VV186 RC2

Zhu Jing

UM-SJTU Joint Institute

2018

Bound

1.5.2. Definition. We say that a set $U \subset \mathbb{Q}$ is ***bounded*** if there exists a constant $c \in \mathbb{Q}$ such that

$$|x| \leq c \qquad \text{for all } x \in U.$$

If this is not the case, we say that U is ***unbounded***.

Numbers c_1 and c_2 such that

$$c_1 \leq x \leq c_2 \qquad \text{for all } x \in U$$

are called ***lower*** and ***upper bounds*** for U, respectively.

We say that a set $U \in \mathbb{Q}$ is ***bounded above*** if there exists an upper bound for U, and ***bounded below*** if there exists a lower bound for U.

Extrema

1.5.4. Definition. Let $U \subset \mathbb{Q}$ be a subset of the rational numbers. We say that a number $x_1 \in U$ is the ***minimum*** of U if

$$x_1 \leq x, \qquad \text{for all } x \in U$$

and we write $x_1 =: \min U$.

Similarly, we say that $x_2 \in U$ is the ***maximum*** of U if

$$x_2 \geq x, \qquad \text{for all } x \in U$$

and write $x_2 =: \max U$.

Note: the extrema are unique while the bounds are not. Thus we can define the greatest lower bound and least upper bound.

Supreme and Infimum

Maxima of subsets of $\mathbb{Q}$ are unique (if they exist), whereas bounds are not. However, we can define the *least upper bound* and the *greatest lower bound* as follows.

Definition

- If c_1 is an upper bound for $U \subset \mathbb{Q}$, and there's no upper bound that is less than c_1, we say c_1 is the *least upper bound* of U, writing
$$c_1 =: \sup U.$$
- If c_2 is a lower bound for $U \subset \mathbb{Q}$, and there's no lower bound that is greater than c_2, we say c_2 is the *greatest lower bound* of U, writing
$$c_2 =: \inf U.$$

Extension to real numbers

Not every bounded set in rational number has a an infimum or supremum.

We thus generate real number as the smallest extension of rational numbers that if A is a bounded set, then there exists a least upper bound for A in $\mathbb{R}$.

The 13th Property

*P*13. If $A \subset \mathbb{R}, A \neq \varnothing$ is bounded above, then there exists a least upper bound for A in $\mathbb{R}$.

Subsets of real numbers

Some important concepts you need to know about:

interval (definition, open, close, half-open)

interior point

exterior point

boundary point

accumulation point

Elaboration of accumulation point

We call $x \in \mathbb{R}$ an ***accumulation point of A*** if for every $\varepsilon > 0$ the interval $(x - \varepsilon, x + \varepsilon) \cap A \setminus \{x\} \neq \emptyset$.

Accumulation point must be interior points or boundary points. But boundary points may not be accumulation points.

Example

- All rational numbers are accumulation points of $\mathbb{Q}$.
- All integers are boundery points of $\mathbb{Z}$, but none of them is an accumulation point.

Open and closed sets

1.5.13. Definition.

(i) A set $A \subset \mathbb{R}$ is called ***open*** if all points of A are interior points, i.e., if

$$A = \operatorname{int} A.$$

(ii) A set $A \subset \mathbb{R}$ is called ***closed*** if $\mathbb{R} \setminus A$ is open.

(iii) The set

$$\overline{A} := A \cup \partial A$$

is called the ***closure*** of A. It is the smallest closed set that contains A.

Note: Open intervals are open sets. Closed intervals are closed sets.

Complex numbers

Definition

We define the set of complex numbers $\mathbb{C}$ by

$$\mathbb{C} := \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2$$

Addition and multiplication are defined by

$$(a1, b1) + (a2, b2) := (a1 + a2, b1 + b2)$$
$$(a1, b1) \cdot (a2, b2) := (a1a2 - b1b2, a1b2 + a2b1)$$

1.6.1. Theorem. Every polynomial equation of the form $\sum_{k=0}^{n} a_k x^k = 0$, where $a_0, \dots, a_n$ are fixed complex numbers, has precisely n complex solutions $x_1, \dots, x_n \in \mathbb{C}$.

