VV186 RC5

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Limits of functions

2.4.1. Definition. Let f be a real- or complex-valued function defined on a subset of $\mathbb{R}$ that includes some interval (a, ∞), $a \in \mathbb{R}$. Then f converges to $L \in \mathbb{C}$ as $x \to \infty$, written

$$\lim_{x\to\infty} f(x) = L \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{C>0}{\exists}\ \underset{x>C}{\forall}\ |f(x) - L| < \varepsilon. \tag{2.4.1}$$

The limit of a function as $x \to -\infty$ is defined similarly.

2.4.3. Definition. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. Then the limit of f as $x \to x_0$ is equal to $L \in \mathbb{C}$, written

$$\lim_{x\to x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall}\ |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Limits of functions

Example:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Let $g : \mathbb{R} \to \mathbb{R}, g(x) := \begin{cases} |x|, & x \neq 2 \\ 3, & x = 2 \end{cases}$ Then $\lim_{x\to 2} g(x) = 2$

2.4.5. Theorem. Let f and g be real- or complex-valued functions and x_0 an accumulation point of $\operatorname{dom} f \cap \operatorname{dom} g$ such that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist. Then

1. $\lim_{x\to x_0} (f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$,
2. $\lim_{x\to x_0} (f(x) \cdot g(x)) = \left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right)$,
3. $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$ if $\lim_{x\to x_0} g(x) \neq 0$.

One-sided limit

2.4.6. Definition. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω.

Then the limit of f as x converges to x_0 from above is equal to $L \in \mathbb{C}$,

$$\lim_{x \searrow x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{x\in\Omega\setminus\{x_0\}}{\forall} \ 0 < x - x_0 < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Analogously, the limit of f as x converges to x_0 from below is equal to $L \in \mathbb{C}$,

$$\lim_{x \nearrow x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{x\in\Omega\setminus\{x_0\}}{\forall} \ 0 < x_0 - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Limits of Functions Using Sequence

2.4.9. Theorem. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. Then

$$\lim_{x \to x_0} f(x) = L \qquad \Leftrightarrow \qquad \underset{\substack{(a_n) \\ a_n \in \Omega \setminus \{x_0\}}}{\forall} \left(a_n \xrightarrow{n \to \infty} x_0 \Rightarrow f(a_n) \xrightarrow{n \to \infty} L\right)$$

A similar result holds for $x_0 = \pm\infty$.

Remark:

If you wish to prove the convergence of function using sequence, you have to show that **EVERY** sequence that converges to x_0 have their function values converging to L.

The Big-O Landau Symbol

2.4.12. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

$$f(x) = O(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\exists}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| \le C|\phi(x)| \tag{2.4.2}$$

2.4.15. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ containing the interval (L, ∞) for some $L \in \mathbb{R}$. We say that

$$f(x) = O(\phi(x)) \qquad \text{as } x \to \infty$$

if and only if

$$\underset{C>0}{\exists}\ \underset{M>L}{\exists} \quad x > M \quad \Rightarrow \quad |f(x)| \le C|\phi(x)|$$

The Little-O Landau Symbol

2.4.17. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

$$f(x) = o(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\forall}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| < C|\phi(x)| \tag{2.4.3}$$

2.4.20. Definition. Let f, ϕ be a real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ containing the interval (L, ∞) for some $L \in \mathbb{R}$. We say that

$$f(x) = o(\phi(x)) \qquad \text{as } x \to \infty$$

if and only if

$$\underset{C>0}{\forall}\ \underset{M>L}{\exists} \quad x > M \quad \Rightarrow \quad |f(x)| < C|\phi(x)|$$

Landau Symbol Using Limits

Theorem

Let f, ϕ be a real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. If $x_0 \in \Omega$, we require $\phi(x_0) \neq 0$. Suppose that exists some $C \geq 0$ such that

$$\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = C$$

Then $f(x) = O(\phi(x))$ as $x \to x_0$.
Let f, ϕ be a real- or complex-valued functions defined on an interval $I \subset \mathbb{R}$ and let $x_0 \in \overline{I}$. Then

$$\lim_{x \to x_0} \frac{|f(x)|}{|\phi(x)|} = 0 \quad \Leftrightarrow \quad f(x) = o(\phi(x)) \text{ as } x \to x_0$$

