

Single Edge Crack

This example deals with the stability of a plate with an edge crack that is subjected to a tensile load. To analyze the stability of existing cracks, you can apply the principles of fracture mechanics.

A common parameter in fracture mechanics is the so-called stress intensity factor K_I, which provides a means to predict if a specific crack may lead to a fracture. When K_I becomes equal to the critical fracture toughness of the material, K_{Ic} (a material property), then usually catastrophic fracture occurs.

Determining the stress intensity factor directly from the local state at the crack tip is problematic, since the stresses are singular there. Therefore, more indirect energy-based methods are attractive. In this example, K_I is computed using the J-integral and from the energy release rate.

In addition, the crack growth rate and number of cycles needed to propagate the crack a certain distance are computed.

Model Definition

A plate with a width w = 1.5 m and height h = 4.5 m has a single horizontal edge-crack at the middle of the left vertical edge. The length of the crack is varied from a = 0.5 m to a = 0.7 m, see Figure 1. An external load is pulling the plate such that the top and bottom edges experience tensile stress, σ , of 20 MPa.

Because of the symmetry, only half of the plate is modeled. There are three paths for computing the J-integral:

- A circle around the crack tip with the radius being half the crack length.
- **2** A circle around the crack tip with the radius being 0.7 times the crack length.
- **3** The external boundaries, excluding the crack surface.

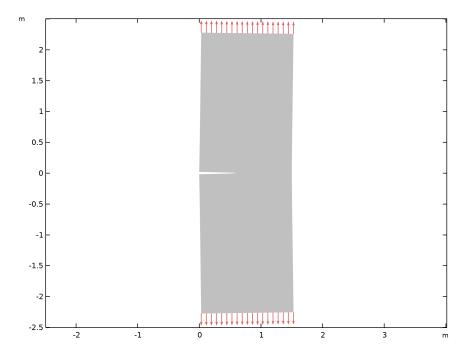


Figure 1: Plate geometry and loading (before applying symmetry conditions).

You apply a tensile load to the upper horizontal edge, while the lower horizontal edge is constrained in the *y* direction using a symmetry condition. One point is constrained in the horizontal direction in order to suppress rigid body motion.

MATERIAL MODEL

The material properties are representative for steel.

TABLE I: MATERIAL DATA.

Quantity	Name	Expression
Young's modulus	E	206 GPa
Poisson's ratio	ν	0.3
Coefficient in Paris' law	Α	I.4·10 ⁻¹¹ (K _I unit system: MN/m ^{3/2})
Exponent in Paris' law	m	3.1

THE J-INTEGRAL

In this model, you determine the stress intensity factor K_I using the so-called J-integral.

The J-integral is a two-dimensional path independent line integral along a counterclockwise contour, Γ , surrounding the crack tip. For a crack extending along the positive x-axis, the J-integral is defined as

$$J = \int_{\Gamma} W dy - T_{i} \frac{\partial u}{\partial x}^{i} ds = \int_{\Gamma} \left(W n_{x} - T_{i} \frac{\partial u}{\partial x}^{i} \right) ds$$

where W is the strain energy density

$$W = \frac{1}{2}(\sigma_x \cdot \varepsilon_x + \sigma_y \cdot \varepsilon_y + \sigma_{xy} \cdot 2 \cdot \varepsilon_{xy})$$

and T is the traction vector defined as

$$\mathbf{T} = \begin{bmatrix} \sigma_x \cdot n_x + \sigma_{xy} \cdot n_y \\ \sigma_{xy} \cdot n_x + \sigma_y \cdot n_y \end{bmatrix}$$

 σ_{ij} denotes the stress components, ϵ_{ij} the strain components, and n_i the normal vector components.

The J-integral has the following relation to the Mode I stress intensity factor for a plane stress case and a linear elastic material:

$$J = \frac{K_I^2}{E} \tag{1}$$

where *E* is Young's modulus.

