EE364a HW2

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2-9

(a) Consider set $S_i = \{x \in R^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2\}$, which is a half-space with boundary $\|x - x_0\|_2 = \|x - x_i\|$.

And $V = \bigcap_i S_i$, hence V is a polyhedron.

(b) ① Pick any point $x_0 \in P$.

② For each surface A_i of P, define x_i as the mirroring image of x_0 against A_i.

③ P is the Voronoi region of x_0 with respect to $x_1, \ldots, x_k$.

2.15. Apparently, P itself is convex.

(a) $Ef(x) = \sum_i p_i f(a_i)$, i.e., $\alpha \leq \sum_i p_i f(a_i) \leq \beta$, which is a slab.

$\therefore$ The intersection with P is convex.

(b) $P(x > \alpha) = \sum_{\{i \mid a_i > \alpha\}} p_i \leq \beta$. $\therefore$ it is convex.

(c) $E|x^3| \leq \alpha E|x| \Rightarrow \sum_i p_i a_i^2 |a_i| \leq \alpha \sum_i p_i |a_i|$

$\Rightarrow \sum_i p_i (|a_i^3| - \alpha |a_i|) \leq 0$ $\therefore$ it is convex.

(d) $E x^2 = \sum_i p_i a_i^2 \leq \alpha$ $\therefore$ it is convex.

(e) ~~Same~~ Similar to (d), it is convex.

(f) $Var(x) = Ex^2 - (Ex)^2 = \sum_i p_i a_i^2 - (\sum_i p_i a_i)^2$

$= -p^T a a^T p + b^T p \leq \alpha$, where $b_i = a_i^2$

$aa^T \succeq 0$ $\therefore$ The above inequality doesn't give a convex set in general.

(g) Similar to (f), the inequality is $p^T a a^T p - b^T p + \alpha \le 0$

$aa^T \succeq 0$ $\therefore$ From 2.10 (a), it is convex.

(h) For the definition of $\text{quartile}(x) = \inf\{\beta \mid P(x \le \beta) \ge 0.25\}$

Assume $\sum_{i=1}^{k-1} p_i < \frac{1}{4}$ and $\sum_{i=1}^{k} p_i \ge \frac{1}{4}$ ~~$\therefore \beta = a$~~

$\therefore \text{quartile}(x) = a_k$

For condition $\text{quartile}(x) \ge \alpha$, assume $a_{k'-1} < \alpha$ and $a_{k'} \ge \alpha$

$\Rightarrow \sum_{i=1}^{k'-1} p_i < \frac{1}{4}$ $\therefore$ it is convex.

(i) Similar to (h), for the same k'. $\sum_{i=1}^{k'-1} p_i \ge \frac{1}{4}$.

3.18 (a) $f(X) = tr(X^{-1})$. Let's look at $g(t) = tr((X+tA)^{-1})$

$$g(t) = tr\left(\left(X^{\frac{1}{2}}(I + tX^{-\frac{1}{2}}AX^{\frac{1}{2}})X^{\frac{1}{2}}\right)^{-1}\right)$$

$$= tr\left(X^{-\frac{1}{2}}(I + tX^{-\frac{1}{2}}AX^{\frac{1}{2}})^{-1}X^{-\frac{1}{2}}\right)$$

Since $tr(ABC) = tr(CAB) = tr(BCA)$

$$\therefore g(t) = tr[(I + tX^{-\frac{1}{2}}AX^{\frac{1}{2}})X'] = \sum_i [(1+t\lambda_i)^{-1} \cdot X'_{ii}]$$

where λ_i are eigenvalues of matrix $X^{-\frac{1}{2}}AX^{\frac{1}{2}}$, X' is another matrix that doesn't involve t. $\therefore g(t)$ is convex since $\lambda_i > 0$.

(b) Similarly, define $g(t) = (\det(X+tA))^{1/n}$

$$= \left(\det X^{\frac{1}{2}} \det(I + tX^{-\frac{1}{2}}AX^{-\frac{1}{2}}) \det X^{\frac{1}{2}}\right)^{1/n}$$

$$= (\det X)^{\frac{1}{n}} \left(\prod_{i}^{n}(1+t\lambda_i)\right)^{1/n}$$

where λ_i are eigenvalues of matrix $X^{-\frac{1}{2}}AX^{-\frac{1}{2}}$.

Since the geometric mean is concave, $g(t)$ is concave. $f(X)$ is concave.

3.24.

(a) $Ex = a^T p$ $\therefore$ it is convex, concave, quasiconvex and quasiconcave.

(b) $P(x \geq \alpha) = \sum_{i=k}^{n} p_i$ for some k which is fixed as long as a_i are given. $\therefore$ It's convex, concave, quasiconvex and quasiconcave.

(c) $P(\alpha \leq x \leq \beta) = \sum_{i=k_1}^{k_2} p_i$ for some fixed k_1 and k_2

$\therefore$ It's convex, concave, quasiconvex and quasiconcave.

