Created in COMSOL Multiphysics 6.2

The Black–Scholes Equation

Introduction

There are different types of stock options:

- A *call option* is the right to buy a security at a specified price (called the exercise or strike price) during a specified period of time.
- A *put option* is the right to sell a security at a specified price during a specified period of time.

American options can be exercised at any time up to and including the day the option expires. European options can be exercised only on the day the option expires.

The famous Black–Scholes equation computes the cost u of a European stock option

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2 x^2\frac{\partial^2 u}{\partial x^2}+rx\frac{\partial u}{\partial x} = ru$$

with the following parameters:

- x, the underlying asset price
- r, the continuous compounding rate of interest
- σ, the standard deviation of the asset's rate of return (also known as volatility)

A put option's value on the exercise day is

$$u(T, x) = \max(K-x, 0)$$

where K is the strike price. The problem domain is infinite and consists of the entire real axis across the time domain $0 \le t \le T$.

The assumptions made in deriving the Black-Scholes equation are:

- The underlying stock pays no dividends.
- The price of the stock, one period ahead, has a log-normal distribution with mean and standard deviation that are constant over the life of the option.
- The existence of a risk-free interest rate which is constant over the life of the option.
- You can lend and borrow at the risk-free interest rate.

Black and Scholes derived an analytical expression for the solution to the above problem. However, the formula works only for certain cases. For instance, you cannot use it when σ and r are functions of x and t. Using the PDE formulation, you can determine the price for such cases.

Model Definition

Because you work within a finite domain $0 \le x \le X$, it is necessary to specify not only the boundary conditions for $t = T$ but also for $x = 0$ and $x = X$. It is therefore necessary to analyze the problem's characteristics to determine the location of the input and output boundaries.

EQUATION DEFINITION

To put the equation in coefficient form, rewrite the equation as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}\sigma^2 x^2 \frac{\partial u}{\partial x}\right) + \left(rx - \frac{\partial}{\partial x}\left(\frac{1}{2}\sigma^2 x^2\right)\right)\frac{\partial u}{\partial x} - rc = 0.$$

In the following, denote

$$\bar{r} = rx - \frac{\partial}{\partial x}\left(\frac{1}{2}\sigma^2 x^2\right).$$

To reduce the problems with inflow boundaries, start by considering a put option: Study the value of a put option at a strike price $K = 40$ with $\sigma = 0.3$ and $r = 0.12$.

BOUNDARY CONDITIONS AND INITIAL CONDITIONS

Make the domain be $0 \le x \le 80$ with time running from 12 to 0. Then the initial condition at $t = 12$ and $x = 80$ is 0 based on the put option's value. The initial condition in the region $0 \le x \le 40$ varies linearly from 0 to 40. At the end of the simulation domain, the boundary is free (use a homogeneous boundary condition).

Notes About the COMSOL Implementation

Model the Black-Scholes equation using the following approach:

- Create a 1D time-dependent model, using the time-stepping algorithm to solve for c as a function of x and t, the time. The time steps go backward in time. Using a variable substitution to reverse the sign of the time, the d_a coefficient becomes -1.
- To model the initial condition, use the logical expression `(x<40)*(40-x)`. This means that in the areas where $x > 40$, the initial value is zero.

Application Library path: `COMSOL_Multiphysics/Equation_Based/black_scholes_put`

Modeling Instructions

From the **File** menu, choose **New**.

NEW

In the **New** window, click **Model Wizard**.

MODEL WIZARD

1 In the **Model Wizard** window, click **1D**.

2 In the **Select Physics** tree, select **Mathematics>PDE Interfaces>Coefficient Form PDE (c)**.

3 Click **Add**.

4 Click **Study**.

5 In the **Select Study** tree, select **General Studies>Time Dependent**.

6 Click **Done**.

ROOT

1 In the **Model Builder** window, click the root node.

2 In the root node's **Settings** window, locate the **Unit System** section.

3 From the **Unit system** list, choose **None**.

Keeping track of units is not important in this model; by turning off unit support, you avoid the need to specify dimensions for equation coefficients and coordinates to get rid of unit warnings.

