EE364a HW4

Wentao Jiang
wentao@stanford
06165416

5.27 The Lagrangian is

$$L(x,\nu) = \|Ax-b\|^2 + \nu^T(Gx-h)$$
$$= x^TA^TAx - 2b^TAx + b^Tb + \nu^TGx - \nu^Th$$
$$= x^T(A^TA)x + (\nu^TG - 2b^TA)x + b^Tb - \nu^Th$$

The minimum over x is realized at $x^* = -(A^TA)^{-1}(\nu^TG - 2b^TA)^T/2$

$\therefore$ The dual function $g(\nu) = L(x^*,\nu)$

$$= -\frac{1}{4}(\nu^TG - 2b^TA)(A^TA)^{-1}(G^T\nu - 2A^Tb) + b^Tb - \nu$$

From KKT conditions: $Gx^* = h$

$$\nabla_x L = 2A^TAx^* + G^T\nu^* - 2A^Tb = 0$$

Also $x^* = -(A^TA)^{-1}(\nu^TG - 2b^TA)^T/2$ ---- ①

$\therefore$ $-G(A^TA^{-1})(G^T\nu^* - 2A^Tb) = 2h$

$$\nu^* = 2[G(A^TA^{-1})G^T]^{-1}(G(A^TA)^{-1}A^Tb - h) \cdots ②$$

Substitute ② into ① gives the solution of x.

6.2 For l_1 norm, object function is piece wise-linear. Denote the medi of b_i as $\bar{b}$, then for $x < \bar{b}$, the slope is negative or zero, for $x > \bar{b}$, the slope is positive or zero. $\therefore$ the solution is $\bar{b}$.

For b in even dimension space R^{2n}, all $b_{\{n\}} \le x \le b_{\{n+1\}}$ is a solution

For l_2 norm, $\|x\vec{1} - b\| = b^Tb - 2\sum_i b_i x + nx^2$

$\therefore$ Minimum is at $x = \frac{1}{n}\sum_i^n b_i$

For l_∞ norm, $\|x\vec{1} - b\| = \max_i(|b_i - x|)$

$\therefore$ Minimum is at $x = \frac{1}{2}(\min_i b_i + \max_i b_i)$

P1

6.6 (a) In general, the corresponded Lagrangian is

$$L(x, r, \nu) = \sum_i \phi(r_i) + \nu^T (Ax - b - r)$$

$$\therefore g(\nu) = \begin{cases} -b^T\nu + \sum_i \inf_{r_i} (\phi(r_i) - \nu_i r_i) & \nu^T A = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

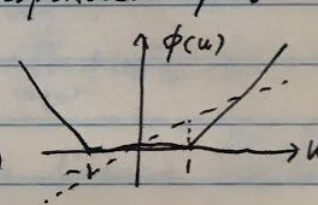
where $\sum_i \inf_{r_i} (\phi(r_i) - \nu_i r_i) = -\sum_i \phi^*(\nu_i)$, $\phi^*(\nu)$ is the conjugate function of $\phi(r)$.

$\therefore$ The dual problem is

$$\begin{aligned} \text{maximize} \quad & -b^T\nu + \sum_i \phi^*(\nu_i) \\ \text{subject to} \quad & A^T\nu = 0 \end{aligned}$$

$\therefore$ The problems are effectively finding the corresponded ϕ^*.

(a) $\phi(u) = \begin{cases} 0 & |u| \le 1 \\ |u| - 1 & |u| > 1. \end{cases}$ which looks like:

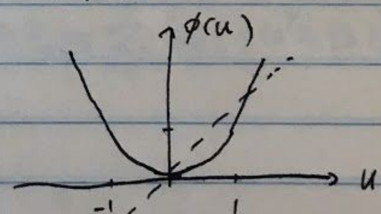


~~For $\sup(\nu u - \phi(u))$~~ $\therefore$ For $\phi^*(\nu) = \sup_u (\nu u - \phi(u))$