(d) $f(x) = x \log x$ is convex on R_+, $\therefore \sum_i p_i \log p_i$ is convex and quasiconvex.

For $n=2$, the expression is $p \log p + (1-p)\log(1-p)$ which has a minimum at $p = \frac{1}{2}$ $\therefore$ the superlevel sets are not convex. $\therefore$ It's not concave and not quasiconcave.

(e) $Var(x) = Ex^2 - (Ex)^2 = -p^T a a^T p + b^T p$ $\quad aa^T \succeq 0$

$\therefore$ It's concave and quasiconcave.

For $n=2$ and $a_1 = 0, a_2 = 1$, $Var(x) = -p_2^2 + p_2$, which is not convex and not quasiconvex.

(f) From 2.15 (h) and (i), the sublevels and superlevels are ~~convex~~ convex sets.

$\therefore$ The function is quasiconvex and quasiconcave.

For $n=2$, if $p_1 = 0$, $p_2 = 1$, then quartile $(x) = a_2$

if $p_2 = 0$, $p_1 = 1$, then quartile $(x) = a_1$

If $p_1 = p_2 = \frac{1}{2}$, quartile $(x) = a_1$, if $p_1 = \frac{1}{5}$, $p_2 = \frac{4}{5}$, quartile$(x) = a_2$

$\therefore$ The function ~~could be either con~~ is not convex and not concave.

~~since a_1 and a_2~~

(g) If we sort p_i by their magnitude $p_{i_1} > p_{i_2} > \cdots > p_{i_n}$

Denote the function as $f(x)$, then $f(x) = \inf\{k \mid \sum_{j=1}^{k} p_{i_j} \geq 90\%\}$

It's better to denote it as $f(p)$, since values a_i doesn't matter.

If we define a new random variable y with $p'_j = p_{i_j}$ and values $a'_j = j$ $\therefore f(p')$ is now very similar to the quartile function except it's $\inf\{\beta \mid P(y \leq \beta) \geq 0.9\}$

$\therefore$ According to (f), it's not convex, not concave, quasiconvex and quasiconcave.

(h) $P(\alpha \le x \le \beta) = P(a_i \le x \le a_j)$ where $i = \min \{ i \mid a_i \ge \alpha \}$
$j = \max \{ j \mid a_j \le \beta \}$

Denote the function as $f(x)$.

$\therefore$ If $\sum_{k=i}^{j} P_k \ge 0.9$, then $f(x) = a_j - a_i$

If $f(x) \ge b$, then $\sum_{k=i}^{j} P_k < 0.9$ for all i, j such that $a_j - a_i < b$

which is a convex set $\therefore f(x)$ is quasiconcave.

For $n=2$, $\vec{p} = (1,0)$, $f(x)=0$, $\vec{p}=(0,1)$, $f(x=0)$

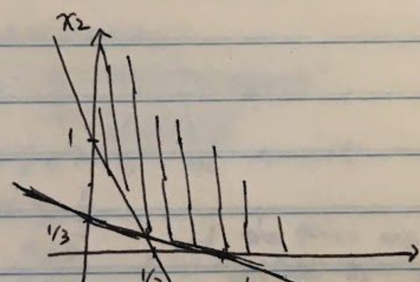
$\vec{p} = (\frac{1}{2}, \frac{1}{2})$, $f(x) = a_2 - a_1 > 0$ $\therefore f(x)$ is not convex.

$f(x)$ is also not quasiconvex, since $f(x) < \frac{a_2 - a_1}{2}$ gives

~~$0 \le p_1 \le 0.1$~~ $p_1 \in [0, 0.1] \cup [0.9, 1]$ which is not convex.

4.1 Conditions $\begin{cases} x_1 \ge 0, \ x_2 \ge 0 \\ x_1 + 3x_2 \ge 1 \\ 2x_1 + x_2 \ge 1 \end{cases}$

$\therefore$ The feasible set is :



(a) The slope of $x_1 + x_2 = 0$ is between $x_1 + 3x_2 = 0$ and $2x_1 + x_2 = 0$

$\therefore$ minimum happens at $\begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 + x_2 = 1 \end{cases} \Rightarrow x = (\frac{2}{5}, \frac{1}{5})$

$\min f_0(x_1, x_2) = \frac{3}{5}$

(b) f_0 is unbounded below

(c) $x_1 \ge 0$ $\therefore \min f_0 = 0$, optimal set is $\{ (0, x_2) \mid x_2 \ge 1 \}$.

