VV186 RC9

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Some concepts of integrals

Partition: a **finite** ordered tuple $(a_0, a_1, a_2 ... a_n)$

Step functions: a function that has same values over partitions of the domain.

Bounded function: A function whose supremum value is finite over an interval. We donote the set of all bounded functions by $L^{\infty}(I)$, where I is the interval domain.

Regulated function: A **bounded** function f that is defined on a closed interval and is the uniform limit of some step functions. Equivalently, for any $\epsilon > 0$, there exists a step function g such that sup $|f(x)g(x)| < \epsilon$ over the domain.

Some concepts of integrals

Any continuous function defined on a closed interval is regulated.

Piecewise continuous function: A function that can be partitioned into a **finite** number of continuous functions, except at **finitely** many boundary points of the partition.

Any **piecewise** continuous function defined on a closed interval is regulated.

Some concepts of integrals

Intergral of regulated function: The limit of the integral of a sequence of step functions that converge uniformly to the given regulated function.

The uniform limit of a sequence of regulated function is also a regulated function. What's more, the intergral of this sequence of function converges to the intergral of the limit function.

Upper area: The infimum of integrals of step functions that are no less than the given **bounded** function.

Some concepts of integrals

Darboux integral: a **bounded** function whose upper area and lower area exist and coincide. Note that the upper and lower areas do **NOT** depend on the partition.

A **regulated** function defined on a **closed** interval is always **Darnoux**-integrable, and its Darboux integral is equal to the regulated integral.

Step function with respect to a tagged partition (P,$\equiv$): The step function with respect to P, where $\equiv$ is the set of uniform function values on P.

Some concepts of integrals

Mesh size: the maximum of the inteval length over the partition.

Riemann sum of a function with tagged partition: Just the normal integral of this step function.

Riemann-integrable: A bounded function f defined on a closed interval is said to be Riemann-integrable, if for any $\epsilon > 0$, there exists δ , such that for any tagged partition with mesh size less than δ, the Riemann sum of f with the partition lies in $B_\epsilon(F)$. We then define F as the Riemann sum of f.

Some concepts of integrals

A bounded real function is Reimann-integrable if and only if it is Darboux-integrable. Then its Reimann integral is equal to the Darboux integal.

Some methods of doing integrals

Pay attention that the following theorem puts **NO** restrict on a and b. In other words, you can define the integral from any point a as you like, and the derivative at x is still the same. We then say F is the primitive of f.

4.2.1. Theorem. Let $f\colon [a,b] \to \mathbb{C}$ be continuous and set

$$F\colon [a,b] \to \mathbb{C}, \qquad F(x) := \int_a^x f.$$

Then F is differentiable on (a,b) and

$$F'(x) = f(x), \qquad x \in (a,b).$$

Some methods of doing integrals

4.2.4. Substitution Rule. Let $f \in \mathrm{Reg}([\alpha, \beta])$ and $g\colon [a, b] \to [\alpha, \beta]$ continuously differentiable. Then

$$\int_a^b (f \circ g)(x) g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy.$$

4.2.8. Theorem. Let $f, g \in C^1([a, b], \mathbb{C})$. Then

$$\int_a^b f'(x) g(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b f(x) g'(x)\, dx.$$

Improper integral

4.2.10. Definition. Assume that $b \leq \infty$ and that $f\colon [a,b) \to \mathbb{C}$ is regulated on any closed subinterval $[a,x]$, $x < b$. Then

$$\int_a^b f(t)\,dt$$

is called an ***improper integral*** and is said to ***converge*** or ***exist*** if

$$\lim_{x \nearrow b} \int_a^x f(t)\,dt =: I$$

exists.

Improper integral

4.2.13. Cauchy Criterion. Let $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ be integrable on every interval $[a, x]$, $x \in \mathbb{R}$. The improper integral

$$\int_a^\infty f(x)\,dx$$

converges if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{R>0}{\exists}\ \underset{x,y>R}{\forall}\ \left|\int_x^y f(t)\,dt\right| < \varepsilon.$$

Reference

1. VV186 Slide and previous RC Slide