MATH 318 Homework 7

Problem 1

(a) The event E that the walker returns to the origin in $2n$ steps is precisely the event that there are equally many left and right steps, as well as equally many up and down steps. Let E_k be the event that there are k left and right steps, and consequently $n-k$ up and down steps. Notice that we require $k \geq 0$ and $n-k \geq 0 \therefore k \leq n$. It follows that $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E = \cup_{k=0}^{n} E_k$.

By symmetry, each permutation of steps is equally likely. There are $\frac{1}{4^{2n}}$ total permutations, since there are $2n$ steps with four options per step. The number of permutations in the event E_k is simply the number of ways to arrange k left and right steps, and $n-k$ up and down steps. The multinomial coefficient gives us

$$\frac{(2n)!}{(k!)^2((n-k)!)^2} = \left(\frac{(2n)!}{(n!)^2}\right)\left(\frac{(n!)^2}{(k!)^2((n-k)!)^2}\right) = \binom{2n}{n}\binom{n}{k}^2$$

Thus

$$P(E_k) = \frac{1}{4^{2n}}\binom{2n}{n}\binom{n}{k}^2$$

By Kolmogorov's third axiom of probability,

$$p_{2n} = P(E) = \sum_{k=0}^{n} P(E_k) = \frac{1}{4^{2n}}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2$$

□

(b) As a small flex, we take a different approach from the hint. Consider the polynomial

$$(1+x)^{2n} = (1+x)^n(1+x)^n$$

The coefficient on the x^n term is $\binom{2n}{n}$. Meanwhile, the coefficient on the x^k term for the polynomial $(1+x)^n$ is $\binom{n}{k}$. But by multiplying $(1+x)^n$ with itself, we find that the coefficient on the x^n term is also given by the convolutional sum

$$\binom{2n}{n}x^n = \sum_{k=0}^{n}\left(\binom{n}{k}x^k\right)\left(\binom{n}{n-k}x^{n-k}\right) = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}x^n$$

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}\binom{n}{k}^2$$

□

(c) We use the identity from part (b) to find

$$P(E) = \frac{1}{4^{2n}}\binom{2n}{n}^2$$

Let M be the number of returns to the origin. If we denote X_{2n} as an indicator RV for returning to the origin after $2n$ steps, we have by linearity of expectaion

$$\mathbb{E}[M] = \sum_{n=0}^{\infty}\mathbb{E}[X_{2n}] = \sum_{n=0}^{\infty} p_{2n} = \sum_{n=0}^{\infty}\frac{1}{4^{2n}}\binom{2n}{n}^2$$

The convergence of this sum depends only on the behaviour of the summand for large n. In this regime, we use Stirling's approximation:

$$\mathbb{E}[M] \sim \sum_{n=0}^{\infty}\frac{1}{4^{2n}}\left(\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{n^{2n}e^{-2n}(2\pi n)}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{4^{2n}}\left(\frac{2^{2n}}{\sqrt{\pi n}}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{\pi n}$$

This is a harmonic series, which diverges to infinity. Hence $\mathbb{E}[M] = +\infty$, so the transient walk is recurrent. $\square$

Problem 2

Let Y_i be the indicator RV that the ith sampled ball is black. For a given X, the probability of this occuring is then $X/8$. Now notice that $Y = \sum_{i=1}^{1} 0Y_i$. By linearity of expectation,

$$\boxed{\mathbb{E}[Y|X] = \sum_{i=1}^{10}\frac{X}{8} = \frac{5}{4}X}$$

Since $Y|X$ is the sum of 10 Bernoulli trials, we have

$$P(Y=y|X=x) \sim \text{Binom}(10, x/8) = \left(\frac{x}{8}\right)^y\left(\frac{8-x}{8}\right)^{10-y}\binom{10}{y}$$

