Topological Aspects of Game Theory

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"lolz I should rlly start on alg topo final project" - Khunpob

1 Introduction

Formally, a *Game* between n players with $m_1, m_2, \dots, m_n \in \mathbb{N}$ possible choices for each player respectively can be defined via some *payoff matrix* $A \in (\mathbb{R}^n)^{m_1 \times m_2 \times \dots \times m_n}$. If each player i picks some choice $c_i \in [m_i]$, the corresponding payoff for each player i is then $(A_{c_1,c_2,\dots,c_n})_i$, i.e. the ith entry of the $(c_1, \dots, c_n)$ entry of the matrix A. Obviously for any player, it is in their self-interest to maximize their own payoff.

In a game, each player picks their choice independently of the others, and a *strategy* for a player i is just some discrete probability distribution $(p_{i,1}, p_{i,2}, \dots, p_{i,m_i}) \in [0,1]^{m_i}$ s.t. $\sum_{j=1}^{m_i} p_{i,j} = 1$. This represents player i playing choice $j \in [m_i]$ with probability $p_{i,j}$. A strategy is called deterministic if it is the case that $\exists j \in [m_i]$ s.t. $p_{i,j} = 1$, and mixed otherwise.

In the case of two person games, suppose that player 1 has payoff matrix A and player 2 has payoff matrix B. Moreover, suppose that they play the strategies $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ respectively. The expected payoff for player 1 is then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_i y_j = x^T A y$$

and analogously for player 2, their expected payoff is $x^T B y$. These definitions can naturally extended to the n player case.

One topic of interest in Game Theory is the notion of an *equilibrium point*. For simplicity, we define it in terms of two person games, but its definition naturally extends to n player games. Informally, it is when player 1 and player 2 play strategies x_*, y_* respectively, and it is in neither players interest to adjust their strategy. Formally, if A, B denote the playoff matrices of players 1 and 2, the pair x_*, y_* denotes an equilibrium point of strategies if

$$x_*^T A y_* = \max_{x \in X} x^T A y_* \text{ and } x_*^T B y_* = \max_{y \in Y} x_*^T B y$$

where X, Y denote the strategy spaces of player 1 and 2 respectively.

In 1928, Von Neumann published and proved his Minimax Theorem for a class of games called two person zero-sum games. It asserted that in that particular case, an equilibrium point always exists (though not necessarily a deterministic strategy). Later, John Nash famously published in 1950 *Equilibrium points in n-person games*, establishing the existence of equilibrium points in general n-person games with no restrictions. In it, he used several topological results, specifically Brouwer's Fixed Point Theorem, to prove the existence of such equilibrium points. The result was most impactful in fields like Economic Theory, but still has applications to all sorts of mathematics even today.

2 Two Person Zero-Sum Games

We now deal with the case when $n = 2$, and when such a game is called a *zero-sum* game. Formally, a two person game can be defined between players 1 and 2 with m_1, m_2 choices respectively via some $m_1 \times m_2$ matrix $(\mathbb{R} \times \mathbb{R})^{m_1 \times m_2}$. In a *zero-sum* game, the payoff matrix A satisfies the additional property that if $(a, b) = A_{i,j}$, then $b = -a$. In other words, for a player to maximize their own payoff it is also in their interest to minimize their opponent's.

In 1928 Von Neumann published and proved the Minimax Theorem for two person zero-sum games. Formally, the theorem is as follows:

Theorem 1. *(Von Neumann Minimax, 1928) Let $X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$ be compact and convex. If $f : X \times Y \to \mathbb{R}$ is a continuous concave-convex function (i.e. concave in the first argument and convex in the second), then it is the case that*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y)$$

In particular, if A is the payoff matrix for player 1 in a zero-sum game (so that player's 2 payoff is represented by $-A$), we have that

$$\max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y$$

Proof: First note that the min/max are well-defined since f is continuous. We will utilize Zorn's lemma and the following lemma:

