# Cavity Optomechanics

Wentao Jiang

## Contents



# 1 Introduction

Historical review:

- Ashkin, focused laser beams can trap and control dielectric particles; Laser cooling;
  Application: optical atomic clocks, precision measurements
- Braginsky, dynamical influence of radiation pressure; quantum fluctuations of it, established the standard quantum limit for continuous position detection
- theoretical: squeezing of light, QND detection of the light intensity, quantum nonlinearities for extremely strong optomechanical coupling, give rise to nonclassical and entangled states of the light field and the mechanics
- experimental: optical feedback cooling; feedback damping, self-induced oscillations
- systems: membranes; nanorods; microdisks; microspheres; optical waveguides; nanomechanical beam inside a superconducting transmission line microwave cavity
- motivations: sensitive optical detection of small forces, displacements, masses and accelerations; interconvert information between solid-state qubits and flying photonic qubits

# 2 Optical Cavities and Mechanical Resonators

## 2.1 Optical resonators

F-P resonator (etalon):

- angular frequency: $\omega_{\mathrm{cav},m} \approx m \cdot \pi(c/L)$
- free spectral range (FSR): $\Delta\omega_{\mathrm{FSR}} = \pi \frac{c}{L}$
- optical finesse: $\mathcal{F} \equiv \Delta\omega_{\mathrm{FSR}}/\kappa$
- quality factor: $Q_{\mathrm{opt}} = \omega_{\mathrm{cav}}\tau$
- total cavity loss rate: $\kappa = \kappa_{\mathrm{ex}} + \kappa_0$
  where $\kappa_{\mathrm{ex}}$ refers to the input coupling loss rate
  photons going into the $\kappa_0$ decay channel won't be recorded

## 2.2 Input-output formalism

Master equations: only internal dynamics is of interest
Input-output theory: include the light field being emitted by the cavity, formulated on the level of Heisenberg equations of motion, describing the time evolution of the field amplitude $\hat{a}$ inside the cavity:

$$\dot{\hat{a}} = -\frac{\kappa}{2}\hat{a} + i\Delta\hat{a} + \sqrt{\kappa_{\mathrm{ex}}}\hat{a}_{\mathrm{in}} + \sqrt{\kappa_0}\hat{f}_{\mathrm{in}} \tag{2.1}$$

where laser detuning $\Delta = \omega_L - \omega_{\mathrm{cav}}$, $\hat{a}_{\mathrm{in}}$ should be regarded as a stochastic quantum field. The field is normalized that

$$P = \hbar\omega_L \left\langle \hat{a}_{\mathrm{in}}^{\dagger}\hat{a}_{\mathrm{in}} \right\rangle \tag{2.2}$$

is the input power launched into the cavity. The same kind of description holds for the "unwanted" channel $\hat{f}_{\mathrm{in}}$.

The field reflected from the F-P resonator is given by

$$\hat{a}_{\mathrm{out}} = \hat{a}_{\mathrm{in}} - \sqrt{\kappa_{\mathrm{ex}}}\hat{a} \tag{2.3}$$

After taking average of eqt 2.1, 2.3 and $\left\langle \hat{f}_{\mathrm{in}} \right\rangle = 0$:

$$\langle\hat{a}\rangle = \frac{\sqrt{\kappa_{\mathrm{ex}}}\,\langle\hat{a}_{\mathrm{in}}\rangle}{\kappa/2 - i\Delta} \tag{2.4}$$

$$\chi_{\mathrm{opt}}(\omega) \equiv \frac{1}{-i(\omega+\Delta)+\kappa/2} \tag{2.5}$$

This is adequate as long as $\kappa \ll \Delta\omega_{\mathrm{FSR}}$, i.e., $Q \gg 1$

The steady state cavity population

$$\bar{n}_{\rm cav} = |\langle \hat{a} \rangle|^2 = \frac{\kappa_{\rm ex}}{\Delta^2 + (\kappa/2)^2} \frac{P}{\hbar\omega_L} \tag{2.6}$$

