Nonlinear Acoustics — Modeling of the 1D Westervelt Equation

Introduction

Although linear acoustics does an outstanding job of explaining most acoustical phenomena, there are an increasing number of applications that require abandoning the small-signal assumption and applying the wave equation for finite-amplitude sound or nonlinear acoustics. For example, the effects of finite amplitude propagation can be seen in almost every medical use of ultrasound. The use of high intensity ultrasound to induce tissue heating for cancer therapy or bleeding control, and the use of shock waves in extracorporeal and endoscopic lithotripsy of kidney stones and gallstones provide additional examples. Even in the field of diagnostic ultrasound, nonlinear acoustics is of interest because the higher harmonics emerged during wave propagation can be exploited to produce a better image quality.

The nonlinear phenomenon of wave distortion is both a local effect and a cumulative effect due to variation of propagation speed over the waveform, which causes distortion that accumulates with propagation distance. The local effect is usually small compared to cumulative distortion and can be neglected once the propagation distance becomes much greater than a wavelength; see Ref. 1. Therefore, a transient analysis is necessary to model the cumulative distortion along with the wave propagation.

This example shows how to model nonlinear propagation of finite-amplitude acoustic waves in fluids using the Pressure Acoustics, Transient physics interface of the Acoustics Module. The nonlinear effects are taken into account by adding the Nonlinear Acoustics (Westervelt) domain feature. Thus the linear wave equation transforms into the Westervelt equation which is an approximation of the full 2nd-order wave equation when cumulative nonlinear effects dominate local nonlinear effects; see Ref. 1. The current model simulates a finite-amplitude wave propagating in 1D along an interval greater than the shock formation distance. The computed numerical solution is compared to an analytical solution before and after shock formation.

Model Definition

The full Westervelt equation reads

$$\frac{1}{\rho_0 c_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla\cdot\left(\frac{1}{\rho_0}\left(\nabla p + \frac{\delta}{c_0^2}\frac{\partial(\nabla p)}{\partial t}\right)\right) = \frac{\beta}{\rho_0^2 c_0^4}\frac{\partial^2 p^2}{\partial t^2} \qquad (1)$$

where p is the total acoustic pressure, ρ_0 and c_0 are the density and the speed of sound, $\beta = 1 + B/2A$ is the coefficient of nonlinearity, and δ is the sound diffusivity (see Ref. 2). This is the equation solved when the Nonlinear Acoustics (Westervelt) domain feature is added and the **General dissipation** is selected as **Fluid model** is selected on the main Transient Pressure Acoustics Model node.

For a piston vibrating in a 1D tube with the velocity $u(t) = u_0 \sin\omega t$, the so-called shock formation distance is

$$x_{\mathrm{sh}} = \frac{1}{\beta\varepsilon k} = \frac{c_0^2}{\omega\beta u_0},$$

There are two classical analytical solutions to Equation 1 available in this case (see Ref. 1). Both of them have the form of a series

$$p(x,t) = p_0 \sum_{n=1}^{\infty} B_n(\sigma)\sin\left(n\omega\left(t-\frac{x}{c}\right)\right), \tag{2}$$

where $\sigma = x/x_{\mathrm{sh}}$ is the dimensionless spatial coordinate and $p_0 = \rho_0 c_0 u_0$ is the source pressure amplitude.

The first one is known as the Fubini solution and is valid for the distances up to x_{sh} ($\sigma \leq 1$). The harmonic amplitudes B_{n} are defined as

$$B_n(\sigma) = \frac{2}{n\sigma} J_n(n\sigma),$$

with J_n being the Bessel function of the first kind of order n. The second one is known as the Fay solution and it is applied for $\sigma \geq 3.5$. The harmonic amplitudes are

$$B_n(\sigma) = \frac{2}{\Gamma \sinh[n(1+\sigma)/\Gamma]}$$

with $\Gamma = 2\beta u_0/\omega\delta$ is the Goldberg number, which is a measure of the strength of nonlinearity relative to that of dissipation (see Ref. 1). A solution provided by Blackstock exists in the transition region for $1 \leq \sigma \leq 3.5$, but this solution is not considered here.

Table 1 shows the material properties of water and some critical parameters used in the simulation. Given those numbers, the shock formation distance for the current model is about 0.1 m. The simulation domain is the interval $0 \leq x \leq 4.5x_{\mathrm{sh}}$, as shown in Figure 1. A sinusoidal pressure source of amplitude p_0 is applied at $x = 0$, and the Plane Wave Radiation boundary condition is applied at $x = 4.5x_{\mathrm{sh}}$ to model the propagation of the wave without reflections. The diffusivity of sound present in the model is due to viscosity and is defined as $\delta = 4\mu/3\rho_0$. It corresponds to the acoustic attenuation $\alpha = 8.1\cdot10^{-5}$ Np/m.

TABLE 1: MATERIAL PROPERTIES AND SOME CRITICAL MODEL PARAMETERS.

