MATH 406, Fall 2021, Wetton
Assignment #1 - due Monday, September 27

Notes:

- The second assignment will be given out the Friday before this is due. Assignments will overlap every other weekend.
- No late assignments will be accepted. If you can't finish, hand in whatever you have on the due date.
- Collaborating on assignments is encouraged, but make sure you write up your answers on your own, using your own words. Make a note of the nature of the collaboration on the assignments.
- Submission will be done by pdf files and Jupyter notebooks on the canvas course page.
- Total marks for the assignment: 50.

1. [5 marks] Estimate the following quantities using interpolation of the given values of a function f. Without knowing more information about the function f (*i.e.* how large its derivatives are), it is impossible to know what the best technique to use is, but in this question, make use of all the given data in your answers.

(a) Estimate $f(1/2)$ if $f(0) = 0$ and $f(1) = 1$.

(b) Estimate $f(1/2)$ if $f(0) = 0$, $f(1) = 1$ and $f(-1) = -1/2$.

(c) Estimate $f'(0)$ if $f(0) = 0$, $f(1) = 1$, $f(-1) = -1/2$ and $f(2) = 2$.

(d) Estimate the location of a root of f if $f(0) = 1/2$, $f(1) = 1$, $f(-1) = -1$.

2. [5] How many terms of the Taylor series for $\tan^{-1}$ based at $x = 0$ need to be taken to ensure a *relative* accuracy of 1% for all $x \in [0, 1/2]$?

3. [5] Find the optimal four interpolation points for cubic approximation in $[0, 1]$. That is, consider a cubic function $F(x)$ that matches function f data at distinct points. The cubic will have optimal error bounds of the form

$$\max_{x\in[0,1]} |f(x) - F(x)| \leq CK_4h^4.$$

Find the points x_1, x_2, x_3 and x_4 that give the smallest constant C.

4. [5] Find the optimal interpolation points for cubic interpolation in $[0, 1]$ assuming that the end points $x = 0$ and $x = 1$ are used (so there's only two free points to pick). Find the corresponding "C" value as above.

5. [5] Compare to your answers to the two previous questions. Given that you can reuse the end values between subintervals, which method is more data efficient for a given accuracy when used to approximate a function on many subintervals?

6. [5] The function $\tan^{-1} x$ can be approximated by the bilinear function

$$B(x) = \frac{\pi}{2} \frac{x}{1+x}$$

Determine the maximum error in this approximation *for all* $x > 0$. Consider a more generalized approximation of the form

$$P(x) = \frac{C_0 + C_1 x + C_2 x^2}{1 + A_1 x + A_2 x^2}$$

where C_0, C_1, C_2, A_1, A_2 are constants. Determine values for these constants so that $P(x)$ has a smaller maximum error to $\tan^{-1} x$ for $x > 0$ than $B(x)$.

7. [5] Wolfram Alpha does not seem to be able to evaluate the indefinite integral

$$\int \frac{\exp(x^2)\cos(x)}{x^{2/3}} dx$$

analytically but it gives a numerical result for

$$\int_0^1 \frac{\exp(x^2)\cos(x)}{x^{2/3}} dx$$

with six significant digits. Evaluate the integral numerically with an error less than 1×10^{-3}. Describe your approach and how you convinced yourself that your answer was accurate enough (without using the Wolfram Alpha result).

8. [5] Use the grid values $F_{i+2}, F_{i+1}, F_i, F_{i-1}, F_{i-2}$ to derive formulas:

(a) A fourth order accurate approximation of the second derivative at a grid point x_i.

(b) A second order accurate approximation of the fourth derivative at a grid point x_i.

9. [10] Implement the adaptive Trapezoidal rule described in the notes. Test on the integral

$$\int_0^7 \sin(x^2)dx$$

for which an approximate answer accurate to six digits can be found using Wolfram Alpha. For this question *hand in your code* as well as the description and details of how you tested that it was working properly. Include a graph of the resulting grid used to evaluate the integral with a tolerance of $\delta = 10^{-4}$. I encourage you to submit this question as a Jupyter notebook.