ENERGY RELEASE RATE

For a linear elastic material it is possible to compute the value of the J-integral without using the path integrals. The reason is that its value equals the value of the energy release rate, G,

$$G = -\frac{1}{t} \frac{\partial U}{\partial a} \tag{2}$$

Here, U is the potential energy, a is the crack length, and t is the thickness. The potential energy of an elastic body is

$$U = \frac{1}{2} \int_{\Omega} \sigma : \varepsilon \ dV - \frac{1}{2} \int_{\Omega} (\mathbf{F_v} \cdot \mathbf{u} \ dV) - \int_{\partial \Omega} \mathbf{T} \cdot \mathbf{u} \ dS$$

The first term is the strain energy in the volume, and the second and third terms are the potential of the body load and prescribed tractions, respectively.

CRACK PROPAGATION

When subjected to a periodic load, the crack growth rate (in meters per load cycle) can be expressed by Paris' law:

$$\frac{da}{dN} = A(\Delta K_I)^m \tag{3}$$

Here, A and m are material parameters and $\Delta K_{\rm I}$ is the range of the stress intensity factor. It is assumed that the load varies between 0 and 20 MPa, so that $\Delta K_{\rm I}$ equals the computed $K_{\rm I}$. Note that the value of the coefficient A depends on the unit for the stress intensity factor in a rather complex way, because of the power m, which in general is noninteger.

Results

Figure 2 shows the stress singularity at the crack tip.

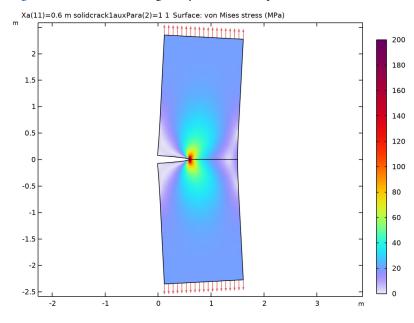


Figure 2: von Mises stress and the deformed shape of the plate when the crack length is 0.6 m. The displacement is exaggerated to illustrate the deformation under the applied load.

Based on Ref. 1 an analytical solution for the stress intensity factor is

$$K_{\text{Ia}} = \sigma \cdot \sqrt{\pi \cdot a} \cdot f\left(\frac{a}{w}\right)$$

where $\sigma = 20$ MPa, and f is a correction factor given in Ref. 1.

$$f\left(\frac{a}{w}\right) = \frac{\sqrt{\frac{2w}{\pi a} \tan\frac{\pi a}{2w}} \left(0.752 + 2.02\frac{a}{w} + 0.37\left(1 - \sin\frac{\pi a}{2w}\right)^3\right)}{\cos\frac{\pi a}{2w}} \tag{4}$$

The expression in Equation 4 assumes that the plate is long, so that the height is significantly larger than the width. The stress field in Figure 2 indicates that this assumption is fulfilled.

The calculated stress intensity factors for the three different contours, tabulated in the evaluation group Cracks (solid), show excellent agreement with the reference value. The largest deviation for any of the studied crack lengths and integration paths is less than 0.3%.

When using the built-in J-integral computation, you should ascertain that none of the contours used for the integration passes outside the computational domain, or encloses another crack. In the default plot **Cracks**, all integration paths are visualized. In Figure 3 and Figure 4 the contours are shown for the shortest and longest cracks, respectively.

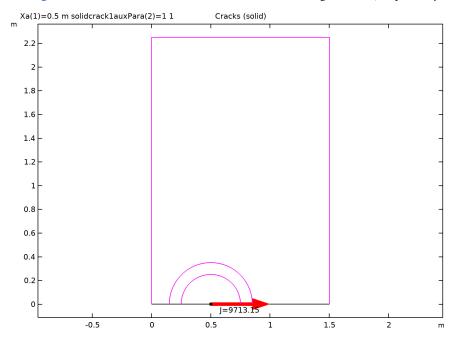


Figure 3: The contours used for integration when a = 0.5 m.