NAME	VALUE	DESCRIPTION
ρ_0	999.6 kg/m^3	Density at 20°C and 1 atm
c_0	1481.44 m/s	Speed of sound at 20°C and 1 atm
μ_0	$1.0016\cdot10^{-3}$ Pa·s	Viscosity at 20°C and 1 atm
β	10	Coefficient of nonlinearity
f_0	0.1 MHz	Source driving frequency
p_0	5 MPa	Source pressure amplitude
N_0	8	Number of harmonics to resolve

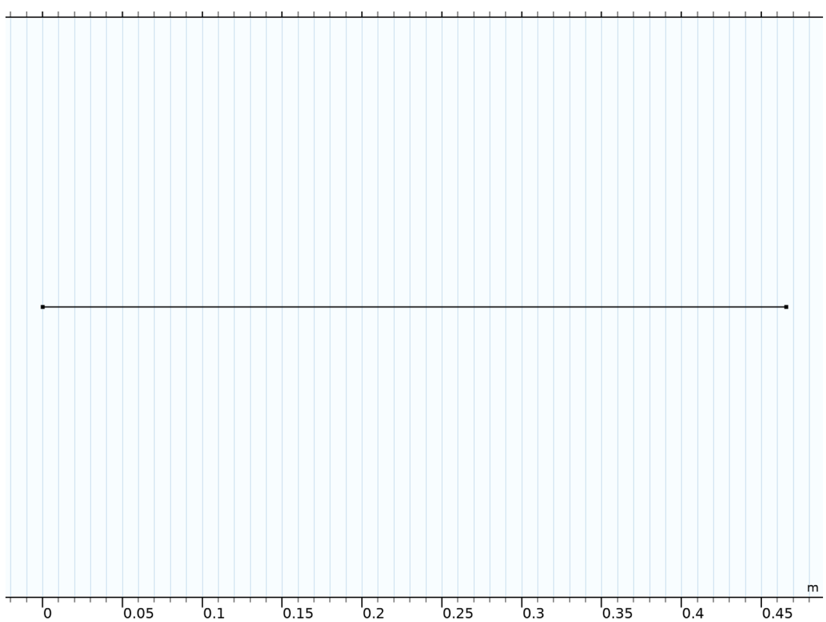


Figure 1: 1D geometry of the model with the source located at $x = 0$.

Results and Discussion

The increasing distortion of the waveform when the wave travels away from the acoustic source is illustrated in Figure 2. The nonlinear effects are apparent by comparing the solution with the linear analytical solution. Initially, the waveform distortion is caused by the dependence of propagation speed on the pressure or particle velocity. The peaks of the wave travel faster than the troughs, and the waveform gradually turns into a sawtooth-like shape clearly seen near the right end of the interval. After the shock formation (the red vertical line in Figure 2) the sound dissipation grows. This is due to the intrinsic frequency dependency of the attenuation. It is proportional to the frequency squared.

The distortion of the waveform results in the generation of higher harmonic components. The further the wave travels, the more energy is transferred to the higher harmonic components from the fundamental frequency of the harmonic source signal. This effect is demonstrated in the plots of Figure 3 through Figure 5. The plots compare the model solution (blue lines) with the analytical solutions (green line) for both waveform and frequency spectrum at $x = 0.5x_{sh}$, x_{sh}, and $3.5x_{sh}$. The Fourier transforms used to determine the spectrum are applied for the time interval $\Delta t = 5T_0$ after the wave arrives at those locations.

At $x = 0.5x_{sh}$ in Figure 3, the 2nd and the 3rd harmonic components start to show up; at $x = x_{sh}$ in Figure 4, more than ten harmonics appear in the frequency spectrum. Their contribution grows with the distance as seen in Figure 5 at $x = 3.5x_{sh}$. These results clearly show how the wave deforms when it travels away from the source and how the acoustic energy is pumped into higher harmonics from the fundamental frequency.

Note: The Fubini solution is a solution to Equation 1 with no dissipation ($\delta = 0$). Therefore, the difference in amplitudes of the numerical and the analytical solution grows as x comes closer to x_{sh} (compare plots in Figure 3 and Figure 4).

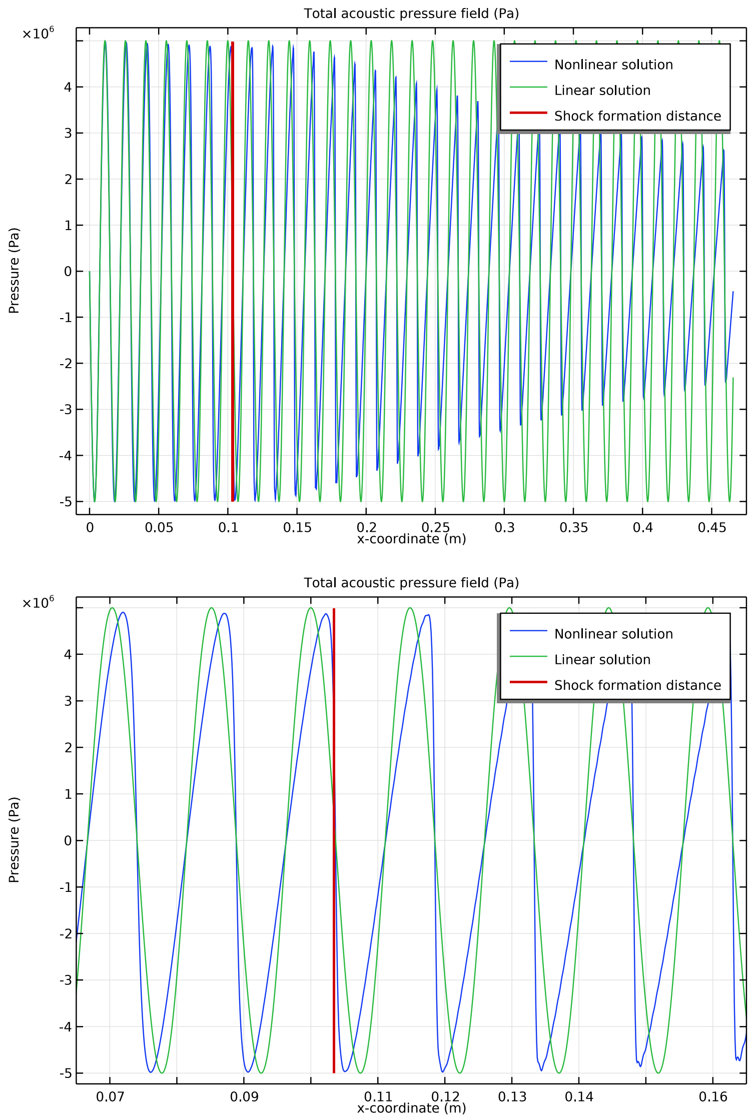


Figure 2: Comparison of the nonlinear numerical solution (blue) with the linear analytical solution (green): the full propagation domain in the top plot and the area around the shock formation distance in the bottom plot. The red vertical line indicates the shock formation distance.