GEOMETRY 1

Interval 1 (i1)

1 In the **Model Builder** window, under **Component 1 (comp1)** right-click **Geometry 1** and choose **Interval**.

2 In the **Settings** window for **Interval**, locate the **Interval** section.

3 In the table, enter the following settings:

Coordinates
0
80

4 Click **Build Selected**.

GLOBAL DEFINITIONS

Parameters 1

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.

2 In the **Settings** window for **Parameters**, locate the **Parameters** section.

3 In the table, enter the following settings:

Name	Expression	Value	Description
r	0.12	0.12	Continuous compounding interest rate
sigma	0.3	0.3	Volatility

COEFFICIENT FORM PDE (C)

Coefficient Form PDE 1

1 In the **Model Builder** window, under **Component 1 (comp1)>Coefficient Form PDE (c)** click **Coefficient Form PDE 1**.

2 In the **Settings** window for **Coefficient Form PDE**, locate the **Diffusion Coefficient** section.

3 In the c text field, type `1/2*sigma^2*x^2`.

4 Locate the **Absorption Coefficient** section. In the a text field, type `r`.

5 Locate the **Source Term** section. In the f text field, type `0`.

6 Locate the **Damping or Mass Coefficient** section. In the d_a text field, type `-1`.

7 Click to expand the **Convection Coefficient** section. In the β text field, type `(-r+sigma^2)*x`.

Initial Values 1

1 In the **Model Builder** window, click **Initial Values 1**.

2 In the **Settings** window for **Initial Values**, locate the **Initial Values** section.

3 In the u text field, type `(x<40)*(40-x)`.

Flux/Source 1

1 In the **Physics** toolbar, click **Boundaries** and choose **Flux/Source**.

2 Select Boundary 1 only.

Dirichlet Boundary Condition 1

1 In the **Physics** toolbar, click **Boundaries** and choose **Dirichlet Boundary Condition**.

2 Select Boundary 2 only.

MESH 1

Edge 1

In the **Mesh** toolbar, click **Edge**.

Size

1 In the **Model Builder** window, click **Size**.

2 In the **Settings** window for **Size**, locate the **Element Size** section.

3 Click the **Custom** button.

4 Locate the **Element Size Parameters** section. In the **Maximum element size** text field, type `2`.

5 Click **Build All**.

The mesh consists of 40 elements.

STUDY 1

Step 1: Time Dependent

1 In the **Model Builder** window, under **Study 1** click **Step 1: Time Dependent**.

2 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

3 In the **Output times** text field, type `range(12,-0.5,0)`.

4 In the **Home** toolbar, click **Compute**.

RESULTS

To see the plot of `u` as a Line Graph at time = 0, follow the steps given below.

Coefficient Form PDE

1 In the **Model Builder** window, under **Results** click **Coefficient Form PDE**.

2 In the **Settings** window for **1D Plot Group**, locate the **Data** section.

3 From the **Time selection** list, choose **From list**.

4 In the **Times (s)** list, select **0**.

5 Click to expand the **Title** section. Locate the **Plot Settings** section.

6 Select the **x-axis label** check box. In the associated text field, type `x`.

7 Select the **y-axis label** check box. In the associated text field, type `u`.

Line Graph 1

1 In the **Model Builder** window, expand the **Coefficient Form PDE** node, then click **Line Graph 1**.

2 In the **Settings** window for **Line Graph**, click **Replace Expression** in the upper-right corner of the **x-Axis Data** section. From the menu, choose **Component 1 (comp1)>Geometry>Coordinate>x - x-coordinate**.

3 Click to expand the **Legends** section. Select the **Show legends** check box.

4 Find the **Prefix and suffix** subsection. In the **Prefix** text field, type `Time = `.

5 In the **Coefficient Form PDE** toolbar, click  **Plot**.