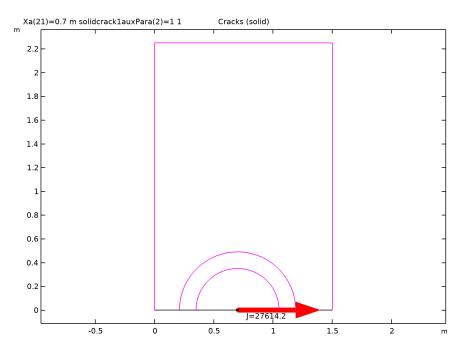


Figure 4: The contours used for integration when a = 0.7 m.

The two different ways of computing the energy release rate, and thus K_I, are compared in Figure 5. As can be seen, these two methods give essentially the same values.

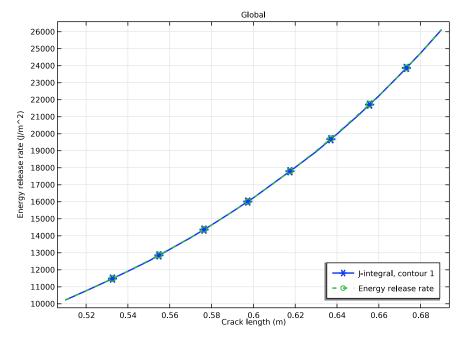


Figure 5: J-integral compared with energy release rates computed using numerical differentiation.

Finally, the crack growth speed can be investigated. In Figure 6, the crack growth speed is shown as function of the crack length. The dependence is quite strong: an increase in crack length from 0.5 m to 0.7 m (40%), increases the crack growth rate by a factor of 5. According to the constants used in Paris' law, the crack growth rate is proportional to the stress intensity factor raised to the power of 3.1. As can be seen from the previous results, the stress intensity factor increases strongly with the crack length, and this combination results in an increasing crack growth rate.

In practice, Paris' law may not be applicable when $K_{\rm I}$ approaches the critical value $K_{\rm Ic}$.

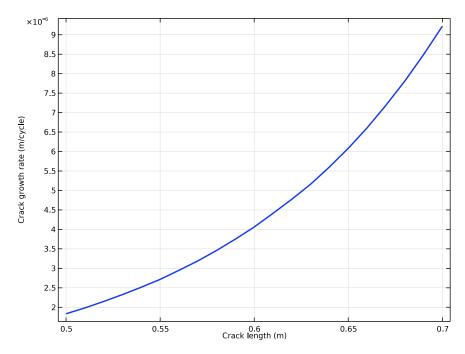


Figure 6: Crack propagation rate as function of the crack length.

Notes about the COMSOL Implementation

In this analysis you compute the J-integral for three different contours traversing three different regions around the crack tip. Note that the boundaries along the crack are not included in the J-integral because they do not give any contribution to the J-integral. This is due to the following facts: for an ideal crack, dy is zero along the crack faces, and all traction components are also zero $(T_i = 0)$ as the crack faces are not loaded.

To evaluate the J-integral along three different paths, you need to add one Crack node with three **I-integral** subnodes. Since the **Symmetric** option is used in the **Crack** node, the values of the J-integrals and stress intensity factors automatically take the full structure into account. When you add a **J-integral** node, the $K_{\rm I}, K_{\rm II}$, and (in 3D) $K_{\rm III}$ stress intensity factors are also computed.

To evaluate the energy release rate, you need to add Virtual Crack Extension subnode under the Crack node. The subnode has an action button Create Deformed Geometry and Study to set up the deformed geometry and stationary study with sensitivity node. Since the

Symmetric option is used in the **Crack** node, the values of the energy release rate automatically take the full structure into account.

Reference

1. H Tada, P.C. Paris, and G.R. Irwin, *The Stress Analysis of Cracks Handbook*, Del Research Corp, Press, 1973.

Application Library path: Structural_Mechanics_Module/Fracture_Mechanics/single_edge_crack

Modeling Instructions

From the File menu, choose New.

NEW

In the New window, click Model Wizard.

MODEL WIZARD

- I In the Model Wizard window, click 9 2D.
- 2 In the Select Physics tree, select Structural Mechanics>Solid Mechanics (solid).
- 3 Click Add.
- 4 Click **Done**.

