Speed of Sound in a Solid with Lagrangian Mechanics

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Introduction

In this problem we derive an expression for the longitudinal speed of sound in a homogeneous isotropic elastic solid by modelling the solid as a three-dimensional cubic lattice. We first determine the speed of sound in terms of microscopic parameters of the material; subsequently we shall express the speed of sound in macroscopic material properties. For simplicity, we consider the case of a rectangular prism.

Part I

Consider a solid rectangular prism of dimensions $a \times b \times c$ at rest. We tie our inertial reference frame to a corner of the prism with axes along those of the prism as shown in **Figure 1**. We may model this solid prism as a $M \times N \times P$ cubic lattice of point masses,

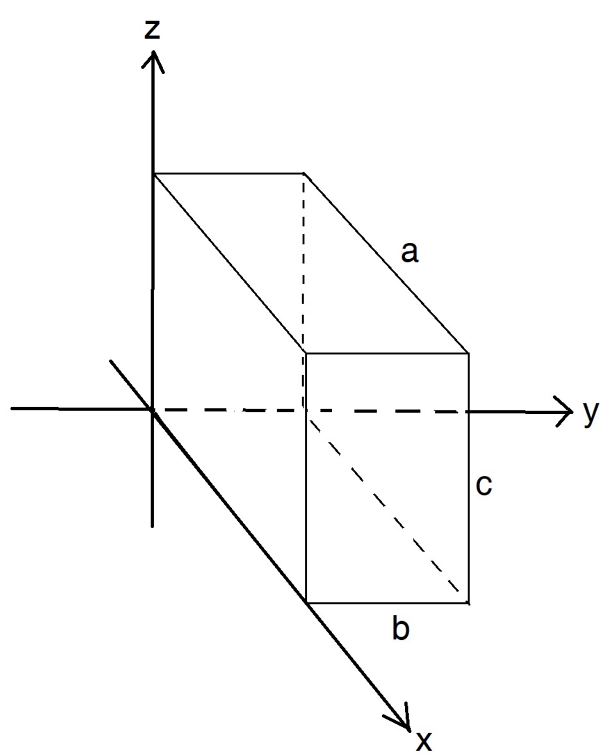
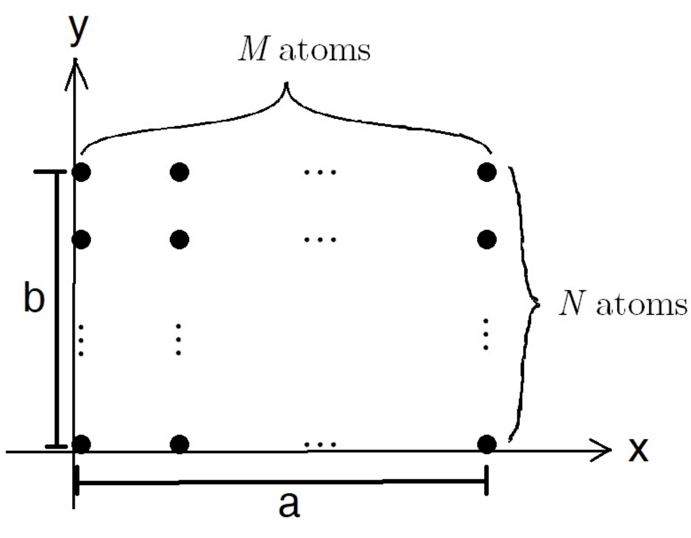


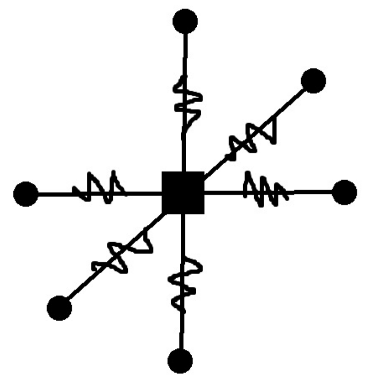
Figure 1: Sketch of the solid prism and our inertial reference frame for **Part I**. The solid is of side lengths a, b, and c along the x-, y-, and z-axes, respectively.

each with mass m. These point masses represent the solid's atoms. This is illustrated in

Figure 2a. The bond potential energy between two adjacent atoms experiences a local



(a)



(b)

Figure 2: (a) Sketch of the crystal lattice projected on the xy-plane. The black dots represent atoms; each atom in the lattice is spaced out by a distance l_0 at equilibrium. In total there are M, N, and P atoms along the x-, y-, and z-axes, respectively. Although this is not depicted, this lattice repeats similarly along the z-axis. (b) Sketch of the interaction between an interior atom (square) and its six neighbours (circles). This interaction is modelled as a spring of equilibrium length l_0 and spring constant K. It follows that these springs lie along the x-, y-, and z-axes (two on each).

minimum at an equilibrium length l_0. Applying the small oscillations approximation to this stable equilibrium, we model the interaction between two adjacent atoms as a harmonic oscillator (i.e. spring) of equilibrium displacement l_0 and spring constant K. Furthermore, we take the interaction between non-adjacent atoms to be negligible. For interior (i.e. not on the solid's boundary) atoms, this is depicted in **Figure 2b** (as we shall see, the boundary atoms and their interactions with the external environment determine the boundary conditions). Finally, we neglect gravity throughout this problem.

To get started, how many degrees of freedom in this system? What set of generalized coordinates could you use to describe this system?

Part II

We may label each atom in the lattice with indices i, j, and k. We shall label the corner atom at the origin with $i = 1$, $j = 1$, and $k = 1$. Then our indices range over $1 \leq i \leq M$, $1 \leq j \leq N$, and $1 \leq k \leq P$. The atom adjacent in the x-direction to the origin corner atom, for example, would have indices $(i, j, k) = (2, 1, 1)$. Two atoms above this atom in the z-direction would have indices $(i, j, k) = (2, 1, 3)$, etc. We may denote the (global i.e. when every atom is at equilibrium) equilibrium x, y, and z positions of the (i, j, k) atom as $\tilde{x}_{ijk}$, $\tilde{y}_{ijk}$, and $\tilde{z}_{ijk}$, respectively.

Determine $\tilde{x}_{ijk}$, $\tilde{y}_{ijk}$, and $\tilde{z}_{ijk}$ in terms of i, j, k, and l_0. What are the dimensions a, b, and c of the prism in terms of M, N, P, and l_0?

Part III

We similarly may denote the (current) x, y, and z positions of the (i, j, k) particle as x_{ijk}, y_{ijk}, and z_{ijk}, respectively. In terms of these coordinates, find the total kinetic and potential energies of the system. What is the Lagrangian?

Part IV

Find the Euler-Lagrange equations for the (i, j, k) interior atom ($2 \leq i \leq M - 1$, $2 \leq j \leq N - 1$, $2 \leq k \leq P - 1$) i.e. for x_{ijk}, y_{ijk}, and z_{ijk}.