CS31400 Homework 5

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1 Section 1

$$A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$$

Perform $R_2 = R_2 + \frac{R_1}{4}$ and $R_3 = R_3 + \frac{R_1}{4}$

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & -\frac{5}{4} & \frac{15}{4} \end{bmatrix}$$

Perform $R_3 = R_3 + \frac{R_2}{3}$

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & 0 & \frac{10}{3} \end{bmatrix}$$

The U matrix is equal to the new A matrix, and the L matrix simply becomes the operations that we performed.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 \end{bmatrix}$$

2 Section 2

2.1 Part A

We can use the above LU decomposition to simply our equation. As such, we take

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 \end{bmatrix}, U = \begin{bmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & 0 & \frac{10}{3} \end{bmatrix}$$

We can then solve the equation $LUx = B$ where $Ly = B$ and $Ux = Y$.

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y_3 = 1$$

$$\begin{bmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & 0 & \frac{10}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ 0 & \frac{15}{4} & -\frac{5}{4} & 0 \\ 0 & 0 & \frac{10}{3} & 1 \end{bmatrix}$$

$$\frac{10}{3}x_3 = 1$$

$$x_3 = \frac{3}{10}$$

$$\frac{15}{4}x_2 - \frac{5}{4}x_3 = 0$$

$$x_2 = \frac{1}{10}$$

$$4x_1 - x_2 - x_3 = 0$$

$$x_1 = \frac{1}{10}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{10} \\ \frac{3}{10} \end{bmatrix}$$

2.2 Part B

To solve part B, we again take the LU decomposition that we determined earlier.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 \end{bmatrix}, U = \begin{bmatrix} 4 & -1 & -1 \\ 0 & \frac{15}{4} & -\frac{5}{4} \\ 0 & 0 & \frac{10}{3} \end{bmatrix}$$

We again use the equation $LUx = B$ where $Ly = B$ and $Ux = Y$.

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -\frac{1}{4} & 1 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{3} & 1 & 1 \end{bmatrix}$$

$$y_1 = 1$$

$$-\frac{1}{4} + y_2 = 0$$

$$y_2 = \frac{1}{4}$$

$$-\frac{1}{4}(1) - \frac{1}{3}(\frac{1}{4}) + y_3 = 1$$

$$y_3 = \frac{4}{3}$$

Now, we use the U matrix to solve for the x vector.

$$\begin{bmatrix} 4 & -1 & -1 & 1 \\ 0 & \frac{15}{4} & -\frac{5}{4} & \frac{1}{4} \\ 0 & 0 & \frac{10}{3} & \frac{4}{3} \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & -1 & 1 \\ 0 & 15 & -5 & 1 \\ 0 & 0 & 10 & 4 \end{bmatrix}$$

$$10x_3 = 4$$

$$x_3 = \frac{2}{5}$$

$$15x_2 - 5(\frac{2}{5}) = 1$$

$$x_2 = \frac{1}{5}$$

$$4x_1 - \frac{1}{5} - \frac{2}{5} = 1$$

$$x_1 = \frac{2}{5}$$

3 Section 3

3.1 Part A

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

3.2 Part B

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

4 Section 4

4.1 Part A

$$||v||_1 = |1| + |-1| = 2$$
$$||v||_2 = (1^2 + -1^2)^{1/2} = \sqrt{1}$$
$$||v||_3 = (1^3 + -1^3)^{1/3} = 0$$
$$||v||_\infty = max(|1|, |-1|) = 1$$

4.2 Part B

$$||v||_1 = |1| + |2| + |-2| = 5$$
$$||v||_2 = (1^2 + 2^2 + -2^2)^{1/2} = 3$$
$$||v||_3 = (1^3 + 2^3 + -2^3)^{1/3} = 1$$
$$||v||_\infty = max(|1|, |2|, |-2|) = 2$$

4.3 Part C

$$||v||_1 = |1| + |-1| + |2| + |-2| = 6$$
$$||v||_2 = (1^2 + -1^2 + 2^2 + -2^2)^{1/2} = \sqrt{10}$$
$$||v||_3 = (1^3 + -1^3 + 2^3 + -2^3)^{1/3} = 0$$
$$||v||_\infty = max(|1|, |-1|, |2|, |-2|) = 2$$

5 Section 5

5.1 Part A

$$||A||_1 = max((|1| + |1|), (|2| + |-2|)) = 4$$
$$||A||_\infty = max((|1| + |2|), (|1| + |-2|)) = 3$$

5.2 Part B

$$||A||_1 = max((|1| + |2| + |1|), (|2| + |-2| + |1|), (|1| + |0| + |3|)) = 5$$
$$||A||_\infty = max((|1| + |2| + |1|), (|2| + |-2| + |0|), (|1| + |1| + |3|)) = 5$$