GLOBAL DEFINITIONS

Parameters 1

- I In the Model Builder window, under Global Definitions click Parameters I.
- 2 In the Settings window for Parameters, locate the Parameters section.
- 3 Click Load from File.
- **4** Browse to the model's Application Libraries folder and double-click the file single_edge_crack_parameters.txt.

DEFINITIONS

Variables 1

- I In the Model Builder window, under Component I (compl) right-click Definitions and choose Variables.
- 2 In the Settings window for Variables, locate the Variables section.
- **3** In the table, enter the following settings:

Name	Expression	Unit	Description
dadN	AP* (solid.crack1.jint1.KI/ 1e6)^mP		Crack growth rate (m/cycle)

GEOMETRY I

Rectangle I (rI)

- I In the Geometry toolbar, click Rectangle.
- 2 In the Settings window for Rectangle, locate the Size and Shape section.
- 3 In the Width text field, type Wp.
- 4 In the Height text field, type Hp.
- 5 Click Build All Objects.

Add a point at the crack tip.

Point I (ptl)

- I In the **Geometry** toolbar, click **Point**.
- 2 In the Settings window for Point, locate the Point section.
- 3 In the x text field, type Xa.

Form Union (fin)

- I In the Model Builder window, click Form Union (fin).
- 2 In the Settings window for Form Union/Assembly, click Paul Build Selected.

MATERIALS

Steel

- I In the Model Builder window, under Component I (comp I) right-click Materials and choose Blank Material.
- 2 In the Settings window for Material, type Steel in the Label text field.

3 Locate the **Material Contents** section. In the table, enter the following settings:

Property	Variable	Value	Unit	Property group
Young's modulus	E	206[GPa]	Pa	Young's modulus and Poisson's ratio
Poisson's ratio	nu	0.3	I	Young's modulus and Poisson's ratio
Density	rho	7850	kg/m³	Basic

SOLID MECHANICS (SOLID)

- I In the Model Builder window, under Component I (compl) click Solid Mechanics (solid).
- 2 In the Settings window for Solid Mechanics, locate the 2D Approximation section.
- 3 From the list, choose Plane stress.
- **4** Locate the **Thickness** section. In the d text field, type Th.

Symmetry I

- I In the Physics toolbar, click Boundaries and choose Symmetry.
- 2 Select Boundaries 2 and 4 only.

It does not matter whether you select boundary 2 as a symmetry boundary or not. If selected, it will be overridden when the **Crack** node is added.

Boundary Load 1

- I In the Physics toolbar, click Boundaries and choose Boundary Load.
- 2 Select Boundary 3 only.
- 3 In the Settings window for Boundary Load, locate the Force section.
- **4** Specify the $\mathbf{F}_{\mathbf{A}}$ vector as

0	x
20[MPa]	у

Prescribed Displacement I

- I In the **Physics** toolbar, click **Points** and choose **Prescribed Displacement**. Suppress rigid body motion.
- 2 Select Point 4 only.
- 3 In the Settings window for Prescribed Displacement, locate the Prescribed Displacement section.
- 4 From the Displacement in x direction list, choose Prescribed.

Crack I

- I In the Physics toolbar, click Boundaries and choose Crack.
- 2 In the Settings window for Crack, locate the Crack Definition section.
- 3 From the Crack surface list, choose Symmetric.
- **4** Select Boundary 2 only.
- 5 Click to expand the Crack Front section. Click Clear Selection.
- **6** Select Point 3 only.

|-Integral |

- I In the Physics toolbar, click _ Attributes and choose J-Integral.
- 2 Right-click J-Integral I and choose Duplicate.

J-Integral 2

- I In the Model Builder window, click J-Integral 2.
- 2 In the Settings window for J-Integral, locate the J-Integral section.
- **3** In the r_{Γ} text field, type solid.crack1.crackSize*0.7.

Crack I

In the Model Builder window, click Crack 1.

I-Integral 3

- I In the Physics toolbar, click Attributes and choose J-Integral.
- 2 In the Settings window for J-Integral, locate the J-Integral section.
- 3 From the Integration path list, choose On edges.
- **4** Select Point 3 only.
- 5 Locate the Integration Path section. Click to select the Im Activate Selection toggle button.
- **6** Select Boundaries 1, 3, and 5 only.

