VV186 RC7

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Vector Space

3.3.1. Definition. A triple $(V, +, \cdot)$ is called a ***real vector space*** (or ***real linear space***) if

1. V is any set;
2. $+ \colon V \times V \to V$ is a map (called addition) with the following properties:
   - $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$ (***associativity***),
   - $u + v = v + u$ for all $u, v \in V$ (***commutativity***),
   - there exists an element $e \in V$ such that $v + e = v$ for all $v \in V$ (***existence of a unit element***),
   - for every $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = e$;
3. $\cdot \colon \mathbb{R} \times V \to V$ is a map (called scalar multiplication) with the following properties:
   - $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$ for all $\lambda \in \mathbb{R}$, $u, v \in V$,
   - $(\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$,
   - $(\lambda\mu) \cdot u = \lambda \cdot (\mu \cdot u)$ for all $\lambda, \mu \in \mathbb{R}$, $u \in V$.

If we replace $\mathbb{R}$ with $\mathbb{C}$, we say that $(V, +, \cdot)$ is a ***complex vector (or linear) space***.

Subspace

3.3.4. Definition. Let $(V, +, \cdot)$ be a real or complex vector space. If $U \subset V$ and $(U, +, \cdot)$ is also a vector space, then we say that $(U, +, \cdot)$ is a ***subspace*** of $(V, +, \cdot)$.

3.3.6. Lemma. Let $(V, +, \cdot)$ be a real (complex) vector space and $U \subset V$. If $u_1 + u_2 \in U$ for $u_1, u_2 \in U$ and $\lambda u \in U$ for all $u \in U$ and $\lambda \in \mathbb{R}$ $(\mathbb{C})$, then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$.

Normed Vector Space

3.3.8. Definition. Let V be a real (complex) vector space. Then a map $\|\cdot\|: V \to \mathbb{R}$ is called a norm if for all $u, v \in V$ and all $\lambda \in \mathbb{R}$ $(\mathbb{C})$,

1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$,
2. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$,
3. $\|u + v\| \leq \|u\| + \|v\|$.

The pair $(V, \|\cdot\|)$ is called a normed vector space or a normed linear space.

Normed Vector Space: Example

1. $\mathbb{R}^n$ with $\|x\|_2 = \left(\sum_{j=1}^{n} x_i^2\right)^{1/2}$,
2. $\mathbb{R}^n$ with $\|x\|_p = \left(\sum_{j=1}^{n} |x_i|^p\right)^{1/p}$ for any $p \in \mathbb{N} \setminus \{0\}$,
3. $\mathbb{R}^n$ with $\|x\|_\infty = \max_{1 \le k \le n} |x_k|$,
4. l^∞ with $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$,
5. c_0 with $\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$,
6. $C([a, b])$, $[a, b] \subset \mathbb{R}$, with $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$

Sequence of Functions

1. ***pointwise convergence***: For every $x \in [-1, 1]$,

$$f_n(x) \xrightarrow{n\to\infty} f(x) \qquad :\Leftrightarrow \qquad |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0$$

2. ***uniform convergence***: Each f_n is an element of the normed vector space $C([-1, 1])$, and so is f. Then

$$f_n \xrightarrow{n\to\infty} f \qquad :\Leftrightarrow \qquad \|f_n - f\|_\infty \xrightarrow{n\to\infty} 0.$$

Sequence of Functions

Remark:

Pointwise convergence describes the property of single points respectively while uniform convergence describes the property of the entire function.

Sequence of Functions

3.4.1. Definition. Let $\Omega \subset \mathbb{R}$ and (f_n) be a sequence of functions $f_n \colon \Omega \to \mathbb{C}$. We say that the sequence (f_n) converges pointwise to the function $f \colon \Omega \to \mathbb{C}$ if

$$\underset{x\in\Omega}{\forall} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0.$$

If f is the pointwise limit of (f_n), we say that (f_n) converges ***uniformly*** to f on Ω if

$$\sup_{x\in\Omega} |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0.$$

Pointwise convergence is the basis of uniform convergence.

Uniform convergence implies pointwise convergence.

