Ion Cyclotron Motion

Introduction

If a charged particle is placed in a uniform magnetic field, it moves in a helical pattern about a fixed gyro radius. The gyro radius, which is also known as the Larmor or cyclotron radius, is given by the simple equation:

$$r_{\mathrm{L}} = \frac{mv_{\perp}}{ZeB}$$

- r_{L} (SI unit: m) is the Larmor radius,
- $v_{\perp}$ (SI unit: m/s) is the velocity component orthogonal to the magnetic field,
- Z (dimensionless) is the particle charge number,
- $e = 1.602176634 \times 10^{-19}$ C is the elementary charge,
- m (SI unit: kg) is the particle mass, and
- B (SI unit: T) is the magnitude of the magnetic flux density.

This model computes the trajectory of an ion in a uniform magnetic field using the Newtonian, Lagrangian, and Hamiltonian formulations available in the Mathematical Particle Tracing interface.

Model Definition

The equations of motion for a charge in a magnetic field can be determined from the Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) = \frac{\partial L}{\partial \mathbf{q}} \tag{1}$$

where $\mathbf{v}$ is the particle velocity, $\mathbf{q}$ is the particle position, and L (SI unit: J) is the Lagrangian, which is defined as:

$$L = \frac{m(\mathbf{v}\cdot\mathbf{v})}{2} + q(\mathbf{v}\cdot\mathbf{A})$$

This form of the Lagrangian is valid for nonrelativistic particles; that is, the particle velocity is much less than the speed of light. The contribution due to the electric potential is neglected. The Hamiltonian is related to the Lagrangian via:

$$H = \mathbf{v}\cdot\frac{\partial L}{\partial \mathbf{v}} - L$$

Introducing the generalized momentum of the particle, $\mathbf{P}$ (SI unit: kg·m/s), the Hamiltonian becomes:

$$H = \frac{(\mathbf{P} - q\mathbf{A})^2}{2m}$$

In order to derive the equations of motion for the Newtonian formulation, start with the right-hand side of Equation 1:

$$\frac{\partial L}{\partial \mathbf{q}} = \nabla L = q\nabla(\mathbf{A} \cdot \mathbf{v}) = q(\mathbf{v} \cdot \nabla)\mathbf{A} + q(\mathbf{v} \times \nabla \times \mathbf{A}) \tag{2}$$

The left-hand side of Equation 1 becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \mathbf{v}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} + q\mathbf{A}) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} + q(\mathbf{v} \cdot \nabla)\mathbf{A} + q\frac{\partial \mathbf{A}}{\partial t}. \tag{3}$$

Equating Equation 2 and Equation 3 and canceling like terms yields

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{v} \times \mathbf{B}) \tag{4}$$

for a stationary magnetic field. Here, the magnetic flux density has been introduced as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5}$$

When the particle velocity is small compared to the speed of light Equation 4 yields the classical equation of motion for a charged particle in a stationary, uniform magnetic field

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = q(\mathbf{v} \times \mathbf{B})$$

Results and Discussion

The model is solved in COMSOL using the Lagrangian, Hamiltonian, and Newtonian formulations. The Larmor radius is compared to the analytic solution and given in Table 1. All formulations agree with the analytic expression to within 0.05 %. The two coupled first-order differential equations give a slightly different result from a single second-order differential equation for each coordinate.

TABLE 1: TABLE COMPARING THE LARMOR RADIUS FOR THE DIFFERENT FORMULATIONS.

	ANALYTIC	LAGRANGIAN	HAMILTONIAN	NEWTONIAN, 2ND ORDER	NEWTONIAN, 1ST ORDER
Larmor radius (µm)	414.57	414.55	414.55	414.55	414.57

The particle trajectories for the three different formulations are plotted below.