$|\nu| > 1$, $\phi^*(\nu) = \infty$, $|\nu| \le 1$, $\phi^*(\nu) = |\nu|$

i.e. $\phi^*(\nu) = \begin{cases} |\nu| & |\nu| \le 1 \\ \infty & |\nu| > 1 \end{cases}$

(b) $\phi(u) = \begin{cases} u^2 & |u| \le 1 \\ 2|u| - 1 & |u| > 1 \end{cases}$:



$\therefore$ For $\phi^*(\nu) = \sup_u (\nu u - \phi(u))$

when $|\nu| > 2$, $\phi^*(\nu) = \infty$, when $|\nu| \le 2$, $|u| = \frac{|\nu|}{2}$, $\phi^*(\nu) = \frac{\nu^2}{4}$

i.e., $\phi^*(\nu) = \begin{cases} \nu^2/4 & |\nu| \le 2 \\ \infty & |\nu| > 2 \end{cases}$

(c) $\phi^*(\nu) = \sup_{|u|<1} (\nu u + \log(1 - u^2))$, $\nu = \frac{2u}{1-u^2}$, i.e., $u = -\frac{1}{\nu} \pm \sqrt{\frac{1}{\nu^2} + 1}$

Only $u = -\frac{1}{\nu} + \sqrt{\frac{1}{\nu^2} + 1}$ satisfy $|u| < 1$. $\therefore 1 - u^2 = \frac{2u}{\nu}$, $\nu u = \sqrt{1 + \nu^2} - 1$

$\therefore \phi^*(\nu) = \sqrt{1+\nu^2} - 1 + \log \frac{2u\nu}{\nu^2} = \sqrt{1+\nu^2} - 1 + \log 2 + \log(\sqrt{1+\nu^2} - 1) - 2\log|\nu|$

6.9 $p(t_i)$ and $q(t_i)$ are affine functions of a and b,

$\therefore \frac{p(t_i)}{q(t_i)} - y_i$ is convex when $q(t) > 0$

The objective function is a point-wise maximum of convex functions, and is hence convex.

The domain is a intersection of convex sets, and is hence convex.

$\therefore$ The problem is covex.

7.3 The likelihood function is $P(a, b, \{y_i\}) = \prod_{i=1}^{q} P(a^T u_i + b + v \leq 0) \times \prod_{i=q+1}^{N} P(a^T u_i + b + v > 0)$.

where $\{y_i\}$ is sorted such that $y_i = 1$ for $i \leq q$ and $y_i = 0$ for $i \geq q+1$.

Since v follow a zero mean unit variance Gaussian distribution.

$$\therefore P(v > x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-v^2/2} dv$$

$$= f(x) \quad \text{(for short)}$$

$\therefore$ The log-likelihood function

$$l(a, b) = \sum_{i=1}^{q} \log[f(a^T u_i + b)] + \sum_{i=q+1}^{N} \log[f(-a^T u_i + b)]$$

3.18 See the attached figures and codes.

P3

A3.20 (a) The constraint on arrival times τ could be reformulated in travel time in each segment t_i:

$$\cancel{\tau_i^{min} - \tau_{i-1}^{max} \le t_i \le \tau_i^{max} - \tau_{i-1}^{min}}$$

$$\tau_i^{min} \le \sum_{j}^{i} t_j \le \tau_i^{max}$$

Constraints on speed are now $\dfrac{d_i}{s_i^{max}} \le t_i \le \dfrac{d_i}{s_i^{min}}$

Objective $\sum_i t_i \Phi\left(\frac{d_i}{t_i}\right)$ is convex.

$\therefore$ The problem is convex.

See the attached figure and codes for the solution.

