VV186 RC3

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Function: Definition

2.1.1. Definition. Let X and Y be sets and let P be a predicate with domain $X \times Y$. Let

$$f := \{(x, y) \in X \times Y : P(x, y)\}$$

and assume that P has the property that

$$\underset{(x_1,y_1)\in f}{\forall} \ \underset{(x_2,y_2)\in f}{\forall} \quad x_1 = x_2 \ \Rightarrow \ y_1 = y_2. \tag{2.1.1}$$

Let

$$\operatorname{dom} f := \Big\{x \in X : \underset{y\in Y}{\exists} : (x, y) \in f\Big\}$$

Then we say that f is a ***function from*** $\operatorname{dom} f$ ***to*** Y. The set $\operatorname{dom} f \subset X$ is called the ***domain*** of f and Y is called the ***codomain*** of f. We also define the ***range*** of f by

$$\operatorname{ran} f := \Big\{y \in Y : \underset{x\in X}{\exists} : (x, y) \in f\Big\}.$$

Function: Note

We can represent a function in three ways:

$$f = \{(x, y) \in V \times W : P(x, y)\}$$
$$f = \{(x, y) \in V \times W : y = f(x)\}$$
$$f : V \to W \qquad y = f(x)$$

Note that codomain and range are different:

Domain: V, the set of all the paired x
Range: ran f, the set of all the paired y
Codomain: W
Range is a subset of codomain.
$P(x, y)$ is (uniquely) defined for every x in V, but not necessarily every y in W.

Function: Note

If given x, then the function is uniquely defined. So the following example cannot be seen as a function

2.1.2. Example. Let $X, Y = \mathbb{Z}$ and $P(x,y)\colon x^2 = y$. Then

$$f = \{(x,y) \in \mathbb{Z}^2 \colon y = x^2\} = \{(0,0), (-1,1), (1,1), (-2,4), (2,4), \ldots\}.$$

Function Operation

$$\begin{aligned} f : V \to W \qquad & y = f(x) \\ g : V \to W \qquad & y = g(x) \\ f + g : V \to W \qquad & y = f(x) + g(x) \\ f \cdot g : V \to W \qquad & y = f(x) \cdot g(x) \end{aligned}$$

$$\begin{aligned} f : V \to W \qquad & y = f(x) \\ g : W \to Y \qquad & y = g(x) \\ f \circ g : V \to Y \qquad & y = f(g(x)) \end{aligned}$$

Sequence Definition

2.2.1. Definition. Let $\Omega \subset \mathbb{N}$. A map $\Omega \to \mathbb{R}$ is called a real sequence, a map $\Omega \to \mathbb{C}$ a complex sequence.

2.2.2. Notation. We denote the values of a sequence by a_n, i.e., a sequence maps $n \mapsto a_n$, $n \in \Omega \subset \mathbb{N}$. Instead of $\{(n, a_n) \colon n \in \Omega\}$ we denote a sequence by any of the following,

$$(a_n)_{n\in\Omega} = (a_n) = a_0, a_1, a_2, \dots .$$

The values a_n are called ***terms*** of the sequence.

Sequence Convergence

If a_n is a real (or complex) sequence, a is a real (or complex) number,

$$\underset{\epsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ |a_n - a| < \epsilon$$

then a_n is said to be *convergent towards the limit a*.
We denote this by

$$\lim_{n\to\infty} a_n = a \qquad \text{or} \qquad a_n \to a \text{ as } a \to \infty$$

A sequence that is not convergent is called divergent.

Sequence Convergence Example

$a_n = n$ is divergent to infinity

$a_n = \dfrac{1}{n}$ converges to zero

$a_n = (-1)^n$ diverges

Sequence

A bounded sequence is a sequence whose range is bounded.

Lemma: A convergent sequence is bounded.

