COA 620 Introduction to Bayesian Statistics in Ecology

Lecture 6 – Components of Bayes' Theorem

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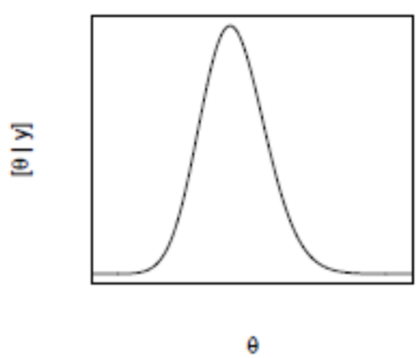
Bayesian inference

All unobserved quantities are treated as random variables. We seek to understand the characteristics of the probability distributions governing the behavior of these random variables.



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Exercise

Assume we have two jointly distributed random variables: θ and y. The random variable θ represents unobserved quantities of interest. The random variable y represents observations, which become fixed after they are observed.

Derive Bayes' theorem

$$[\theta \mid y] = \frac{[y \mid \theta][\theta]}{[y]}$$

Using your knowledge of laws of probability, particular the definition of conditional probability.

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We will often make use of the equivalent equation

$$[\theta \mid y] = \frac{[y, \theta]}{[y]}$$

As starting point for developing hierarchical models by factoring [y,θ] into ecologically sensible components that can be treated in Markov Chain Monte Carlo simulation as univariate distributions.

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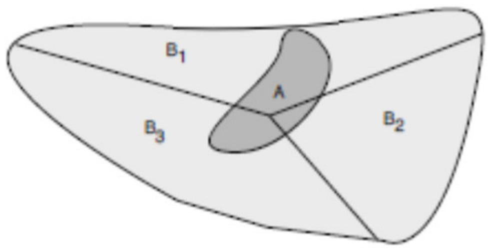
What is [y]

Recall the law of total probability for discrete random variables



$$[A] = \sum_i [A | B_i][B_i]$$

and for continuous random variables

$$[\mathrm{A}] = \int_B [A \mid B][B]dB$$

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What is [y]

It follows that

$$[y] = \sum_{\theta_i \in \{\Theta\}} [y \mid \theta_i][\theta_i] \text{ for discrete parameters}$$

$$[\mathrm{y}] = \int_\theta [y \mid \theta][\theta]d\theta \text{ for continuous parameters}$$

$$[\theta \mid \mathrm{y}] = \frac{[\mathrm{y} \mid \theta_\mathrm{i}][\theta_i]}{\sum_{\theta_i \in \{\Theta\}} [y \mid \theta_i][\theta_i]} \text{ for discrete valued parameters}$$

$$[\theta \mid \mathrm{y}] = \frac{[\mathrm{y} \mid \theta_\mathrm{i}][\theta_i]}{\int_\theta [y \mid \theta][\theta]d\theta} \text{ for parameters that are continous}$$

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The components of Bayes theorem

$$\overbrace{[\theta|y]}^{\text{Posterior}} = \frac{\overbrace{[y|\theta]}^{\text{likelihood}} \; \overbrace{[\theta]}^{\text{prior}}}{\underbrace{\int_\theta [y|\theta]\,[\theta]\,d\theta}_{\text{marginal distribution of data}}}$$

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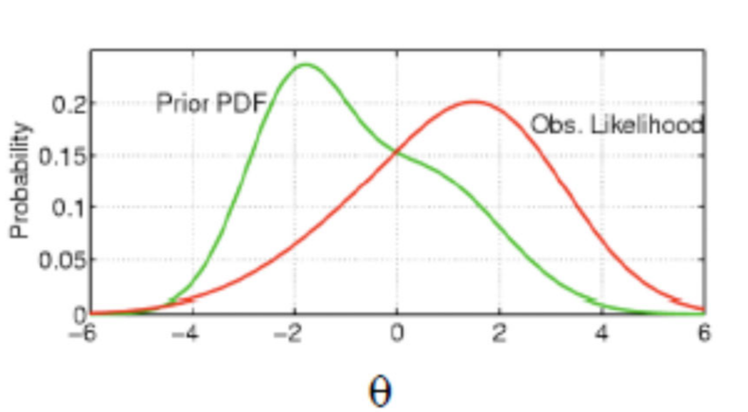
The components of Bayes theorem



Courtesy of Chris Wikle, University of Missouri

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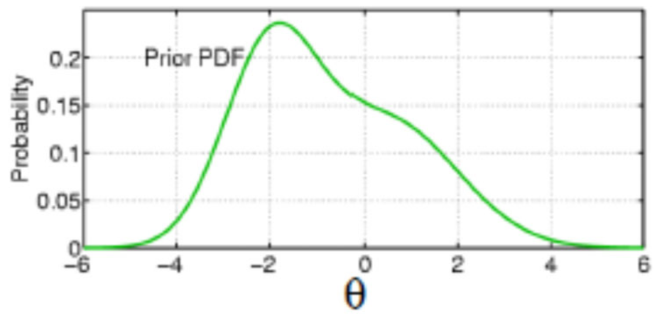
$$[\theta \mid y] = \frac{[y \mid \theta][\theta]}{\int_{\theta} [y \mid \theta][\theta] d\theta}$$

The prior, [θ], can be informative or vague.



