Q: Is $\mathcal{R}(A)$ a subspace?

$A \in \mathbb{R}^{\overset{3}{m} \times \overset{5}{n}}$

$$\boxed{\mathcal{R}(A) = \{ y \in \mathbb{R}^{\overset{3}{m}} : \exists\, x \in \mathbb{R}^{\overset{5}{n}},\ Ax = y \}}$$

$\mathcal{R}(A) \subseteq \mathbb{R}^{\overset{3}{m}}$

$y_1, y_2 \in \mathcal{R}(A)$

need to see

$y_1 + y_2 \in \mathcal{R}(A)$

$y \in \mathcal{R}(A)$

$\alpha \in \mathbb{R}$

n2c

$\alpha y \in \mathcal{R}(A)$

n2c

$0_m \in \mathcal{R}(A)$

$Ax_1 = y_1$

$Ax_2 = y_2$

$+$

$$Ax_1 + Ax_2 = y_1 + y_2$$

$$A(x_1 + x_2) = y_1 + y_2$$

Hence

$$y_1 + y_2 \in R(A)$$

$$\alpha A x = \alpha y$$

$$A(\alpha x) = \alpha y$$

$$\alpha y \in R(A)$$

$N(A)$ is vector space

$$\boxed{N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0_m \}}$$

$$x_1, \; x_2 \in N(A)$$

$$Ax_1 = 0$$

$$+ \quad Ax_2 = 0$$

$$A(x_1 + x_2) = 0$$

$$\alpha A x = \alpha 0$$

$$A(\alpha x) = 0$$

$$\alpha x \in N(A)$$

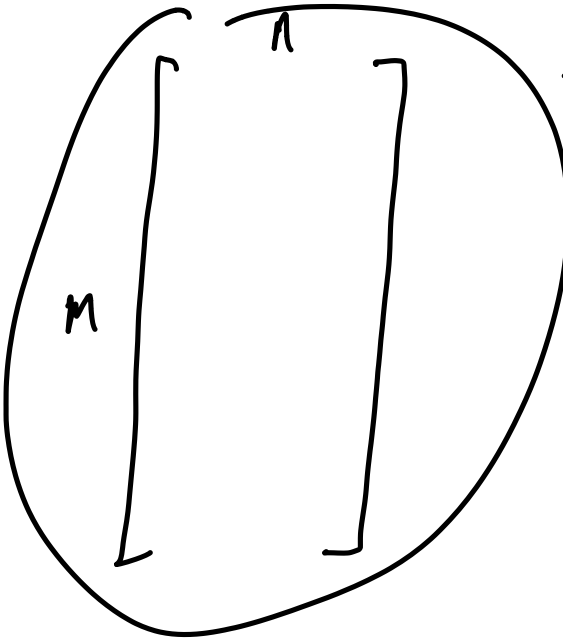
$x_1 + x_2 \in N(A)$

$0 \in N(A)$

$A \in \mathbb{R}^{m \times n}$

[VML S] Gram Matrix

$\boxed{G = A^T A} \neq \cancel{A A^T}$



$$G^T = (A^T A)^T$$

$$= (A^T A^{T^T})$$

$$= G$$

G is symmetric

$G \in \mathbb{R}^{n \times n}$

$\in \mathbb{R}^{m \times n}$ Claim $\in \mathbb{R}^{1 \times n}$

$$N(A) = N(A^TA)$$

$$\mathbb{R}^n \qquad \mathbb{R}^n$$

$$Ax = 0_m \qquad A^TAx = 0_n$$

$$\{0\} = N(A) \qquad N(A^TA) = \{0\}$$

Not LI $\Longrightarrow$ "At least one redundent vector"

$\{v_1, \ldots, v_k\}$ not LI

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

and NOT all $\alpha_i = 0$

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix} \end{array} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} n$$

$$A$$

$\exists i \quad \alpha_i \neq 0$

$$v_i = -\left(\frac{\alpha_1}{\alpha_i}\right) v_1 - \left(\frac{\alpha_2}{\alpha_i}\right) v_2 \cdots\cdots - \left(\frac{\alpha_{i-1}}{\alpha_i}\right) v_{i-1}$$

$$-\left(\frac{\alpha_{i+1}}{\alpha_i}\right) v_{i+1} - \cdots\cdots - \left(\frac{\alpha_k}{\alpha_i}\right) v_k$$

$\{v_1, \ldots, v_k\}$ LI

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ | \end{bmatrix}$$

$\alpha = 0$

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$$

$$x = \alpha_1 v_1 + \dots + \alpha_k v_k$$

$$x = \beta_1 v_1 + \dots \dots + \beta_k v_k$$

$-$

$A\alpha = x$

$A\beta = x$

$n = 3$

3-vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$k = 2$

$$x = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 2e_1 + 3e_2$$

$$0 = \underbrace{(\alpha_1 - \beta_1)}_{= 0} v_1 + \dots + \underbrace{(\alpha_k - \beta_k)}_{= 0} v_k$$

$\alpha_1 = \beta_1 \quad \alpha_2 = \beta_2 \quad \dots \quad \alpha_k = \beta_k$

$$\boxed{Ax = b}$$

A is "zero null space"

If there is a solution then it is unique.

A set of orthonormal vectors is LI
ON

$v_1, \ldots, v_k$ are ON

$$\boxed{\alpha_1 v_1 + \ldots + \alpha_k v_k = 0}$$

NZS

$$\boxed{\alpha_i = 0 \quad \forall i}$$

L mult by v_i^T

$$\overset{0}{\underbrace{v_i^T \alpha_1 v_1}_{\alpha_1 \underbrace{v_i^T v_1}_{0}}} + \dots + \underbrace{v_i^T \alpha_i v_i}_{\boxed{\alpha_i \underbrace{v_i^T v_i}_{1}}} + \dots + \overset{0}{\underbrace{v_i^T \alpha_k v_k}_{0}} =$$

$$0+0+0 \dots 0+\alpha_i 1 + 0+0+0 \dots = 0$$

$$\alpha_i = 0$$