MATH 318 Homework 7

Problem 1

(a) The event E that the walker returns to the origin in $2n$ steps is precisely the event that there are equally many left and right steps, as well as equally many up and down steps. Let E_k be the event that there are k left and right steps, and consequently $n-k$ up and down steps. Notice that we require $k \geq 0$ and $n-k \geq 0 \therefore k \leq n$. It follows that $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E = \cup_{k=0}^{n} E_k$.

By symmetry, each permutation of steps is equally likely. There are $\frac{1}{4^{2n}}$ total permutations, since there are $2n$ steps with four options per step. The number of permutations in the event E_k is simply the number of ways to arrange k left and right steps, and $n-k$ up and down steps. The multinomial coefficient gives us

$$\frac{(2n)!}{(k!)^2((n-k)!)^2} = \left(\frac{(2n)!}{(n!)^2}\right)\left(\frac{(n!)^2}{(k!)^2((n-k)!)^2}\right) = \binom{2n}{n}\binom{n}{k}^2$$

Thus

$$P(E_k) = \frac{1}{4^{2n}}\binom{2n}{n}\binom{n}{k}^2$$

By Kolmogorov's third axiom of probability,

$$p_{2n} = P(E) = \sum_{k=0}^{n} P(E_k) = \frac{1}{4^{2n}}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2$$

□

(b) As a small flex, we take a different approach from the hint. Consider the polynomial

$$(1+x)^{2n} = (1+x)^n(1+x)^n$$

The coefficient on the x^n term is $\binom{2n}{n}$. Meanwhile, the coefficient on the x^k term for the polynomial $(1+x)^n$ is $\binom{n}{k}$. But by multiplying $(1+x)^n$ with itself, we find that the coefficient on the x^n term is also given by the convolutional sum

$$\binom{2n}{n}x^n = \sum_{k=0}^{n}\left(\binom{n}{k}x^k\right)\left(\binom{n}{n-k}x^{n-k}\right) = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}x^n$$

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}\binom{n}{k}^2$$

□

(c) We use the identity from part (b) to find

$$P(E) = \frac{1}{4^{2n}} \binom{2n}{n}^2$$

Let M be the number of returns to the origin. If we denote X_{2n} as an indicator RV for returning to the origin after $2n$ steps, we have by linearity of expectaion

$$\mathbb{E}\left[M\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[X_{2n}\right] = \sum_{n=0}^{\infty} p_{2n} = \sum_{n=0}^{\infty} \frac{1}{4^{2n}} \binom{2n}{n}^2$$

The convergence of this sum depends only on the behaviour of the summand for large n. In this regime, we use Stirling's approximation:

$$\mathbb{E}\left[M\right] \sim \sum_{n=0}^{\infty} \frac{1}{4^{2n}} \left(\frac{(2n)^{2n} e^{-2n} \sqrt{4\pi n}}{n^{2n} e^{-2n} (2\pi n)} \right)^2 = \sum_{n=0}^{\infty} \frac{1}{4^{2n}} \left(\frac{2^{2n}}{\sqrt{\pi n}} \right)^2 = \sum_{n=0}^{\infty} \frac{1}{\pi n}$$

This is a harmonic series, which diverges to infinity. Hence $\mathbb{E}\left[M\right] = +\infty$, so the transient walk is recurrent. $\square$