VV186 RC1

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Some words

This recitation class mainly focuses on the slides and assignments. If you have any question with regard to the assignment or slides, feel free to ask.

Notation

Define A as B

$$A := B$$

Define B as A

$$A =: B$$

Don't use $:=$ or $=:$ in tautologies, *i.e.* compound statements that are always correct.

Don't use $\because$ or $\therefore$ in homework, projects or exams.

Basic concept

Statement: anything that we can either regard as true or false

1.1.2. Examples.

- A: 4 is an even number.
- $B: 2 > 3$.
- $A(n): 1 + 2 + 3 + ... + n = n(n+1)/2$.

Quantifier

- the ***universal quantifier***, denoted by the symbol $\forall$, read as "for all" and
- the ***existential quantifier***, denoted by $\exists$, read as "there exists."

1.1.12. Definition. Let M be a set and $A(x)$ be a predicate. Then we define the quantifier $\forall$ by

$$\underset{x \in M}{\forall} A(x) \quad \Leftrightarrow \quad A(x) \text{ is true for all } x \in M$$

We define the quantifier $\exists$ by

$$\underset{x \in M}{\exists} A(x) \quad \Leftrightarrow \quad A(x) \text{ is true for at least one } x \in M$$

We may also write $\forall x \in M \colon A(x)$ instead of $\underset{x \in M}{\forall} A(x)$ and similarly for $\exists$.

Note: Hanging Quantifier

Basic concept

Unary Operation: Negation

Binary Operation: Conjunction

Binary Operation: Disjunction

Binary Operation: Implication

Binary Operation: Equivalence

Binary Operation: Contraposition

Operation on sets

If $A = \{x \colon P_1(x)\}$, $B = \{x \colon P_2(x)\}$ we define the ***union***, ***intersection*** and ***difference*** of A and B by

$$A \cup B := \{x \colon P_1(x) \vee P_2(x)\}, \qquad A \cap B := \{x \colon P_1(x) \wedge P_2(x)\},$$
$$A \setminus B := \{x \colon P_1(x) \wedge (\neg P_2(x))\}.$$

Let $A \subset M$. We then define the ***complement*** of A by

$$A^c := M \setminus A.$$

If $A \cap B = \emptyset$, we say that the sets A and B are ***disjoint***.

Operation on sets: Proof

Proof for $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$

$x \in A$	$x \in B$	$x \in C$	$x \in (A \cup B) \cap C$	$x \in (A \backslash C) \cup (B \backslash C)$
T	T	T	F	F
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	T	T
F	F	T	F	F
F	F	F	F	F

Natural number

1. $a + (b + c) = (a + b) + c$ (***Associativity***)
2. $a + 0 = 0 + a = a$ (***Existence of a neutral element***)
3. $a + b = b + a$ (***Commutativity***)

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (***Associativity***)
2. $a \cdot 1 = 1 \cdot a = a$ (***Existence of a neutral element***)
3. $a \cdot b = b \cdot a$ (***Commutativity***)

$$a \cdot (b + c) = a \cdot b + a \cdot c. \qquad (\textit{\textbf{Distributivity}})$$

Notation

For any numbers $a_1, a_2, \dots, a_n$ we define the notation

$$a_1 + a_2 + \cdots + a_n =: \sum_{j=1}^{n} a_j =: \sum_{1 \le j \le n} a_j$$

and

$$a_1 \cdot a_2 \cdots a_n =: \prod_{j=1}^{n} a_j =: \prod_{1 \le j \le n} a_j.$$

Mathematical Induction

First mathematical induction:

1. prove that the statement $A(n)$ is true for $n = 0$.
2. suppose $A(n)$ is true for $n = k$ prove that $A(n)$ is true for $n = k + 1$.
3. conclude $A(n)$ is true for all natural numbers.

Second mathematical induction:

1. prove that the statement $A(n)$ is true for $n = 0$.
2. suppose $A(n)$ is true for $n <= k$ prove that $A(n)$ is true for $n = k + 1$.
3. conclude $A(n)$ is true for all natural numbers.

You can use second mathematical induction to prove for Fibonacci sequence.

Mathematical Induction: Example

Binomial formula

Exercise

1. Suppose that

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 8 \\ 4(a_{n-2} + a_{n-1}) &= 3a_n + 5n^2 - 24n + 20 \ (n \geq 3) \end{aligned}$$

Prove that $a_n = 2^n + n^2$ $(n \geq 3)$.

Mathematical Induction: Solution

$$a(3) = \frac{4 \cdot (3+8) - (5 \cdot 3^2 - 24 \cdot 3 + 20)}{3} = 17 = 3^2 + 2^3.$$

Suppose $a_n = n^2 + 2^n, \ \forall n \leq k, \ n \geq 3$, then

$$\begin{aligned}
a_{k+1} &= \frac{4 \cdot (a_k + a_{k-1}) - (5 \cdot (k+1)^2 - 24 \cdot (k+1) + 20)}{3} \\
&= \frac{4 \cdot (k^2 + 2^k + (k-1)^2 + 2^{k-1})}{3} \\
&\quad - \frac{(5 \cdot (k+1)^2 - 24 \cdot (k+1) + 20)}{3} \\
&= (k+1)^2 + 2^{k+1}
\end{aligned}$$

Hence $a(n) = n^2 + 2^n, \ \forall n \geq 3$.

Rational Numbers

Properties of Addition

$P1.$	$a+(b+c)=(a+b)+c$	(Associativity)
$P2.$	$a+0=0+a=a$	(Existence of a neutral element)
$P3.$	$(-a)+a=a+(-a)=0$	(Existence of an inverse element)
$P4.$	$a+b=b+a$	(Commutativity)

Properties of Multiplication

$P5.$	$a\cdot(b\cdot c)=(a\cdot b)\cdot c$	(Associativity)
$P6.$	$a\cdot 1=1\cdot a=a$	(Existence of a neutral element)
$P7.$	$a\cdot a^{-1}=a^{-1}\cdot a=1$	(Existence of an inverse element)
$P8.$	$a\cdot b=b\cdot a$	(Commutativity)

Rational Numbers

$$
\begin{array}{lll}
P9. & a \cdot (b + c) = a \cdot b + a \cdot c. & \text{(Distributivity)} \\
P10. & \begin{cases} a = 0 \\ a \in P \\ -a \in P \end{cases} & \text{(Trichotomy law)} \\
P11. & a \in P, b \in P \quad \Rightarrow a + b \in P & \\
P12. & a \in P, b \in P \quad \Rightarrow a \cdot b \in P &
\end{array}
$$

Triangle Inequality

$$\big||a-b|\big| \leqslant |a+b| \leqslant |a|+|b|$$

Reference

1. VV186 Slide and previous RC Slide