Reflection amplitude:

$$\mathcal{R} = \frac{\langle \hat{a}_{\rm out} \rangle}{\langle \hat{a}_{\rm in} \rangle} = \frac{(\kappa_0 - \kappa_{\rm ex})/2 - i\Delta}{(\kappa_0 + \kappa_{\rm ex})/2 - i\Delta} \tag{2.7}$$

If the external coupling dominates the cavity losses ($\kappa_{\rm ex} \approx \kappa \gg \kappa_0$), the cavity is called "**overcoupled**" and $|\mathcal{R}|^2 \approx 1$. The case where $\kappa_{\rm ex} = \kappa_0$ is called "**critical coupling**", where $\mathcal{R}(\Delta = 0) = 0$ on resonance. This implies that the input power is either fully dissipated within the resonator or fully transmitted. The situation $\kappa_{\rm ex} \ll \kappa_0$ is referred to as "**undercoupling**" and is associated with cavity losses dominated by intrinsic losses, leading to an effective loss of information.

## 2.3 Mechanical resonators

- displacement field: $\vec{u}_n(\vec{r})$, $n$ denotes the mode
- vibration frequency: $\Omega_m$
- energy damping rate: $\Gamma_m$
- mechanical quality factor: $Q_m = 1/\delta\Phi = \Omega_m/\Gamma_m$
- global amplitude: $x(\vec{r})$
- a suitable mode function: $\vec{u}(\vec{r}, t) = x(t) \cdot \vec{u}(\vec{r})$

then the temperal evolution of $x(t)$:

$$m_{\rm eff} \frac{d^2x(t)}{dt^2} + m_{\rm eff} \frac{dx(t)}{dt} + m_{\rm eff}\Omega_m^2 x(t) = F_{\rm ex}(t) \tag{2.8}$$

$F_{\rm ex}(t)$ denotes the sum of all forces acting on the mechanical oscillator. In the absence of external forces, it is given by the Langevin force.

Eqt 2.8 can be solved easily in frequency space:

$$x(\omega) = \int dt e^{i\omega t} x(t) \tag{2.9}$$

$$x_m(\omega) = \left[m_{\rm eff}\left(\Omega_m^2 - w^2\right) - i m_{\rm eff}\Gamma_m\omega\right]^{-1} \tag{2.10}$$

The quantum mechanical treatment:

$$\hat{H} = \hbar\Omega_m \hat{b}^\dagger \hat{b} + \frac{1}{2}\hbar\Omega_m \tag{2.11}$$

$$\hat{x} = x_{\rm XPF}(\hat{b} + \hat{b}^\dagger) \tag{2.12}$$

$$\hat{p} = -i m_{\rm eff}\Omega_m x_{\rm XPF}(\hat{b} - \hat{b}^\dagger) \tag{2.13}$$

where

$$x_{\rm ZPF} = \sqrt{\frac{\hbar}{2m_{\rm eff}\Omega_m}} \tag{2.14}$$

is the zero-point fluctuation amplitude.

Coupled to a high-temperature bath:

$$\frac{d}{dt}\bar{n} = -\Gamma_m(\bar{n} - \bar{n}_{\rm th}) \tag{2.15}$$

$$\frac{d}{dt}\bar{n}(t=0) = \Gamma_m \bar{n}_{\rm th} \approx \frac{k_B T_{\rm bath}}{\hbar Q_m} \tag{2.16}$$

### 2.3.1 Mechanical dissipation

- viscous damping
- clamping losses
- fundamental anharmonic effects
- materials-induced losses

$$\frac{1}{Q_{\rm total}} = \sum \frac{1}{Q_i} \tag{2.17}$$

### 2.3.2 Susceptibility, noise spectra, fluctuation-dissipation theorem

Gated Fourier transform:

$$\tilde{x}(\omega) = \frac{1}{\sqrt{\tau}} \int_0^\tau x(t) e^{i\omega t} dt \tag{2.18}$$