Lemma: Let K_1, K_2 be nonempty convex, compact subsets of X s.t. $X = K_1 \cup K_2$. Then it is the case that

$$\max \left(\min_{y \in Y} \max_{x \in K_1} f(x, y), \min_{y \in Y} \max_{x \in K_2} f(x, y) \right) = \min_{y \in Y} \max_{x \in X} f(x, y)$$

Proof: Let $\alpha = \max\left(\min_{y \in Y} \max_{x \in K_1} f(x, y), \min_{y \in Y} \max_{x \in K_2} f(x, y)\right), \beta = \min_{y \in Y} \max_{x \in X} f(x, y)$ for simplicity. Now we can pick $z_1, z_2 \in Y$ s.t. $\alpha = \max(\max_{x \in K_1} f(x, z_1), \max_{x \in K_2} f(x, z_2))$. Then, letting $C = \{z_1 t + z_2(1 - t) : t \in [0, 1]\}$ denote the convex hull of z_1, z_2, it must follow that

$$\alpha \geq \max_{x \in K_1} f(x, z_1), \max_{x \in K_2} f(x, z_2) \implies \alpha \geq \max_{x \in K_1} \min_{y \in C} f(x, y), \max_{x \in K_2} \min_{y \in C} f(x, y) \implies \alpha \geq \max_{x \in X} \min_{y \in C} f(x, y)$$

Now suppose FTSOC that $\min_{y \in C} \max_{x \in X} f(x, y) > \alpha$. Then it must be that $\exists \gamma, \min_{y \in C} \max_{x \in X} f(x, y) > \gamma > \alpha$. If we let $L_\gamma(x) := \{y \in Y : f(x, y) \leq \gamma\}, U_\gamma(y) := \{x \in X : f(x, y) \geq \gamma\}$, it follows by the concave-convexity of f that both U_γ, L_γ are closed and convex. Now consider both $\cap_{x \in X}(L_\gamma(x) \cap C), \cap_{y \in I} U_\gamma(y)$ and suppose FTSOC that they are nonempty. If $\exists y \in \cap_{x \in X}(L_\gamma(x) \cap C)$, then $\exists y \in C$ s.t. $\forall x \in X$

$$f(x, y) \leq \gamma \implies \min_{y \in C} \max_{x \in X} f(x, y) \leq \gamma. \Rightarrow\Leftarrow$$

Similarly if $\exists x \in \cap_{y \in C} U_\gamma(y)$ then $\exists x \in X$ s.t. $\forall y \in C$

$$f(x, y) \geq \gamma \implies \max_{x \in X} \min_{y \in C} f(x, y) \geq \gamma > \alpha. \Rightarrow\Leftarrow$$

So by the finite intersection property of closed sets (Helly's Theorem in 1 dimension), there must exist $x_1, x_2 \in X$ with $L_\gamma(x_1) \cap L_\gamma(x_2) \cap C = \emptyset$. From this, letting $D = \{x_1 t + x_2(1 - t) : t \in [0, 1]\}$, we see that $\cap_{y \in C}(U_\gamma(y) \cap D) = \emptyset \implies \exists y_1, y_2 \in C$ with $U_\gamma(y_1) \cap U_\gamma(y_2) \cap D = \emptyset$, once again by Helly. Note that the second intersection implies that for x_1, $x_1 \notin U_\gamma(y_1)$ or $U_\gamma(y_2)$ s.t. $f(x_1, y_1)$ or $f(x_1, y_2) < \gamma$. We get a similar result for x_2, and conclude that for $i = 1, 2$ we must have $L_\gamma(x_i) \cap \{y_1, y_2\} \neq \emptyset$.

Now let $E = \{y_1 t + y_2(1 - t) : t \in [0, 1]\}$. Since the L_γ are closed and E is path-connected, there must exist some $y_0 \in E \setminus (L_\gamma(x_1) \cup L_\gamma(x_2))$. Similarly, there must exist some $x_0 \in D \setminus (U_\gamma(y_1) \cup U_\gamma(y_2))$. But then $f(x_1, y_0), f(x_2, y_0) > \gamma \implies \exists \delta > \gamma$ with $\{x_1, x_2\} \subset U_\delta(y_0)$. This also implies $f(x_0, y_0) > \delta$ by the concave-convexity of f. However, $x_0 \in D \setminus (U_\gamma(y_1) \cup U_\gamma(y_2)) \implies f(x_0, y_1), f(x_0, y_2) < \delta \implies f(x_0, y_0) < \delta$

once again by concave-convexity. As such, we have a contradiction, and $\min_{y\in C}\max_{x\in X} f(x,y)=\alpha$.

