EPFL | MGT-418 : Convex Optimization | Project 6

Questions – Fall 2021

Lyapunov Stability

Description

Asserting stability is a major task both in dynamical system analysis and in control system design. The dynamics of a system with state $x(t) \in \mathbb{R}^n$ is defined through the equation $\dot{x}(t) := \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t))$ where f can be any Lipschitz continuous function (this ensures existence and uniqueness of solutions). A point $x_{\mathrm{eq}} \in \mathbb{R}^n$ at which $f(x_{\mathrm{eq}}) = 0$ is called an *equilibrium* because if the system reaches x_{eq}, it will remain in that state forever. Observe that, whenever an equilibrium exists, without loss of generality one can choose coordinates such that the equilibrium is situated at the origin, *i.e.*, $x_{\mathrm{eq}} = 0$.

Loosely speaking, an equilibrium is called *globally asymptotically stable* if the trajectories evolving from any initial state remain in a bounded neighborhood of the equilibrium for all times and converge to the equilibrium as time goes to infinity. This property is of interest in many settings. For instance, if the aim is to make a drone follow a reference trajectory, one is interested in keeping the difference between the reference position and the actual position of the drone as small as possible (thus bounded) and ideally making it converge to zero as time progresses.

One way of asserting the stability properties of an equilibrium is via the stability theory developed by Lyapunov. In this project, we focus on the following sufficient result for global asymptotic stability.

Theorem. *Consider a dynamical system $\dot{x}(t) = f(x(t))$ with equilibrium $x_{\mathrm{eq}} = 0$. If there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ with corresponding $\dot{V} : \mathbb{R}^n \to \mathbb{R}$ defined through $\dot{V}(x) := (\nabla_x V(x))^\top f(x)$ so that*

- $V(0) = 0, \quad V(x) > 0$ *for all* $x \neq 0, \quad V(x) \to \infty$ *as* $\|x\| \to \infty,$
- $\dot{V}(0) = 0, \quad \dot{V}(x) < 0$ *for all* $x \neq 0,$

then the equilibrium $x_{\mathrm{eq}} = 0$ is globally asymptotically stable.

Intuitively, one can think of $V(x)$ as the energy stored in the system (which is zero at the origin, positive everywhere else, and radially unbounded in all directions). By the chain rule, we can see that the time derivative of $V(x(t))$ along a trajectory $x(t)$ corresponds to $\dot{V}(x(t))$, that is,

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} = \big(\nabla_x V(x(t))\big)^\top \dot{x}(t) = \big(\nabla_x V(x(t))\big)^\top f(x(t)) = \dot{V}(x(t)).$$

Accordingly, the conditions on $\dot{V}(x)$ ensure that energy is dissipated along all trajectories, except at the origin. As the system is constantly losing energy and the origin is the point with the lowest energy level, one intuitively expects any trajectory to remain bounded and gradually approach the latter. For proofs and a more in depth discussion see Chapter 5 of Sastry, S. 1999. *Nonlinear Systems.*

Finding a suitable Lyapunov function $V(x)$, or even proving its existence, for a given dynamical system is generally difficult. However, in this project we will see how convex optimization can be used to automatically find Lyapunov functions for dynamical systems with polynomial dynamics. The key idea behind our approach is as follows: Observe that any pair of functions $V(x)$ and $\dot{V}(x)$ fulfilling

$$V(0) = 0,\ V(x) - \|x\|_2^2 \geq 0 \ \ \forall x \in \mathbb{R}^n \quad \text{and} \quad \dot{V}(0) = 0,\ \ -\dot{V}(x) - \|x\|_2^2 \geq 0 \ \ \forall x \in \mathbb{R}^n \qquad (\star)$$

immediately satisfies the conditions of the theorem, therefore asserting stability. Observe further that if $V(x)$ is chosen to be a polynomial and the system dynamics $f(x)$ are polynomial, then $\dot{V}(x)$ is a polynomial as well. In this case checking the conditions $(\star)$ therefore boils down to establishing the nonnegativity of polynomials, which, as seen in class, can be done via semidefinite programming.

Questions

1. **Hanging Pendulum (Stable Linear System)**: Figure 1 below displays a planar mechanical pendulum consisting of a point mass m that is attached to a rigid, massless rod of length L. The system state consists of the angular position $\varphi(t)$ and the angular velocity $\dot{\varphi}(t)$ of the pendulum. The system moves under the influence of a constant, downward pointing gravitational force mg and experiences a slight damping torque $\delta\dot{\varphi}(t)$ due to friction in the pivoting joint.

