Math 318 Assignment 7: Due Friday, 2021-03-26, 15:00

I. Problems to be handed in:

Problem 1. Consider simple symmetric random walk in two dimensions.

(a) Show that the probability p_{2n} that a walker returns to its starting place at the origin after $2n$ steps is given by $p_{2n} = (\frac{1}{4})^{2n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2$.

(b) Show that $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$. Hint: Think of choosing n balls from a box that has $2n$ balls, n white and n black.

(c) Use this to give another proof that the two-dimensional walk is recurrent.

Problem 2. Let X be uniform in $\{0, 1, 2, \ldots, 8\}$. We put X black balls and $8 - X$ white balls in a bag. Now sample 10 balls out of the bag (returning each ball before sampling the next). Let Y be the number of black balls out of the 10 samples.
What is $\mathbf{E}(Y|X)$? What is $\mathbf{E}(X|Y)$?

Problem 3. In each of (a)–(d), determine whether or not the Markov chain with given transition matrix is irreducible, identify each state as recurrent or transient, and as periodic or aperiodic. In (a) and (b), the state space is $S = \{1, \ldots, 5\}$.

(a) $$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $$P = \begin{pmatrix} 0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) The simple symmetric random walk on $\mathbb{Z}^d$ (answer for each $d \geq 1$).

(d) The random walk on $\mathbb{Z}$ with probability 1/3 of moving right and 2/3 of moving left.

Problem 4. Consider the Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Suppose $P(X_0 = 1) = P(X_0 = 2) = \frac{1}{2}$. What is $P(X_3 = 3)$?

Problem 5. There are n coins on the table. At each step we choose at random one of the coins toss it and place it back (with the resulting side facing up). Let X_m be the number of heads showing. Show that this chain has transition probabilities

$$P_{ii} = \frac{1}{2}, \quad P_{i,i-1} = \frac{i}{2n}, \quad P_{i,i+1} = \frac{n-i}{2n}.$$

Problem 6. (a) For the gambler's ruin problem, let M_k denote the expected number of games that will be played when Mark initially has \$k, and stops at 0 or n. Let p be the probability of winning each bet, and $q = 1 - p$. Show that $M_0 = M_n = 0$ and

$$M_k = 1 + pM_{k+1} + qM_{k-1}$$

for $0 < k < n$. (Hint: Compute the expectation of the number of games X by conditioning on the outcome of the first game. If A is the event that Mark wins the first game,

$$EX = E[X|A]P(A) + E[X|A^c]P(A^c).$$

(b) Solve the equations in (a) to obtain

$$M_k = k(n-k) \text{ if } p = 1/2,$$

and

$$M_k = \frac{k}{q-p} - \frac{n}{q-p}\frac{1-\alpha^k}{1-\alpha^n} \text{ if } p = 1/2,$$

where $\alpha = q/p$. To do this, proceed as follows. First, find the general solution to the homogeneous equation $M_k = pM_{k+1} + qM_{k-1}$ (as done in class). Next, find a particular solution to the inhomogeneous equation $M_k = 1 + pM_{k+1} + qM_{k-1}$ (try $M_k = ck^2$ for $p = 1/2$ and $M_k = ck$ for $p \neq 1/2$). Add the general solution of the homogeneous equation to the particular solution of the inhomogeneous equation. Finally, solve for the two unknown constants in the general solution by using the boundary conditions.

Problem 7. Simulate the process of problem 5 with 1000 coins. Start with 0 heads, and run the process for 20000 steps. Keep track of how many times each state is visited, from time 0 to time 1000, from time 1000 to 2000, and finally, in times 10000-20000. Submit your code and histograms of those.

II. Extra practice problems (do not hand in)

(a) Chapter 4: 2,13,14,15,38,57

(b) Write down a 6×6 stochastic matrix and determine its irreducible classes, recurrence and periodicity of states.

(c) 25 measurements are made of the splitting tensile stress (lb/in^2) of concrete cylinders. The following table shows the frequency of each measured value, with the strength on the first line and the frequency on the second line. Assuming a normal distribution, determine a 99% confidence interval for the mean splitting tensile stress of the population from which the sample was drawn.

320	330	340	350	360	370	380	390
1	1	3	3	8	3	5	1

(d) Suppose weather evolves as follows: whether it rains today depends only on the last two days. If it rained in both, then it rains today with probability 0.7; If it rained yesterday but not the day before, the probability is 0.5; if it rained the day before but not yesterday then it is 0.4, and if it has not rained in the last two days it is 0.1. Let $X_n = 1$ if it rains on day n and 0 if not. Show that this is not a Markov chain. However, if you let Y_n be the pair (X_{n-1}, X_n) (with 4 possibilities), show that this is a Markov chain, and find its transition Matrix.