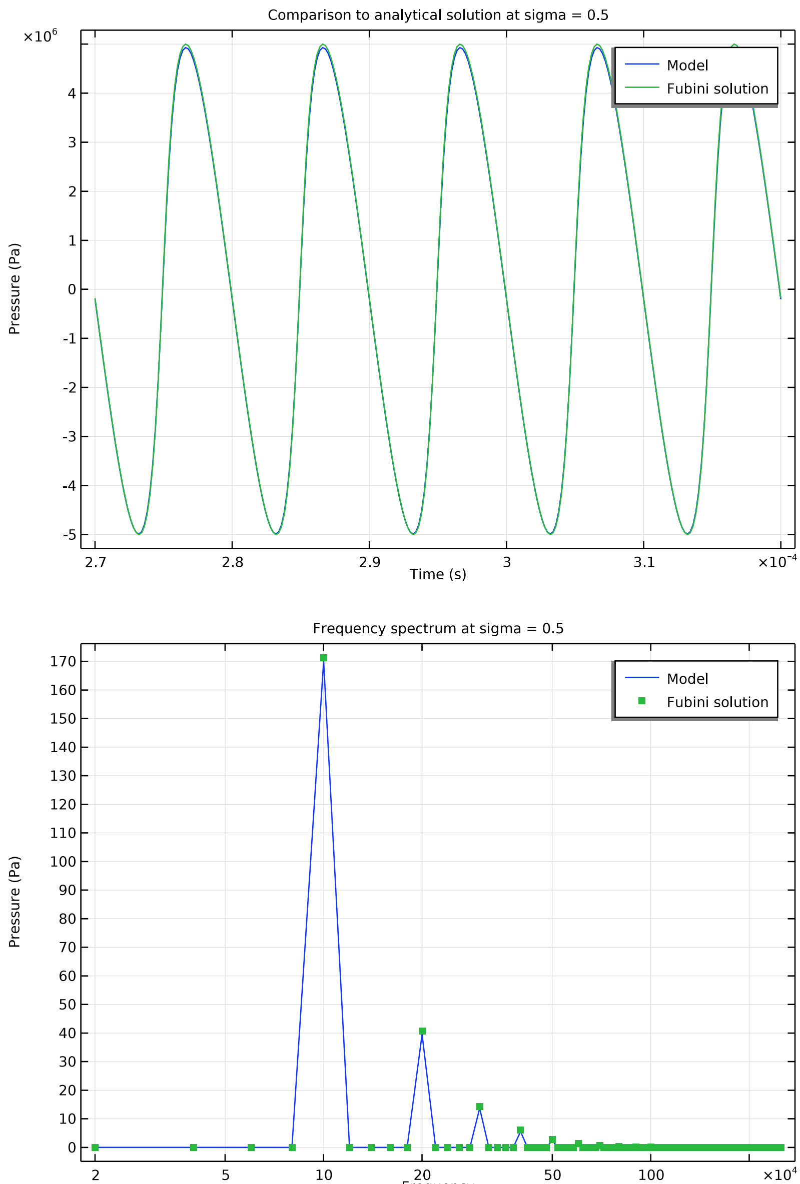
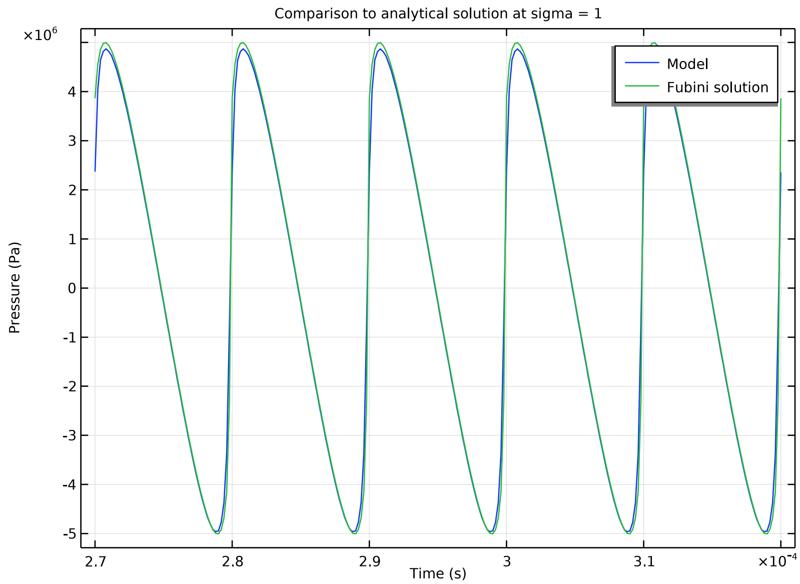


Figure 3: Comparison of model solution (blue) with Fubini nonlinear analytical solution (green) at $x = 0.5x_{\mathrm{sh}}$. The top plot shows the pressure profiles and the bottom plot shows the frequency spectra.