A5.2 The problem is to find minimum γ where $|f(t_i) - y_i| \le \gamma$ for $i = 1, \cdots, k$.

which can be solved by bisection + LP feasibility problem of:

Find a_0, a_1, a_2, b_1, b_2

subject to $|(a_0 + a_1 t_i + a_2 t_i^2) - y_i(1 + b_1 t_i + b_2 t_i^2)|$

$\le \gamma(1 + b_1 t_i + b_2 t_i^2)$

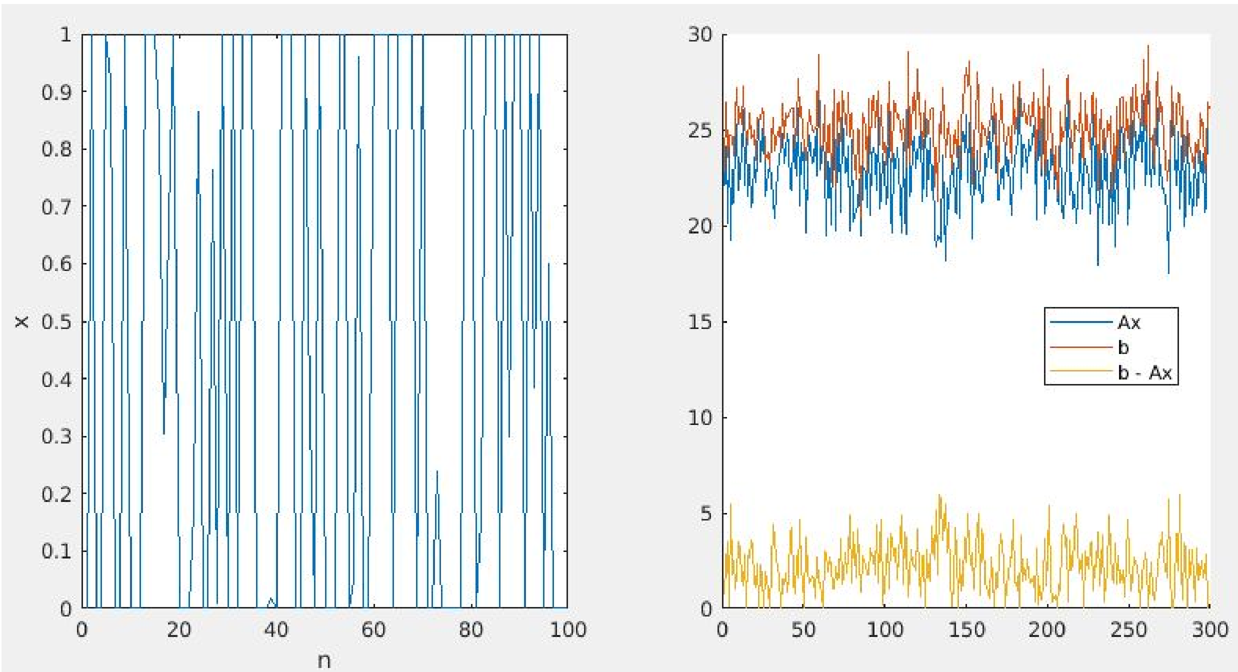
Since $f(t) \le e^3 \therefore \gamma \in [0, e^3]$.

See the attached figure and codes for the solution

A8.3 See the attached figure and code.