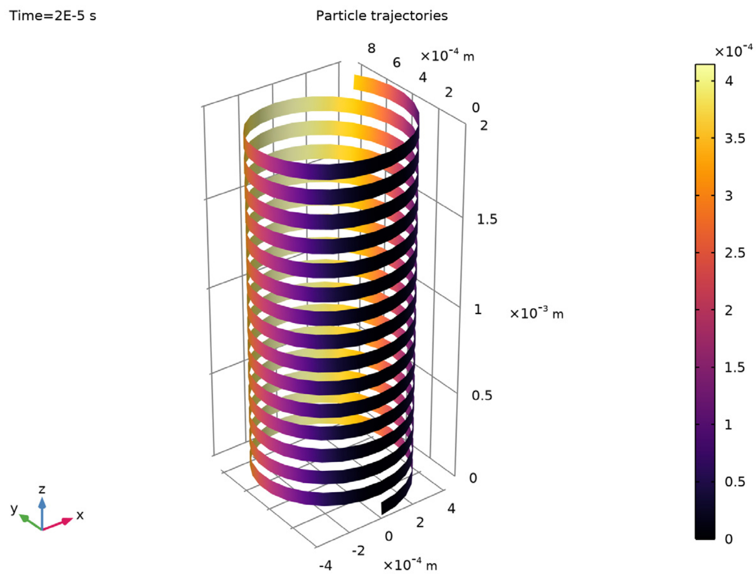

Figure 1: Plot of the ion trajectory for the Lagrangian formulation.

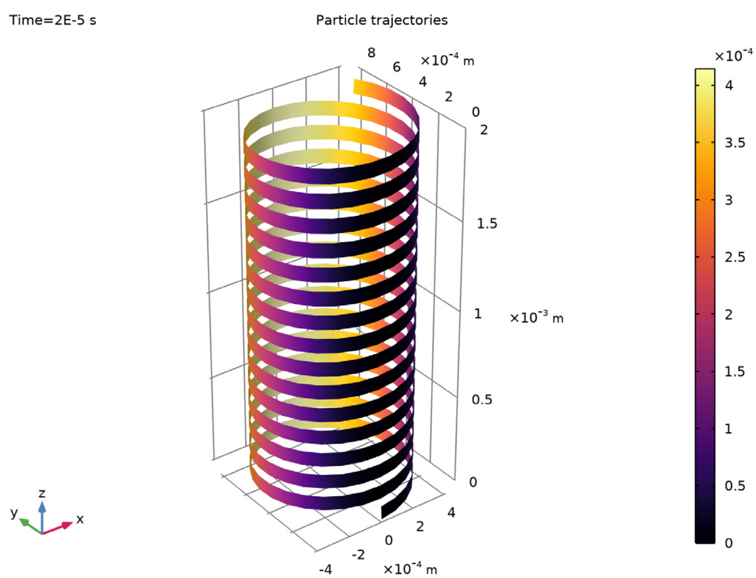

Figure 2: Plot of the ion trajectory for the Hamiltonian formulation.

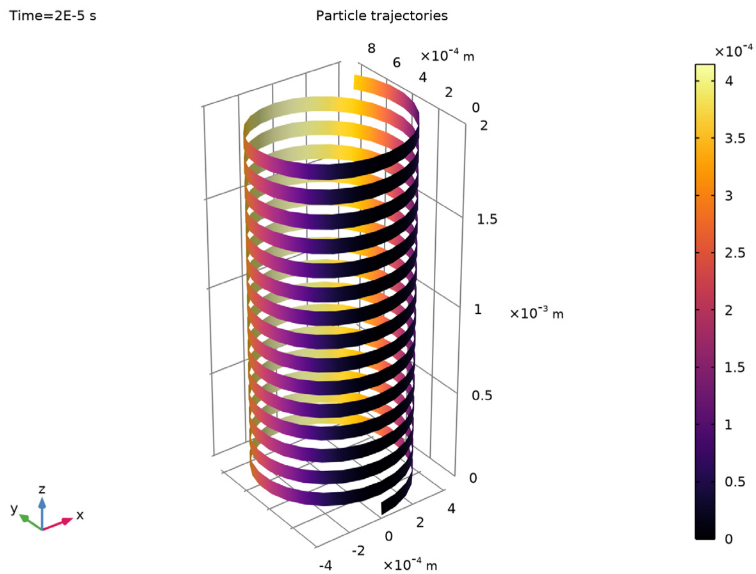

Figure 3: Plot of the particle trajectory for the Newtonian formulation.

