VV186 RC8

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Formal Power Series

3.6.1. Definition. For any complex sequence (a_k), the expression

$$\sum_{k=0}^{\infty} a_k z^k$$

is called a (formal) complex power series.

3.6.4. Definition. A formal power series $\sum a_k z^k$ is said to be ***is (absolutely) convergent at*** $z_0 \in \mathbb{C}$ if the series $\sum_{k=0}^{\infty} a_k z_0^k$ converges (absolutely).

Radius of Convergence

3.6.6. Definition and Theorem. Let $\sum a_k z^k$ be a complex power series. Then there exists a unique number $\varrho \in [0, \infty]$ such that

(i) $\sum a_k z^k$ is absolutely convergent if $|z| < \varrho$,

(ii) $\sum a_k z^k$ diverges if $|z| > \varrho$.

This number is called the ***radius of convergence*** of the power series. It is given by ***Hadamard's formula***,

$$\varrho = \begin{cases} \frac{1}{\overline{\lim}\sqrt[k]{|a_k|}}, & 0 < \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} < \infty, \\ 0, & \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} = \infty, \\ \infty, & \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} = 0. \end{cases} \tag{3.6.1}$$

3.6.7. Remark. No information is given about the convergence of $\sum a_k z^k$ if $|z| = \varrho$. In these cases the series may converge or diverge, depending on the point z.

Radius of Convergence

3.6.9. Lemma. Let $\sum a_k z^k$ be a complex power series with radius of convergence ϱ. Then the series $\sum k a_k z^{k-1}$ has the same radius of convergence ϱ.

3.6.10. Lemma. If $\sum a_k z^k$ is a complex power series with radius of convergence ϱ, then for any $R < \varrho$ the series converges uniformly on $B_R(0) = \{z : |z| < R\}$.

3.6.11. Corollary. A power series $\sum_{k=0}^{\infty} a_k x^k$ with radius of convergence ϱ defines a continuous function

$$f : B_\varrho(0) \to \mathbb{C}, \qquad f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Radius of Convergence

3.6.14. Theorem. The real or complex power series $f(z) := \sum a_k z^k$ with radius of convergence ϱ defines a differentiable function $f\colon B_\varrho(0) \to \mathbb{C}$. Furthermore,

$$f'(z) = \sum k a_k z^{k-1} \tag{3.6.2}$$

where the series has the same radius of convergence as f.

The Exponential Function

3.7.1. Definition. We define the ***exponential function***

$$\exp\colon \mathbb{C} \to \mathbb{C}, \qquad \exp z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{3.7.1}$$

We similarly define its restriction to the real numbers,

$$\exp\colon \mathbb{R} \to \mathbb{R}, \qquad \exp x := \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{3.7.2}$$

Note that $\exp(0) = 1$ from the series representation

$$\exp(0) = 1 + 0 + \frac{0^2}{2!} + \cdots = 1.$$

Taylor expansion of common functions

$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$

$\sin x = x - \frac{x^3}{3!} + ... + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + ...$

$\cos x = 1 - \frac{x^2}{2!} + .. + (-1)^n\frac{x^{2n}}{(2n)!} + ...$

$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^{n-1}\frac{x^n}{n} + ...$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + ... + (-1)^n\frac{x^{2n+1}}{2n+1} + ...$

$(1 + x)^\alpha = 1 + \Sigma_{n=1}^{\infty} C_\alpha^n x^n$

Initial Value Function

In fact, it turns out that exp is the only function satisfying the ***initial value problem***

$$y'(x) = y(x), \qquad y(0) = 1. \tag{3.7.3}$$

Here the equation on the left is called a ***differential equation***; its solution is a function y such that the derivative of y is equal to y. The second equation is called an ***initial condition***, because it specifies the value of y at for some $x_0 \in \mathbb{R}$.

the Euler Number

3.7.3. Definition. The number

$$e := \exp(1)$$

is called the ***Euler number***.

It can be proven that e is irrational and transcendental (not the solution of an algebraic equation). An approximate value is

$$e \approx 2.7182818284590$$

3.7.4. Lemma. The Euler number is the monotonic limit of the sequence in Proposition 3.7.2, i.e.,

$$\left(1+\frac{1}{n}\right)^n \nearrow e \qquad \text{as } n \to \infty.$$

the Real Exponents, Logarithm

The exponential function is continuous (even C^∞) on $\mathbb{R}$, and for rational x coincides with e^x. We do not yet have a definition for e^x when x is real, so it is logical to define

$$e^x := \exp x \qquad\qquad \text{for } x \in \mathbb{R}$$

We say that $\exp x$ is a ***continuous extension*** of e^x to the real numbers. It is automatically the only such extension.

We have seen that the function $\exp\colon \mathbb{R} \to \mathbb{R}_+$ ($\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$) is increasing and hence bijective. Thus there exists an inverse function, which we call the ***(natural) logarithm*** and denote by $\ln\colon \mathbb{R}_+ \to \mathbb{R}$.

the Euler Relation

We first note that we define

$$e^z := \exp z \qquad\qquad \text{for } z \in \mathbb{C}.$$

We then introduce the well-known trigonometric cosine and sine functions $\cos, \sin\colon \mathbb{R} \to \mathbb{R}$ by

$$\cos(x) := \operatorname{Re} e^{ix} = \frac{e^{ix} + e^{-ix}}{2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

$$\sin(x) := \operatorname{Im} e^{ix} = \frac{e^{ix} - e^{-ix}}{2i} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

The equation

$$e^{ix} = \cos(x) + i \sin(x)$$

is sometimes called the ***Euler relation***.

the Trigonometric Functions

3.8.1. Lemma. Suppose that $y\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and that

$$y'' + y = 0, \qquad y(0) = 0, \qquad y'(0) = 0. \tag{3.8.3}$$

Then $y(x) = 0$ for all $x \in \mathbb{R}$.

3.8.2. Theorem. Suppose that $y\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable and that for some $a, b \in \mathbb{R}$

$$y'' + y = 0, \qquad y(0) = a, \qquad y'(0) = b. \tag{3.8.4}$$

Then

$$y(x) = a\cos x + b\sin x.$$

the Hyperbolic Trigonometric Functions

We conclude this part by introducing a final family of functions, the ***hyperbolic trigonometric functions***:

We define the ***hyperbolic sine*** and ***hyperbolic cosine***, $\sinh, \cosh\colon \mathbb{C} \to \mathbb{C}$, by

$$\sinh(x) := \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) := \frac{e^x + e^{-x}}{2}$$

A comparison with the definition of the sine and cosine functions immediately shows that

$$\sinh(ix) = i\sin(x), \qquad \cosh(ix) = \cos x.$$

From the definition, we see that

$$\cosh(x) + \sinh(x) = e^x.$$

Reference

1. VV186 Slide and previous RC Slide