Noise power spectral density:

$$S_{xx} \equiv \int_{-\infty}^{+\infty} \langle x(t)x(0)\rangle\, e^{i\omega t} dt \tag{2.19}$$

Wiener-Khinchin theorem:

$$\lim_{\tau\to\infty} \left\langle |\tilde{x}(\omega)|^2 \right\rangle = S_{xx}(\omega) \tag{2.20}$$

The experimentally measured mechanical noise spectrum yields the variance of the mechanical displacement $\langle x^2 \rangle$:

$$\int_{-\infty}^{+\infty} S_{xx}(\omega) \frac{d\omega}{2\pi} = \left\langle x^2 \right\rangle \tag{2.21}$$

The fluctuation-dissipation theorem (FDT):

$$S_{xx}(\omega) = 2\frac{k_B T}{\omega} \mathrm{Im}\chi_m(\omega) \tag{2.22}$$

where $\chi_m(\omega)$ denotes the mechanical susceptibility.

The quantum FDT:

$$S_{xx}(\omega) = \frac{\hbar}{1 - \exp(-\hbar\omega/k_B T)} \mathrm{Im}\chi_{xx}(\omega) \tag{2.23}$$

it implies $S_{xx}(\omega) = 0$ for $\omega < 0$ at $T = 0$, means the $T = 0$ bath is not able to supply energy.

# 3 Principles of Optomechanical Coupling

## 3.1 The radiation-pressure force and optomechanical coupling

The fundamental mechanism that couples the properties of the cavity radiation field to the mechanical motion is the momentum transfer of photons, i.e., radiation pressure.

$$\langle \hat{F} \rangle = 2\hbar k \frac{\langle \hat{a}^\dagger \hat{a} \rangle}{\tau_c} = \hbar \frac{\omega_{\rm cav}}{L} \left\langle \hat{a}^\dagger \hat{a} \right\rangle \tag{3.1}$$

$\tau_c = 2L/c$: cavity round-trip time

$\hbar\omega_{\rm cav}/L$: r-p force caused by one intracavity photon

$G = \omega_{\rm cav}/L$: the change of cavity resonance frequency with position, i.e., the frequency pull parameter

More general:

- by direct momentum transfer via reflection
- by coupling via a dispersive shift of the cavity frequency
- by optical near-field effects

Gradient forces:

$$U = \frac{1}{2}\vec{p} \cdot \vec{E} = \frac{1}{2}\alpha \left|\vec{E}(\vec{r})\right|^2 \tag{3.2}$$

$$\vec{F} = -\vec{\nabla} U \tag{3.3}$$

## 3.2 Hamiltonian formulation

Restrict to:

- one of the many optical mode
- one of the many mechanical normal modes

$$\hat{H}_0 = \hbar\omega_{\mathrm{cav}}\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} \tag{3.4}$$

The cavity resonance frequency is modulated by the mechanical amplitude:

$$\omega_{\mathrm{cav}}(x) \approx \omega_{\mathrm{cav}} + x\frac{\partial\omega_{\mathrm{cav}}}{\partial x} + ... \tag{3.5}$$

For most realizations, it suffices to keep the linear term.
$G = -\partial\omega_{\mathrm{cav}}/\partial x$: optical frequency shift per displacement

For a simple cavity of length $L$, $G = \omega_{\mathrm{cav}}/L$. The minus sign of $G$ reflects the fact that we choose $x > 0$ to increase the cavity length, leading to a decrease in $\omega_{\mathrm{cav}}$ if $G > 0$.

$$\therefore \hbar\omega_{\mathrm{cav}}(x)\hat{a}^\dagger\hat{a} \approx \hbar(\omega_{\mathrm{cav}} - G\hat{x})\hat{a}^\dagger\hat{a} \tag{3.6}$$

The interaction part of the Hamiltonian:

$$\hat{H}_{\mathrm{int}} = -\hbar g_0\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger) \tag{3.7}$$

$g_0 = Gx_{\mathrm{ZPF}}$: vacuum optomechanical coupling strength.