Since X is uniform, $P(X=x) = 1/9$ for $0 \le x \le 8$. By Bayes' theorem,

$$\begin{aligned}
P(X=x|Y=y) &= \frac{P(Y=y|X=x)P(X=x)}{\sum_{k=0}^{8}P(Y=y|X=k)P(X=k)} = \frac{\left(\frac{x}{8}\right)^y\left(\frac{8-x}{8}\right)^{10-y}}{\sum_{k=0}^{8}\left(\frac{k}{8}\right)^y\left(\frac{8-k}{8}\right)^{10-y}} \\
&= \frac{x^y(8-x)^{10-y}}{\sum_{k=0}^{8}k^y(8-k)^{10-y}} \\
\mathbb{E}[X|Y] &= \sum_{x=0}^{8} x\frac{x^y(8-x)^{10-y}}{\sum_{k=0}^{8}k^y(8-k)^{10-y}}
\end{aligned}$$

$$\boxed{\mathbb{E}[X|Y] = \frac{\sum_{k=0}^{8}k^{y+1}(8-k)^{10-y}}{\sum_{k=0}^{8}k^y(8-k)^{10-y}}}$$

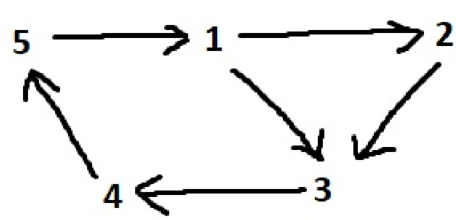
Problem 3

(a) Nonzero transistion probabilities are described graphically as follows:

Since the states $\{1, 2, 3\}$ and $\{4, 5\}$ are in different communicating classes, the Markov chain is $\boxed{\text{not irreducible}}$. Looking at the graph, we also characterize each state:

$$\boxed{\begin{cases} 1\text{: recurrent and periodic with period 2} \\ 2\text{: transient and periodic with period } \infty \\ 3\text{: recurrent and periodic with period 2} \\ 4\text{: transient and aperiodic} \\ 5\text{: recurrent and aperiodic} \end{cases}}$$

(b) Nonzero transistion probabilities are described graphically as follows:



Because we have the loop $1 \to 2 \to 3 \to 4 \to 5 \to 1$, the Markov chain is $\boxed{\text{irreducible}}$. Hence, $\boxed{\text{all states are recurrent}}$. Finally, notice that $P_{11}^4 > 0$ because we have the chain $1 \to 3 \to 4 \to 5 \to 1$ but also $P_{11}^5 > 0$ because of the chain $1 \to 2 \to 3 \to 4 \to 5 \to 1$. Since 4 and 5 are relatively prime, this state is aperiodic. But because the Markov chain is irreducible, $\boxed{\text{all states are aperiodic}}$.

(c) Every pair of states is communicating. To see how, we consider the path taken by incrementing/decrementing a single dimension's coordinate, repeating for each dimension, until the target node is reached from the source node. This path has a probability of $\frac{1}{(2d)^M} > 0$, where M is the Manhattan distance between the two points. The transition probability P_{ij}^M between these two states over M steps is therefore at least $\frac{1}{(2d)^M} > 0$. Hence $\boxed{\text{all symmetric random walks are irreducible}}$. The origin is recurrent for $d \leq 2$ and transient otherwise. Since recurrence is a class property, $\boxed{\text{all states are reccurent for } d \leq 2 \text{ and transient otherwise}}$. Finally, returning to a given node requires an even number of steps. To see why, notice that every step flips the parity of the sum of all coordinates. Furthermore, it is possible to return to a state by moving to the right and back i.e. in 2 steps. Thus $\boxed{\text{all states are periodic with period 2}}$.

(d) Every pair of states is communicating. Suppose we have coordinates $i < j$. Then we can go from i to j in $j - i$ steps with nonzero probability $\left(\frac{2}{3}\right)^{j-i}$ and go from j to i in $j - i$ steps with nonzero probability $\left(\frac{1}{3}\right)^{j-i}$. Thus the Markov chain is $\boxed{\text{irreducible}}$. Furthermore, $\boxed{\text{every state is transient}}$. To see why, recall that the asymmetric random walk on $\mathbb{Z}$ has a finite expected number of returns to the origin. Because the Markov chain is irreducible, every state must be transient like the origin is. Finally, we may employ the parity logic used in part (c) to find that $\boxed{\text{all states are periodic with period 2}}$.

Problem 4

We compute

$$(P^T)^3 \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4028 \\ 0.1759 \\ 0.4213 \end{pmatrix}$$

Thus $\boxed{P(X_3 = 3) = 0.4213}$.

Problem 5

Suppose we have i heads on the table. Selecting a coin with uniform probability, we have an i/n probability of selecting a coin with heads and an $(n-i)/n$ probability of selecting a coin with tails. Denote these events as h and t, respectively.