To compute the energy release rate, add a Virtual Crack Extension feature. Use the action buttons in the feature to generate the deformed geometry and study.

Crack I

In the Model Builder window, click Crack 1.

Virtual Crack Extension I

- In the Physics toolbar, click Attributes and choose Virtual Crack Extension.

 To create a parametric study, use options in the feature's Advanced section. To see this section, activate advanced physics settings as follows.
- 2 Click the Show More Options button in the Model Builder toolbar.
- **3** In the **Show More Options** dialog box, in the tree, select the check box for the node **Physics>Advanced Physics Options**.
- 4 Click OK.
- 5 In the Settings window for Virtual Crack Extension, click to expand the Advanced section.
- 6 From the Add parametric sweep list, choose Yes.
- 7 In the Parameters table, enter the following settings:

Index	Parameter name	Parameter value list	Parameter unit
1	Xa	range(0.5,da,0.7)	m

8 Click Automated Model Setup in the upper-right corner of the Sensitivity section. From the menu, choose Create Deformed Geometry and Study.

MESH I

- I In the Model Builder window, under Component I (compl) click Mesh I.
- 2 In the Settings window for Mesh, locate the Physics-Controlled Mesh section.
- 3 From the Element size list, choose Finer.
- **4** Locate the **Sequence Type** section. From the list, choose **User-controlled mesh**.

Size 1

- I In the Model Builder window, under Component I (compl)>Mesh I click Size I.
- 2 In the Settings window for Size, locate the Element Size section.
- **3** Click the **Custom** button.
- 4 Locate the Element Size Parameters section.
- 5 Select the Maximum element size check box. In the associated text field, type 0.01.
- 6 Click III Build All.

VIRTUAL CRACK EXTENSION STUDY

- I In the Home toolbar, click **Compute**.
- 2 Click the **Zoom Extents** button in the **Graphics** toolbar.

RESULTS

Mirror 2D I

- I In the Results toolbar, click More Datasets and choose Mirror 2D.
- 2 In the Settings window for Mirror 2D, locate the Axis Data section.
- 3 In row Point 2, set X to 1.
- 4 In row Point 2, set Y to 0.
- 5 Locate the Data section. From the Dataset list, choose Virtual Crack Extension Study/ Solution 2 (solidcrack I solp).

Stress (solid)

- I In the Model Builder window, under Results click Stress (solid).
- 2 In the Settings window for 2D Plot Group, locate the Data section.
- 3 From the Dataset list, choose Mirror 2D 1.
- 4 From the Parameter value (Xa (m)) list, choose 0.6.
- 5 Locate the Plot Settings section. Clear the Plot dataset edges check box.

Surface I

- I In the Model Builder window, expand the Stress (solid) node, then click Surface I.
- 2 In the Settings window for Surface, locate the Expression section.
- 3 From the Unit list, choose MPa.
- 4 Click to expand the Range section. Select the Manual color range check box.
- 5 In the Maximum text field, type 200.

Stress (solid)

Right-click Results>Stress (solid)>Surface I and choose Arrow Line.

Boundary Load

- I In the Settings window for Arrow Line, type Boundary Load in the Label text field.
- 2 Click Replace Expression in the upper-right corner of the Expression section. From the menu, choose Component I (compl)>Solid Mechanics>Load>solid.F_Ax,solid.F_Ay -Load (spatial frame).
- 3 Click to expand the **Title** section. From the **Title type** list, choose **None**.
- 4 Locate the Arrow Positioning section. From the Placement list, choose Mesh nodes.
- 5 Locate the Coloring and Style section.
- 6 Select the Scale factor check box. In the associated text field, type 1e-8.

- 7 Click to expand the Inherit Style section. From the Plot list, choose Surface 1.
- 8 Clear the Color and data range check box.
- **9** Clear the **Color** check box.
- 10 Clear the Arrow scale factor check box.