Landau Symbol Using Limits

We do sacrifice the symmetry of the "=" sign, however. For example,

$$O(x^3) = O(x^2) \quad \text{as } x \to 0$$

means "any function f such that $|f(x)| < C|x|^3$ for all $|x| < \varepsilon$ for some ε, C also satisfies $|f(x)| < C'|x|^2$ for all $|x| < \varepsilon'$ for some ε', C'." Clearly,

$$O(x^2) \stackrel{?}{=} O(x^3) \quad \text{as } x \to 0 \qquad \text{is false.}$$

Landau symbols can be combined with each other and with functions. For example, as $x \to 0$,

$$c \cdot O(x^n) = O(x^n), \qquad x^n O(x^m) = O(x^{n+m}),$$
$$O(x^n) + O(x^m) = O(x^{\min(n,m)}), \qquad O(x^n)O(x^m) = O(x^{n+m}).$$

Continuity

2.5.1. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $f : \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x_0 \in \Omega$. We say that f is ***continuous at*** x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If $U \subset \Omega$, we say that f is ***continuous on*** U if f is continuous at every $x_0 \in U$.

We say that f is ***continuous on its domain***, or simply ***continuous***, if f is continuous at every $x_0 \in \Omega$.

2.5.2. Remark. For a function f to be continuous at a point x_0, three conditions have to be fulfilled:

(i) f needs to have a limit at x_0 ($\lim_{x \to x_0} f(x)$ must exist);

(ii) f needs to be defined at x_0 ($f(x_0)$ must exist);

(iii) the value of f must coincide with its limit ($f(x_0) = \lim_{x \to x_0} f(x)$).

Continuity

2.5.4. Theorem. Let $\Omega \subset \mathbb{R}$ be any set and $f \colon \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x_0 \in \Omega$. Then the following are equivalent:

1. f is continuous at x_0;
2. for any real sequence (a_n) with $a_n \to x_0$, $\lim\limits_{n\to\infty} f(a_n) = f(x_0)$;
3. $\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in \operatorname{dom} f}{\forall} : |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$

Continuous Extension

2.5.6. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $\widetilde{\Omega} \supset \Omega$. Suppose that $f\colon \Omega \to \mathbb{R}$ and $\widetilde{f}\colon \widetilde{\Omega} \to \mathbb{R}$ are continuous functions. If $\widetilde{f}(x) = f(x)$ for all $x \in \Omega$, we say that $\widetilde{f}$ is a ***continuous extension*** of f to $\widetilde{\Omega}$.

2.5.8. Remark. Suppose that $\Omega \subset \mathbb{R}$, $x_0 \in \Omega$ and $f\colon \Omega \setminus \{x_0\} \to \mathbb{R}$ is continuous and has the property that $\lim\limits_{x \to x_0} f(x)$ exists. Then $\widetilde{f}\colon \Omega \to \mathbb{R}$,

$$\widetilde{f}(x) = \lim_{y \to x} f(y), \qquad\qquad x \in \Omega,$$

defines the unique continuous extension of f to Ω.

Continuity

2.5.9. Theorem. Let f and g be two real functions and $x \in (\operatorname{dom} f) \cap (\operatorname{dom} g)$. Assume that both f and g are continuous at x. Then

(i) $f + g$ is continuous at x and

(ii) $f \cdot g$ is continuous at x.

Furthermore, if $g(x) \neq 0$, the function h defined by $h(x) = 1/g(x)$ is continuous at x.

Continuity

2.5.10. Theorem. Let f, g be real functions such that $\lim_{x \to x_0} g(x) = L$ exists and f is continuous at $L \in \operatorname{dom} f$. Then

$$\lim_{x \to x_0} f(g(x)) = f(L).$$

Proof

Let $\epsilon > 0$. Since f is continous at L, we know that there exists some $\delta > 0$ such that for all $y \in \text{dom } f$

$$|y - L| < \delta \quad \Leftrightarrow \quad |f(y) - f(L)| < \epsilon$$

Fix such $\delta > 0$. Since $\lim\limits_{x \to x_0} g(x) = L$, there exists some $\widetilde{\delta} > 0$ such that for all $z \in \text{dom } g \backslash \{x_0\}$

$$|z - x_0| < \widetilde{\delta} \quad \Leftrightarrow \quad |g(z) - L| < \delta$$

Hence, for $z \neq x_0$, if $|z - x_0| < \widetilde{\delta}$, then $|f(g(z)) - f(L)| < \epsilon$. This proves that the limits of $f(g(x))$ at x_0 equals $f(L)$.

Important Lemma about Continuity

2.5.11. Lemma. Let $\Omega \subset \mathbb{R}$ be some set, $f\colon \Omega \to \mathbb{R}$ a function that is continuous at some point $x_0 \in \Omega$ and assume that $f(x_0) > 0$. Then there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap \Omega$.