Now denote the events of flipping a heads H and flipping a tails T. Then $P(H|h) = P(H|t) = P(T|h) = P(T|t) = 1/2$. Now,

$$P_{ii} = P(H, h) + P(T, t) = P(H|h)P(h) + P(T|t)P(t) = \frac{i}{2n} + \frac{n-i}{2n} = \frac{n}{2n} = \frac{1}{2}$$

$$P_{i,i-1} = P(T, h) = P(T|h)P(h) = \frac{i}{2n}$$

$$P_{i,i+1} = P(H, t) = P(H|t)P(t) = \frac{n-i}{2n}$$

□

Problem 6

(a) If Mark initially has 0 or n dollars, no games will be played (with certainty) as the process terminates immediately. Hence $M_0 = M_n = 0$. Now if $0 < k < n$, Mark will play a game, which he will either win or lose. Let W be the event he wins. Then the event he loses is W^c. If X is the number of games played, then $\mathbb{E}[X|W] = 1 + M_{k+1}$ since we add the game that Mark just played, in addition to the expected number of games he will be playing, now starting with $k+1$ dollars. Similarly, $\mathbb{E}[X|W^c] = 1 + M_{k-1}$. Thus

$$\begin{aligned} M_k &= \mathbb{E}[X] = \mathbb{E}[X|W]P(W) + \mathbb{E}[X|W^c]P(W^c) = p(1+M_{k+1}) + q(1+M_{k-1}) \\ &= (p+q) + pM_{k+1} + qM_{k-1} = 1 + pM_{k+1} + qM_{k-1} \end{aligned}$$

(b) First we consider the case $p = q = 1/2$. Then the homogeneous equation is

$$M_k = \frac{M_{k+1} + M_{k-1}}{2} \ \therefore\ M_k - M_{k-1} = M_{k+1} - M_k$$

This implies that the homogeneous solution is linear: $M_k = a + bk$. For the inhomogeneous equation, we guess $M_k = ck^2$. Then

$$ck^2 = 1 + \frac{c(k+1)^2 + c(k-1)^2}{2} = 1 + ck^2 + c \ \therefore\ c = -1$$

That gives the general solution $M_k = a + bk - k^2$. Plugging in the boundary conditions gives us $M_0 = a = 0$ and $M_n = bn - n^2 = 0 \ \therefore\ b = n$. This gives us $M_k = nk - k^2 = k(n-k)$ for $p = 1/2$. □

Now let us consider the case that $p \neq 1/2 \ \therefore\ p \neq q$. We make the ansatz that $M_k = x^k$ for the homogeneous equation. That gives us

$$x^k = px^{k+1} + qx^{k-1}$$

This yields the quadratic

$$px^2 - x + (1-p) = 0$$

$$\begin{aligned} x &= \frac{1}{2p}\left(1 \pm \sqrt{1 - 4p(1-p)}\right) = \frac{1}{2p}\left(1 \pm \sqrt{4p^2 - 4p + 1}\right) = \frac{1}{2p}\left(1 \pm (1-2p)\right) \\ &= 1,\ \frac{1-p}{p} = 1,\ \alpha \end{aligned}$$

Since $p \neq 1/2$, $1 \neq \frac{1-p}{p}$, so these are two distinct roots. The general homogeneous solution is thus $M_k = a + b\left(\frac{1-p}{p}\right)^k$. Now we guess $M_k = ck$ for the inhomogeneous equation:

$$ck = 1 + pc(k+1) + qc(k-1) = 1 + (p-q)c + (p+q)ck = 1 + (p-q)c + ck$$

$$1 + (p - q)c = 0 \ \therefore \ c = \frac{1}{q - p}$$

Recall that $p \neq q$ for this case. We now have the general solution $M_k = \frac{k}{q-p} + a + b\alpha^k$. Plugging in boundary conditions gets us $M_0 = a + b = 0 \ \therefore \ b = -a$. Thus $M_k = \frac{k}{q-p} + a\left(1 - \alpha^k\right)$. Now, $M_n = \frac{n}{q-p} + a\left(1 - \alpha^n\right) = 0$, giving us $a = -\frac{n}{q-p}\frac{1}{(1-\alpha^n)}$. Hence, our solution is

$$M_k = \frac{k}{q - p} - \frac{n}{q - p}\frac{1 - \alpha^k}{1 - \alpha^n}$$

for $p \neq 1/2$. $\square$