

## MATH 318 Homework 7

### Problem 1

- (a) The event  $E$  that the walker returns to the origin in  $2n$  steps is precisely the event that there are equally many left and right steps, as well as equally many up and down steps. Let  $E_k$  be the event that there are  $k$  left and right steps, and consequently  $n - k$  up and down steps. Notice that we require  $k \geq 0$  and  $n - k \geq 0 \therefore k \leq n$ . It follows that  $E_i \cap E_j = \emptyset$  for  $i \neq j$  and  $E = \cup_{k=0}^n E_k$ .

By symmetry, each permutation of steps is equally likely. There are  $\frac{1}{4^{2n}}$  total permutations, since there are  $2n$  steps with four options per step. The number of permutations in the event  $E_k$  is simply the number of ways to arrange  $k$  left and right steps, and  $n - k$  up and down steps. The multinomial coefficient gives us

$$\frac{(2n)!}{(k!)^2((n-k)!)^2} = \left( \frac{(2n)!}{(n!)^2} \right) \left( \frac{(n!)^2}{(k!)^2((n-k)!)^2} \right) = \binom{2n}{n} \binom{n}{k}^2$$

Thus

$$P(E_k) = \frac{1}{4^{2n}} \binom{2n}{n} \binom{n}{k}^2$$

By Kolmogorov's third axiom of probability,

$$p_{2n} = P(E) = \sum_{k=0}^n P(E_k) = \frac{1}{4^{2n}} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^2$$

□

- (b) As a small flex, we take a different approach from the hint. Consider the polynomial

$$(1+x)^{2n} = (1+x)^n (1+x)^n$$

The coefficient on the  $x^n$  term is  $\binom{2n}{n}$ . Meanwhile, the coefficient on the  $x^k$  term for the polynomial  $(1+x)^n$  is  $\binom{n}{k}$ . But by multiplying  $(1+x)^n$  with itself, we find that the coefficient on the  $x^n$  term is also given by the convolutional sum

$$\begin{aligned} \binom{2n}{n} x^n &= \sum_{k=0}^n \left( \binom{n}{k} x^k \right) \left( \binom{n}{n-k} x^{n-k} \right) = \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} x^n \\ \binom{2n}{n} &= \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^n \binom{n}{k}^2 \end{aligned}$$

□

(c) We use the identity from part (b) to find

$$P(E) = \frac{1}{4^{2n}} \binom{2n}{n}^2$$

Let  $M$  be the number of returns to the origin. If we denote  $X_{2n}$  as an indicator RV for returning to the origin after  $2n$  steps, we have by linearity of expectation

$$\mathbb{E}[M] = \sum_{n=0}^{\infty} \mathbb{E}[X_{2n}] = \sum_{n=0}^{\infty} p_{2n} = \sum_{n=0}^{\infty} \frac{1}{4^{2n}} \binom{2n}{n}^2$$

The convergence of this sum depends only on the behaviour of the summand for large  $n$ . In this regime, we use Stirling's approximation:

$$\mathbb{E}[M] \sim \sum_{n=0}^{\infty} \frac{1}{4^{2n}} \left( \frac{(2n)^{2n} e^{-2n} \sqrt{4\pi n}}{n^{2n} e^{-2n} (2\pi n)} \right)^2 = \sum_{n=0}^{\infty} \frac{1}{4^{2n}} \left( \frac{2^{2n}}{\sqrt{\pi n}} \right)^2 = \sum_{n=0}^{\infty} \frac{1}{\pi n}$$

This is a harmonic series, which diverges to infinity. Hence  $\mathbb{E}[M] = +\infty$ , so the transient walk is recurrent.  $\square$

## Problem 2

Let  $Y_i$  be the indicator RV that the  $i$ th sampled ball is black. For a given  $X$ , the probability of this occurring is then  $X/8$ . Now notice that  $Y = \sum_{i=1}^{10} Y_i$ . By linearity of expectation,

$$\mathbb{E}[Y|X] = \sum_{i=1}^{10} \frac{X}{8} = \frac{5}{4}X$$

Since  $Y|X$  is the sum of 10 Bernoulli trials, we have

$$P(Y = y|X = x) \sim \text{Binom}(10, x/8) = \binom{10}{y} \left(\frac{x}{8}\right)^y \left(\frac{8-x}{8}\right)^{10-y}$$

