $P(B)$ is calculated, we must assume that A has already occurred. Formally, $P(A \text{ and } B) = P(A) \times P(B|A)$ where $P(B|A)$ is read as "the probability of B given A." This is the probability that B occurs knowing that A occurs.

Example: Suppose that a bowl contains 3 red marbles and 2 blue marbles. Two marbles are selected from the bowl without replacement. What is the probability that both marbles are blue?

Solution: The probability that the first marble is blue is $\frac{2}{5}$. After one blue marble has been selected, there are 4 marbles left in the bowl, and one of these is blue. Thus the probability that the second is blue is $\frac{1}{4}$. So

$$P(\text{first blue and second blue}) = P(\text{first blue}) \times P(\text{second blue}) = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}.$$

Example: Suppose that a bowl contains 3 red marbles and 2 blue marbles. Two marbles are selected from the bowl without replacement. What is the probability that at least one marble is red?

Solution: For questions of "at least one" we will usually use complements:

$$P(\text{at least one red}) = P(\overline{\text{no red}}) = P(\overline{\text{both blue}}) = 1 - P(\text{both blue}) = \frac{9}{10}.$$

Disjoint: Two events A and B are disjoint if they cannot occur at the same time.

Example: In the three-flips example the events of the first flip being an H and the first flip being a T are disjoint. The events of the first flip being an H and the last flip being a T are not disjoint.

Or – Addition Rule, Disjoint Events: If A and B are disjoint, then $P(A \text{ or } B) = P(A) + P(B)$.

Example: Suppose that a bowl contains 3 red marbles and 2 blue marbles. Two marbles are selected from the bowl with replacement. What is the probability that the first marble is red or that both are blue?

Solution: Let A be the event that the first marble is red, and let B be the event that both marbles are blue. These two events cannot happen at the same time, so they are independent. We already have $P(B)$ from an earlier example. $P(A) = \frac{3}{5}$ since there are 3 red marbles and 5 total marbles. Then

$$P(A \text{ or } B) = P(A) + P(B) = \frac{4}{25} + \frac{3}{5} = \frac{19}{25}.$$

Union − Addition Rule: When events are not disjoint, we have this formal addition rule:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

If we think of our calcuations as counting, we can try to find $P(A \text{ or } B)$ by counting everything in A and then counting everything in B and adding. This gives $P(A) + P(B)$. But in counting this way, we have counted those events in A and B twice, so we subtract them off again.

Example: In the three-flips example, let A be the event that the first flip is a T, and let B be the event that the third flip is a T. Then we already know that $P(A) = P(B) = \frac{1}{2}$ and that $P(A \text{ and } B) = \frac{1}{4}$. Then the probability that either the first flip is a T or the last is a T (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}.$$

And's and Or's: Associate "and" with multiplication and "or" with addition.

REDUNDANCY

A certain type of alarm clock works on only 80% of mornings. If Bob uses three of these alarm clocks, what is the probability that at least one works on a given morning?

Solution: Since this problems asks about "at least one" we will use complements.

$$P(\text{at least one works}) = P(\overline{\text{all three fail}}) = 1 - P(\text{all three fail}).$$

Since the clocks are independent of each other,

$$P(\text{all three fail}) = P(\text{first fails}) \times P(\text{second fails}) \times P(\text{third fails})$$

We can use the probability of a clock working to find the probability it fails. For each clock, we have:

$$P(\text{fails}) = 1 - P(\text{works}) = 1 - 0.8 = 0.2.$$

Now we have

$$P(\text{all three fail}) = 0.2 \times 0.2 \times 0.2 = .008.$$

So

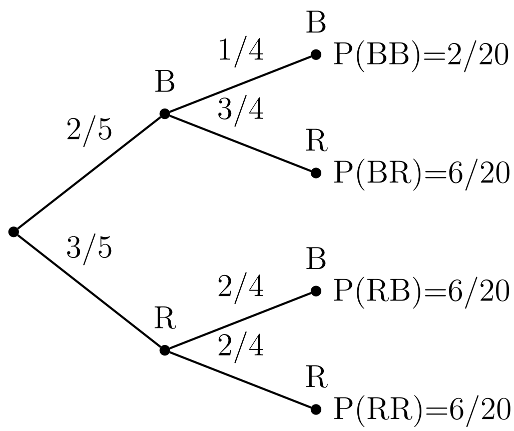
$$P(\text{at least one works}) = 1 - .001 = .992.$$

Thus, independent components which are not very good can be combined into a redundant system that works quite well.

TREES

We can draw trees which allow us to calculate probabilities associated with selections, multiple flips, redundancies, etc.

Example: Suppose that a bowl contains 3 red marbles and 2 blue marbles. Two marbles are selected from the bowl without replacement. We can draw a tree that depicts every outcome of the experiment:



Each node in the tree represents a selection. Branching up indicates that a blue marble was selected. Branching down indicates that a red marble was selected. Each segment of each branch is labeled with the probability at that selection of getting that color. To find the probability of following a particular branch to its end, we multiply down the branch.

Example: To find the probability that at least one marble selected is red, we simply list the outcomes that correspond to this event: RR, RB, BR, and we add their probabilities:

$$P(\text{at least one red}) = P(RR \text{ or } RB \text{ or } BR) = P(RR)+P(RB)+P(BR) = \frac{6}{20}+\frac{6}{20}+\frac{6}{20} = \frac{9}{10}.$$

This result agrees with our earlier example.