(d) ~~For fixed x_1, $\min(\max(x_1, x_2)) = \begin{cases} x_2 & x_2 \ge x_1 \\ x_2 \end{cases}$~~

$\min(\max(x_1, x_2)) = \begin{cases} \min x_1 & \text{if } x_1 \ge x_2 \\ \min x_2 & \text{if } x_2 \ge x_1 \end{cases}$

$\Rightarrow \begin{cases} \min x_1 \ge x_2 \\ \min x_2 \ge x_1 \end{cases}$ $\therefore \min f_0$ must happen at line $x_1 = x_2$

$\therefore x = (\frac{1}{3}, \frac{1}{3})$, $\min f_0 = \frac{1}{3}$

(e) $\nabla f_0 = (2x_1, 18x_2) > 0$ on the feasible set

$\therefore$ minimum happens on the boundary, where ∇f_0 is perpendicular to the boundary. $\therefore$ It must be on either $x_1 + 3x_2 = 1$ or $2x_1 + x_2 = 1$.

Try to solve $\begin{cases} x_1 + 3x_2 = 1 \\ \frac{2x_1}{18x_2} = \frac{1}{3} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2} \\ x_2 = \frac{1}{6} \end{cases}$, which satisfies $2x_1 + x_2 \geq 1$

$\begin{cases} 2x_1 + x_2 = 1 \\ \frac{2x_1}{18x_2} = 2 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{18}{37} \\ x_2 = \frac{1}{37} \end{cases}$, which doesn't satisfy $x_1 + 3x_2 \geq 1$.

$\therefore \min f_0 = \frac{1}{2}$, $x = (\frac{1}{2}, \frac{1}{6})$.

A3.2. See attached codes.

A3.3. (a) Left hand side is not affine.

Norm $= 0 \Rightarrow$ all components are zero $\therefore \begin{cases} x + 2y = 0 \\ x - y = 0 \end{cases}$

(e) ~~Left hand side is not convex.~~

~~$(x+y)^{1/4} \leq x - y \iff \begin{cases} x+y \leq (x-y)^4 \\ x+y \geq 0 \end{cases}$~~

Left hand side of $xy \geq 1$ is not concave.

It's equivalent to $\begin{cases} x \geq \frac{1}{y} \\ x \geq 0 \\ y \geq 0 \end{cases}$, and the first one should be written as `x >= inv_pos(y)`

(f) $(x+y)^2/\sqrt{y}$ will not be recognized as a convex expression.

quad-over_lin should be used, i.e.,

`quad-over_lin(x+y, sqrt(y)) <= x-y+5`

sqrt is nondecreasing and concave $\therefore$ satisfies composition rules.

(g) The left hand side $x^3 + y^3$ is convex only on R^2_{+}.

Hence should use `pow_abs(x, 3) + pow_abs(y, 3) <= 1`

A10.2

(a) $f(u,v) = (u,v)^T X (u,v) = u^TAu + u^TBv + v^TB^Tu + v^TCv$

$\nabla_u f(u,v) = 2Au + 2Bv$

If $A \succ 0$, then $\inf_u f(u,v) = f(u,v)\big|_{\nabla_u f(u,v)=0}$

$= f(-A^{-1}Bv, v)$

$= v^TB^TA^{-1}AA^{-1}Bv - 2v^TB^TA^{-1}Bv + v^TCv$

$= v^TSv$, where $S = C - B^TA^{-1}B$.

(b) For the first theory: $X \succ 0$ iff $A \succ 0$ and $S \succ 0$

We see from (a) that $g(v) = \inf_u f(u,v) = v^TSv$ if $A \succ 0$.

If further $S \succ 0$, then $g(v) > 0$, i.e., $f(u,v) > 0$

i.e., $X \succ 0$.

Code for A 3.2 with CVX:

```
%% 4.1 a
cvx_begin
variables x1 x2
minimize(x1+x2)
subject to
2 * x1 + x2 >= 1
x1 + 3 * x2 >= 1
x1 >= 0
x2 >= 0
cvx_end
% Status: Solved
% Optimal value (cvx_optval): +0.6

%% 4.1 b
cvx_begin
variables x1 x2
minimize(-x1-x2)
subject to
```

```
2 * x1 + x2 >= 1
x1 + 3 * x2 >= 1
x1 >= 0
x2 >= 0
cvx_end
% Status: Unbounded
% Optimal value (cvx_optval): -Inf

%% 4.1 c
cvx_begin
variables x1 x2
minimize(x1)
subject to
2 * x1 + x2 >= 1
x1 + 3 * x2 >= 1
x1 >= 0
x2 >= 0
cvx_end
% Status: Solved
% Optimal value (cvx_optval): +8.45293e-10

%% 4.1 d
cvx_begin
variables x1 x2
minimize(max(x1,x2))
subject to
2 * x1 + x2 >= 1
x1 + 3 * x2 >= 1
x1 >= 0
x2 >= 0
cvx_end
% Status: Solved
% Optimal value (cvx_optval): +0.333333

%% 4.1 e
cvx_begin
variables x1 x2
minimize(x1^2 + 9*x2^2)
subject to
2 * x1 + x2 >= 1
x1 + 3 * x2 >= 1
x1 >= 0
x2 >= 0
```

```
cvx_end
% Status: Solved
% Optimal value (cvx_optval): +0.5
```

They all give the same values as analytically obtained (expect 4.1 c and 4.1 d).