The Hamiltonian reveals that the interaction is fundamentally **nonlinear**, involving three operators (three-wave mixing).

Radiation-pressure force:

$$\hat{F} = -\frac{d\hat{H}_{\mathrm{int}}}{d\hat{x}} = \hbar G\hat{a}^\dagger\hat{a} = \hbar\frac{g_0}{x_{\mathrm{ZPF}}}\hat{a}^\dagger\hat{a} \tag{3.8}$$

Applying the unitary transformation $\hat{U} = \exp(i\omega_L\hat{a}^\dagger\hat{a}t)$ makes the driving terms time independent[1] and generate a new Hamiltonian of the form

$$\hat{H} = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} - \hbar g_0\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger) + ... \tag{3.9}$$

$\Delta = \omega_L - w_{\mathrm{cav}}$: laser detuning

Driving term (omitted in the above Hamiltonian):

$$\hat{H}_{\mathrm{drive}} = i\hbar\sqrt{\kappa_{\mathrm{ex}}}\hat{a}^\dagger\alpha_{\mathrm{in}} + H.c. \tag{3.10}$$

for a laser of amplitude $\alpha_{\mathrm{in}}$

"Linerized" approximate description:

$$\hat{a} = \bar{\alpha} + \delta\hat{a} \tag{3.11}$$

[1] $\hat{U}(\hat{a}^\dagger e^{-i\omega_L t} + \hat{a}e^{+i\omega_L t})\hat{U}^\dagger = \hat{a}^\dagger + \hat{a}$

expand the interactive part of the Hamiltonian in powers of $\bar{\alpha}$:
$-\hbar g_0|\bar{\alpha}|^2(\hat{b} + \hat{b}^\dagger)$: an average r-p force $\bar{F} = \hbar G|\bar{\alpha}|^2$, can be onitted after:
an appropriate shift of the displacement's origin: $\delta\bar{x} = \bar{F}/m_{\mathrm{eff}}\Omega_m^2$
a modified detuning: $\Delta_{\mathrm{new}} = \Delta_{\mathrm{old}} + G\delta\bar{x}$

The term we keep:

$$-\hbar g_0(\bar{\alpha}^*\delta\hat{a} + \bar{\alpha}\delta\hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger) \tag{3.12}$$

Further assume $\bar{\alpha} = \sqrt{\bar{n}_{\mathrm{cav}}}$ is real, thus the Hamiltonian

$$\hat{H} \approx -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{b}^\dagger\hat{b} + \hat{H}_{\mathrm{int}}^{(\mathrm{lin})} \tag{3.13}$$

$$\hat{H}_{\mathrm{int}}^{(\mathrm{lin})} = -\hbar g_0\sqrt{\bar{n}_{\mathrm{cav}}}(\delta\hat{a} + \delta\hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger) \tag{3.14}$$

$g = g_0\sqrt{\bar{n}_{\mathrm{cav}}}$: optomechanical coupling strength

The linearized description can be good even if the average photon number inside the cavity is not large, because the mechanical system may not be able to resolve the individual photon for large decay rate $\kappa$.