Sequence of Functions

3.4.3. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval. Let (f_n) be a sequence of continuous functions defined on $[a, b]$ such that $f_n(x)$ converges to some $f(x) \in \mathbb{R}$ as $n \to \infty$ for every $x \in [a, b]$. If the sequence (f_n) converges uniformly to the thereby defined function $f : [a, b] \to \mathbb{R}$, then f is continuous.

3.4.4. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval and $C([a, b])$ the vector space of continuous functions on $[a, b]$, endowed with the metric

$$\varrho(f, g) = \|f - g\|_\infty = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Then the metric space $(C([a, b]), \varrho)$ is complete, i.e., every Cauchy sequence in the space converges.

Series: Convergence

3.5.1. Definition. Let (a_n) be a sequence in a normed vector space $(V, \|\cdot\|)$. Then we say that (a_n) is ***summable*** with sum $s \in V$ if

$$\lim_{n\to\infty} s_n = s, \qquad\qquad s_n := \sum_{k=0}^{n} a_k.$$

We call s_n the ***nth partial sum*** of (a_n). We use the notation

$$\sum_{k=0}^{\infty} a_k \qquad \text{or simply} \qquad \sum a_k \tag{3.5.1}$$

to denote not only s, but also the "procedure of summing the sequence (a_n)." We call (3.5.1) an ***infinite series*** and we say that the series converges if (a_n) is summable. If (s_n) does not converge, we say that $\sum a_k$ diverges.

Cauchy Criterion

3.5.4. Cauchy Criterion. Let $\sum a_k$ be a series in a **complete** vector space $(V, \|\cdot\|)$. Then

$$
\begin{aligned}
\sum a_k \text{ converges} \quad &\Leftrightarrow \quad (s_n)_{n\in\mathbb{N}} \text{ converges, } s_n = \sum_{k=0}^{n} a_k \\
&\Leftrightarrow \quad (s_n) \text{ is Cauchy} \\
&\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m>n>N}{\forall}\ \|s_m - s_n\| < \varepsilon \\
&\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m>n>N}{\forall}\ \left\| \sum_{k=n+1}^{m} a_k \right\| < \varepsilon
\end{aligned}
$$

Corollary given by Cauchy Criterion

3.5.5. Corollary. If the series $\sum_{k=0}^{\infty} a_k$ converges, then the sequence $a_k \to 0$ as $k \to \infty$. (Take $m = n + 1$ in the Cauchy Criterion.)

3.5.6. Corollary. If the series $\sum_{k=0}^{\infty} a_k$ converges, then the sequence (A_n) given by

$$A_n := \sum_{k=n}^{\infty} a_k$$

converges to 0 as $n \to \infty$. (Let $m \to \infty$ in the Cauchy Criterion.)

Absolute Convergence

3.5.9. Definition. A series $\sum a_k$ in a normed vector space $(V, \|\cdot\|)$ is called ***absolutely convergent*** if $\sum \|a_k\|$ converges.

A sequence (a_k) in a normed vector space $(V, \|\cdot\|)$ is called ***absolutely summable*** if $\sum a_k$ converges absolutely.

3.5.10. Theorem. An absolutely convergent series $\sum a_k$ in a **complete** vector space $(V, \|\cdot\|)$ is convergent.

Comparison Test

The following criteria are used to establish the absolute convergence of a series.

3.5.13. Comparison Test. Let (a_k) and (b_k) be real-valued sequences with $0 \le a_k \le b_k$ for sufficiently large k. Then

$$\sum b_k \text{ converges} \quad \Rightarrow \quad \sum a_k \text{ converges.}$$

Remark: It is the most general test. You are recommended to try this method with the help of basic inequalities to judge convergence of series first.

the Weierstrass M-Test

3.5.17. Weierstraß M-test. Let $\Omega \subset \mathbb{R}$ and (f_k) be a sequence of functions defined on Ω, $f_k \colon \Omega \to \mathbb{C}$, satisfying

$$\sup_{x \in \Omega} |f_k(x)| \le M_k, \qquad k \in \mathbb{N} \tag{3.5.9}$$

for a sequence of real numbers (M_k). Suppose that $\sum M_k$ converges. Then the limit

$$f(x) := \sum_{k=0}^{\infty} f_k(x) \qquad \text{exists for every } x \in \Omega.$$

Furthermore, the sequence (F_n) of partial sums

$$F_n(x) = \sum_{k=0}^{n} f_k(x)$$

converges uniformly to f.

the Root Test

3.5.20. Root Test. Let $\sum a_k$ be a series of positive real numbers $a_k \geq 0$.

(i) Suppose that there exists a $q < 1$ such that

$$\sqrt[k]{a_k} \leq q \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ converges.