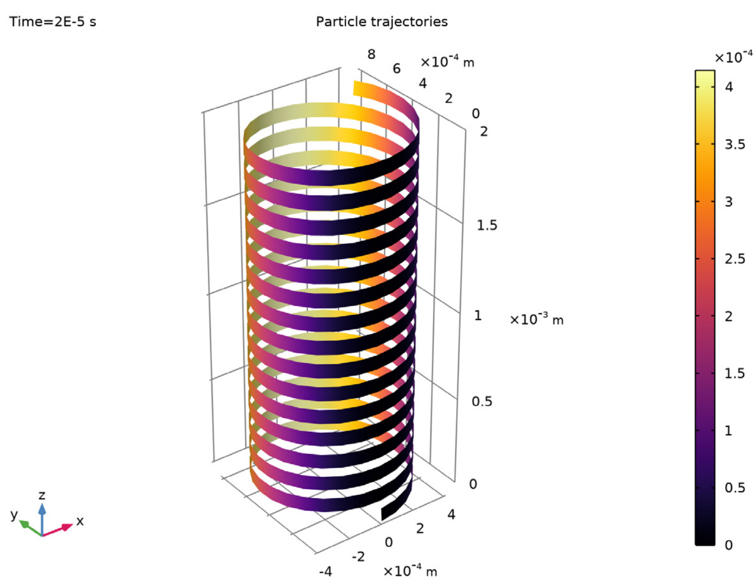

Figure 4: Plot of the particle trajectory for the first-order Newtonian formulation.

Reference

1. L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed., Elsevier, 2005.

Application Library path: `Particle_Tracing_Module/Charged_Particle_Tracing/ion_cyclotron_motion`

Modeling Instructions

From the **File** menu, choose **New**.

NEW

In the **New** window, click **Model Wizard**.

MODEL WIZARD

1 In the **Model Wizard** window, click **3D**.

2 In the **Select Physics** tree, select **Mathematics>Mathematical Particle Tracing (pt)**.

3 Click **Add**.

4 Click **Study**.

5 In the **Select Study** tree, select **General Studies>Time Dependent**.

6 Click **Done**.

GEOMETRY 1

Cylinder 1 (cyl1)

1 In the **Geometry** toolbar, click **Cylinder**.

2 In the **Settings** window for **Cylinder**, locate the **Size and Shape** section.

3 In the **Radius** text field, type `2e-3`.

4 In the **Height** text field, type `2e-3`.

5 In the **Geometry** toolbar, click **Build All**.

GLOBAL DEFINITIONS

Define parameters for the particle mass, magnetic flux density, initial particle velocity, and Larmor radius. The Larmor radius is only used during results processing.

Parameters 1

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.

2 In the **Settings** window for **Parameters**, locate the **Parameters** section.

3 In the table, enter the following settings:

Name	Expression	Value	Description
mp	0.04[kg/mol]/N_A_const	6.6422E-26 kg	Ion mass
B	2[T]	2 T	Magnetic flux density
v0	2E3[m/s]	2000 m/s	Particle velocity, perpendicular to the magnetic field
rL	mp*v0/(e_const*B)	4.1457E-4 m	Larmor radius

DEFINITIONS

Define an analytic expression for the magnetic vector potential, which results in a uniform magnetic field in the *z* direction.