Now we have $\alpha=\min_{y\in C}\max_{x\in X} f(x,y)\geq \min_{y\in Y}\max_{x\in X} f(x,y)=\beta$. But also clearly we must have that for $i=1,2$, $\min_{y\in Y}\max_{x\in K_i} f(x,y)\leq \min_{y\in Y}\max_{x\in X} f(x,y)=\beta \implies \alpha\leq\beta$. Then it follows that $\alpha=\beta$, as desired. $\square$

We now show Von Neumann's theorem itself. Fix some finite subset $A\subset X$ and let $\beta=\min_{y\in Y}\max_{x\in C_A} f(x,y)$ where C_A denotes the convex hull of A. Moreover, we may let $\mathcal{K}=\{K\subseteq C_A : K \text{ closed, convex, s.t. } \beta=\min_{y\in Y}\max_{x\in K} f(x,y)\}$, and let it be partially ordered via inclusion. By definition, it is clear that $\forall K\in\mathcal{K}, y\in Y$ we must have that $U_\beta(y)\cap K\neq\emptyset$. Therefore, if $\mathcal{C}$ is an arbitrary chain of $\mathcal{K}$, it follows via finite intersection property of closed sets that $U_\beta(y)\cap(\cap_{C\in\mathcal{C}}C)\neq\emptyset \implies \cap_{C\in\mathcal{C}}C\in\mathcal{K}$ is a lower bound of $\mathcal{C}$.

As such, via Zorn's lemma, there is some minimal element $K_0\in\mathcal{K}$. If it is the case that K_0 can be decomposed into $K_0=K_1\cup K_2$ where $K_1,K_2\neq\emptyset$ and are proper convex, closed subsets of K_0, then by the lemma, one of $K_1,K_2\in\mathcal{K}$. But this contradicts the minimality of K_0, and so K_0 cannot be decomposed, meaning that it is a singleton $K_0=\{x_0\}$ for $x_0\in X$. As such, we have

$$\min_{y\in Y} f(x_0,y)=\beta=\min_{y\in Y}\max_{x\in C_A} f(x,y)$$

But now clearly $\alpha:=\max_{x\in X}\min_{y\in Y} f(x,y)\geq\beta\geq\min_{y\in Y}\max_{x\in A} f(x,y) \implies \cap_{x\in A}L_\alpha(x)\neq\emptyset$ for any finite subset $A\subset X$. But now we can employ finite intersection property again, since each $L_\alpha(x)\subseteq Y$ is compact, and therefore $\cap_{x\in X}L_\alpha(x)\neq\emptyset \implies \alpha\geq\min_{y\in Y}\max_{x\in X} f(x,y)$. But also $\alpha\leq\min_{y\in Y}\max_{x\in X} f(x,y)$ since $\forall x\in X, \min_{y\in Y} f(x,y)\leq\min_{y\in Y}\max_{x\in X} f(x,y) \implies \alpha\leq\min_{y\in Y}\max_{x\in X} f(x,y)$. It follows then that

$$\alpha=\max_{x\in X}\min_{y\in Y} f(x,y)=\min_{y\in Y}\max_{x\in X} f(x,y).\ \square$$

As a corollary, one can note that letting $X=\{(x_1,x_2,\ldots,x_m)\in\mathbb{R}^m : \sum_{i=1}^m x_i=1, x_i\geq 0\}$ and $Y=\{(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n : \sum_{i=1}^n y_i=1, y_i\geq 0\}$, which are obviously convex and compact, and $f(x,y)=x^TAy$, which is obviously concave-convex (it is linear), f satisfies the minimax theorem. As such

$$\max_{x\in X}\min_{y\in Y} x^TAy=\min_{y\in Y}\max_{x\in X} x^TAy$$

But recall that if A is the payoff matrix for player 1 in a two person zero-sum game, then x^TAy is the expected payoff for player 1 given that player 1 plays strategy x and player 2 plays strategy y. This corollary tells us then that two person zero-sum games admit an *equilibrium point*. They could both publish their strategies to the other player, and provided that they are both rational players it would be in neither players interest to change their strategy.