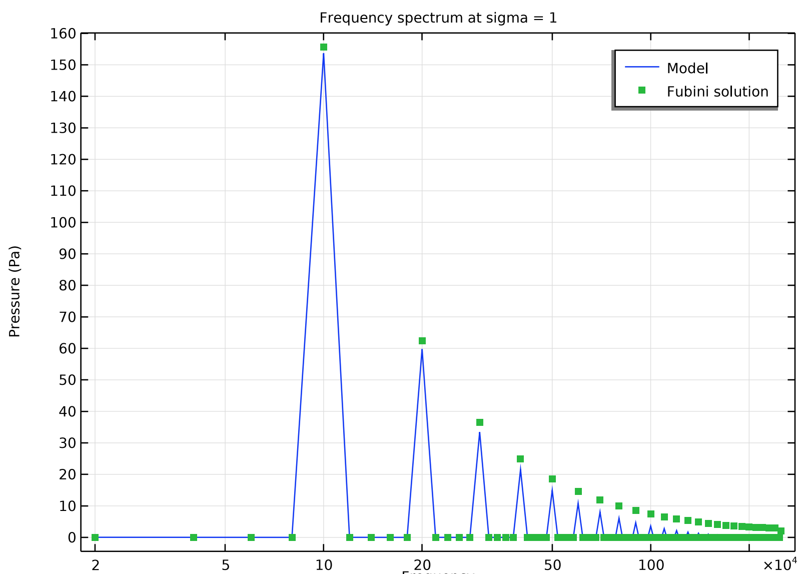


Figure 4: Comparison of model solution (blue) with Fubini nonlinear analytical solution (green) at $x = x_{sh}$. The top plot shows the pressure profiles and the bottom plot shows the frequency spectra.

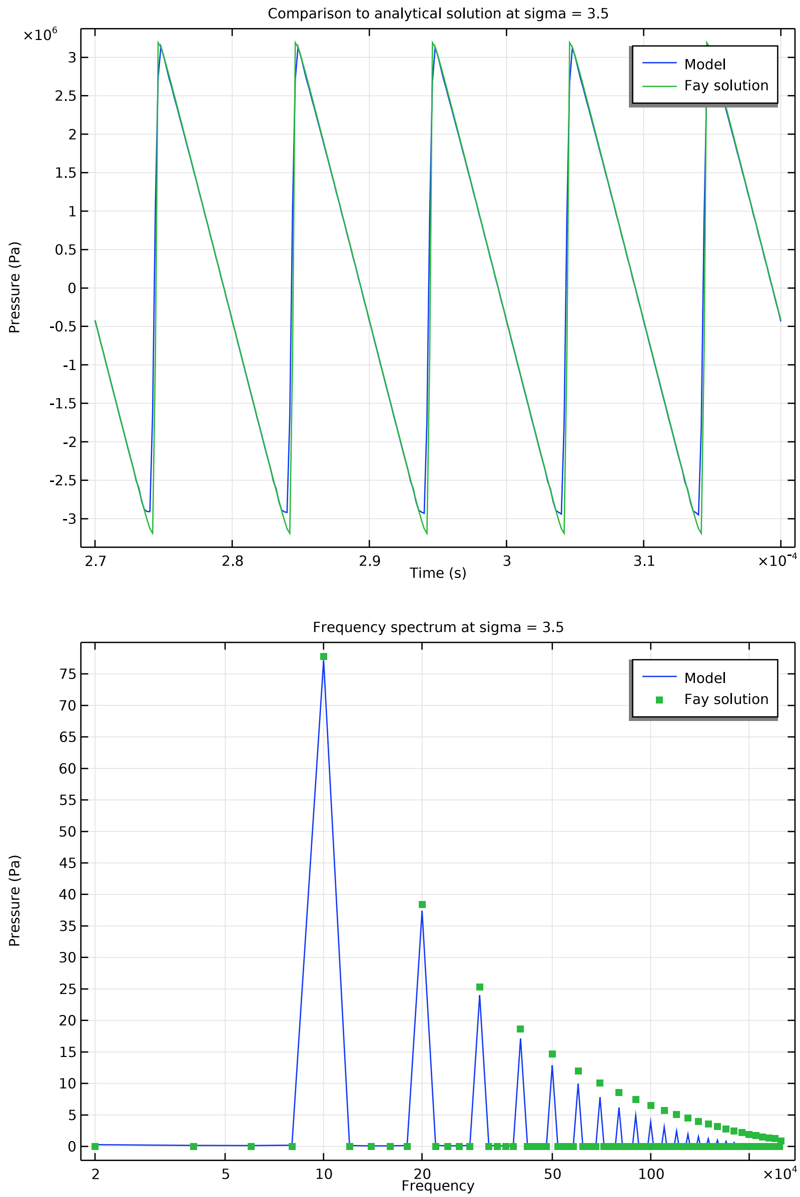


Figure 5: Comparison of model solution (blue) with Fay nonlinear analytical solution (green) at $x = 3.5x_{\mathrm{sh}}$. The top plot shows the pressure profiles and the bottom plot shows the frequency spectra.

MESH AND FINITE ELEMENT DISCRETIZATION

The mesh is required to resolves the frequency content of the signal. That means resolving higher harmonics along the wave propagation direction. The specified number of harmonics to resolve N_0, should therefore contribute to the mesh element size. To accurately resolve the acoustic pressure, use at least second-order (quadratic) elements. Then the mesh element size is defined as $dx = c_0/(6N_0f_0)$.

TIME STEPPING

The time stepping depends on the maximum frequency to resolve. It is specified in **Transient Solver Settings** section at the top physics level. In the model it is selected to resolve the desired number of harmonics as $f_{\mathrm{max}} = N_0f_0$. The Generalized alpha time stepping method generated as the default transient solver will automatically use an appropriate time step to resolve up to f_{max}.

Note that when the Nonlinear Acrostics (Westervelt) feature is used the default solver should be regenerated if, for example, a linear model was solved previously. This is to ensure that a proper solve is set up. The addition of the nonlinear feature will trigger the Automatic (Newton) method for solving the boundary value problem.