Notation

We can then split any complex number into two components:

$$(a, b) = (a, 0) + (0, b) = a \cdot (1, 0) + b \cdot (0, 1).$$

The pair $(1, 0) \in \mathbb{C}$ corresponds to $1 \in \mathbb{R}$, while the pair $(0, 1) \in \mathbb{C}$ is often denoted by the letter i. Hence we usually write

$$\mathbb{C} \ni z = (a, b) = a \cdot 1 + b \cdot i = a + bi, \qquad a, b \in \mathbb{R},$$

where $a =: \operatorname{Re} z$ is called the ***real part*** and $b =: \operatorname{Im} z$ the ***imaginary part*** of a complex number $z = a + bi$.

Complex Numbers

1.6.3. Definition. We define the ***modulus*** or ***absolute value*** of a complex number $z = a + bi$ by

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}}.$$

Here $\bar{z} = a - bi$ is called the ***complex conjugate*** of z.

1.6.4. Definition. Let $z_0 \in \mathbb{C}$. Then we define the ***open ball of radius*** $R > 0$ ***centered at*** z_0 by

$$B_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}.$$

We say that a set $\Omega \subset \mathbb{C}$ is ***bounded*** if there exists some $R > 0$ such that $\Omega \subset B_R(0)$.

Complex Numbers: Calculation

$$(a+bi)+(c+di)=(a+b)+(c+d)i$$

$$(a+bi)(c+di)=(ac-bd)(ad+bc)i$$

$$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

Complex Numbers: Triangular form

$$z_1 = r_1(cos\theta_1 + isin\theta_1),\ z_2 = r_2(cos\theta_2 + isin\theta_2)$$

$$z_1z_2 = r_1r_2(cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2))$$

$$z_1/z_2 = r_1/r_2(cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2))$$

We can use mathematical induction to get

$$z^n = r^n(cos(n\theta) + isin(n\theta))$$

$$\sqrt[n]{z} = \sqrt[n]{r}(cos(\tfrac{\theta+2k\pi}{n}) + isin(\tfrac{\theta+2k\pi}{n})), \quad 0 \leq k \leq n-1, \quad k \in \mathbb{N}$$

Complex Numbers: Exercise

Recall that

$$z := (a, b) = a + bi = r(cos\theta + i\ sin\theta)$$

$$r\text{: absolute value of } z,$$
$$\theta\text{: } \textit{argument} \text{ of } z, \quad \theta \in [0, 2\pi)$$

Solve the equation $z^7 = 1, z \in \mathbb{C}$.
Hint: calculate $z_1 \cdot z_2$ in the general form.

Complex Numbers: Solution

Let $z := r(cos\theta + i\ sin\theta)$ solve the equation $z^7 = 1$, then

$$
\begin{aligned}
(r(cos\theta + i\ sin\theta))^7 = 1 \quad &\Leftrightarrow \quad r^7(cos7\theta + i \cdot 7sin\theta) = 1 \\
&\Leftrightarrow \quad (r = 1) \wedge (7\theta = 2k\pi), \theta \in [0, 2\pi)
\end{aligned}
$$

Hence the solution set is $\{1, cos\frac{2\pi}{7} + i\ sin\frac{2\pi}{7}, cos\frac{4\pi}{7} + i\ sin\frac{4\pi}{7}, cos\frac{6\pi}{7} + i\ sin\frac{6\pi}{7}, cos\frac{8\pi}{7} + i\ sin\frac{8\pi}{7}, cos\frac{10\pi}{7} + i\ sin\frac{10\pi}{7}, cos\frac{12\pi}{7} + i\ sin\frac{12\pi}{7}\}$.

Reference

1. VV186 Slide and previous RC Slide