Speed of Sound in a Solid with Lagrangian Mechanics

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Introduction

In this problem we derive an expression for the longitudinal speed of sound in a homogeneous isotropic elastic solid by modelling the solid as a three-dimensional cubic lattice. We first determine the speed of sound in terms of microscopic parameters of the material; subsequently we shall express the speed of sound in macroscopic material properties. For simplicity, we consider the case of a rectangular prism.

Part I

Consider a solid rectangular prism of dimensions $a \times b \times c$ at rest. We tie our inertial reference frame to a corner of the prism with axes along those of the prism as shown in **Figure 1**. We may model this solid prism as a $M \times N \times P$ cubic lattice of point masses,

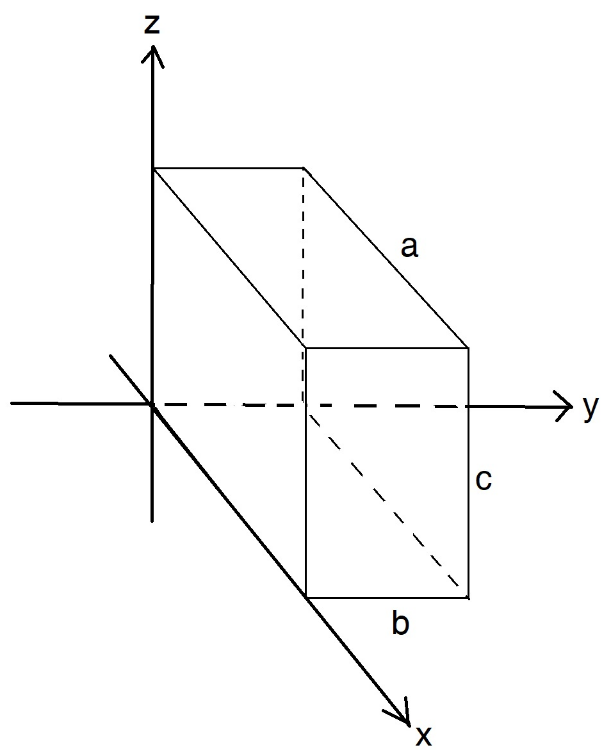
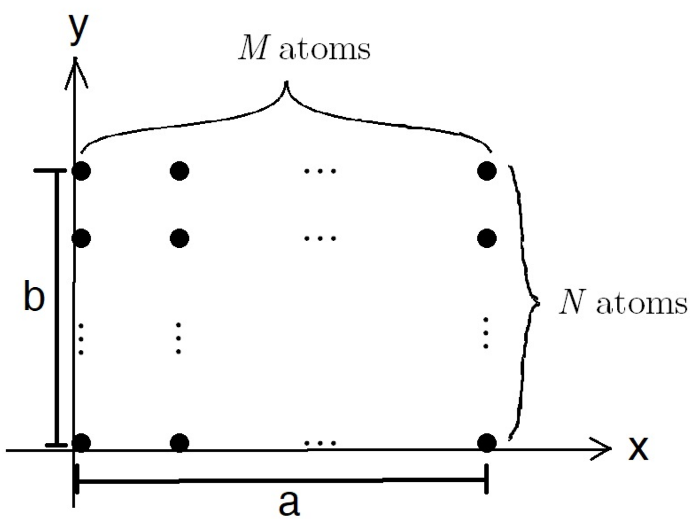


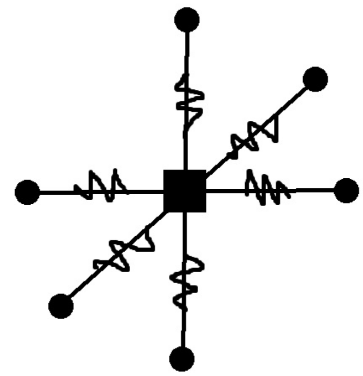
Figure 1: Sketch of the solid prism and our inertial reference frame for **Part I**. The solid is of side lengths a, b, and c along the x-, y-, and z-axes, respectively.

each with mass m. These point masses represent the solid's atoms. This is illustrated in

Figure 2a. The bond potential energy between two adjacent atoms experiences a local



(a)



(b)

Figure 2: (a) Sketch of the crystal lattice projected on the xy-plane. The black dots represent atoms; each atom in the lattice is spaced out by a distance l_0 at equilibrium. In total there are M, N, and P atoms along the x-, y-, and z-axes, respectively. Although this is not depicted, this lattice repeats similarly along the z-axis. (b) Sketch of the interaction between an interior atom (square) and its six neighbours (circles). This interaction is modelled as a spring of equilibrium length l_0 and spring constant k. It follows that these springs lie along the x-, y-, and z-axes (two on each).

minimum at an equilibrium length l_0. Applying the small oscillations approximation to this stable equilibrium, we model the interaction between two adjacent atoms as a harmonic oscillator (i.e. spring) of equilibrium displacement l_0 and spring constant k. Furthermore, we take the interaction between non-adjacent atoms to be negligible. This is depicted in **Figure 2b**. Finally, we neglect gravity throughout this problem.

To get started, how many degrees of freedom in this system? What set of generalized coordinates could you use to describe this system?