SHOCK-CAPTURING STABILIZATION

Since the model discusses a subject of nonlinear wave propagation beyond the shock-formation distance, a special treatment is required to resolve the discontinuities of the shock. The Nonlinear Acoustics (Westervelt) feature contains a built-in Shock-Capturing Stabilization technique available when **Stabilization** is enabled in the **Show** view. The stabilization is turned off per default as it requires manual tuning.

Whenever the **Enable q-Laplacian relaxation** is enabled, an extra nonlinear term

$$-\delta\kappa\left(1 - \left|\frac{\partial}{\partial t}\nabla p\right|^{q-1}\right)$$

is added to the sound diffusivity, δ. This nonlinear term introduces additional dissipation that is maximal where the acoustic pressure endures discontinuities, that is, at shocks. The parameters κ and q must be tuned so as not to introduce too much or too little dissipation. The choice depends on the material properties and the frequency of the input signal. In this model, $\kappa = 0.01$ and $q = 1.35$. This simple 1D model can be used to tune the stabilization parameters (for other materials and frequencies) for higher dimension models as it is relatively fast to solve.

References

1. M.F. Hamilton and D.T. Blackstock, eds., *Nonlinear Acoustics*, Academic Press, San Diego, CA, 1998.

2. D.T. Blackstock, *Fundamentals of Physical Acoustics*, John Wiley & Sons, 2000.

Application Library path: `Acoustics_Module/Nonlinear_Acoustics/nonlinear_acoustics_westervelt_1d`

Modeling Instructions

From the **File** menu, choose **New**.

NEW

In the **New** window, click **Model Wizard**.

MODEL WIZARD

1 In the **Model Wizard** window, click **1D**.

2 In the **Select Physics** tree, select **Acoustics>Pressure Acoustics>Pressure Acoustics, Transient (actd)**.

3 Click **Add**.

4 Click **Study**.

5 In the **Select Study** tree, select **General Studies>Time Dependent**.

6 Click **Done**.

GLOBAL DEFINITIONS

Load the parameters used in the model from a file. Some of the parameters are presented in Table 1.

Parameters 1

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.

2 In the **Settings** window for **Parameters**, locate the **Parameters** section.

3 Click **Load from File**.

4 Browse to the model's Application Libraries folder and double-click the file `nonlinear_acoustics_westervelt_1d_parameters.txt`.

Now, define the amplitudes B_n used in the Fubini and Fay analytical solutions to Equation 1.

Analytic 1 (an1)

1 In the **Home** toolbar, click f(X) **Functions** and choose **Global>Analytic**.

2 In the **Settings** window for **Analytic**, type `Pn_fubini` in the **Function name** text field.

3 Locate the **Definition** section. In the **Expression** text field, type `1/n*besselj(n, n*sigma)*sin(n*omega0*(t - sigma*x_sh/c0))`.

4 In the **Arguments** text field, type `sigma, t, n`.

5 Locate the **Units** section. In the table, enter the following settings:

Argument	Unit
sigma	1
t	s
n	1

6 In the **Function** text field, type `1`.

Analytic 2 (an2)

1 In the **Home** toolbar, click f(X) **Functions** and choose **Global>Analytic**.

2 In the **Settings** window for **Analytic**, type `Pn_fay` in the **Function name** text field.

3 Locate the **Definition** section. In the **Expression** text field, type `1/sinh(n*(sigma + 1)/Gamma)*sin(n*omega0*(t - sigma*x_sh/c0))`.

4 In the **Arguments** text field, type `sigma, t, n`.

5 Locate the **Units** section. In the table, enter the following settings:

Argument	Unit
sigma	1
t	s
n	1

6 In the **Function** text field, type `1`.

Define variables for the linear and nonlinear analytical solutions to the problem. Use the `sum()` operator with 100 terms to approximate the expression given in Equation 2.

DEFINITIONS

Variables 1

1 In the **Model Builder** window, under **Component 1 (comp1)** right-click **Definitions** and choose **Variables**.

2 In the **Settings** window for **Variables**, locate the **Variables** section.

3 In the table, enter the following settings:

Name	Expression	Unit	Description
sigma	x/x_sh		Relative distance
p_fubini	2*P0/sigma* sum(Pn_fubini(sigma, t, n), n, 1, 100)	Pa	Fubini solution
p_fay	2*P0/Gamma* sum(Pn_fay(sigma, t, n), n, 1, 100)	Pa	Fay solution
p_linear	P0*sin(omega0*(t - x/ c0))	Pa	Linear solution

GEOMETRY 1

Interval 1 (i1)

1 In the **Model Builder** window, under **Component 1 (comp1)** right-click **Geometry 1** and choose **Interval**.

2 In the **Settings** window for **Interval**, locate the **Interval** section.

3 In the table, enter the following settings:

Coordinates (m)
0
L

4 Click **Build All Objects**.

The geometry should look like the one presented in Figure 1.

PRESSURE ACOUSTICS, TRANSIENT (ACTD)

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Pressure Acoustics, Transient (actd)**.

2 Locate the **Transient Solver and Mesh Settings** section. In the **Maximum frequency to resolve** field enter `N0*f0`. It will give the maximal time step for the Transient Solver required to resolve up to N_0-harmonics. It will also be used by the mesh.

Transient Pressure Acoustics Model 1

1 In the **Model Builder** window, under **Component 1 (comp1)>Pressure Acoustics, Transient (actd)** click **Transient Pressure Acoustics Model 1**.

2 In the **Settings** window for **Transient Pressure Acoustics Model**, locate the **Transient Pressure Acoustics Model** section.