Variables 1

1 In the **Model Builder** window, expand the **Component 1 (comp1)>Definitions** node.

2 Right-click **Definitions** and choose **Local Variables**.

3 In the **Settings** window for **Variables**, locate the **Variables** section.

4 In the table, enter the following settings:

Name	Expression	Unit	Description
Ax	1[Wb/m]*y[1/m]	Wb/m	Magnetic vector potential, x-component
Ay	-1[Wb/m]*x[1/m]	Wb/m	Magnetic vector potential, y-component
Az	0[Wb/m]	Wb/m	Magnetic vector potential, z-component
Bx	d(Az,y)-d(Ay,z)	T	Magnetic flux density, x-component
By	d(Ax,z)-d(Az,x)	T	Magnetic flux density, y-component
Bz	d(Ay,x)-d(Ax,y)	T	Magnetic flux density, z-component

MATHEMATICAL PARTICLE TRACING (PT)

Release a single particle at the origin with an initial velocity in the x direction so that the Lorentz force is nonzero. Also add a small initial velocity in the z direction so that you can clearly see the particle trajectory after solving.

Release from Grid 1

1 In the **Model Builder** window, under **Component 1 (comp1)** right-click **Mathematical Particle Tracing (pt)** and choose **Release from Grid**.

2 In the **Settings** window for **Release from Grid**, locate the **Initial Velocity** section.

3 Specify the $\mathbf{v}_0$ vector as

v0	x
0	y
1e2	z

The first formulation you will use is **Lagrangian**. The Lagrangian for a particle in a magnetic field is the sum of the particle kinetic energy, which is here defined as `pt.Ep`, and the dot product of the particle velocity and the magnetic potential, multiplied by the particle charge.

4 In the **Model Builder** window, click **Mathematical Particle Tracing (pt)**.

5 In the **Settings** window for **Mathematical Particle Tracing**, locate the **Particle Release and Propagation** section.

6 From the **Formulation** list, choose **Lagrangian**.

Particle Properties 1

1 In the **Model Builder** window, click **Particle Properties 1**.

2 In the **Settings** window for **Particle Properties**, locate the **Particle Mass** section.

3 In the m_p text field, type `mp`.

4 Locate the **Lagrangian** section. In the L text field, type `pt.Ep+e_const*(pt.vx*Ax+pt.vy*Ay+pt.vz*Az)`.

MESH 1

Use a coarse mesh. The field is entered using an analytic expression, so the accuracy of the solution is independent of the mesh element size.

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Mesh 1**.

2 In the **Settings** window for **Mesh**, locate the **Physics-Controlled Mesh** section.

3 From the **Element size** list, choose **Extra coarse**.

4 Click **Build All**.

STUDY 1

Step 1: Time Dependent

1 In the **Model Builder** window, under **Study 1** click **Step 1: Time Dependent**.

2 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

3 In the **Output times** text field, type `range(0,5.0e-8,2.0e-5)`.

For all of the studies in this example, a tight user-defined tolerance will be used to ensure particle kinetic energy is conserved.

4 From the **Tolerance** list, choose **User controlled**.

5 In the **Relative tolerance** text field, type `1.0E-6`.

6 In the **Model Builder** window, click **Study 1**.

7 In the **Settings** window for **Study**, type `Lagrangian Study` in the **Label** text field.

8 In the **Home** toolbar, click **Compute**.

RESULTS

Lagrangian Results

In order to be able to see the radius of the particle orbit, plot the y-component of the particle location as a color expression.

1 In the **Settings** window for **3D Plot Group**, type `Lagrangian Results` in the **Label** text field.

2 Locate the **Plot Settings** section. Clear the **Plot dataset edges** check box.

Particle Trajectories 1

Render the particle trajectory as a ribbon. The default ribbon orientation is in the direction of the unit binormal, or the direction out of the plane tangent to the curved trajectory.