   The equations of motion of the pendulum are $mL^2\ddot{\varphi}(t) + \delta\dot{\varphi}(t) + mgL\sin(\varphi(t)) = 0$ which linearized around the hanging equilibrium $(\varphi^{\mathrm{h}}_{\mathrm{eq}}, \dot{\varphi}^{\mathrm{h}}_{\mathrm{eq}}) = (0, 0)$ results in the linear dynamics

$$\dot{x}(t) = A^{\mathrm{h}}x(t), \quad \text{where} \quad x(t) = (\varphi(t), \dot{\varphi}(t))^\top, \; A^{\mathrm{h}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{\delta}{mL^2} \end{bmatrix}. \tag{1}$$

   1.1 Consider the dynamical system (1) and the candidate Lyapunov function $V(x) = x^\top Px$ with $P \in \mathbb{S}^2$. Note that $V(0) = 0$ and $\dot{V}(0) = 0$ by construction. Show that in this context, the conditions $(\star)$ are equivalent to

$$P - \mathbf{I} \succeq 0 \qquad \text{and} \qquad PA^{\mathrm{h}} + (A^{\mathrm{h}})^\top P + \mathbf{I} \preceq 0.$$

   As a consequence, if there exists a P satisfying the above two conditions, then the hanging equilibrium of the pendulum is asserted to be globally asymptotically stable in (1).

   1.2 Implement the above semidefinite program in CVXPY and solve it with MOSEK. As this is a feasibility problem, the objective function can be set to zero. Once a feasible P is found, plot some level curves of the Lyapunov function together with a few trajectories starting at different initial conditions. A skeleton of the code containing the numerical values of all parameters and the necessary plotting commands is provided in the Python file `p5q1.py`.

2. **Standing Pendulum (Stabilized Linear System)**: Consider again the pendulum displayed in Figure 1 below, where the joint is additionally equipped with a motor that allows to apply a torque $bu(t)$ to the pendulum. Here, $u(t) \in \mathbb{R}$ denotes our control signal, and b represents the motor's transmission factor. With this addition, there is hope to stabilize the pendulum not only at its hanging equilibrium but also at its (otherwise unstable) standing equilibrium.

   The controlled pendulum's motion is governed by $mL^2\ddot{\varphi}(t) + \delta\dot{\varphi}(t) + mgL\sin(\varphi(t)) + bu(t) = 0$ which linearized around the standing equilibrium $(\varphi^{\mathrm{s}}_{\mathrm{eq}}, \dot{\varphi}^{\mathrm{s}}_{\mathrm{eq}}) = (\pi, 0)$ of the free pendulum results in the linear dynamics

$$\dot{x}(t) = A^{\mathrm{s}}x(t) + Bu(t) \quad \text{where} \quad x(t) = (\varphi(t) - \pi, \dot{\varphi}(t))^\top, \; A^{\mathrm{s}} = \begin{bmatrix} 0 & 1 \\ +\frac{g}{L} & -\frac{\delta}{mL^2} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ -\frac{b}{mL^2} \end{bmatrix}. \tag{2}$$

   To stabilize the pendulum at its standing equilibrium, we apply a static state feedback policy, *i.e.*, we set $u(t) = Kx(t)$ where $K = \begin{bmatrix} k & 0 \end{bmatrix}$ and k is a suitably chosen, static control gain.

   2.1 Consider the dynamical system (2), controlled by the described static feedback policy, and the candidate Lyapunov function $V(x) = x^\top Px$ with $P \in \mathbb{S}^2$. Observe that $V(0) = 0$ and $\dot{V}(0) = 0$ by construction. Show that in this context, the conditions $(\star)$ are equivalent to

$$P - \mathbf{I} \succeq 0 \qquad \text{and} \qquad P(A^{\mathrm{s}} + BK) + (A^{\mathrm{s}} + BK)^\top P + \mathbf{I} \preceq 0.$$

   As a consequence, if there exists a P satisfying the above two conditions, then the standing equilibrium of the pendulum is asserted to be globally asymptotically stable in (2).

   2.2 Implement the above semidefinite program in CVXPY and solve it with MOSEK. As this is a feasibility problem, the objective function can be set to zero. Once a feasible P is found, plot some level curves of the Lyapunov function together with a few trajectories starting at different initial conditions. A skeleton of the code containing the numerical values of all parameters and the necessary plotting commands is provided in the Python file `p5q2.py`.

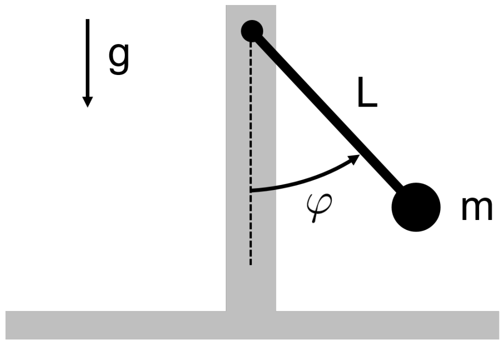

Figure 1: A planar mechanical pendulum.