3 From the **Fluid model** list, choose **General dissipation**.

4 From the c list, choose **User defined**. In the associated text field, type `c0`.

5 From the ρ list, choose **User defined**. In the associated text field, type `rho0`.

6 From the δ list, choose **User defined**. In the associated text field, type `d_diff`.

Nonlinear Acoustics (Westervelt) Contributions 1

1 In the **Physics** toolbar, click **Domains** and choose **Nonlinear Acoustics (Westervelt) Contributions**.

2 Select Domain 1 only.

3 In the **Settings** window for **Nonlinear Acoustics (Westervelt) Contributions**, locate the **Nonlinear Acoustics (Westervelt) Contributions** section.

4 From the **Specify** list, choose **Coefficient of nonlinearity**.

5 In the β text field, type `beta`.

Since the computational domain is larger than the shock formation distance, shocks form as the wave passes x_{sh}. Therefore it is required to enable the shock-capturing stabilization to resolve the shocks.

6 Click the **Show More Options** button in the **Model Builder** toolbar.

7 In the **Show More Options** dialog box, in the tree, select the check box for the node **Physics>Stabilization**.

8 Click **OK**.

9 In the **Settings** window for **Nonlinear Acoustics (Westervelt) Contributions**, click to expand the **Shock-Capturing Stabilization** section.

10 Select the **Enable q-Laplacian relaxation** check box.

11 In the q text field, type `1.35`.

12 In the κ text field, type `0.01`.

Pressure 1

1 In the **Physics** toolbar, click **Boundaries** and choose **Pressure**.

2 Select Boundary 1 only.

3 In the **Settings** window for **Pressure**, locate the **Pressure** section.

4 In the p_0 text field, type `P0*sin(omega0*t)`.

Plane Wave Radiation 1

1 In the **Physics** toolbar, click **Boundaries** and choose **Plane Wave Radiation**.

2 Select Boundary 2 only.

MESH 1

Proceed and generate the mesh using the **Physics-controlled mesh** functionality. The frequency controlling the maximum element size is per default taken **From study**, that is, from the **Maximum frequency to resolve**. In general, 5 to 6 second-order elements per wavelength are needed to resolve the waves. For more details, see *Meshing (Resolving the Waves)* in the *Acoustics Module User's Guide*. In this model, use 6 elements per wavelength; the default **Automatic** is to have 5.

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Mesh 1**.

2 In the **Settings** window for **Mesh**, locate the **Pressure Acoustics, Transient (actd)** section.

3 From the **Number of mesh elements per wavelength** list, choose **User defined**.

4 In the text field, type `6`.

5 Click **Build All**.

STUDY 1

Step 1: Time Dependent

1 In the **Model Builder** window, under **Study 1** click **Step 1: Time Dependent**.

2 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

3 In the **Output times** text field, type `range(0,T0/50,Nt*T0)`.

4 In the **Home** toolbar, click **Compute**.

RESULTS

Acoustic Pressure (actd)

1 In the **Settings** window for **1D Plot Group**, locate the **Data** section.

2 From the **Time selection** list, choose **Last**.

3 Locate the **Plot Settings** section. Select the **x-axis label** check box.

4 Select the **y-axis label** check box. In the associated text field, type `Pressure (Pa)`.

5 Click to expand the **Title** section. From the **Title type** list, choose **Manual**.

6 In the **Title** text area, type `Total acoustic pressure field (Pa)`.

Line Graph 1

1 In the **Model Builder** window, expand the **Acoustic Pressure (actd)** node, then click **Line Graph 1**.

2 In the **Settings** window for **Line Graph**, click to expand the **Legends** section.

3 Select the **Show legends** check box.

4 From the **Legends** list, choose **Manual**.

5 In the table, enter the following settings:

Legends
`Nonlinear solution`

Line Graph 2

1 In the **Model Builder** window, right-click **Acoustic Pressure (actd)** and choose **Line Graph**.

2 Select Domain 1 only.

3 In the **Settings** window for **Line Graph**, locate the **y-Axis Data** section.

4 In the **Expression** text field, type `p_linear`.

5 Locate the **Legends** section. Select the **Show legends** check box.

6 From the **Legends** list, choose **Manual**.

7 In the table, enter the following settings:

Legends
`Linear solution`

8 Locate the **x-Axis Data** section. From the **Parameter** list, choose **Expression**.

9 In the **Expression** text field, type `x`.

Line Graph 3

1 Right-click **Acoustic Pressure (actd)** and choose **Line Graph**.

2 Select Domain 1 only.

3 In the **Settings** window for **Line Graph**, locate the **x-Axis Data** section.

4 From the **Parameter** list, choose **Expression**.

5 In the **Expression** text field, type `x_sh`.

6 Click to expand the **Coloring and Style** section. From the **Width** list, choose **2**.

7 Locate the **Legends** section. Select the **Show legends** check box.

8 From the **Legends** list, choose **Manual**.

9 In the table, enter the following settings:

Legends
`Shock formation distance`

10 In the **Acoustic Pressure (actd)** toolbar, click **Plot**.

The plot should look like the top plot in Figure 2.

You can examine the plot in greater detail by zooming around parts of the plot using the **Zoom Box** tool as shown in the bottom plot in Figure 2.

Create **Cut Points** to compare the numerical solution with the analytical ones at $x = 0.5x_{sh}$, x_{sh}, and $3.5x_{sh}$.