$g > \kappa$: Strong coupling regime

$g_0 > \kappa$: nonlinear quantum effects become observable

Three regime depending on the detuning:

- $\kappa \ll \Omega_m$ sideband-resolved regime
- $\Delta \approx -\Omega_m$: red-detuned regime
- $\Delta \approx +\Omega_m$: blue-detuned regime

For $\Delta \approx -\Omega_m$(red-detuned regime), we have two harmonic oscillators of equal frequency that can interchange quanta: the mechanical oscillator and the driven cavity mode. The terms in the interactive Hamiltonian describing this are

$$-\hbar g(\delta\hat{a}^\dagger\hat{b} + \delta\hat{a}\hat{b}^\dagger) \tag{3.15}$$

those term that create or destroy two quanta at the same time ( $\delta\hat{a}^\dagger\hat{b}^\dagger, \delta\hat{a}\hat{b}$ ) can be omitted because they are strongly "nonresonant". Keeping only terms in 3.15 is known as the rotating-wave approximation (RWA). This case is the one relevant for cooling and for quantum state transfer between light and mechanics, also referred to as a "beam-splitter" interaction.

For $\Delta \approx +\Omega_m$ (blue-detuned regime), the dominant terms in RWA

$$\hbar g(\delta\hat{a}^\dagger\hat{b}^\dagger + \delta\hat{a}\hat{b}) \tag{3.16}$$

represent a "two-mode squeezing" interaction that lies at the heart of parametric amplification. Without dissipation, this leads to an exponential growth of energies in bothvibrational mode and the driven optical mode, with strong quantum correlations between, thus can be used for entangling both modes.

When $\Delta = 0$, the whole interaction term means that the mechanical position $\hat{x} \propto \hat{b} + \hat{b}^\dagger$ leads to a phase shift of the light field, which is used in optomechanical displacement detection.

### 3.3 Optomechanical equation of motion

Optomechanical "backaction":
mechanical motion $\rightarrow$
optical resonance frequency shift $\rightarrow$
change of circulating light intensity $\rightarrow$
change of radiation-pressure force

Input-output formalism in the form of quantum Langevin equations:

$$\dot{\hat{a}} = -\frac{\kappa}{2}\hat{a} + i(\Delta + G\hat{x})\hat{a} + \sqrt{\kappa_{\mathrm{ex}}}\hat{a}_{\mathrm{in}} + \sqrt{\kappa_0}\hat{f}_{\mathrm{in}} \tag{3.17}$$

$$\dot{\hat{b}} = \left(-i\Omega_m - \frac{\Gamma_m}{2}\right)\hat{b} + ig_0\hat{a}^\dagger\hat{a} + \sqrt{\Gamma_m}\hat{b}_{\mathrm{in}} \tag{3.18}$$

correct as long as $\Omega_m \gg \Gamma_m$

The noise correlators:(omitted)

Classical, averaged version:

$$\dot{\alpha} = -\frac{\kappa}{2}\alpha + i(\Delta + Gx)\alpha + \sqrt{\kappa_{\mathrm{ex}}}\alpha_{\mathrm{in}} \tag{3.19}$$

$$m_{\mathrm{eff}}\ddot{x} = -m_{\mathrm{eff}}\Omega_m^2 x - m_{\mathrm{eff}}\Gamma_m\dot{x} + \hbar G|\alpha|^2 \tag{3.20}$$

neglecting all fluctuations. These are the basis for discussion of nonlinear phenomena, particularly the optomechanical parametric instability.

Linearize around some steady-state solution:

$$\delta\dot{\hat{a}} = (i\Delta - \frac{\kappa}{2})\delta\hat{a} + ig(\hat{b} + \hat{b}^\dagger) + \sqrt{\kappa_{\mathrm{ex}}}\delta\hat{a}_{\mathrm{in}}(t) + \sqrt{\kappa_0}\hat{f}_{\mathrm{in}}(t) \tag{3.21}$$

$$\dot{\hat{b}} = (-i\Omega_m - \frac{\Gamma_m}{2})\hat{b} + ig(\delta\hat{a} + \delta\hat{a}^\dagger) + \sqrt{\Gamma_m}\hat{b}_{\mathrm{in}}(t) \tag{3.22}$$

The quantum and the classical equations can all be displayed in frequency space.

# 4 Experimental Realization and Optomechanical Parameters