(ii) Suppose that

$$\sqrt[k]{a_k} > 1 \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ diverges.

3.5.21. Remark. Note that the existence of a $q < 1$ so that $\sqrt[k]{a_k} < q$ is crucial; this is not the same as requiring $\sqrt[k]{a_k} < 1$.

3.5.24. Root Test. Let a_k be a sequence of positive real numbers $a_k \geq 0$. Then

$$\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} < 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{converges,}$$

$$\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} > 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{diverges.}$$

3.5.25. Remarks.

(i) No statement is possible if $\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} = 1$.

(ii) If $\lim_{k\to\infty} \sqrt[k]{a_k}$ exists, it equals $\overline{\lim_{k\to\infty}} \sqrt[k]{a_k}$. This will be the case in many applications.

the Ratio Test

3.5.26. Ratio Test. Let $\sum a_k$ be a series of strictly positive real numbers $a_k > 0$.

(i) Suppose that there exists a $q < 1$ such that
$$\frac{a_{k+1}}{a_k} \leq q \qquad \text{for all sufficiently large } k.$$
Then $\sum a_k$ converges.

(ii) Suppose that
$$\frac{a_{k+1}}{a_k} \geq 1 \qquad \text{for all sufficiently large } k.$$
Then $\sum a_k$ diverges.

Remark: Every

the Ratio Test Using Limits

3.5.28. Ratio Test. Let (a_k) be a sequence of strictly positive real numbers $a_k > 0$. Then

$$\overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} < 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \quad \text{converges,}$$

$$\underline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} > 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \quad \text{diverges.}$$

Remark: Any problem that can be solved by the ratio test can also be solved by the root test. The root test is a more powerful tool.

the Ratio Comparison Test

3.5.29. Ratio Comparison Test. Let (a_k) and (b_k) be sequences of strictly positive real numbers $a_k, b_k > 0$. Suppose that $\sum b_k$ converges. If

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k} \qquad \text{for sufficiently large } k,$$

then $\sum a_k$ converges.

Remark: if the condition of the ration comparison test is satisfied, then the sequence cannot increase rapidly.

Raabe's test

3.5.32. Raabe's Test. Let $\sum a_k$ be a series of positive real numbers $a_k \geq 0$. Suppose that there exists a number $p > 1$ such that

$$\frac{a_{k+1}}{a_k} \leq 1 - \frac{p}{k} \qquad \text{for sufficiently large } k.$$

Then the series $\sum a_k$ converges.

Remark: For some sequences, if we cannot find a suitable q for the ratio test, we can try to use Raabe's test.

Some comments about the five tests.

Pay attention to the p-Series.

the Leibniz Theorem

3.5.38. Leibniz Theorem. Let $\sum \alpha_k$ be a complex series whose partial sums are bounded but need not converge. Let (a_k) be a decreasing convergent sequence with limit zero, $a_k \searrow 0$. Then the series

$$\sum \alpha_k a_k \qquad \text{converges.}$$

A typical application of the Leibniz Theorem 3.5.38 are alternating series, for which $\alpha_k = (-1)^k$. In particular, the Leibniz series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converges by the Leibniz Theorem.

Cauchy Product

3.5.40. Theorem. Let $\sum a_k$ and $\sum b_k$ be absolutely convergent series. Then the ***Cauchy product*** $\sum c_k$ given by

$$c_k := \sum_{i+j=k} a_i b_j$$

converges absolutely and $\sum c_k = \left(\sum a_k\right)\left(\sum b_k\right)$.

3.5.41. Remark. If $a = (a_k)$ and $b = (b_k)$ are two absolutely summable sequences, the sequence

$$a * b := (c_k), \qquad c_k := \sum_{i+j=k} a_i b_j,$$

is called the ***convolution*** of a and b.

Reference

1. VV186 Slide and previous RC Slide