The theory of equilibria beyond zero-sum and two player games was relatively undeveloped for the next twenty years, until John Nash famously introduced the concept of a *Nash Equilibrium* and proved its existence in two separate papers, the first using Kakutani's Fixed Point Theorem, and the second using Brouwer's fixed point theorem. Both theorems, their proofs, and how they can be applied to prove existence of equilibiria will be discussed in later sections.

3 Equilibrium Points in Finite Games

We now formally define an *equilibrium point* for generalized games. Suppose once more that we have n players with $m_1,m_2,\ldots,m_n\in\mathbb{N}$ possible choices, and that our payoff matrix is $(\mathbb{R}^n)^{m_1\times m_2\times\ldots\times m_n}$. Each of the n players is equipped with some strategy $s_i=(s_{i,1},\ldots,s_{i,m_i})\in[0,1]^{m_i}$ where $\sum_{j=1}^{m_i} s_{i,j}=1$, and write $\overrightarrow{s}$ to denote the n-tuple of strategies. Subsequently, for any n-tuple of strategies $\overrightarrow{s}$, we can define the payoff function of player i as

$$p_i(\overrightarrow{s})=\sum_{(i_1,i_2,\ldots,i_n)\in[m_1]\times[m_2]\times\ldots\times[m_n]}(A_{i_1,i_2,\ldots,i_n})_i\left(\prod_{j=1}^n s_{j,i_j}\right)$$

In other words, it is the expected payoff for player i given the n-tuple of strategies $\overrightarrow{s}$. Now given $\overrightarrow{s} = (s_1, s_2, \ldots, s_n)$, and some strategy $t_i \in [0,1]^{m_i}$ for player i, we can define the n-tuple of strategies

$$(\overrightarrow{s}; t_i) = (s_1, s_2, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$$

An n-tuple of strategies $\overrightarrow{s}$ is now an *equilibrium point* if

$$p_i(\overrightarrow{s}) = \max_{t_i \in [0,1]^{m_i}, \sum t_{i,j}=1} p_i(\overrightarrow{s}; t_i)$$

In other words, $\overrightarrow{s}$ is an equilibrium point if it is in no player's interest to change their own strategy. One can note that in the $n = 2$ zero-sum case, Von Neumann's Minimax theorem proves the existence of an equilibrium point. If X, Y deonte the strategy spaces for the row and column player respectively, and (x', y') denote the extrema satisfied in the equation

$$v = \max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y$$

then it is not in the interest of the row player to change their strategy, since x' is chosen s.t. $v = (x')^T A(y') = \max_{x \in X} x^T A(y')$. Likewise it is not in the interest of column player to change their strategy, since y' is chosen s.t. $v = (x')^T A(y') = \min_{y \in Y} (x')^T A y$.

In 1950, Nash famously published his paper *Equilibrium points in n-person games*, establishing the existence of equilibrium points in generalized n-person games, outside of the previous $n = 2$ zero-sum case discussed by Von Neumann. To do this, he utilized both the Kakutani and Brouwer Fixed Point theorem in two separate papers. To motivate this, note that for any strategy $\overrightarrow{s}$ not in equilibrium, there is some function $f(\overrightarrow{s})$ s.t. $f(\overrightarrow{s})_i$ is the best strategy for player i in response to players $j \neq i$ playing strategies s_j. In other words,

$$p_i(\overrightarrow{s}; f(\overrightarrow{s})_i) = \max_{t_i \in [0,1]^{m_i}, \sum t_{i,j}=1} p_i(\overrightarrow{s}; t_i)$$

Fixed points of f then correspond exactly to equilibrium points. It can be shown that f is continuous, and clearly strategy spaces are compact and convex. Thus, one can expect a theorem from Algebraic Topology like Brouwer to work well in showing the existence of equilibrium points.