Color Expression 1

- I Right-click Boundary Load and choose Color Expression.
- 2 In the Settings window for Color Expression, click Replace Expression in the upper-right corner of the Expression section. From the menu, choose Component I (compl)> Solid Mechanics>Load>solid.F_A_Mag Load magnitude N/m².
- 3 Locate the Coloring and Style section. From the Coloring list, choose Gradient.
- **4** From the **Top color** list, choose **Red**.
- 5 From the Bottom color list, choose Gray.
- **6** Clear the **Color legend** check box.

Deformation I

Right-click Boundary Load and choose Deformation.

Line 1

- I In the Stress (solid) toolbar, click Line.
- 2 In the Settings window for Line, locate the Expression section.
- **3** In the **Expression** text field, type 1.
- 4 Click to expand the **Title** section. From the **Title type** list, choose **None**.
- 5 Locate the Coloring and Style section. From the Coloring list, choose Uniform.
- **6** From the **Color** list, choose **From theme**.
- 7 Click to expand the Quality section. From the Smoothing list, choose None.

Deformation I

In the Stress (solid) toolbar, click Tolormation.

Line 1

- I In the Model Builder window, click Line I.
- 2 Click to expand the Inherit Style section. From the Plot list, choose Surface 1.
- **3** Clear the **Color** check box.
- 4 Clear the Color and data range check box.

Deformation

- I In the Model Builder window, expand the Results>Stress (solid)>Surface I node, then click **Deformation**.
- 2 In the Settings window for Deformation, locate the Scale section.
- 3 Select the Scale factor check box. In the associated text field, type 200.
- 4 In the Stress (solid) toolbar, click **Plot**.
- 5 Click the **Zoom Extents** button in the **Graphics** toolbar.

Check the placement of integration contours. In particular, ensure that circular paths remain inside the domain.

ADD PREDEFINED PLOT

- I In the Home toolbar, click Add Predefined Plot to open the Add Predefined Plot window.
- 2 Go to the Add Predefined Plot window.
- 3 In the tree, select Virtual Crack Extension Study/Solution 2 (solidcrack Isolp)> Solid Mechanics>Cracks (solid).
- 4 Click Add Plot in the window toolbar.

RESULTS

Cracks (solid)

- I In the Cracks (solid) toolbar, click **Plot**.
- 2 In the Settings window for 2D Plot Group, click **cycle_plot_level**.
- 3 Click ► Plot First.

Check if there are any differences between the results for the three contours by adding predefined evaluation group. That could indicate a too coarse mesh.

ADD PREDEFINED PLOT

- I Go to the Add Predefined Plot window.
- 2 In the tree, select Virtual Crack Extension Study/Solution 2 (solidcrack Isolp)> Solid Mechanics>Fracture Mechanics Results (solid).
- 3 Click Add Plot in the window toolbar.
- 4 In the Home toolbar, click Add Predefined Plot to close the Add Predefined Plot window.

Since this is a pure Mode I case, remove the evaluation of the Mode II stress intensity factors

RESULTS

Stress Intensity Factors, Mode 2

- I In the Model Builder window, expand the Results>Fracture Mechanics Results (solid) node.
- 2 Right-click Stress Intensity Factors, Mode 2 and choose Delete.

Stress Intensity Factors, Mode I

Add a comparison between the values from the second integration path and the reference stress intensity factor.

- I In the Model Builder window, under Results>Fracture Mechanics Results (solid) click Stress Intensity Factors, Mode I.
- 2 In the Settings window for Global Evaluation, locate the Expressions section.
- **3** In the table, enter the following settings:

Expression	Unit	Description
K1r	1	Reference stress intensity factor
(solid.crack1.jint2.KI-K1r)/ K1r*100	1	Percent difference from reference value

Fracture Mechanics Results (solid)

- I In the Model Builder window, click Fracture Mechanics Results (solid).
- 2 In the Settings window for Evaluation Group, locate the Data section.
- 3 From the Parameter selection (solidcrack laux Para) list, choose Last.
- 4 In the Fracture Mechanics Results (solid) toolbar, click **= Evaluate**.

Compare the J-integral with the energy release rate.