Useful Lemma:

1. The sequence (a_n) converges to a if and only if the sequence $(b_n) := (a_n - a)$ converges to zero, i.e.,
$$a_n \to a \qquad \Leftrightarrow \qquad a_n - a \to 0$$
2. The sequence (a_n) converges to 0 if and only if the sequence $(b_n) = (|a_n|)$ converges to zero, i.e.,
$$a_n \to 0 \qquad \Leftrightarrow \qquad |a_n| \to 0$$

Sequence: General Result

1. A convergent sequence has precisely one limit.
2. The addition and multiplication of limits

2.2.15. Proposition. Let (a_n) and (b_n) be convergent real or complex sequences with $a_n \to a$ and $b_n \to b$ for some $a, b \in \mathbb{C}$. Then

1. $\lim(a_n + b_n) = a + b$,
2. $\lim(a_n \cdot b_n) = ab$,
3. $\lim \frac{a_n}{b_n} = a/b$ if $b \neq 0$.

Sequence: Proof of the third statement

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{b \cdot a_n - a \cdot b_n}{b \cdot b_n}\right| \\
&= \left|\frac{b \cdot a_n - b \cdot a + a \cdot b - a \cdot b_n}{b \cdot b_n}\right| \\
&= \left|\frac{b(a_n - a) + a(b - b_n)}{b \cdot b_n}\right| \\
&\leq \left|\frac{1}{b_n}\right| \cdot |a_n - a| + \left|\frac{a}{b \cdot b_n}\right| \cdot |b - b_n|
\end{aligned}$$

Because $|a_n - a| \to 0$, $|b_n - b| \to 0$, $\left|\frac{1}{b_n}\right|$ and $\left|\frac{a}{b \cdot b_n}\right|$ are bounded, we have $\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \to 0$.

Sequence: Application

1. Squeeze theorem:

   If $\lim a_n = \lim b_n = a$, and if $a_n < c_n < b_n$ for all $n > N \in \mathbb{N}$, then we have $\lim c_n = a$.

2. Corollary for complex numbers:

   2.2.17. Corollary. Let (r_n) be a real sequence with $\lim r_n = 0$ and let (z_n) be a complex sequence with

   $$|z_n| < r_n \qquad \text{for sufficiently large } n.$$

   Then $z_n \to 0$.

Sequence: Application

3. Bernoulli's inequality:

   2.2.19. Lemma. For $x > -1$ and $n \in \mathbb{N}$,

   $$(1+x)^n > nx.$$

4. Proposition

   2.2.20. Proposition. Let $q \in \mathbb{C}$, $|q| < 1$. Then $\lim_{n\to\infty} q^n = 0$.

5. Complex sequence:

   2.2.21. Lemma. Let (z_n) be a complex sequence, $z_n = x_n + iy_n$ and let $z = x + iy \in \mathbb{C}$. Then

   $$z_n \to z \qquad \Longleftrightarrow \qquad (x_n \to x \quad \text{and} \quad y_n \to y).$$

Real Sequence

2.2.22. Definition. A real sequence (a_n) is called

- ***increasing*** if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$.
- ***decreasing*** if $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly increasing*** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.
- ***strictly decreasing*** if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.
- ***monotonic*** if it is either increasing or decreasing.

Important theorem:

2.2.24. Theorem. Every monotonic and bounded (real) sequence (a_n) is convergent. More precisely,

$$\begin{aligned} &a_n \nearrow \sup\{a_n \colon n \in \mathbb{N}\} && \text{if } (a_n) \text{ is increasing,} \\ &a_n \searrow \inf\{a_n \colon n \in \mathbb{N}\} && \text{if } (a_n) \text{ is decreasing,} \end{aligned}$$

Sequence Exercise

Calculate

$$\lim(\frac{n^2+n+1}{2n^3-1}+\frac{n^2+n+2}{2n^3-4}+...+\frac{n^2+n+n}{2n^3-n^2})$$

Which of the solution is correct and why?

Extra exercises

See blackboard

Reference

1. VV186 Slide and previous RC Slide