4 Kakutani Fixed Point Theorem

We first discuss Nash's 1950 paper *Equilibrium points in n-person games*, which used Kakutani's Fixed Point theorem to establish the existence of equilibrium points in n-person games. Kakutani's theorem deals with fixed points of set-valued functions $\phi : S \to 2^S$ (i.e. functions which take on subsets of S as their values).

Formally, a fixed point of such a set-valued function ϕ is some $a \in S$ s.t. $a \in \phi(a)$. Moreover, a set function $\phi : S \to 2^S$ is said to have a *closed graph* if the set $\{(x, y) : y \in \phi(x)\}$ is a closed subset of $S \times S$ in the product topology.

Theorem 2. *(Kakutani Fixed Point Theorem, 1941) Let $S \subseteq \mathbb{R}^n$ be nonempty, compact, and convex. Then if $\phi : S \to 2^S$ is a set-valued function which has a closed graph and satisfies the property that $\phi(x)$ is nonempty and convex $\forall x \in S$, then ϕ has a fixed point.*

Proof: We deal with the case $S = [0,1]$, and the proof generalizes to $S \subseteq \mathbb{R}^n$ via Barycentric subdivisions. We construct a sequence as follows. First let $a_0 = 0, b_0 = 1$, and $p_0 \in \phi(0), q_0 \in \phi(1)$. We now construct a sequence of $\{(a_i, b_i, p_i, q_i)\}_i$ with $a_i \leq p_i, q_i \leq b_i, p_i \in \phi(a_i), q_i \in \phi(b_i)$, and $0 \leq b_i - a_i \leq 2^{-i}$ inductively as follows:

1. Let $m = \frac{a_i + b_i}{2}$. If $\exists r \in \phi(m)$ s.t. $r \geq m$, then let $a_{i+1} = m, b_{i+1} = b_i, p_{i+1} = m, q_{i+1} = q_i$. It is easy to check that these verify the necessary properties.

2. If $\nexists r \in \phi(m)$ with $r \geq m$, then since each $\phi(m) \neq \emptyset$, $\exists r \in \phi(m)$ with $r < m$. Then let $a_{i+1} = a_i, b_{i+1} = m, p_{i+1} = p_i, q_{i+1} = r$.

It is easily seen that this sequence satisfies the necessary conditions of the sequence. Moreover, since $[0,1]^4$ is compact, by Bolzanno Weierstrass we may extract a convergent subsequence $\{(a_{i_k}, b_{i_k}, p_{i_k}, q_{i_k})\}_k$ that converges to some $(a,b,p,q) \in [0,1]^4$. Clearly we have $a = b, a \leq p, q \leq b$. Moreover since ϕ has a closed graph, we may also note that $p \in \phi(a), q \in \phi(b)$.

Clearly if $p = a$ or $q = b$, we have a fixed point. Suppose not then, and that $q < a = b < p$. In this case, since $q, p \in \phi(a) = \phi(b)$, and $\phi(a)\phi(b)$ is convex, it follows that $a = b \in \phi(a) = \phi(b)$ since $a = b$ is clearly a convex combination of q and p. As such, in any case we have that ϕ admits a fixed point. $\square$

To generalize this proof to compact n-simplexes, we can instead take Barycentric subdivisons at each step as opposed to cutting up $[0,1]$. The proof of this theorem also has similar flavors to the proof of Brouwer's Fixed Point theorem with Sperner's Lemma and the Banach Fixed Point Theorem (contractions in Banach Spaces have unique fixed points). They all use Bolzanno Weierstrass with shrinking distances between points in some way to find the existence of fixed points.

We now show the main result of Nash's 1950 paper, the existence of equilibria via Kakutani's theorem.

