VV186 RC4

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Subsequence: Definition

2.2.25. Definition. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. Then the composition

$$(a_{n_k})_{k\in\mathbb{N}} := (a_n)_{n\in\mathbb{N}} \circ (n_k)_{k\in\mathbb{N}} = (a_{n_1}, a_{n_2}, a_{n_3}, \dots)$$

is called a subsequence of (a_n).

2.2.27. Definition. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and $A \subset \mathbb{N}$ any infinite set of natural numbers. We define the subsequence

$$(a_n)_{n\in A}$$

to be the composition of (a_n) and the sequence (n_k), where $n_0 = \min A$ and $n_{k+1} > n_k$, $n_k \in A$, for all $k \in \mathbb{N}$ (n_k is the kth-smallest element of A).

Subsequence: Important lemma

2.2.29. Lemma. Let (a_n) be a convergent sequence with limit a. Then any subsequence of (a_n) is convergent with the same limit.

2.2.30. Lemma. Every real sequence has a monotonic subsequence.

2.2.31. Definition. Let (a_n) be a sequence. Then a number a is called an ***accumulation point*** of (a_n) if

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\forall}\ \underset{n>N}{\exists} |a_n - a| < \varepsilon.$$

If a sequence converges, then the limit is the only accumulation point.

Divergent sequence can also have accumulation point.

Subsequence

2.2.34. Lemma. A number a is an accumulation point of a sequence (a_n) if and only if there exists a subsequence of (a_n) that converges to a.

2.2.35. Theorem of Bolzano-Weierstraß. Every bounded real sequence has an accumulation point.

The Bolzano-weierstrass theorem can also be said as Every bounded real sequence in $\mathbb{R}$ has a convergent subsequence.

Generalizing Convergence

2.2.36. Definition. Let M be a set. A map $\varrho\colon M \times M \to \mathbb{R}$ is called a ***metric*** if

(i) $\varrho(x,y) \geq 0$ for all $x, y \in M$ and $\varrho(x,y) = 0$ if and only if $x = y$.

(ii) $\varrho(x,y) = \varrho(y,x)$

(iii) $\varrho(x,z) \leq \varrho(x,y) + \varrho(y,z)$.

The pair (M, ϱ) is then called a ***metric space***.

Generalized Convergence

$$\lim_{n\to\infty} a_n = a \qquad :\Leftrightarrow \qquad \underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ a_n \in B_\epsilon(a)$$

where

$$B_\epsilon(a) := \{y \in M : \rho(y,a) < \epsilon\}, \qquad \epsilon > 0, \qquad a \in M$$

Generalizing Boundedness and Cauchy sequence

Generalized boundedness:

(a_n) is called bounded if

$$\underset{R>0}{\exists}\ \underset{n\in\mathbb{N}}{\forall}\ a_n \in B_R(x)$$

Boundness defined in such way is called *well-defined* because it does not depend on the choice of x.

2.2.40. Definition. A sequence (a_n) in a metric space (M, ϱ) is called a ***Cauchy sequence*** if

$$\underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m,n>N}{\forall}\ \varrho(a_m, a_n) < \varepsilon.$$

Note: Every convergent sequence is a Cauchy sequence.

Cauchy sequence

Note: Convergence and Cauchy sequence both depends on the choice of metric space. Different metric will lead to different result. Do not take $\rho = |x - y|$ for granted.

2.2.42. Lemma. Every Cauchy sequence in a metric space (M, ϱ) is bounded.

2.2.43. Theorem. Every Cauchy sequence in $\mathbb{R}$ with the metric

$$\varrho \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad \varrho(x, y) = |x - y|,$$

is convergent.

Complete metric space

2.2.44. Definition. A metric space (M, ϱ) is called ***complete*** if every Cauchy sequence converges in M.

The metrix space $(\mathbb{N}, \rho)$ is not complete with

$$\rho(x, y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|$$

Notice that when proving this you should first check if $(\mathbb{N}, \rho)$ is indeed a metric space. This example shows that the same metric in a complete metric space can be incomplete in another metric space.

Complete metric space: Exercise

Prove that the metric space $(\mathbb{C}, \rho)$ with $\rho(x, y) = |x - y|$ is complete.
Hint: the metric space $(\mathbb{R}, \rho)$ is complete.

Solution

Proof

Suppose (a_n) is a complex Cauchy sequence, $(a_n) := (x_n) + (y_n)i$, where (x_n) and (y_n) are complex sequences. Then we know

$$\underset{\epsilon>0}{\forall}\ \underset{N>0}{\exists}\ \underset{m,n>N}{\forall} |(x_n - x_m) + (y_n - y_m)i| < \epsilon$$

$$|x_n - x_m| < |(x_n - x_m) + (y_n - y_m)i| < \epsilon$$
$$|(y_n - y_m)i| = |y_n - y_m| < |(x_n - x_m) + (y_n - y_m)i| < \epsilon$$

So (x_n) and (y_n) are both real Cauchy sequences, and they are convergent. Let $x := \lim x_n$, and let $y := \lim y_n$, then for properly chosen $n > N(\epsilon)$

$$|(x_n + y_n)i - (x + y \cdot i)| \leq |x_n - x| + |(y_n - y)i| < \epsilon$$

So (a_n) is convergent to $x + y \cdot i$.

Real Functions

IMPORTANT: Remember that we denote a function by the symbol f, and the values of the function at a some point x by $f(x)$. It is important not to confuse these two notations; f is a function (set of pairs), while $f(x)$ is a (real) number.

You need to know:

polynomial functions

power functions

rational functions

piecewise functions

periodic functions

Function manipulation

Reference

1. VV186 Slide and previous RC Slide