MATH 318 Homework 7

Problem 1

(a) The event E that the walker returns to the origin in $2n$ steps is precisely the event that there are equally many left and right steps, as well as equally many up and down steps. Let E_k be the event that there are k left and right steps, and consequently $n-k$ up and down steps. Notice that we require $k \geq 0$ and $n-k \geq 0 \therefore k \leq n$. It follows that $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E = \cup_{k=0}^{n} E_k$.

By symmetry, each permutation of steps is equally likely. There are $\frac{1}{4^{2n}}$ total permutations, since there are $2n$ steps with four options per step. The number of permutations in the event E_k is simply the number of ways to arrange k left and right steps, and $n-k$ up and down steps. The multinomial coefficient gives us

$$\frac{(2n)!}{(k!)^2((n-k)!)^2} = \left(\frac{(2n)!}{(n!)^2}\right)\left(\frac{(n!)^2}{(k!)^2((n-k)!)^2}\right) = \binom{2n}{n}\binom{n}{k}^2$$

Thus

$$P(E_k) = \frac{1}{4^{2n}}\binom{2n}{n}\binom{n}{k}^2$$

By Kolmogorov's third axiom of probability,

$$p_{2n} = P(E) = \sum_{k=0}^{n} P(E_k) = \frac{1}{4^{2n}}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2$$

$\square$

(b) As a small flex, we take a different approach from the hint. Consider the polynomial

$$(1+x)^{2n} = (1+x)^n(1+x)^n$$

The coefficient on the x^n term is $\binom{2n}{n}$. Meanwhile, the coefficient on the x^k term for the polynomial $(1+x)^n$ is $\binom{n}{k}$. But by multiplying $(1+x)^n$ with itself, we find that the coefficient on the x^n term is also given by the convolutional sum

$$\binom{2n}{n}x^n = \sum_{k=0}^{n}\left(\binom{n}{k}x^k\right)\left(\binom{n}{n-k}x^{n-k}\right) = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}x^n$$

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}\binom{n}{k}^2$$

$\square$

(c) We use the identity from part (b) to find

$$P(E) = \frac{1}{4^{2n}}\binom{2n}{n}^2$$

Let M be the number of returns to the origin. If we denote X_{2n} as an indicator RV for returning to the origin after $2n$ steps, we have by linearity of expectaion

$$\mathbb{E}\left[M\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[X_{2n}\right] = \sum_{n=0}^{\infty} p_{2n} = \sum_{n=0}^{\infty} \frac{1}{4^{2n}}\binom{2n}{n}^2$$

The convergence of this sum depends only on the behaviour of the summand for large n. In this regime, we use Stirling's approximation:

$$\mathbb{E}\left[M\right] \sim \sum_{n=0}^{\infty} \frac{1}{4^{2n}}\left(\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{n^{2n}e^{-2n}(2\pi n)}\right)^2 = \sum_{n=0}^{\infty} \frac{1}{4^{2n}}\left(\frac{2^{2n}}{\sqrt{\pi n}}\right)^2 = \sum_{n=0}^{\infty} \frac{1}{\pi n}$$

This is a harmonic series, which diverges to infinity. Hence $\mathbb{E}\left[M\right] = +\infty$, so the transient walk is recurrent. □

Problem 2

Let Y_i be the indicator RV that the ith sampled ball is black. For a given X, the probability of this occuring is then $X/8$. Now notice that $Y = \sum_{i=1}^{1} 0Y_i$. By linearity of expectation,

$$\boxed{\mathbb{E}\left[Y|X\right] = \sum_{i=1}^{10} \frac{X}{8} = \frac{5}{4}X}$$

Since $Y|X$ is the sum of 10 Bernoulli trials, we have

$$P(Y = y|X = x) \sim \text{Binom}\left(10, x/8\right) = \left(\frac{x}{8}\right)^y \left(\frac{8-x}{8}\right)^{10-y}\binom{10}{y}$$

Since X is uniform, $P(X = x) = 1/9$ for $0 \le x \le 8$. By Bayes' theorem,

$$\begin{aligned}
P(X = x|Y = y) &= \frac{P(Y = y|X = x)P(X = x)}{\sum_{k=0}^{8} P(Y = y|X = k)P(X = k)} = \frac{\left(\frac{x}{8}\right)^y\left(\frac{8-x}{8}\right)^{10-y}}{\sum_{k=0}^{8}\left(\frac{k}{8}\right)^y\left(\frac{8-k}{8}\right)^{10-y}} \\
&= \frac{x^y(8-x)^{10-y}}{\sum_{k=0}^{8} k^y(8-k)^{10-y}} \\
\mathbb{E}\left[X|Y\right] &= \sum_{x=0}^{8} x\frac{x^y(8-x)^{10-y}}{\sum_{k=0}^{8} k^y(8-k)^{10-y}}
\end{aligned}$$

$$\boxed{\mathbb{E}\left[X|Y\right] = \frac{\sum_{k=0}^{8} k^{y+1}(8-k)^{10-y}}{\sum_{k=0}^{8} k^y(8-k)^{10-y}}}$$