VV186 Mathematical proof

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Some words

Here are some guidance to help you write some mathematical proof.

Proof

A proof is a series of statements, each of which follows logically from what has gone before. It starts with things we are assuming to be true. It ends with the thing we are trying to prove.

So, like a good story, a proof has a beginning, a middle and an end.

- Beginning: things we are assuming to be true, including the definitions of the things we're talking about
- Middle: statements, each following logically from the stuff before it
- End: the thing we're trying to prove

Proof

One of the difficult things about writing a proof is that the order in which we write it is often not the order in which we thought it up. In fact, we often think up the proof backwards.

For example, you will write some proof using $\epsilon - \delta$ language. When writing the proof, you may need to set ϵ equals to some value first. But what ϵ equals is actually found at last.

Example

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Sequences - General Results

2.2.14. Lemma. A convergent sequence has precisely one limit.

Obviously a convergent sequence has a limit; the statement of the lemma is that it can not have more than one limit.

Proof.
Let $a \neq b$ and assume that $a_n \to a$ and $a_n \to b$. Then for every $\varepsilon > 0$ there exists some $N(\varepsilon)$ such that $|a_n - a|, |a_n - b| < \varepsilon/2$ for all $n > N(\varepsilon)$. Let $\varepsilon = |a-b|/2$ and choose $n > N(|a-b|/2)$. Then

$$|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \frac{|a-b|}{2},$$

which is a contradiction. $\square$

Examples of a bad proof

1. Begin at the end and end at the beginning

This is a really, really terrible thing to do. This might be even worse than leaving out gaps in the middle. Because if you begin at the end and end at the beginning you *monumentally* haven't got where you're trying to go. Here's an example of this for Example 1 from Section 4:

$$\begin{aligned}
a(b-c) &= ab - ac \\
ab + a(-c) &= ab - ac \\
a(-c) &= -ac \\
ac + a(-c) &= 0 \\
a(c + (-c)) &= 0 \\
a.0 &= 0 \\
0 &= 0 \qquad \square
\end{aligned}$$

Try comparing this with the *good* proof given in Section 4 – you'll see that all the correct steps are there, but they're all in the wrong order.

Examples of a bad proof

2. Take flying leaps instead of earthbound steps.

This category includes leaping from one statement to another

- without justifying the leap
- leaving out too many steps in between
- using a profound theorem without proving it
- (worse) using a profound theorem without even mentioning it

Examples of a bad proof

3. Take flying leaps and land flat on your face in the mud

By which I mean making steps that are actually wrong. The end may well justify the means in some worlds, but in mathematics if you use the wrong means to get to the right end, you haven't actually got to the end at all. You just think you have. But it's a figment of your imagination. Here's an example of a very imaginitive "proof" that is definitely flat on its face in the mud:

$$\begin{aligned} a(b-c) &= ab + a(-c) \\ &= ab + a(-c) + a.1 \\ &= ab + a(1-c) \\ &= ab - ac \qquad \square \end{aligned}$$

Of course, it's even worse if you do something illegal and thereby reach a conclusion that isn't even true. Like

$$x^2 = y^2 \Longrightarrow x = y$$

or

$$x^2 < y^2 \Longrightarrow x < y.$$

What is wrong with these two "deductions"?

Examples of a bad proof

4. Handwaving

Handwaving is when you arrive at a statement by some not-very-mathematical means. The step isn't necessarily wrong, but you haven't arrived at it in a good logical manner. Perhaps you had to resort to writing a few sentences of prose in English rather than Mathematics-speak. This is often a sign that you've got the right idea but you haven't worked out how to express it. Spot the handwaving here – you can see it from a mile off:

$$\begin{aligned} a(b-c) &= ab + a(-c) \\ a(-c) &= -ac \qquad \text{because if you add ac to} \\ & \qquad\qquad \text{both sides then both sides vanish} \\ & \qquad\qquad \text{which means they're inverse} \\ \therefore ab + a(-c) &= ab - ac \qquad \square \end{aligned}$$

Handwaving is bad but is not ultimately catastrophic – you just need to learn how to translate from English into Mathematics. This is probably easier to learn than the problem of coming up with the right idea in the first place.

Examples of a bad proof

5. Incorrect logic

This includes the two great classics

- negating a statement incorrectly
- proving the converse of something instead of the thing itself

What is the negation of the following statement:

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } \forall x \text{ satisfying } 0 < |x-a| < \delta, \quad |f(x) - l| < \varepsilon$$

The correct answer is at the bottom of the page[1]. If you get it wrong, you go directly to Jail. Do not pass Go. Do not collect $200.