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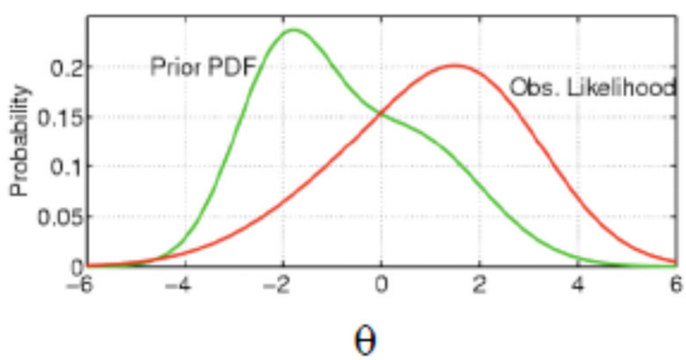
$$[\theta \mid y] = \frac{[y \mid \theta][\theta]}{\int_{\theta} [y \mid \theta][\theta] d\theta}$$

The likelihood (data distribution [y|θ])



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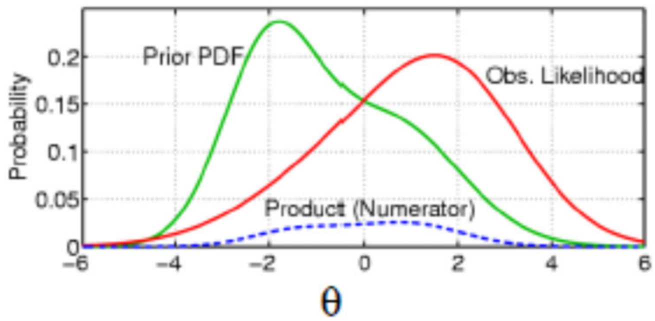
$$[\theta \mid y] = \frac{[y,\theta]}{[y]} = \frac{[y \mid \theta][\theta]}{\int_\theta [y \mid \theta][\theta] d\theta}$$

The product of the prior and the likelihood, [y|θ][θ], is the joint distribution of the parameters and the data, [y,θ]



What is the maximum likelihood estimate of θ?

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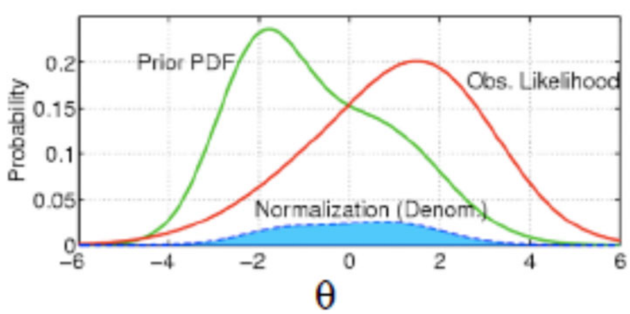
$$[\theta \mid y] = \frac{[y \mid \theta][\theta]}{\int_\theta [y \mid \theta][\theta] d\theta}$$

The marginal distribution of the data (the denominator) is the area under the joint distribution.



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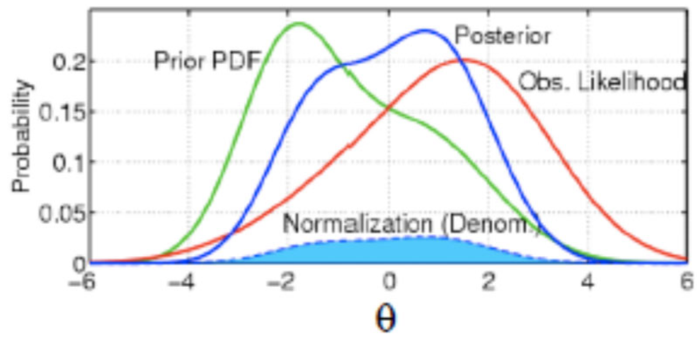
What are we seeking: The Posterior distribution [θ|y]

$$[\theta \mid y] = \frac{[y \mid \theta][\theta]}{\int_\theta [y \mid \theta][\theta] d\theta}$$

- Posterior distribution represents a balance between the information contained in the likelihood and the information contained in the prior distribution.
- An informative prior influences the posterior distribution. A vague prior exerts minimal influence on posterior distribution.



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Likelihood [y|θ]

Probability of two whites on three draws conditional on θ_i

Parameter	Likelihood $[y \mid \theta_i]$
$\theta_1 = 5/6$	.347
$\theta_2 = 1/2$	.375
$\theta_3 = 1/6$	.069
$\sum_{i=1}^{3} [y \mid \theta_i] =$	.791

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Posterior [θ|y]

Probability of two whites on three draws conditional on θ_i

Probability of θ_i conditional on two whites on three draws

Parameter	Prior $[\theta_i]$	Likelihood $[y\|\theta_i]$	Joint $[y\|\theta_i]\,[\theta_i]$	Posterior $\frac{[y\|\theta_i][\theta_i]}{[y]} = [\theta_i\|y]$
θ_1	0.333	0.347	0.115	0.439
θ_2	0.333	0.375	0.125	0.474
θ_3	0.333	0.069	0.023	0.087
		$[y] = \sum_{i=1}^{3} [y\|\theta_i]\,[\theta_i] =$	0.261	$\sum_{i=1}^{3} [\theta_i\|y] = 1$

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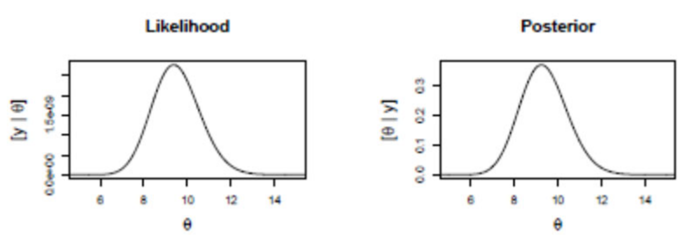
[y] is critical to Bayes

Y=(5,10,11,12,14,9,8,6)

$$likelihood = \prod_{i=1}^{8} Poisson(y_i \mid \theta)$$

$$posterior = \frac{\prod_{i=1}^{8} Poisson(y_i \mid \theta) gamma(\theta \mid .0001,.0001)}{[y]}$$



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