Cut Point - 0.5 Shock

1 In the **Results** toolbar, click **More Datasets** and choose **Cut Point 1D**.

2 In the **Settings** window for **Cut Point 1D**, locate the **Point Data** section.

3 In the **X** text field, type `0.5*x_sh`.

4 In the **Label** text field, type `Cut Point - 0.5 Shock`.

Cut Point - 1 Shock

1 In the **Results** toolbar, click **More Datasets** and choose **Cut Point 1D**.

2 In the **Settings** window for **Cut Point 1D**, locate the **Point Data** section.

3 In the **X** text field, type `x_sh`.

4 In the **Label** text field, type `Cut Point - 1 Shock`.

Cut Point - 3.5 Shock

1 In the **Results** toolbar, click **More Datasets** and choose **Cut Point 1D**.

2 In the **Settings** window for **Cut Point 1D**, locate the **Point Data** section.

3 In the **X** text field, type `3.5*x_sh`.

4 In the **Label** text field, type `Cut Point - 3.5 Shock`.

Plot the numerical and the analytical solutions at **Cut Points**.

Acoustic Pressure at sigma = 0.5

1 In the **Results** toolbar, click **1D Plot Group**.

2 In the **Settings** window for **1D Plot Group**, type `Acoustic Pressure at sigma = 0.5` in the **Label** text field.

3 Locate the **Data** section. From the **Dataset** list, choose **Cut Point - 0.5 Shock**.

4 From the **Time selection** list, choose **Interpolated**.

5 In the **Times (s)** text field, type `range((Nt - 5)*T0, T0/50, Nt*T0)`.

6 Locate the **Plot Settings** section.

7 Select the **y-axis label** check box. In the associated text field, type `Pressure (Pa)`.

8 Locate the **Title** section. From the **Title type** list, choose **Manual**.

9 In the **Title** text area, type `Comparison to analytical solution at sigma = 0.5`.

Point Graph 1

1 Right-click **Acoustic Pressure at sigma = 0.5** and choose **Point Graph**.

2 In the **Settings** window for **Point Graph**, click to expand the **Legends** section.

3 Select the **Show legends** check box.

4 From the **Legends** list, choose **Manual**.

5 In the table, enter the following settings:

Legends
`Model`

Point Graph 2

1 In the **Model Builder** window, right-click **Acoustic Pressure at sigma = 0.5** and choose **Point Graph**.

2 In the **Settings** window for **Point Graph**, locate the **y-Axis Data** section.

3 In the **Expression** text field, type `p_fubini`.

4 Locate the **Legends** section. Select the **Show legends** check box.

5 From the **Legends** list, choose **Manual**.

6 In the table, enter the following settings:

Legends
`Fubini solution`

7 In the **Acoustic Pressure at sigma = 0.5** toolbar, click **Plot**.

The plot should look like the top plot in Figure 3.

Acoustic Pressure at sigma = 1

1 In the **Home** toolbar, click **Add Plot Group** and choose **1D Plot Group**.

2 In the **Settings** window for **1D Plot Group**, type `Acoustic Pressure at sigma = 1` in the **Label** text field.

3 Locate the **Data** section. From the **Dataset** list, choose **Cut Point - 1 Shock**.

4 From the **Time selection** list, choose **Interpolated**.

5 In the **Times (s)** text field, type `range((Nt - 5)*T0, T0/50, Nt*T0)`.

6 Locate the **Plot Settings** section.

7 Select the **y-axis label** check box. In the associated text field, type `Pressure (Pa)`.

8 Locate the **Title** section. From the **Title type** list, choose **Manual**.

9 In the **Title** text area, type `Comparison to analytical solution at sigma = 1`.

Point Graph 1

1 Right-click **Acoustic Pressure at sigma = 1** and choose **Point Graph**.

2 In the **Settings** window for **Point Graph**, locate the **Legends** section.

3 Select the **Show legends** check box.

4 From the **Legends** list, choose **Manual**.

5 In the table, enter the following settings:

Legends
Model

Point Graph 2

1 In the **Model Builder** window, right-click **Acoustic Pressure at sigma = 1** and choose **Point Graph**.

2 In the **Settings** window for **Point Graph**, locate the **y-Axis Data** section.

3 In the **Expression** text field, type `p_fubini`.

4 Locate the **Legends** section. Select the **Show legends** check box.

5 From the **Legends** list, choose **Manual**.

6 In the table, enter the following settings:

Legends
Fubini solution

7 In the **Acoustic Pressure at sigma = 1** toolbar, click **Plot**.

The plot should look like the top plot in Figure 4.

Acoustic Pressure at sigma = 3.5

1 In the **Home** toolbar, click **Add Plot Group** and choose **1D Plot Group**.

2 In the **Settings** window for **1D Plot Group**, type `Acoustic Pressure at sigma = 3.5` in the **Label** text field.

3 Locate the **Data** section. From the **Dataset** list, choose **Cut Point - 3.5 Shock**.

4 From the **Time selection** list, choose **Interpolated**.

5 In the **Times (s)** text field, type `range((Nt - 5)*T0, T0/50, Nt*T0)`.

6 Locate the **Plot Settings** section.

7 Select the **y-axis label** check box. In the associated text field, type `Pressure (Pa)`.

8 Locate the **Title** section. From the **Title type** list, choose **Manual**.

9 In the **Title** text area, type `Comparison to analytical solution at sigma = 3.5`.

Point Graph 1

1 Right-click **Acoustic Pressure at sigma = 3.5** and choose **Point Graph**.

2 In the **Settings** window for **Point Graph**, locate the **Legends** section.

3 Select the **Show legends** check box.

4 From the **Legends** list, choose **Manual**.

5 In the table, enter the following settings:

Legends
`Model`

Point Graph 2

1 In the **Model Builder** window, right-click **Acoustic Pressure at sigma = 3.5** and choose **Point Graph**.

2 In the **Settings** window for **Point Graph**, locate the **y-Axis Data** section.

3 In the **Expression** text field, type `p_fay`.

4 Locate the **Legends** section. Select the **Show legends** check box.

5 From the **Legends** list, choose **Manual**.

6 In the table, enter the following settings:

Legends
`Fay solution`

7 In the **Acoustic Pressure at sigma = 3.5** toolbar, click **Plot**.

The plot should look like the top plot in Figure 5.