2.5.12. Theorem. Let $a < b$ and $f\colon [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Then there exists some $x \in [a, b]$ such that $f(x) = 0$.

2.5.13. Bolzano Intermediate Value Theorem. Let $a < b$ and $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then for $y \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$ there exists some $x \in [a, b]$ such that $y = f(x)$.

Important Lemma about Continuity

2.5.14. Theorem. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function with $\operatorname{ran} f \subset [a, b]$. Then f has a fixed point, i.e., there exists some $x \in [a, b]$ such that $f(x) = x$.

2.5.15. Lemma. Let $\Omega \subset \mathbb{R}$ be some set and $f\colon \Omega \to \mathbb{R}$ a function that is continuous at some point $x_0 \in \Omega$. Then there exists a $\delta > 0$ such that f is bounded above on $(x_0 - \delta, x_0 + \delta) \cap \Omega$.

Basically, this lemma says that $\sup\{f(x)\colon x \in B_\delta(x_0)\}$ exists for some $\delta > 0$.

2.5.17. Theorem. Let $a < b$ and $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then there exists a $y \in [a, b]$ such that $f(x) \leq f(y)$ for all $x \in [a, b]$.

Hence $\max\{f(x)\colon x \in [a, b]\}$ exists. Colloquially, we say that "a continuous function attains its maximum".

Inverse Functions

Surjective Function

$$\text{range} = \text{codomain}$$

If $f : V \to W$ is a surjective function, then all the elements in W are covered by f.

Injective Function

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

x is uniquely determined if given $f(x)$.

Bijective Function

$$\text{Bijective} = \text{Surjective} + \text{Injective}$$

If $f : V \to W$ is a bijective function, then there's a one-to-one relation between all elements in V and all elements in W.

Inverse Functions

Inverse Function

If $f : V \to W$ is an injective function, then

$$f^{-1} : W \to V, \qquad f^{-1}(f(x)) = x$$

is indeed a function. Then we say f^{-1} is an inverse function of f, and f is invertible.

Inverse Functions

Theorem

Let $a, b \in \mathbb{R} \cup -\infty \cup \infty$ with $a < b$. Suppose $f : (a, b) \to \mathbb{R}$ is strictly increasing and continuous. Then

$$\alpha := \lim_{x \searrow a} f(x) \geq -\infty$$

$$\beta := \lim_{x \nearrow a} f(x) \leq \infty$$

exist and $f : (a, b) \to (\alpha, \beta)$ is bijective. Furthermore, f^{-1} is also continuous and strictly increasing and

$$\lim_{y \searrow \alpha} f^{-1}(y) = a$$

$$\lim_{y \nearrow \beta} f^{-1}(y) = b$$

Inverse Functions

2.5.20. Theorem. Let $I \subset \mathbb{R}$ be an interval and $\widetilde{\Omega} \subset \mathbb{R}$ a set. If $f : I \to \widetilde{\Omega}$ is continuous and bijective, then f is strictly monotonic on I.

Image and Pre-Image of Sets

2.5.21. Definition. Let $\Omega \subset \mathbb{R}$ and $f\colon \Omega \to \mathbb{R}$. Then for any $A \subset \Omega$ and $B \subset \operatorname{ran} f$

$$f(A) := \Big\{ y \in \mathbb{R} \colon \underset{x\in A}{\exists} f(x) = y \Big\}$$

is called the ***image of*** A and

$$f^{-1}(B) := \Big\{ x \in \Omega \colon \underset{y\in B}{\exists} f(x) = y \Big\}$$

is called the ***pre-image of*** B. Note that the symbol $f^{-1}(B)$ makes sense whether or not f is invertible.

Uniform Continuity

2.5.23. Definition. Let $I \subset \mathbb{R}$ be an interval and $f : \Omega \to \mathbb{R}$ a function with $I \subset \Omega$. Then f is called ***uniformly continuous on I*** if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in I}{\forall} |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon.$$

You can compare it with the definition of continuity:

$$\underset{\epsilon>0}{\forall}\ \underset{x\in I}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall} |x-u|<\delta \Leftrightarrow |f(x)-f(y)|<\epsilon$$

Remark and Theorem

Uniform Continuity $\Rightarrow$ Continuity

Continuity $+$ Closed Interval $\Rightarrow$ Uniform Continuity

Reference

1. VV186 Slide and previous RC Slide