1 In the **Model Builder** window, expand the **Lagrangian Results** node, then click **Particle Trajectories 1**.

2 In the **Settings** window for **Particle Trajectories**, locate the **Coloring and Style** section.

3 Find the **Line style** subsection. From the **Type** list, choose **Ribbon**.

4 Select the **Width scale factor** check box. In the associated text field, type `4E-5`.

5 Find the **Point style** subsection. From the **Type** list, choose **None**.

Color Expression 1

1 In the **Model Builder** window, expand the **Particle Trajectories 1** node, then click **Color Expression 1**.

2 In the **Settings** window for **Color Expression**, locate the **Expression** section.

3 In the **Expression** text field, type `qy/2`.

4 Locate the **Coloring and Style** section. Click **Change Color Table**.

5 In the **Color Table** dialog box, select **Thermal>Inferno** in the tree.

6 Click **OK**.

7 In the **Lagrangian Results** toolbar, click **Plot**.

8 Click the **Zoom Extents** button in the **Graphics** toolbar. The plot should look like Figure 1.

Particle Evaluation 1

1 In the **Results** toolbar, click **More Derived Values** and choose **Other>Particle Evaluation**.

2 In the **Settings** window for **Particle Evaluation**, locate the **Data** section.

3 From the **Time selection** list, choose **First**.

4 Locate the **Expression** section. In the **Expression** text field, type `rL`.

5 Click the arrow next to the **Evaluate** button and click **New Table**.

Particle Evaluation 2

1 In the **Results** toolbar, click **More Derived Values** and choose **Other>Particle Evaluation**.

2 In the **Settings** window for **Particle Evaluation**, locate the **Data** section.

3 From the **Time selection** list, choose **First**.

4 Locate the **Expression** section. In the **Expression** text field, type `timemax(0,2E-5,qy)/2`.

5 Click the arrow next to the **Evaluate** button and select **Table 1 - Particle Evaluation 1 (rL)**.

Now switch formulation from **Lagrangian** to **Hamiltonian**. When you do this, the particle momentum components are added as additional degrees of freedom. The momentum has three components: p_x, p_y, and p_z. This results in a doubling of the number of degrees of freedom in the model.

MATHEMATICAL PARTICLE TRACING (PT)

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Mathematical Particle Tracing (pt)**.

2 In the **Settings** window for **Mathematical Particle Tracing**, locate the **Particle Release and Propagation** section.

3 From the **Formulation** list, choose **Hamiltonian**.

Particle Properties 1

1 In the **Model Builder** window, under **Component 1 (comp1)> Mathematical Particle Tracing (pt)** click **Particle Properties 1**.

