CS31400 Homework 2

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1 Basic Probability

1.1 Question 1

Given that all arrival times are equally likely, if the bus has not arrived by 8:40, then the probability that it arrives by 8:50 is $\frac{1}{6}$. One can determine this because the total amount of time that an arrival can occur in is 8am to 9am, which is a time period of 60 minutes. If we take out discrete time step to be equal to 1 minute, then that means there are 60 time steps. 10 of those time steps abide by the 8:40 $< t \leq$ 8:50 time restraint proposed in the problem.

1.2 Question 2

Given that we have already popped one balloon from the fair, the new total of balloons is 19. Additionally, because we have already popped one green balloon, we now know that the total number of green balloons is 4. Thus, the probability of the next balloon being popped being green is $\frac{4}{19}$.

1.3 Question 3

To find this combined probability, we must first look at the total combinations possible for the desired outcome to be sufficient. The outcomes that we are looking for are, in order of (red, white) dice: (2, 1), (2, 3), (2, 5). The total number of events possible is equal to $6 * 6 = 36$. As such, the probability that you get a 2 on the red dice and an odd sum of numbers on the two dice is equal to $\frac{3}{36}$.

1.4 Question 4

1.4.1 Part 1

The probability of getting three heads is equal to the probability of getting heads independently on each coin multiplied together. As such, this probability is equal to $0.5 * 0.7 * 0.7 = 0.245$.

1.4.2 Part 2

The probability of two heads and a tail is dependent upon the individual probabilities of each condition. For two heads and a tail to be the result of a toss, the result must be equal to $P(0,1,1) + P(1,0,1) + P(1,1,0)$, with 1 representing a toss of heads and the 2nd and 3rd index representing the coins that are unfair. As a result, the probability is equal to:

$$P(0,1,1) = 0.5 * 0.7 * 0.7 = 0.245$$

$$P(1,0,1) = 0.5 * 0.3 * 0.7 = 0.105$$

$$P(1,1,0) = 0.5 * 0.7 * 0.3 = 0.105$$

$$P(0,1,1) + P(1,0,1) + P(1,1,0) = 0.455$$

1.5 Question 5

1.5.1 Part 1

The probability of getting exactly two heads is equal to $P(0,1,1) + P(1,0,1) + P(1,1,0)$, with each index representing a different flip, and a 1 indicating a toss of heads. As such, the resultant probability is equal to:

$$P(0,1,1) = 0.5 * 0.5 * 0.5 = 0.125$$

$$P(1,0,1) = 0.5 * 0.5 * 0.5 = 0.125$$

$$P(1,1,0) = 0.5 * 0.5 * 0.5 = 0.125$$

$$P(0,1,1) + P(1,0,1) + P(1,1,0) = 0.375$$

1.5.2 Part 2

The probability that the third flip of the coin is a head is equal to 0.5. This is because the third flip of the coin is independent to what the previous flips have been. As such, it is simply equal to the probability of a single toss with probability P(1), with a 1 indicating a toss of heads.

1.5.3 Part 3

The two events are independent if the covariance of the two variables is equal to 0. To find the covariance, we must use $Cov(X,Y) = E(XY) - E(X)E(Y)$. We will utilize X as the condition of getting two heads, and Y as the condition of the third flip being a head. First, we find $E(XY)$. To find $E(XY)$ we must recognize that the only overlapping probabilities are for $P(0,1,1)$ and $P(1,0,1)$, thus $E(XY) = P(0,1,1) + P(1,0,1) = 0.125 + 0.125 = 0.25$. We determined earlier that $E(X) = 0.375$ and $E(Y) = 0.5$. Thus, the covariance is equal to $Cov(X,Y) = 0.25 - 0.375 * 0.5 \neq 0$. Because the covariance is not equal to 0, these events are dependent.

1.6 Question 6

1.6.1 Part 1

If we take a 0 to indicate that the six-sided dice was rolled to be odd, and a 1 to indicate that the six-sided dice was rolled to be even, we can simply find the value of $P(0,0) + P(1,1)$, with one index representing one roll. As such, the resulting probability is equal to:

$$P(0,0) = 0.5 * 0.5 = 0.25$$

$$P(1,1) = 0.5 * 0.5 = 0.25$$

$$P(0,0) + P(1,1) = 0.5$$

1.6.2 Part 2

The probability that the roll of the second dice is ≥ 4 is equal to the probability that we roll a 4, 5, or 6. On a six sided dice, this is equal to $P(4|5|6) = 0.5$.

