CS 314 Numerical Methods Problem Set 6

Deadline: 11:59 PM, November 17, 2023

For each problem, briefly explain/justify how you obtained your answer. This will help us determine your understanding of the problem and whether or not you got the correct answer. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide. **Again, please type your solution and submit it on Gradescope!**

Note 1: This assignment is going to be a programming assignment. Therefore, you are expected to implement the Newton Interpolation algorithm for Problems 1 and 2, Inverse of a Matrix for Problem 3 part (a), and QR Decomposition Algorithm for Problem part (b) from scratch. Any solution using the MATLAB/Python built-in Newton Interpolation, Inverse of a Matrix, and QR Decomposition algorithm will be considered as a No Attempt/Blank solution! Any built-in matrix operation/algorithm other than these three algorithms is allowed to be used.

Note 2: Please put the programs' final output, derivations, and plots in your write-up. Your code is supposed to be submitted to the "Homework 6 Code" portal on Gradescope!

1. *(10 points)* (Newton Interpolation Algorithm) Let $\mathcal{D} = \{(x_i, y_i), i = 0, \cdots, 20\} \in \mathbb{R}^2$ be a set of points defined as the following

$$x_i = -5 + \frac{1}{2} \cdot i \tag{1}$$
$$y_i = 2\sin(x_i) + \cos(3x_i) \tag{2}$$

   Please use the Newton Interpolation algorithm to find the polynomial of degree 20.

2. *(40 points)* (Choice of Interpolating point) Let $f(x) = \sin(x)$, where $x \in [-1, 1]$

   (a) *(5 points)* Consider the following 5 equi-spaced points on $f(x) = \sin(x)$

$$(x_i, y_i) = \left(-1 + \frac{1}{2} \cdot i, \ \ \sin(x_i)\right), i = 0, 1, 2, 3, 4 \tag{3}$$

   Please use the Newton Interpolation formula to find the polynomial of degree 4. **(Recall that given $n + 1$ points, there exists a unique polynomial of degree n passing through these n points)**

(b) *(5 points)* Now, let's consider 5 Chebyshev points on $f(x) = \sin(x)$

$$(x_i, y_i) = \left(\cos\left(\frac{\pi i}{4}\right),\ \sin(x_i)\right), i = 0, 1, 2, 3, 4 \tag{4}$$

Please use the Newton Interpolation formula to find the polynomial of degree 4.

(c) *(5 + 5 points)* In this part, we are going to analytically compute an upper bound of the interpolation error between $f(x) = \sin(x), x \in I = [-1, 1]$ and the interpolation polynomial $p(x)$. Recall that the **polynomial interpolation error theorem** tells us

$$|f(x) - p_n(x)| = \frac{1}{(n+1)!}|f^{(n+1)}(\xi_x)| \left|\prod_{i=0}^{n}(x - x_i)\right|, \text{where } x, \xi_x \in I \tag{5}$$

i. Let's consider the following $n + 1$ equal spaced points

$$x_i = -1 + \frac{(1 - (-1))}{n} \cdot i \tag{6}$$

What is the maximum interpolation error E_1 (i.e., $E_1 = \max_{x\in[-1,1]} |f(x) - p_n(x)|$)?

ii. If we find the interpolation polynomial $p_n(x)$ for $n + 1$ Chebyshev points, i.e.,

$$(x_i, y_i) = \left(\cos\left(\frac{\pi i}{n}\right),\ \sin(x_i)\right), i = 0, 1, \cdots, n \tag{7}$$

what is the maximum interpolation error E_2 (i.e., $E_2 = \max_{x\in[-1,1]} |f(x) - p_n(x)|$)?

(d) *(10 + 10 points)* In part (c), we derive the bound for the interpolation error between $\sin(x)$ and $p_n(x)$. In this part, we are going to plot the maximum error for these two cases. (Note: x-axis represents the degree of the interpolation polynomial, and y-axis represents the maximum error)

i. Plot the maximum error E_1 versus the degree of $p_n(x)$, where $p_n(x)$ is the degree-n interpolating polynomial for $n + 1$ **equal spaced points**, and $n = 3, 4, 5, 6, 7$

ii. Plot the maximum error E_2 versus the degree of $p_n(x)$, where $p_n(x)$ is the degree-n interpolating polynomial for $n + 1$ **Chebyshev points**, and $n = 3, 4, 5, 6, 7$

3. *(50 points)* (Quadratic least square regression) Consider quadratic least square regression in an Euclidean plane on a set of 30 **distinct** data points $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^{30}$ demonstrated below

x_i	y_i
-5	25.198
-4.6552	22.354
-4.3103	11.286
-3.9655	-5.3643
-3.6207	-9.9406
-3.2759	5.5574
-2.931	20.364
-2.5862	10.826
-2.2414	12.018
-1.8966	18.742
-1.5517	-4.5468
-1.2069	-6.2935
-0.86207	11.038
-0.51724	-6.9372
-0.17241	22.685
0.17241	10.281
0.51724	-4.0523
0.86207	-0.26773
1.2069	10.089
1.5517	13.64
1.8966	20.074
2.2414	13.906
2.5862	2.7136
2.931	16.384
3.2759	2.9209
3.6207	27.113
3.9655	30.377
4.3103	29.667
4.6552	22.724
5	48.731

In this problem, we consider the following two approaches,

(a) *(25 points)* Implement a program to derive the least squares quadratic polynomial by solving the normal equations. State the limitation of this approach.

(b) *(25 points)* Implement a program to derive the least square quadratic polynomial using QR factorization that finds the orthogonal projection of **b** onto the column space of A.