Examples of a bad proof

6. Incorrect assumption

You could have all your logic right, you could make a series of perfectly good and sensible steps, but if you start in the wrong place then you're not going to have a good proof. Or, if you use any assumption along the way that simply isn't true, then it's all going to go horribly pear-shaped...

Examples of a bad proof

7. Incorrect use of definitions – or use of incorrect definitions

This is a very, very avoidable error. Especially if it's not a test and so you have all your notes and all the books in the world to consult: getting the definitions wrong is a really

pointless way of going wrong. What's wrong with the following "proof" for the second example from Section 4?

$$\begin{aligned} f(a) = f(a') &\implies a = a' \\ g(a) = g(a') &\implies a = a' \end{aligned}$$

$$\begin{aligned} (g \circ f)(a) = (g \circ f)(a') &\implies g(a) \circ f(a) = g(a') \circ f(a') \\ &\implies a = a' \end{aligned}$$

$$\therefore \quad g \circ f \text{ is injective.} \qquad \square$$

Examples of a bad proof

8. Assuming too much

This is a tricky one, especially when you're a student at the beginning of a course. What are you allowed to assume? How much do you have to justify each step? A good rule of thumb is:

> *You need to justify everything enough for your peers to understand it.*

This is not a hard and fast rule, but it's a guideline that will always remain true however far you progress in mathematics, even if you become an internationally acclaimed Fields-medal-winning mathematician. The point is that as you become more advanced your peers do too, so you are eventually going to be taking bigger steps in your proofs than you do now. i.e. don't worry, you won't be required to write down every use of the distributive law forever!

> *If in doubt, justify things more rather than less.*

Very few people give too much explanation of things. In fact, I have only ever encountered one student who consistently explained things too much.

Examples of a good proof

Sequences - Examples

2.2.6. Theorem. The sequence (a_n), $a_n = \frac{1}{n}$, $n \in \mathbb{N} \setminus \{0\}$, converges towards $a = 0$.

Proof.
We need to show that for any $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|a_n - a| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon.$$

We note that $\frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$. Therefore, given $\varepsilon > 0$, we choose any $N \in \mathbb{N}$ with $N > 1/\varepsilon$. Then for all $n > N$ we have $n > 1/\varepsilon$ and hence $1/n < \varepsilon$. Thus $|a_n - 0| = a_n < \varepsilon$ for all $n > N$, so by the definition of convergence we have

$$\lim_{n\to\infty} \frac{1}{n} = 0. \qquad \square$$

Examples of a good proof

Sequences - General Results

2.2.12. Lemma. A convergent sequence is bounded.

Proof.
Let $a_n \to a$. Then there exists some N such that $|a_n - a| < 1$ for all $n > N$. By the triangle inequality, this means that

$$|a_n| = |a_n - a + a| \le |a_n - a| + |a| < 1 + |a|$$

for $n > N$. It follows that

$$|a_n| \le \max\{|a_1|, \dots, |a_N|, |a| + 1\} =: C$$

for all $n \in \mathbb{N}$, so $\sup|a_n| \le C < \infty$ and the range is bounded. $\square$

Some suggestions

In order to obtain the highest possible score, make sure that you explain your reasoning. Often, simple formulae are not enough to answer a question. *Explain what you are doing!* This will also ensure that you get a large fraction of the total points even if you make a mistake in your calculations. In short, write simple, whole grammatical sentences that include a subject, verb and object.

You are **forbidden to use the symbols $\therefore$ and $\because$** in your writing. Write English sentences instead. You will **lose all points** for an exercise that includes these two symbols, no matter whether the solution is correct or not! On this subject, I would like to quote the great mathematician Serge Lang:

> It seems to me essential that students be required to write their mathematics papers in full and coherent sentences. A large portion of their difficulties with mathematics stems from their slapping down mathematical symbols and formulas isolated from a meaningful sentence and appropriate quantifiers. Papers should also be required to be neat and legible. They should not look as if a stoned fly had just crawled out of an inkwell. Insisting on reasonable standards of expression will result in drastic improvements of mathematical performance.

Serge Lang, *A First Course in Calculus*

Reference

1.Eugenia Cheng How to write proofs: a quick guide

2. VV186 slides