1.6.3 Part 3

As stated in 1.5.3, two events are independent if the covariance of the two variables is equal to 0. The covariance formula is given by $Cov(X, Y) = E(XY) - E(X)E(Y)$. We will take the probability of the sum of the two rolls being even to be represented by X and the roll of the second dice being greater than 4 to be Y. We must first calculate the $E(XY)$ term. The overlapping probabilities are as follows: (1, 5), (2, 4), (2, 6), (3, 5), (4, 4), (4, 6), (5, 5), (6, 4), (6, 6). There are nine possible overlapping probabilities, which gives us $E(XY) = \frac{9}{36} = \frac{1}{4}$. As determined earlier, $E(X) = 0.5$ and $E(Y) = 0.5$. We must now simply plug the result into the covariance formula. $Cov(X, Y) = 0.25 - (0.5)(0.5) = 0$. The two events are independent.

2 Central Limit Theorem

2.1 Question 1

2.1.1 Part 1

The mean of a six-sided dice roll where even numbers are twice as likely as odd numbers can be modeled via

$$\mu = \sum_{x=odd(1...6)}^{6} \frac{x}{9} + \sum_{x=even(1...6)}^{6} \frac{2x}{9}$$

$$\mu = 1 + \frac{8}{3} = 3.667$$

The variance of the rolls proposed is then equal to

$$var(x) = E(X^2) - E(X)^2$$

$$var(x) = (\sum_{x=odd(1...6)}^{6} \frac{x^2}{9} + \sum_{x=even(1...6)}^{6} \frac{2x^2}{9}) - (\sum_{x=odd(1...6)}^{6} \frac{x}{9} + \sum_{x=even(1...6)}^{6} \frac{2x}{9})^2$$

$$var(x) = (\frac{35}{9} + \frac{112}{9}) - (\frac{11}{3})^2$$

$$var(x) = \frac{26}{9}$$

2.1.2 Part 2

After simulating the situation using python, I received the following data-set:

$$1 : 113$$

$$2 : 225$$

$$3 : 116$$

$$4 : 216$$

$$5 : 117$$

$$6 : 213$$

I then calculated the mean, variance, and standard deviation for the above data-set.

Mean: 3.638

Variance: 2.871

Standard Deviation: 1.694

2.1.3 Part 3

The normal distribution corresponding to the variables in part 2 is equal to

$$y(x) = \frac{1}{1.694 * \pi\sqrt{2}} e^{-\frac{(x-3.638)^2}{2*(1.694)^2}}$$

This function follows the standard deviation formula that is introduced in page 9 of the ch3 part 1 notes.

2.1.4 Part 4

To find the probability of a variable being within one standard deviation from the mean of this function is equal to

$$\int_{3.638-1.694}^{3.638+1.694} \frac{1}{1.694 * \pi\sqrt{2}} e^{-\frac{(x-3.638)^2}{2*(1.694)^2}}$$

If we solve this, we get that the probability is equal to 0.385.

3 Expectation, Variance, Standard Deviation

3.1 Question 1

The mean of the sum of two rolls of a fair dice is equal to the sum of the mean of one dice roll. As such, the mean is equal to:

$$2\sum_{x=1}^{6}\frac{1}{6}x = 7$$

The variance is equal to $E(X^2) - (E(X))^2$. Thus, we can simply solve the formula in order to determine the variance:

$$2((\sum_{x=1}^{6}\frac{1}{6}x^2) - (\sum_{x=1}^{6}\frac{1}{6}x)^2) = \frac{35}{6}$$

The standard deviation is equal to the square root of the variance:

$$\sqrt{\frac{35}{6}} = \frac{\sqrt{210}}{6} = 2.415$$

3.2 Question 2

We will use much of the same method as question 1, expect we will exclude the number 1 from the summation in order to account for it being twice as likely as the other numbers. As such, we can find the probability of any of the five numbers being rolled as $2x + 5x = 1$, which solves to $x = \frac{1}{7}$. That means that 1 has a probability of $\frac{2}{7}$, and all other numbers have a probability of $\frac{1}{7}$. We can use this information to solve for the new mean.