Theorem 3. *(Existence of Nash Equilibrium, 1950) Any finite game allowing mixed strategies with n players that have m_i choices each admits an equilibrium point, or a Nash Equilibrium.*

Proof: Again consider an n person game with m_i choices for the ith player. We now define n strategy spaces for n players

$$X_i = \{(x_{i,1}, \dots, x_{i,m_i}) \in [0,1]^{m_i} : \sum_j x_{i,j} = 1\}$$

Clearly X_i is compact and convex, and subsequently if $s_i \in X_i$, we may define $f : X = \prod_{i=1}^n X_i \to 2^X$ as

$$f(\overrightarrow{s}) = \{\overrightarrow{t} \in X : \forall i \in [n], p_i(\overrightarrow{s}; t_i) = \max_{x_i \in X_i} p_i(\overrightarrow{s}; x_i)\}$$

i.e. f maps an n-tuple of strategies $\overrightarrow{s}$ to all "countering" n-tuples of strategies $\overrightarrow{t}$ s.t. if all players $j \neq i$ keep the same strategy s_j and player i changes their strategy to t_i, player i's payoff is now optimized against the s_j. $f(\overrightarrow{s}) \neq \emptyset$ clearly since we can always just set $\overrightarrow{t} = (t_1, \dots, t_n)$ s.t. $p_i(\overrightarrow{s}; t_i)$ is maximized. Moreover, since p_i is linear, any convex combination of $\overrightarrow{t} \in f(\overrightarrow{s})$ s.t. $p_i(\overrightarrow{s}; t_i) = \max_{x_i \in X_i} p_i(\overrightarrow{s}; x_i)$ will still satisfy the maximum condition so that $f(\overrightarrow{s})$ is convex. Finally, the continuity of the payoff function ensures that the graph $\{(\overrightarrow{s}, \overrightarrow{t}) : \overrightarrow{t} \in f(\overrightarrow{s})\}$ is closed.

Now we are in a position to apply Kakutani's Fixed Point theorem, and we see that $\exists \overrightarrow{s} \in \prod_{i=1}^n X_i$ s.t. $\overrightarrow{s} \in f(\overrightarrow{s})$. But then it is the case that $\forall i$ we have $p_i(\overrightarrow{s}) = p_i(\overrightarrow{s}; s_i) = \max_{x_i \in X_i} p_i(\overrightarrow{s}; x_i)$. This by definition is an equilibrium point of the game, as desired. $\square$

5 Existence of Nash Equilbria via Brouwer's Fixed Point Theorem

We begin with the statement and proof$^{\text{TM}}$ of Brouwer's Fixed Point Theorem.

Theorem 4. *(Brouwer's Fixed Point Theorem, 1910) Let $K \subseteq \mathbb{R}^n$ be convex and compact. Then any continuous $f : K \to K$ admits a fixed point, or $\exists x \in K$ s.t. $f(x) = x$*

Proof: Omitted. Done both in class using homotopy and on the homework via Sperner's Lemma. It is also a simplification of the aforementioned Kakutani Fixed Point Theorem by letting $\phi(x) = \{f(x)\}$. $\square$

We now present a simplified proof of Nash's previous existence theorem using Brouwer's Fixed Point theorem instead.

Theorem 5. *(Existence of Nash Equilibrium v2, 1951) Any finite game allowing mixed strategies with n players that have m_i choices each admits an equilibrium point, or a Nash Equilibrium.*

Proof: We once again let $\overrightarrow{s}$ denote the n-tuple of strategies, and p_i denote the payoff function for the ith player. For $j = 1$ to m_i, we also now define $\pi_{ij} = (0, \ldots, 1, \ldots, 0)$ the m_i-tuple of 0's with 1 in the jth position, or the pure strategy for player i in which they always pick option j. From this, we define $p_{ij}(\overrightarrow{s}) = p_i(\overrightarrow{s}; \pi_{ij})$, i.e. the payoff for player i if they switch the pure strategy of always picking option j. From this, the family of continuous function $\phi_{i,j}(\overrightarrow{s}) : \prod_{i=1}^{n} X_i \to \mathbb{R}$ is defined via

$$\phi_{ij}(\overrightarrow{s}) = \max(0, p_{ij}(\overrightarrow{s}) - p_i(\overrightarrow{s}))$$