2 In the **Settings** window for **Particle Properties**, locate the **Hamiltonian** section.

3 In the H text field, type `((px-e_const*Ax)^2+(py-e_const*Ay)^2+(pz-e_const*Az)^2)/(2*pt.mp)`.

ADD STUDY

1 In the **Home** toolbar, click **Add Study** to open the **Add Study** window.

2 Go to the **Add Study** window.

3 Find the **Studies** subsection. In the **Select Study** tree, select **General Studies> Time Dependent**.

4 Click **Add Study** in the window toolbar.

5 In the **Home** toolbar, click **Add Study** to close the **Add Study** window.

STUDY 2

Step 1: Time Dependent

1 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

2 In the **Output times** text field, type `range(0,5.0e-8,2.0e-5)`.

3 From the **Tolerance** list, choose **User controlled**.

4 In the **Relative tolerance** text field, type `1.0E-6`.

5 In the **Model Builder** window, click **Study 2**.

6 In the **Settings** window for **Study**, type `Hamiltonian Study` in the **Label** text field.

7 In the **Home** toolbar, click **Compute**.

RESULTS

Hamiltonian Results

1 In the **Settings** window for **3D Plot Group**, type `Hamiltonian Results` in the **Label** text field.

2 Locate the **Plot Settings** section. Clear the **Plot dataset edges** check box.

Particle Trajectories 1

1 In the **Model Builder** window, expand the **Hamiltonian Results** node, then click **Particle Trajectories 1**.

2 In the **Settings** window for **Particle Trajectories**, locate the **Coloring and Style** section.

3 Find the **Line style** subsection. From the **Type** list, choose **Ribbon**.

4 Select the **Width scale factor** check box. In the associated text field, type `4E-5`.

5 Find the **Point style** subsection. From the **Type** list, choose **None**.

Color Expression 1

1 In the **Model Builder** window, expand the **Particle Trajectories 1** node, then click **Color Expression 1**.

2 In the **Settings** window for **Color Expression**, locate the **Expression** section.

3 In the **Expression** text field, type `qy/2`.

4 Locate the **Coloring and Style** section. Click **Change Color Table**.

5 In the **Color Table** dialog box, select **Thermal>Inferno** in the tree.

6 Click **OK**.

7 In the **Hamiltonian Results** toolbar, click **Plot**.

8 Click the **Go to Default View** button in the **Graphics** toolbar. The plot should look like Figure 2.

Particle Evaluation 3

1 In the **Model Builder** window, expand the **Results>Datasets** node.

2 Right-click **Results>Derived Values** and choose **More Derived Values>Particle Evaluation**.

3 In the **Settings** window for **Particle Evaluation**, locate the **Data** section.

4 From the **Dataset** list, choose **Particle 2**.

5 From the **Time selection** list, choose **First**.

6 Locate the **Expression** section. In the **Expression** text field, type `timemax(0,2E-5,qy)/2`.

7 Click the arrow next to the **Evaluate** button and select **Table 1 - Particle Evaluation 1 (rL)**.

Switch to the **Newtonian** formulation and add the Lorentz force manually.

MATHEMATICAL PARTICLE TRACING (PT)

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Mathematical Particle Tracing (pt)**.
2 In the **Settings** window for **Mathematical Particle Tracing**, locate the **Particle Release and Propagation** section.
3 From the **Formulation** list, choose **Newtonian**.

Force 1

1 In the **Physics** toolbar, click **Domains** and choose **Force**.
2 In the **Settings** window for **Force**, locate the **Domain Selection** section.
3 From the **Selection** list, choose **All domains**.
4 Locate the **Force** section. Specify the **F** vector as

`e_const*(Bz*pt.vy-By*pt.vz)`	x
`e_const*(-Bz*pt.vx+Bx*pt.vz)`	y
`e_const*(By*pt.vx-Bx*pt.vy)`	z

ADD STUDY

1 In the **Home** toolbar, click **Add Study** to open the **Add Study** window.
2 Go to the **Add Study** window.
3 Find the **Studies** subsection. In the **Select Study** tree, select **General Studies> Time Dependent**.
4 Click **Add Study** in the window toolbar.
5 In the **Home** toolbar, click **Add Study** to close the **Add Study** window.

STUDY 3

Step 1: Time Dependent

1 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.
2 In the **Output times** text field, type `range(0,5.0e-8,2.0e-5)`.
3 From the **Tolerance** list, choose **User controlled**.
4 In the **Relative tolerance** text field, type `1.0E-6`.
5 In the **Model Builder** window, click **Study 3**.
6 In the **Settings** window for **Study**, type `Newtonian Study` in the **Label** text field.