$$2 * \frac{2}{7} * 1 + 2\sum_{x=2}^{6}\frac{1}{7}x = 6.286$$

To find the variance, we can use the same method as used in question 1.

$$2((\frac{2}{7} * 1^2 + \sum_{x=2}^{6}\frac{1}{7}x^2) - (\frac{2}{7} * 1 + \sum_{x=2}^{6}\frac{1}{7}x)^2) = \frac{320}{49} = 6.531$$

The standard deviation is then equal to the square root of the variance, thus:

$$\sqrt{\frac{320}{49}} = 2.556$$

3.3 Question 3

The mean is equal to the weighted sum of each of the proportions:

$$P(1) = 1 * 0.267 = 0.267$$

$$P(2) = 2 * 0.336 = 0.672$$

$$P(3) = 3 * 0.158 = 0.474$$

$$P(4) = 4 * 0.137 = 0.548$$

$$P(5) = 5 * 0.063 = 0.315$$

$$P(6) = 6 * 0.024 = 0.144$$

$$P(7) = 7 * 0.015 = 0.105$$

$$0.267 + 0.672 + 0.474 + 0.548 + 0.315 + 0.144 + 0.105 = \mu = 2.525$$

The variance is then equal to $E(X^2) - E(X)^2$. We can then solve this equation for the given problem:

$$E(X^2) - E(X)^2$$

$$E(X^2) = [1^2*0.267+2^2*0.336+3^2*0.158+4^2*0.137+5^2*0.063+6^2*0.024+7^2*0.015] = 8.399$$

$$E(X)^2 = 2.525^2 = 6.375625$$

$$E(X^2) - E(X)^2 = 2.023375$$

The standard deviation is equal to the square of the variance, which is equal to:

$$\sqrt{2.023375} = 1.422$$

4 Discrete Random Variables

4.1 Question 1

4.1.1 Part 1

$P(X \leq 3)$ is equal to the cumulative probability of all of the probabilities when x $\leq$ 3. As such, we can find this by summing $0.1 + 0.2 + 0.3 = 0.6$. Thus, $P(X \leq 3) = 0.6$

4.1.2 Part 2

$P(1 \leq x \leq 4)$ is similar to part 1, with the exception of summing x over the probability interval of [1, 4]. As such, $P(1 \leq x \leq 4) = 0.1+0.2+0.3+0.2 = 0.8$.

4.1.3 Part 3

$E(X)$ is equal to the weighted mean of each of the probabilities. We can calculate this by simply taking $\sum_{i=1}^{6} p_i x_i$. If we perform this operation we get $(0.1 * 1 + 0.2 * 2 + 0.3 * 3 + 0.2 * 4 + 0.1 * 5 + 0.1 * 6) = 3.3$.

4.1.4 Part 4

$Var(X) = E(X^2) - (E(X))^2$. We have already found $E(X)$ in part 3, which was equal to 3.3. To solve for $E(X^2)$ we find $\sum_{i=1}^{6} p_i x_i^2$, which is equal to:

$$E(X^2) = (0.1 * 1^2 + 0.2 * 2^2 + 0.3 * 3^2 + 0.2 * 4^2 + 0.1 * 5^2 + 0.1 * 6^2) = 12.9$$

Thus, we simply solve the equation.

$$Var(X) = 12.9 - 3.3^2 = 2.01$$

4.1.5 Part 5

$E(0.5X)$ is equal to $E(X) * 0.5$. Therefore, $E(0.5X) = 1.65$.

4.1.6 Part 6

$Var(0.5X)$ is equal to $(0.5)^2 Var(X)$. Therefore, $Var(0.5X) = 0.5^2 * 2.01 = 0.5025$.

4.2 Question 2

4.2.1 Part 1

To find $E(Y)$, we must solve the standard expectancy formula for a roll of a dice: $\sum_{1}^{6} p_i x_i$. In the problem statement, we are given that $E(Y) = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$. Due to the fact that each roll of the dice is independent from one another, we can simplify this equation to equal $E(Y) = 6X_1$. $E(X_1) = \sum_1^6 \frac{1}{6}x_i = 3.5$. Thus, $E(Y) = 6 * 3.5 = 21$.