and for each i, we may define the continuous function $T_i : X_i \to X_i$ via

$$T_i(s_i) = \frac{s_i + \sum_j \phi_{ij}(\overrightarrow{s})\pi_{ij}}{1 + \sum_j \phi_{ij}(\overrightarrow{s})}$$

where the RHS is a valid strategy since it is a convex combination of the strategies s_i, π_{ij}. Finally let $T : \prod_{i=1}^{n} X_i \to \prod_{i=1}^{n} X_i$ via

$$T(\overrightarrow{s}) = (T_1(s_1), T_2(s_2), \ldots, T_n(s_n))$$

T is clearly continuous, so by Brouwer, there must exist some fixed point $\overrightarrow{s}$. Now note that since s_i is just a convex combination of some pure strategies π_{ij}, there must exist some "minimal pure strategy" π_{ik} in this convex combination s.t. $p_{ik}(\overrightarrow{s}) - p_i(\overrightarrow{s}) \leq 0$, since if not, taking a convex combination of the j for which the component of π_{ij} in s_i is nonzero, we would get $p_i(\overrightarrow{s}) > p_i(\overrightarrow{s})$ which is absurd. So now $\phi_{ik}(\overrightarrow{s}) = 0$, and subsequently matching the coefficients of π_{ik} in s_i and $T_i(s_i)$ yields

$$\langle s_i, \pi_{ik} \rangle = \frac{\langle s_i, \pi_{ik} \rangle}{1 + \sum_j \phi_{ij}(\overrightarrow{s})}$$

but this is only possible if $\sum_j \phi_{ij}(\overrightarrow{s}) = 0 \implies \phi_{ij}(\overrightarrow{s}) = 0 \ \forall j$. Since this analysis is true $\forall i$, it follows that $T(\overrightarrow{s}) = \overrightarrow{s} \implies \phi_{ij}(\overrightarrow{s}) = 0 \implies p_{ij}(\overrightarrow{s}) \leq p_i(\overrightarrow{s}) \ \forall i, j$. We now show that this condition is sufficient for $\overrightarrow{s}$ to be an equilibrium point. Suppose that $\overrightarrow{s}$ were not an equilibrium point, then it must follow that for some i,

$$p_i(\overrightarrow{s}) < \max_{t_i \in X_i} p_i(\overrightarrow{s}; t_i) \implies p_i(\overrightarrow{s}) < \sum_{j=1}^{m_i} \langle t_i, \pi_{ij} \rangle p_i(\overrightarrow{s}; \pi_{ij})$$

for some $t_i \in X_i$. But then since $\sum_j \langle t_i, \pi_{ij} \rangle = 1$, the RHS is a convex combination of $p_i(\overrightarrow{s}; \pi_{ij})$, and it follows that $\exists j$ s.t. $p_i(\overrightarrow{s}) < p_i(\overrightarrow{s}; \pi_{ij})$. In other words, if $\overrightarrow{s}$ is not an equilibrium point, then for some player i they can change their strategy to some pure strategy π_{ij} which will be better for themself. But then in this case we have $p_i(\overrightarrow{s}) < p_{ij}(\overrightarrow{s})$, which contradicts the required property of $\overrightarrow{s}$ being a fixed point. It follows then that $\overrightarrow{s}$ is an equilibrium point, as desired. $\square$

6 Oddness Theorem

Recall that the results of Sperner's Lemma in n dimensions enforced that in any Sperner coloring, the number of vertices colored with exactly $n + 1$ colors is odd, motivating the idea that Brouwer Fixed points and the Nash equilibria of games may have odd parity, although this is generally not the case. As examples

1. The mapping $f : [0, 1] \to [0, 1]$ defined via $f(x) = x$ has infinite fixed points, so nothing can be said about the parity of fixed points.
2. The mapping $f : [0, 1] \to [0, 1]$ defined via $f(x) = -6x^3 + 9x^2 - 3x + \frac{4}{9}$ has exactly two fixed points at $x = \frac{1}{6}$ and $\frac{2}{3}$.
3. The *Free Money* game defined by the below payoff matrix

$$A = \begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (0,0) \end{pmatrix}$$

has exactly two Nash equilibria, namely when the row and column player both play row/column 1, and when they both play row/column 2.