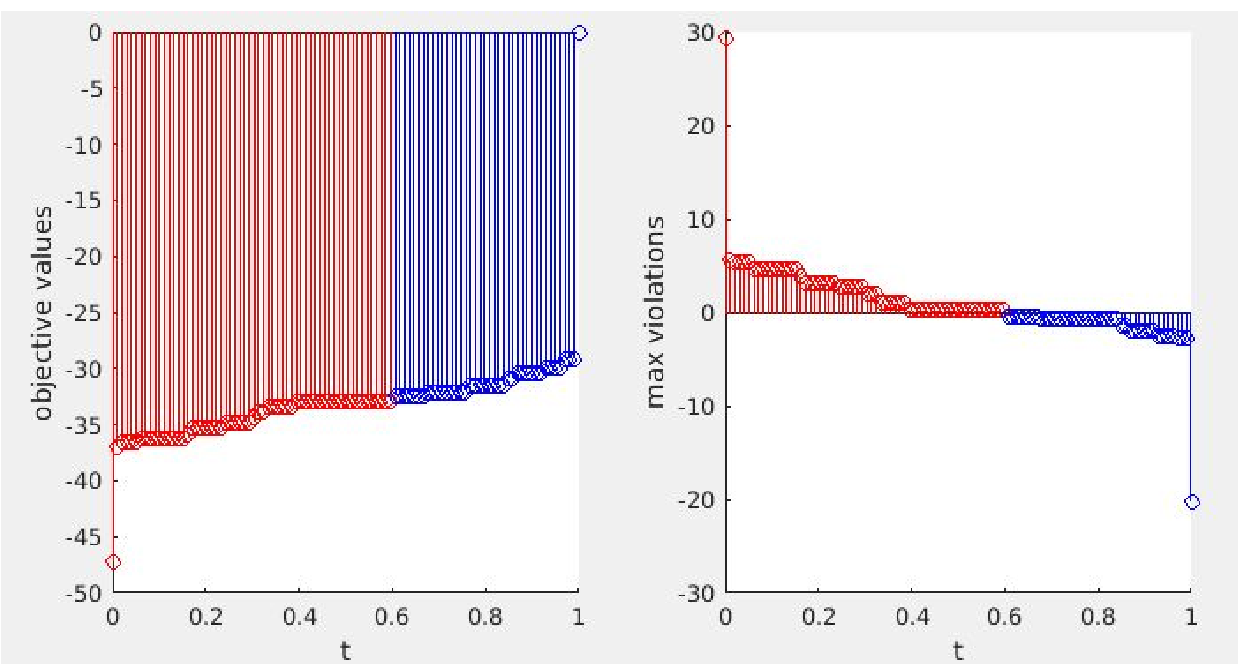
A3.18

Solution of the relaxed problem constraint Ax <= b is satisfied:



Plot of the objective values and max violations after threshold x with t:

- feasible solutions are blue and infeasible solutions are red.



See the last section of the MATLAB code for U and L calculation. The results are:

- L =
- -33.1672
- U =
- -32.4450

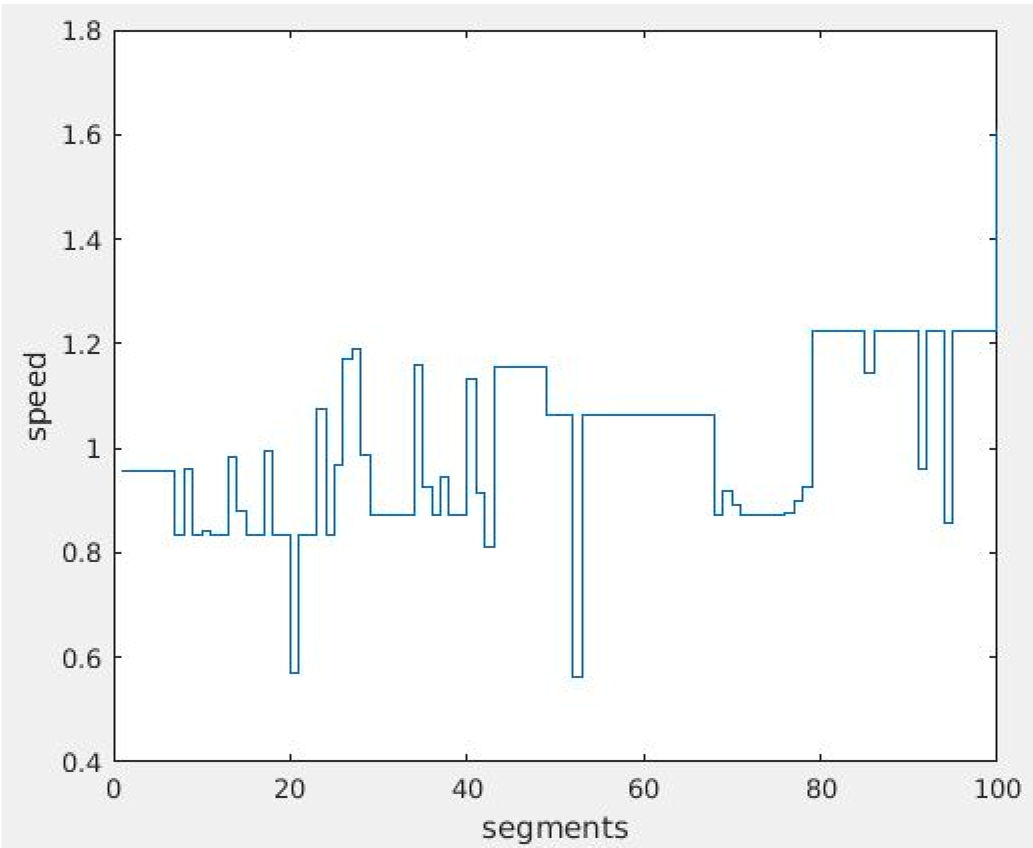
MATLAB code:

```
%% problem A3.18
rand('state',0);
n=100;
m=300;
A=rand(m,n);
b=A*ones(n,1)/2;
c=-rand(n,1);

%% CVX
cvx_begin
variable x(n)
minimize(c'*x)
subject to
0<=x<=1;
A*x<=b;
cvx_end

%% check results
figure;
subplot(1,2,1);
plot(x);
xlabel('n');
ylabel('x');
subplot(1,2,2);
hold all;
plot(A*x);
plot(b);
plot(b-A*x);
legend({'Ax','b','b - Ax'});

%% now threshold x by t and plot objective and max violation
ts = linspace(0,1,100);
```

```
maxVlts = [];
objVals = [];
figure;
subplot(1,2,1);
hold all;
subplot(1,2,2);
hold all;
for ii = 1:length(ts)
    t = ts(ii);
    x_bool = x >= t;
    maxVlt = max(A*x_bool - b);
    maxVlts(ii) = maxVlt;
    objVals(ii) = c'*x_bool;
    if maxVlt > 0
        subplot(1,2,1);
        stem(t, objVals(ii), 'r');
        subplot(1,2,2);
        stem(t, maxVlts(ii), 'r');
    else
        subplot(1,2,1);
        stem(t, objVals(ii), 'b');
        subplot(1,2,2);
        stem(t, maxVlts(ii), 'b');
    end
end

subplot(1,2,1);
xlabel('t');
ylabel('objective values');
subplot(1,2,2);
ylabel('max violations');
xlabel('t');

%% find min obj val where x is still feasible
inds_feasible = maxVlts <= 0;
[minObj, ind_min] = min(objVals(inds_feasible));
ts_feasible = ts(inds_feasible);
t_minObj = ts_feasible(ind_min);
x_minObj = x >= t_minObj;
L = c'*x
U = c'*x_minObj
```

A3.20

Optimal consumption is 2617.83
Speed:



MATLAB code:

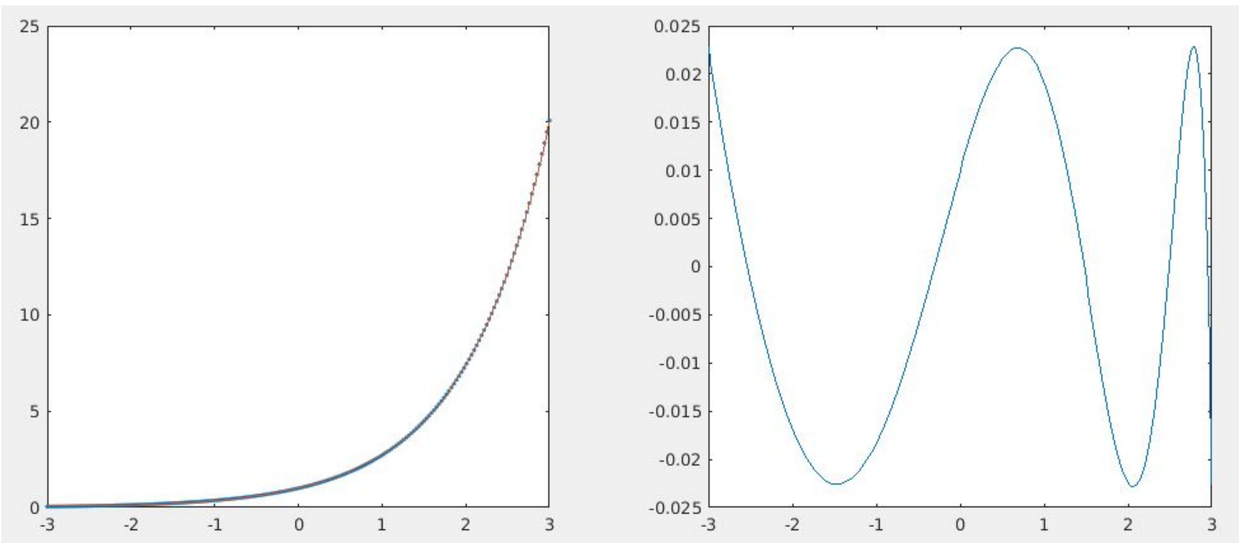
```
%% A3.20
veh_speed_sched_data;
cvx_begin
variable t(n)
minimize(a*(d.^2)'*inv_pos(t) + b*sum(d) + c* sum(t));
subject to
tau_min<=cumsum(t)<=tau_max;
d./smax<=t<=d./smin;
cvx_end

%% plot results
```

```
figure;
stairs(d./t);
xlabel('segments')
ylabel('speed')
```