6 Section 6

6.1 Part A

To find the absolute and relative condition numbers of a matrix, we must first compute the inverse matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -4 & -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$$

6.1.1 Absolute Condition Number

$$||A^{-1}||_1 = max((|\frac{1}{2}| + |\frac{1}{4}|), (|\frac{1}{2}| + |-\frac{1}{4}|)) = \frac{3}{4}$$

$$||A^{-1}||_\infty = max((|\frac{1}{2}| + |\frac{1}{2}|), (|\frac{1}{4}| + |-\frac{1}{4}|)) = 1$$

6.1.2 Relative Condition Number

$$||A^{-1}||_1||A||_1 = (\frac{3}{4})(4) = 3$$

$$||A^{-1}||_\infty||A||_\infty = (1)(3) = 3$$

6.2 Part B

One must first compute the inverse matrix in order to find the absolute and relative condition numbers.

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -6 & -2 & -2 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{2}{3} & -\frac{1}{6} & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -1 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{6}{21} & -\frac{3}{42} & \frac{3}{7} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{7} & \frac{5}{14} & -\frac{1}{7} \\ 0 & 1 & 0 & \frac{3}{7} & -\frac{1}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & -\frac{2}{7} & -\frac{1}{14} & \frac{3}{7} \end{bmatrix}$$

$$A^{-1} = \frac{1}{14} \begin{bmatrix} 6 & 5 & -2 \\ 6 & -2 & -2 \\ -4 & -1 & 6 \end{bmatrix}$$

6.2.1 Absolute Condition Number

$$||A^{-1}||_1 = \frac{1}{14} max((|6|+|6|+|-4|), (|5|+|-2|+|-1|), (|-2|+|-2|+|6|)) = \frac{8}{7}$$

$$||A^{-1}||_\infty = \frac{1}{14} max((|6|+|5|+|-2|), (|6|+|-2|+|-2|), (|-4|+|-1|+|6|)) = \frac{13}{14}$$

6.2.2 Relative Condition Number

$$||A^{-1}||_1 ||A||_1 = (\frac{8}{7})(5) = \frac{40}{7}$$

$$||A^{-1}||_\infty ||A||_\infty = (\frac{13}{14})(5) = \frac{65}{14}$$

7 Section 7

To find the maximum error in x, one must use the relative error in x due to relative error in b formula, or

$$\frac{||\Delta x||}{||x||} \leq \kappa(A) * \frac{||\Delta b||}{||b||}$$

$$\kappa(A) = ||A^{-1}||_\infty ||A||_\infty$$

$$A^{-1} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ -2 & 2 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 10 & 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$||A^{-1}||_\infty = \frac{1}{5} max((|1| + |-2|), (|1| + |\frac{1}{2}|)) = \frac{3}{5}$$

$$||A||_\infty = max((|1| + |4|), (|-2| + |2|)) = 5$$

$$\kappa(A) = ||A^{-1}||_\infty ||A||_\infty = (\frac{3}{5})(5) = 3$$

$$\frac{||\Delta x||}{||x||} \leq 3 * 10^{-7}$$

Thus, the maximum relative error in x is equal to $3 * 10^{-7}$

8 Section 8

$$\text{Vandermonde Form} = \begin{bmatrix} 1 & x_0^1 & x_0^2 \\ 1 & x_1^1 & x_1^2 \\ 1 & x_2^1 & x_2^2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 8 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The polynomial can then be identified as $1 + 2x + x^2$.

9 Section 9

One can determine that n = 3 because there are three points given. As such, the Lagrange form is given as follows:

$$L = y_0 * l_0 + y_1 * l_1 + y_2 * l_2$$

Where

$$l_0 = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{x^2 - 5x + 6}{-2}$$

$$l_1 = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{x^2 - 4x + 3}{-1}$$

$$l_2 = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{x^2 - 3x + 2}{2}$$

$$L = 4(\frac{x^2 - 5x + 6}{-2}) + 8(\frac{x^2 - 4x + 3}{-1}) + 14(\frac{x^2 - 3x + 2}{2})$$

$$L = x^2 + x + 2$$

10 Section 10

$$P = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$P = 1 + 2(x - 1) + 1(x - 1)(x - 2)$$

$$P = 1 + 2x - 2 + x^2 - 3x + 2$$

$$P = x^2 - x + 1$$

11 Section 11

The divided difference example is quite easy to solve for. It is much like the newton form, except the method to determine a_0, a_1, a_2 is different.

$$P = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$a_0 = y_0 = -1$$

$$a_1 = F[x_1, x_0] = 0$$

$$a_2 = F[x_2, x_1, x_0] = 1$$

$$P = -1 + 0(x - 0) + 1(x - 0)(x - 1)$$

$$P = x^2 - x - 1$$