Therefore in general, there are no deterministic results about the parity of fixed points or Nash equilibria. However, in 1971, Robert Wilson published the so called *Oddness Theorem* which probabilistically gave constraints for parity. Specifically,

Theorem 6. *(Oddness Theorem, 1971) The set of games which have an even number of Nash equilibria has measure 0.*

The proof of this is nontrivial and too long in length for this paper, so it will be left as an exercise for the reader. One can also reference Wilson's original paper *Computing Equilibria of N-person Games.*

7 Complexity of finding Nash Equilibria

We'll now discuss the computational problem of finding Nash equilibria, NASH. We assume a computional model in which precise real number solutions to Nash equilibria are not required, but only approximations within some $\epsilon > 0$. In general, computing this does not seem to admit a polynomial time algorithm. In fact, it belongs to its own complexity class known as PPAD, short for polynomial parity argument for directed graphs.

In general a complexity class like NP is defined by the set of problems that can be verified in polynomial time. However, another way to define it is by all problems that reduce to SAT via a Karp Reduction, since SAT is NP-complete. In this sense, we define the class of problems PPAD as the class of problems that reduce via Karp Reduction to

END OF THE LINE: Given a directed graph G and a specified unbalanced vertex of G, output some other unbalanced vertex.

where a vertex is called *unbalanced* if the number of incoming edges differs from the number of outgoing edges. A simple counting argument shows that if one unbalanced vertex exists, another distinct one must as well. Moreover, if the graph G were presented in some input like an adjacency matrix or list, the problem would clearly be polynomial time. In this case, however, we are assuming G has 2^n vertices and is implemented via a query-based representation of G in which one must inquire whether each edge exists, making any obvious solutions run in exponential time.

One can show that NASH, or the problem of computing the Nash equilibrium of a finite game, is PPAD-complete, along with other similar problems like Brouwer point. PPAD is also clearly a subclass of NP which gives us some notion of the hardness of computing Nash equilibria and fixed points. Here we will discuss an outline of the proof that NASH is in PPAD, but not go into full detail. The main steps are as follows:

1. Showing that BROUWER is PPAD via the Sperner's Lemma construction. We first assume that we are given some function $f : [0,1]^n \to [0,1]^n$ satisfying the hypotheses of Brouwer's Theorem. We wish to find some x s.t. $d(x, f(x)) < \epsilon$. To construct the reduction to END OF THE LINE, we subdivide $[0,1]^n$ into cubes, coloring vertices by the direction in which f sends them.

   In the case for $n = 2$, suppose points sent at an angle from 0 to 90 are colored yellow, 90 to 225 are colored blue, and 225 to 360 are colored red. The vertices of the graph will be triangles that have at least one red and yellow colored vertex. There will be a directed edge from triangle T to triangle T' iff they share a red-yellow edge that goes from red to yellow clockwise w.r.t. T. One is able to then conclude that a monochromatic triangle, which corresponds to a Brouwer point, is the sink of some path in the directed graph. Subsequently, BROUWER reduces to END OF THE LINE.

2. The reduction from NASH to BROUWER is now easily seen Nash's 1951 existence theorem referenced in section 5. This gives a direct construction of a Brouwer function that correspond to the Nash equilibrium of some game.

8 References

1. *A simple proof of Sion's Minimax Theorem.*
   https://www.jstor.org/stable/30037472.

2. *Equilibrium Points in N-person Games.*
   https://www.pnas.org/content/pnas/36/1/48.full.pdf

3. *Non-Cooperative Games.*
   https://www.jstor.org/stable/1969529.

4. *Computing Equilibria of N-Person Games*
   https://www.jstor.org/stable/2099844

5. *The Complexity of Computing a Nash Equilibrium.*
   https://people.csail.mit.edu/costis/simplified.pdf.

6. *Fixed Point Theorems.*
   https://cpb-us-w2.wpmucdn.com/sites.wustl.edu/dist/3/2139/files/2019/09/fixedpoints.pdf.