7 In the **Home** toolbar, click **Compute**.

RESULTS

Newtonian Results

1 In the **Settings** window for **3D Plot Group**, type `Newtonian Results` in the **Label** text field.

2 Locate the **Plot Settings** section. Clear the **Plot dataset edges** check box.

Particle Trajectories 1

1 In the **Model Builder** window, expand the **Newtonian Results** node, then click **Particle Trajectories 1**.

2 In the **Settings** window for **Particle Trajectories**, locate the **Coloring and Style** section.

3 Find the **Line style** subsection. From the **Type** list, choose **Ribbon**.

4 Select the **Width scale factor** check box. In the associated text field, type `4E-5`.

5 Find the **Point style** subsection. From the **Type** list, choose **None**.

Color Expression 1

1 In the **Model Builder** window, expand the **Particle Trajectories 1** node, then click **Color Expression 1**.

2 In the **Settings** window for **Color Expression**, locate the **Expression** section.

3 In the **Expression** text field, type `qy/2`.

4 Locate the **Coloring and Style** section. Click **Change Color Table**.

5 In the **Color Table** dialog box, select **Thermal>Inferno** in the tree.

6 Click **OK**.

7 In the **Newtonian Results** toolbar, click **Plot**.

8 Click the **Go to Default View** button in the **Graphics** toolbar. The plot should look like Figure 3.

Particle Evaluation 4

1 In the **Results** toolbar, click 8.85 e-12 **More Derived Values** and choose **Other>Particle Evaluation**.

2 In the **Settings** window for **Particle Evaluation**, locate the **Data** section.

3 From the **Dataset** list, choose **Particle 3**.

4 From the **Time selection** list, choose **First**.

5 Locate the **Expression** section. In the **Expression** text field, type `timemax(0,2E-5,qy)/2`.

6 Click the arrow next to the **Evaluate** button and select **Table 1 - Particle Evaluation 1 (rL)**.

Finally, switch to the **Newtonian, first order** formulation.

MATHEMATICAL PARTICLE TRACING (PT)

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Mathematical Particle Tracing (pt)**.

2 In the **Settings** window for **Mathematical Particle Tracing**, locate the **Particle Release and Propagation** section.

3 From the **Formulation** list, choose **Newtonian, first order**.

ADD STUDY

1 In the **Home** toolbar, click **Add Study** to open the **Add Study** window.

2 Go to the **Add Study** window.

3 Find the **Studies** subsection. In the **Select Study** tree, select **General Studies> Time Dependent**.

4 Click **Add Study** in the window toolbar.

5 In the **Home** toolbar, click **Add Study** to close the **Add Study** window.

STUDY 4

Step 1: Time Dependent

1 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

2 In the **Output times** text field, type `range(0,5.0e-8,2.0e-5)`.

3 From the **Tolerance** list, choose **User controlled**.

4 In the **Relative tolerance** text field, type `1.0E-6`.

5 In the **Model Builder** window, click **Study 4**.

6 In the **Settings** window for **Study**, type `Newtonian, First Order Study` in the **Label** text field.

7 In the **Home** toolbar, click **Compute**.

RESULTS

Newtonian, First Order Results

1 In the **Settings** window for **3D Plot Group**, type `Newtonian, First Order Results` in the **Label** text field.

2 Locate the **Plot Settings** section. Clear the **Plot dataset edges** check box.

Particle Trajectories 1

1 In the **Model Builder** window, expand the **Newtonian, First Order Results** node, then click **Particle Trajectories 1**.

2 In the **Settings** window for **Particle Trajectories**, locate the **Coloring and Style** section.

3 Find the **Line style** subsection. From the **Type** list, choose **Ribbon**.

4 Select the **Width scale factor** check box. In the associated text field, type `4E-5`.

5 Find the **Point style** subsection. From the **Type** list, choose **None**.

Color Expression 1

1 In the **Model Builder** window, expand the **Particle Trajectories 1** node, then click **Color Expression 1**.

2 In the **Settings** window for **Color Expression**, locate the **Expression** section.

3 In the **Expression** text field, type `qy/2`.

4 Locate the **Coloring and Style** section. Click **Change Color Table**.

5 In the **Color Table** dialog box, select **Thermal>Inferno** in the tree.

6 Click **OK**.

7 In the **Newtonian, First Order Results** toolbar, click **Plot**.

8 Click the **Go to Default View** button in the **Graphics** toolbar. The plot should look like Figure 4.

Particle Evaluation 5

1 In the **Results** toolbar, click **More Derived Values** and choose **Other> Particle Evaluation**.

2 In the **Settings** window for **Particle Evaluation**, locate the **Data** section.

3 From the **Dataset** list, choose **Particle 4**.

4 From the **Time selection** list, choose **First**.

5 Locate the **Expression** section. In the **Expression** text field, type `timemax(0,2E-5,qy)/2`.

6 Click the arrow next to the **Evaluate** button and select **Table 1 - Particle Evaluation 1 (rL)**.

The Larmor radius shows good agreement with the analytic expression for all four formulations.