J-Integral and Energy Release Rate

- I In the Home toolbar, click Add Plot Group and choose ID Plot Group.
- 2 In the Settings window for ID Plot Group, type J-Integral and Energy Release Rate in the Label text field.
- 3 Locate the Data section. From the Dataset list, choose Virtual Crack Extension Study/ Solution 2 (solidcrack I solp).
- 4 From the Parameter selection (Xa) list, choose Manual.

- 5 In the Parameter indices (1-21) text field, type range (2,20).
- 6 From the Parameter selection (solidcrack laux Para) list, choose Last.

Global I

- I Right-click J-Integral and Energy Release Rate and choose Global.
- 2 In the Settings window for Global, click Replace Expression in the upper-right corner of the y-Axis Data section. From the menu, choose Component I (compl)>Solid Mechanics> Cracks>solid.crack | .jint | .J - J-integral - J/m2.
- **3** Locate the **y-Axis Data** section. In the table, enter the following settings:

Expression	Unit	Description
solid.crack1.jint1.J	J/m^2	J-integral, contour 1
solid.crack1.vce1.G	J/m^2	Energy release rate

- 4 Locate the x-Axis Data section. From the Axis source data list, choose Xa.
- 5 Click to expand the Coloring and Style section. Find the Line style subsection. From the Line list, choose Cycle.
- **6** From the **Width** list, choose **2**.
- 7 Find the Line markers subsection. From the Marker list, choose Cycle.
- 8 From the Positioning list, choose Interpolated.
- 9 Click to expand the Legends section. Find the Include subsection. Clear the Solution check box.

J-Integral and Energy Release Rate

- I In the Model Builder window, click J-Integral and Energy Release Rate.
- 2 In the Settings window for ID Plot Group, locate the Plot Settings section.
- 3 Select the x-axis label check box. In the associated text field, type Crack length (m).
- 4 Select the y-axis label check box. In the associated text field, type Energy release rate (J/m^2) .
- 5 Locate the Legend section. From the Position list, choose Lower right.

Crack Growth Rate

- I In the Home toolbar, click **Add Plot Group** and choose ID Plot Group.
- 2 In the Settings window for ID Plot Group, type Crack Growth Rate in the Label text field.

- 3 Locate the Data section. From the Dataset list, choose Virtual Crack Extension Study/ Solution 2 (solidcrack I solp).
- 4 From the Parameter selection (solidcrack laux Para) list, choose Last.

Global I

- I Right-click Crack Growth Rate and choose Global.
- 2 In the Settings window for Global, locate the y-Axis Data section.
- **3** In the table, enter the following settings:

Expression	Unit	Description
dadN	1	Crack growth rate (m/cycle)

- 4 Locate the x-Axis Data section. From the Axis source data list, choose Xa.
- 5 Locate the Coloring and Style section. From the Width list, choose 2.

Crack Growth Rate

- I In the Model Builder window, click Crack Growth Rate.
- 2 In the Settings window for ID Plot Group, click to expand the Title section.
- **3** From the **Title type** list, choose **None**.
- 4 Locate the **Plot Settings** section.
- 5 Select the x-axis label check box. In the associated text field, type Crack length (m).
- 6 Select the **y-axis label** check box. In the associated text field, type Crack growth rate (m/cycle).
- 7 Locate the **Legend** section. Clear the **Show legends** check box.
- 8 In the Crack Growth Rate toolbar, click Plot.

 Compute the total number of cycles needed for driving the crack from 0.5 m to 0.7 m.

Number of cycles

- I In the Results toolbar, click (8.5) Global Evaluation.
- 2 In the Settings window for Global Evaluation, type Number of cycles in the Label text field.
- 3 Locate the Data section. From the Dataset list, choose Virtual Crack Extension Study/ Solution 2 (solidcrack I solp).
- **4** Locate the **Expressions** section. In the table, enter the following settings:

Expression	Unit	Description
1/dadN	1	

- **5** Locate the **Data Series Operation** section. From the **Transformation** list, choose **Integral**.
- 6 Click **= Evaluate**.