Since  $X$  is uniform,  $P(X = x) = 1/9$  for  $0 \leq x \leq 8$ . By Bayes' theorem,

$$\begin{aligned} P(X = x|Y = y) &= \frac{P(Y = y|X = x)P(X = x)}{\sum_{k=0}^8 P(Y = y|X = k)P(X = k)} = \frac{\left(\frac{x}{8}\right)^y \left(\frac{8-x}{8}\right)^{10-y}}{\sum_{k=0}^8 \left(\frac{k}{8}\right)^y \left(\frac{8-k}{8}\right)^{10-y}} \\ &= \frac{x^y (8-x)^{10-y}}{\sum_{k=0}^8 k^y (8-k)^{10-y}} \\ \mathbb{E}[X|Y] &= \sum_{x=0}^8 x \frac{x^y (8-x)^{10-y}}{\sum_{k=0}^8 k^y (8-k)^{10-y}} \end{aligned}$$

$$\mathbb{E}[X|Y] = \frac{\sum_{k=0}^8 k^{y+1} (8-k)^{10-y}}{\sum_{k=0}^8 k^y (8-k)^{10-y}}$$

**Problem 3**

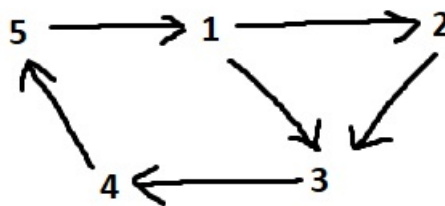
(a) Nonzero transition probabilities are described graphically as follows:



Since the states  $\{1, 2, 3\}$  and  $\{4, 5\}$  are in different communicating classes, the Markov chain is not irreducible. Looking at the graph, we also characterize each state:

- |   |   |
|---|---|
| { | 1: recurrent and periodic with period 2 |
|   | 2: transient and aperiodic              |
|   | 3: recurrent and periodic with period 2 |
|   | 4: transient and aperiodic              |
|   | 5: recurrent and aperiodic              |

(b) Nonzero transition probabilities are described graphically as follows:



Because we have the loop  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ , the Markov chain is irreducible. Hence, all states are recurrent. Finally, notice that  $P_{11}^4 > 0$  because we have the chain  $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$  but also  $P_{11}^5 > 0$  because of the chain  $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ . Since 4 and 5 are relatively prime, this state is aperiodic. But because the Markov chain is irreducible, all states are aperiodic.

(c) Every pair of states is communicating. To see how, we consider the path taken by incrementing/decrementing a single dimension's coordinate, repeating for each dimension, until the target node is reached from the source node. This path has a

probability of  $\frac{1}{(2d)^M} > 0$ , where  $M$  is the Manhattan distance between the two points. The transition probability  $P_{ij}^M$  between these two states over  $M$  steps is therefore at least  $\frac{1}{(2d)^M} > 0$ . Hence all symmetric random walks are irreducible. The origin is recurrent for  $d \leq 2$  and transient otherwise. Since recurrence is a class property, all states are recurrent for  $d \leq 2$  and transient otherwise. Finally, returning to a given node requires an even number of steps. To see why, notice that every step flips the parity of the sum of all coordinates. Furthermore, it is possible to return to a state by moving to the right and back i.e. in 2 steps. Thus all states are periodic with period 2.

- (d) Every pair of states is communicating. Suppose we have coordinates  $i < j$ . Then we can go from  $i$  to  $j$  in  $j - i$  steps with nonzero probability  $(\frac{2}{3})^{j-i}$  and go from  $j$  to  $i$  in  $j - i$  steps with nonzero probability  $(\frac{1}{3})^{j-i}$ . Thus the Markov chain is irreducible. Furthermore, every state is transient. To see why, recall that the asymmetric random walk on  $\mathbb{Z}$  has a finite expected number of returns to the origin. Because the Markov chain is irreducible, every state must be transient like the origin is. Finally, we may employ the parity logic used in part (c) to find that all states are periodic with period 2.

#### Problem 4

We compute

$$(P^T)^3 \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4028 \\ 0.1759 \\ 0.4213 \end{pmatrix}$$

Thus  $P(X_3 = 3) = 0.4213$ .