4.2.2 Part 2

To find $Var(Y)$, we use the standard variance equation: $E(Y^2) - (E(Y))^2$. This problem gets a bit tricky in that the 6 independent events lead to 6 different variances. Thus, to find the variance of Y, we must find the variance of a single dice roll and then multiply it by 6. $E(X_1^2) = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}$. We solved for $E(X_1)$ in part 1, which was equal to 3.5. As such, we can simply plug into the general variance formula with a result of $\frac{91}{6} - 3.5 = \frac{35}{12}$. If we multiply this by 6 we then get $Var(Y) = \frac{35}{12} * 6 = \frac{35}{2}$.

4.3 Question 3

To solve this we need to take the integral of the Poisson distribution equation. $\int_{10}^{15} \frac{10^x e^{-10}}{x!} = 0.4142$.

5 Continuous Random Variables

5.1 Question 1

We can find the expected mean by solving $E(Z) = E(X) + E(Y)$. Plugging the given means into this equation lets us see that $E(Z) = 3 + 2 = 5$. The variance of Z is equal to $Var(Z) = Var(X) + Var(Y)$. We then solve for the variance of Z: $Var(Z) = 1 + 2 = 3$. Determining the distribution is now as simple as plugging in the variance and mean values into the normal distribution equation. If we perform this operation, we get the following equation:

$$p(x) = \frac{1}{3\sqrt{2}\pi} e^{\frac{-(x-5)^2}{2\sqrt{3}}}$$

5.2 Question 2

5.2.1 Part 1

The cumulative distribution function is equal to the integral of the probability distribution function. As such, the following equation can be solved to get the cumulative distribution function.

$$CDF(-1 \leq x \leq 3) = \int_{-1}^{3} (0.125x + 0.125)dt$$

We can solve this integral and find the final value for the cumulative distribution function:

$$CDF(-1 \leq x \leq 3) = 0.0625x^2 + 0.125x]_{-1}^{3}$$

5.2.2 Part 2

The probability that X takes the value of 0.5 $\leq$ X $\leq$ 1 is equal to $CDF(a \leq x \leq b) = 0.0625x^2 + 0.125x]_{0.5}^{1} = 0.109$

5.2.3 Part 3

The PDF of $Z = |X|$ gets changed from 0 to 1. This is because we must account for the probabilities that are denoted from -1 to 0. As a result, the probability from $0 \leq x \leq 1$ becomes 0.25 because we must flip the portion of the original PDF from -1 to 0 across the x axis. As a result, the new PDF is as follows:

$$PDF(Z) = \begin{cases} 0.25 & 0 \leq x \leq 1 \\ 0.125x + 0.125 & 1 \leq x \leq 3 \end{cases}$$

5.3 Question 3

5.3.1 Part 1

To solve for c, we must note that this variable is a continuous random variable. As such, it cannot have a jump in the piece wise function that is presented. Thus, to solve we must set $0.5 = ce^{-1}$. If we solve this equation we can find that $c = 0.5e$.

5.3.2 Part 2

The CDF of X is equal to the integral of the PDF of X. As such, the CDF of X is equal to:

$$CDF(X) = \begin{cases} 0 & x \leq 0 \\ 0.5x & 0 \leq x < 1 \\ 0.5e^{1-x} & x \geq 1 \end{cases}$$

5.3.3 Part 3

To solve for the expected value of a continous random variable, we must find the integral of each portion of the piece wise function multiplied by x.

$$E(X) = \int_0^1 0.5x + \int_1^\infty 0.5xe^{1-x}$$

$$E(X) = 0.25 + 1 = 1.25$$

5.4 Question 4

5.4.1 Part 1

For two uniform random variables, we can assume that each is equally likely to be greater than one another. This means that $E(X) = E(Y)$. As a result, the probability that $X < Y$ is equal to 0.5.

5.4.2 Part 2

The probability that $X < 2Y$ is equal to 0.25. As found in part 1, the probability that Y is greater than X is equal to 0.5. If we shift the 2 to the other side of the equation, we can find that this probability is the same as $\frac{X}{2} < Y$. Thus, we must simply divide the previously found probability of 0.5 by 2.

5.4.3 Part 3

The probability that $X + Y < 0.5$ is equal to 0.0625. This is because if we take the integral of each variable's PDF up to 0.25 (the most likely value for the equation to be true) then we get 0.25 as the resultant probability. We must then multiply them together to find that $P(X + Y < 0.5) = 0.0625$