A5.2

Optimal values of [a0 a1 a2 b1 b2] = [1.0099 0.6119 0.1134 -0.4146 0.0485]
Minimized max deviation = 0.023

Left: plot of data to fit (dot) and fit curve. Right: fitting error



MATLAB code:

```
%% A5.2
k = 201;
is = 1:k;
t = -3 + 6*(is-1)/(k-1);
y = exp(t);
l = 0; u = exp(3);
tol = 0.001;
while (abs(l-u) > tol)
    gamma = (l+u)/2;
    cvx_begin
    variables a0 a1 a2 b1 b2
    subject to
```

```
    abs(a0+a1*t+a2*t.^2-y.*(1+b1*t+b2*t.^2))<=gamma*(1+b1*t+b2*t.^2)
    cvx_end
    if strcmp(cvx_status,'Solved')
        u = gamma;
    else
        l = gamma;
    end
end
[a0 a1 a2 b1 b2]

%% plot
figure;
subplot(1,2,1)
plot(t, y,'.')
hold all
plot(t, (a0+a1*t+a2*t.^2)./(1+b1*t+b2*t.^2))
subplot(1,2,2)
plot(t, (a0+a1*t+a2*t.^2)./(1+b1*t+b2*t.^2) - y)
```

A13.3

(a)

The code tells the variance in different situations:

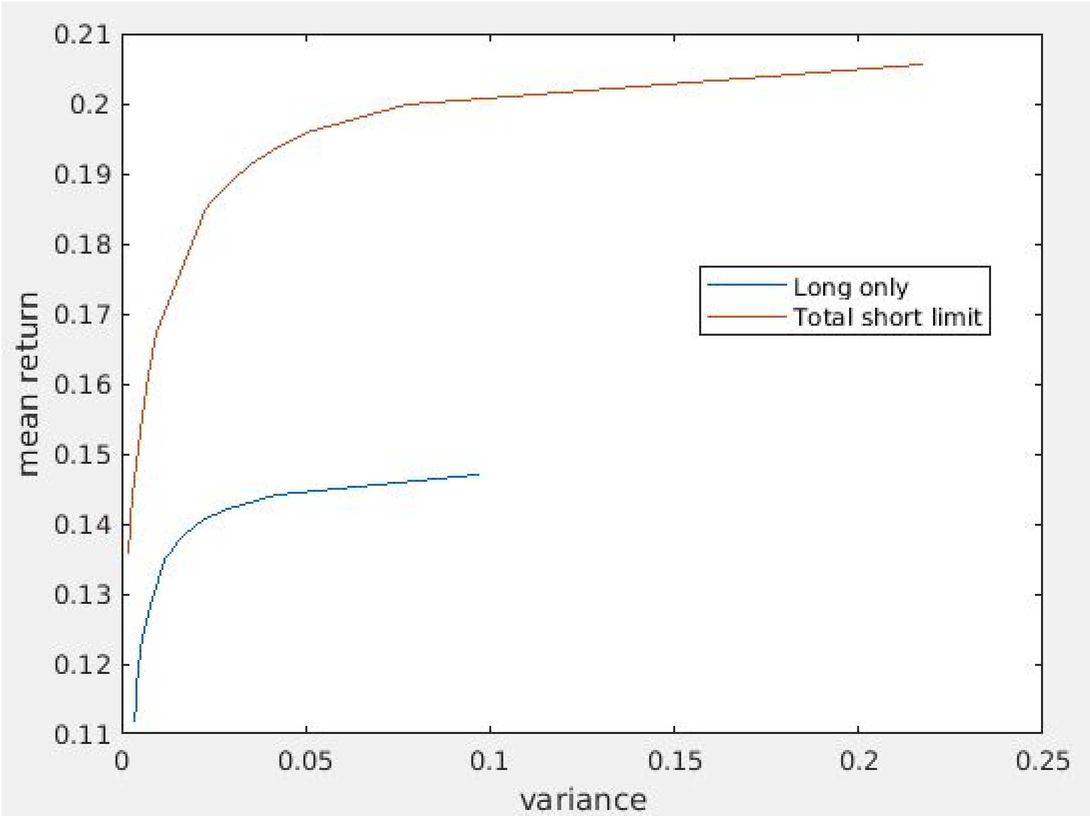
- uniform portfolio: v_UP = 0.0076
- no constraint: v_NC = 3.4622e-4
- long only: v_LO = 0.0026
- Limit on total short position: v_LTS = 4.4078e-4

v_LO > v_LTS > v_NC is expected.

(b)

Optimal risk-return trade-off curve by optimizing

- -pbar'*x+mu*x'*S*x



MATLAB code:

```
%% A13.3
simple_portfolio_data;

% risk of the uniform portfolio
x_unif'*S*x_unif

% no constraint
cvx_begin
variable x(n)
minimize(x'*S*x)
subject to
sum(x) == 1
pbar'*x == pbar'*x_unif
cvx_end
var_noConst = cvx_optval;

% long-only
cvx_begin
```

```
variable x(n)
minimize(x'*S*x)
subject to
pbar'*x == pbar'*x_unif
sum(x) == 1
x >= 0
cvx_end
var_longOnly = cvx_optval;

% limit on total short
cvx_begin
variable x(n)
minimize(x'*S*x)
subject to
pbar'*x == pbar'*x_unif
sum(x) == 1
sum(max([-x, zeros(n,1)]')) <= 0.5
cvx_end
var_totShort = cvx_optval;

%% plot the optimal risk-return trade-off curves
% long only
mus = [0:0.1:1, 2:1:10];
mean_rets = [];
var_rets = [];
for ii = 1:length(mus)
   mu = mus(ii)
   cvx_begin
   variable x(n)
   minimize(-pbar'*x+mu*x'*S*x)
   subject to
   sum(x) == 1
   x >= 0
   cvx_end
   mean_rets(ii) = pbar'*x;
   var_rets(ii) = x'*S*x;
end



%% tot short less than 0.5
mus = [0:0.1:1, 2:1:10];
mean_rets_TSL = [];
```

```
var_rets_TSL = [];
for ii = 1:length(mus)
    mu = mus(ii)
    cvx_begin
    variable x(n)
    minimize(-pbar'*x+mu*x'*S*x)
    subject to
    sum(x) == 1
    sum(max([-x, zeros(n,1)]')) <= 0.5
    cvx_end
    mean_rets_TSL(ii) = pbar'*x;
    var_rets_TSL(ii) = x'*S*x;
end


%% plot
figure;
plot(var_rets, mean_rets);
xlabel('variance');
ylabel('mean return');
hold all;
plot(var_rets_TSL, mean_rets_TSL);
legend({'Long only','Total short limit'})
```