3. **Jet Engine (Stabilized Nonlinear System)**: Stall and surge are undesirable effects that may affect the compressor stage of jet engines. In addition to preventive design procedures, specific control systems are typically in place to stabilize the axial velocity $\phi(t)$ and the circumferential velocity $\psi(t)$ of a fluid traversing the engine at desirable nominal levels ϕ_{ref} and ψ_{ref}, respectively. A simplified description of the resulting system dynamics is

$$\dot{x}(t) = f(x(t)) \quad \text{where} \quad x(t) = (\phi(t) - \phi_{\text{ref}}, \psi(t) - \psi_{\text{ref}})^\top, \;\; f(x) = \begin{bmatrix} -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ 3x_1 - x_2 \end{bmatrix}. \tag{3}$$

Recall that a polynomial $p(x)$ of degree $2m$ can be represented in two ways, namely

$$\text{linearly:} \quad p(x) = p^\top z_{2m}(x), \qquad\qquad \text{quadratically:} \quad p(x) = (z_m(x))^\top P z_m(x),$$

where $p \in \mathbb{R}^{s(n,2m)}$ is the polynomial coefficients vector, $z_{2m}(x) \in \mathbb{R}^{s(n,2m)}$ and $z_m(x) \in \mathbb{R}^{s(n,m)}$ denote the vector of monomials up to degree $2m$ and m, respectively, and $P \in \mathbb{S}^{s(n,m)}$. Observe that, while the linear representation is unique, usually the quadratic representation is not.

3.1 Consider the dynamical system (3) and choose a fourth degree polynomial as candidate Lyapunov function, *i.e.*, set $V(x) = p^\top z_4(x)$ with $p \in \mathbb{R}^{15}$. Compute the polynomial $\dot{V}(x)$ and express it quadratically, *i.e.*, determine some $Q(p) \in \mathbb{S}^{10}$ so that $\dot{V}(x) = (z_3(x))^\top Q(p) z_3(x)$. *Hint:* Observe that $\frac{\mathrm{d}z_4(x)}{\mathrm{d}x_i} = D_i z_3(x)$ and that $f_i(x) = f_i^\top z_3(x)$ for an appropriate matrix $D_i \in \mathbb{R}^{15\times 10}$ and an appropriate vector $f_i \in \mathbb{R}^{10}$, respectively, which should be determined.

3.2 A linear representation of the latter polynomial is $\dot{V}(x) = q^\top z_6(x)$ with $q \in \mathbb{R}^{28}$. Write down conditions on q and $Q(p)$ ensuring that both representations express the same polynomial.

3.3 Enforce that $V(0) = 0$ and $\dot{V}(0) = 0$ through a (very) simple condition on p and q.

3.4 Observe that $V(x) - \|x\|_2^2 = \tilde{p}^\top z_4(x)$ for some $\tilde{p} \in \mathbb{R}^{15}$ and that $-\dot{V}(x) - \|x\|_2^2 = \tilde{q}^\top z_6(x)$ for some $\tilde{q} \in \mathbb{R}^{28}$. Write down the connection between p and $\tilde{p}$, and between q and $\tilde{q}$.

3.5 Define $\tilde{p}(x) := \tilde{p}^\top z_4(x)$ and $\tilde{q}(x) := \tilde{q}^\top z_6(x)$. We are interested in proving the nonnegativity of these polynomials as this would assert that the conditions ($\star$) are satisfied and thus that the nominal flow speeds are globally asymptotically stable in (3). As discussed in class, a sufficient condition for this is the existence of matrices $\tilde{P} \in \mathbb{S}^6$ and $\tilde{Q} \in \mathbb{S}^{10}$ such that

$$\tilde{P} \succeq 0, \;\; \tilde{p}_\alpha = \sum_{\substack{|\beta|\le 2,\, |\gamma|\le 2, \\ \beta+\gamma=\alpha}} \tilde{P}_{\beta\gamma} \;\; \forall |\alpha| \le 4 \qquad \text{and} \qquad \tilde{Q} \succeq 0, \;\; \tilde{q}_\alpha = \sum_{\substack{|\beta|\le 3,\, |\gamma|\le 3, \\ \beta+\gamma=\alpha}} \tilde{Q}_{\beta\gamma} \;\; \forall |\alpha| \le 6,$$

where α, β and γ are double indices. Use the results derived in parts 3.1–3.4 to formulate a semidefinite program with decision variables p, q, $\tilde{P}$ and $\tilde{Q}$ which, if feasible, establishes the desired stability result.

3.6 Implement the semidefinite program derived in part 3.5 using CVXPY and solve it with MOSEK. Retrieve the feasible solution vector p and plot some level curves of the Lyapunov function together with a few trajectories starting at different initial conditions. A skeleton of the code containing the necessary plotting commands is provided in the Python file `p5q3.py`.