Acoustic Pressure at sigma = 0.5

In the **Model Builder** window, under **Results** right-click **Acoustic Pressure at sigma = 0.5** and choose **Duplicate**.

Acoustic Pressure Spectrum at sigma = 0.5

1 In the **Model Builder** window, under **Results** click **Acoustic Pressure at sigma = 0.5.1**.

2 In the **Settings** window for **1D Plot Group**, type `Acoustic Pressure Spectrum at sigma = 0.5` in the **Label** text field.

3 Locate the **Title** section. In the **Title** text area, type `Frequency spectrum at sigma = 0.5`.

4 Click the **x-Axis Log Scale** button in the **Graphics** toolbar.

Point Graph 1

1 In the **Model Builder** window, expand the **Acoustic Pressure Spectrum at sigma = 0.5** node, then click **Point Graph 1**.

2 In the **Settings** window for **Point Graph**, locate the **x-Axis Data** section.

3 From the **Parameter** list, choose **Discrete Fourier transform**.

4 From the **Show** list, choose **Frequency spectrum**.

5 From the **Scale** list, choose **Multiply by sampling period**.

Point Graph 2

1 In the **Model Builder** window, click **Point Graph 2**.

2 In the **Settings** window for **Point Graph**, locate the **x-Axis Data** section.

3 From the **Parameter** list, choose **Discrete Fourier transform**.

4 From the **Show** list, choose **Frequency spectrum**.

5 From the **Scale** list, choose **Multiply by sampling period**.

6 Click to expand the **Coloring and Style** section. Find the **Line style** subsection. From the **Line** list, choose **None**.

7 Find the **Line markers** subsection. From the **Marker** list, choose **Point**.

8 In the **Acoustic Pressure Spectrum at sigma = 0.5** toolbar, click **Plot**.

The plot should look like the bottom plot in Figure 3.

Acoustic Pressure at sigma = 1

In the **Model Builder** window, under **Results** right-click **Acoustic Pressure at sigma = 1** and choose **Duplicate**.

Acoustic Pressure Spectrum at sigma = 1

1 In the **Model Builder** window, under **Results** click **Acoustic Pressure at sigma = 1.1**.

2 In the **Settings** window for **1D Plot Group**, type `Acoustic Pressure Spectrum at sigma = 1` in the **Label** text field.

3 Locate the **Title** section. In the **Title** text area, type `Frequency spectrum at sigma = 1`.

4 Click the **x-Axis Log Scale** button in the **Graphics** toolbar.

Point Graph 1

1 In the **Model Builder** window, expand the **Acoustic Pressure Spectrum at sigma = 1** node, then click **Point Graph 1**.

2 In the **Settings** window for **Point Graph**, locate the **x-Axis Data** section.

3 From the **Parameter** list, choose **Discrete Fourier transform**.

4 From the **Show** list, choose **Frequency spectrum**.

5 From the **Scale** list, choose **Multiply by sampling period**.

Point Graph 2

1 In the **Model Builder** window, click **Point Graph 2**.

2 In the **Settings** window for **Point Graph**, locate the **x-Axis Data** section.

3 From the **Parameter** list, choose **Discrete Fourier transform**.

4 From the **Show** list, choose **Frequency spectrum**.

5 From the **Scale** list, choose **Multiply by sampling period**.

6 Locate the **Coloring and Style** section. Find the **Line style** subsection. From the **Line** list, choose **None**.

7 Find the **Line markers** subsection. From the **Marker** list, choose **Point**.

8 In the **Acoustic Pressure Spectrum at sigma = 1** toolbar, click **Plot**.

The plot should look like the bottom plot in Figure 4.

Acoustic Pressure at sigma = 3.5

In the **Model Builder** window, under **Results** right-click **Acoustic Pressure at sigma = 3.5** and choose **Duplicate**.

Acoustic Pressure Spectrum at sigma = 3.5

1 In the **Model Builder** window, under **Results** click **Acoustic Pressure at sigma = 3.5.1**.

2 In the **Settings** window for **1D Plot Group**, type `Acoustic Pressure Spectrum at sigma = 3.5` in the **Label** text field.

3 Locate the **Title** section. In the **Title** text area, type `Frequency spectrum at sigma = 3.5`.

4 Click the **x-Axis Log Scale** button in the **Graphics** toolbar.

Point Graph 1

1. In the **Model Builder** window, expand the **Acoustic Pressure Spectrum at sigma = 3.5** node, then click **Point Graph 1**.
2. In the **Settings** window for **Point Graph**, locate the **x-Axis Data** section.
3. From the **Parameter** list, choose **Discrete Fourier transform**.
4. From the **Show** list, choose **Frequency spectrum**.
5. From the **Scale** list, choose **Multiply by sampling period**.

Point Graph 2

1. In the **Model Builder** window, click **Point Graph 2**.
2. In the **Settings** window for **Point Graph**, locate the **x-Axis Data** section.
3. From the **Parameter** list, choose **Discrete Fourier transform**.
4. From the **Show** list, choose **Frequency spectrum**.
5. From the **Scale** list, choose **Multiply by sampling period**.
6. Locate the **Coloring and Style** section. Find the **Line style** subsection. From the **Line** list, choose **None**.
7. Find the **Line markers** subsection. From the **Marker** list, choose **Point**.
8. In the **Acoustic Pressure Spectrum at sigma = 3.5** toolbar, click **Plot**.

   The plot should look like the bottom plot in Figure 5.