Mid 1 Review

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Limit of Sequences and Functions

Basic Concepts

- *domf*
- *ranf* VS. codomain *f*
- Real Functions

Limit of Sequences and Functions

Method of Showing Convergence and Calculating Limits

- $N-\epsilon$ method or $\delta-\epsilon$ method
- Limit addition, multiplication, division
- Squeze theorem
- Prove that the given sequence is a Cauchy sequence in a complete metric space to show that it converges
- Monotonic + bounded $\Rightarrow$ convergence, then use $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$ to derive the limit
- Show that $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} a_{2n+1} = a$
- Construct a function to calculate the sequence limit
- Composition of two functions: calculate the limit of the inner function and show that the outer function is continuous

Limit of Functions

Method of Showing Divergence

- Show that $\lim\limits_{x \nearrow x_0} f(x) \neq \lim\limits_{x \searrow x_0} f(x)$
- Show divergence to infinity
- Show that if $x_n \to a, y_n \to a$, and a is an accumulation point of f, $\lim\limits_{x_n \to a} f(x_n) \neq \lim\limits_{y_n \to a} f(y_n)$

Limit of Sequence and Function

Exercise

Sample Exercise 7

Exercise 7. If $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x\to 0} \frac{f(x)}{x} = \alpha$ for some $\alpha \in \mathbb{R}$, calculate

i) $\lim_{x\to 0} \frac{f(2x)}{x}$,

ii) $\lim_{x\to 0} \frac{[f(2x)]^2}{x^2}$,

iii) $\lim_{x\to 0} \frac{[f(2x)]^2}{x}$.

(3 × 1 Mark)

Big- O Landau Symbol

$f(x) = O(\phi(x))$ as $x \to \infty$:

$$\underset{C>0}{\exists}\ \underset{M>L}{\exists}\ x > M \quad \Leftrightarrow \quad |f(x)| \le C|\phi(x)|$$

$f(x) = O(\phi(x))$ as $x \to x_0$ (x_0 is an accumulation point):

$$\underset{C>0}{\exists}\ \underset{\epsilon>0}{\exists}\ \underset{x\in\Omega}{\forall}\ |x - x_0| < \epsilon \quad \Leftrightarrow \quad |f(x)| \le C|\phi(x)|$$

$$\lim_{x\to x_0} \frac{|f(x)|}{|g(x)|} = C > 0$$

Little- O Landau Symbol

$f(x) = o(\phi(x))$ as $x \to \infty$:

$$\underset{C>0}{\forall}\ \underset{M>L}{\exists}\ x > M \quad \Leftrightarrow \quad |f(x)| < C|\phi(x)|$$

$f(x) = o(\phi(x))$ as $x \to x_0$ (x_0 is an accumulation point):

$$\underset{C>0}{\forall}\ \underset{\epsilon>0}{\exists}\ \underset{x\in\Omega}{\forall}\ |x - x_0| < \epsilon \quad \Leftrightarrow \quad |f(x)| < C|\phi(x)|$$

$$\lim_{x\to x_0} \frac{|f(x)|}{|\phi(x)|} = 0$$

Landau Symbols

Remark

- $\leq C$ v.s. $< C$
- > 0 v.s. $= 0$
- $\forall$ v.s. $\exists$
- Clearly indicate $x \to \infty$ or $x \to x_0$

Limit of Sequence and Function

Exercise

1. Define (x_n) by $x_0 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$. Show that $x_n - 2 = o(\frac{1}{3^n})$ as $n \to \infty$.

Limit of Sequence and Function

Exercise

Suppose that f is continuous on $\mathbb{R}$, $f(1) = 2$. For any $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y) \geq 0$. Prove that $f(x) = 2^x, x \in \mathbb{R}$.

Solution

First, for $m \in \mathbb{Z}^+$, we can prove by induction that $f(m) = 2^m$.
Second, $f(1) = f(1)f(0) = 2$, so $f(0) = 1$.
$f(0) = f(-x)f(x) = 1$, so $f(-x) = \dfrac{1}{f(x)}$.
For $m \in \mathbb{Z}^-$, $f(m) = \dfrac{1}{f(-m)} = \dfrac{1}{2^{-m}} = 2^m$.
Hence for any $m \in \mathbb{Z}$, $f(m) = 2^m$ (with $f(0) = 1$).
Moreover, for $1 = \dfrac{n}{n}$, $f(1) = [f(\dfrac{1}{n})]^n = 2$. Hence $f(\dfrac{1}{n}) = 2^{1/n}$.

Limit of Sequence and Function

Solution (Continued)

So for any $p = \dfrac{m}{n} \in \mathbb{Q}$, $f(\dfrac{m}{n}) = (2^{1/n})^m = 2^{m/n}$.
Lastly, since every Cauchy rational sequence has a limit in $\mathbb{R}$, so $\forall x \in \mathbb{R}$ $\exists a_n \xrightarrow{n\to\infty} x$, $a_n \in \mathbb{Q}$.
Then since f is continuous,
$f(x) = \lim\limits_{y\to x} f(y) = \lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} 2^{a_n} = \lim\limits_{a_n\to x} 2^{a_n} = 2^x$
Hence $f(x) = 2^x, x \in \mathbb{R}$.

Thank You and Good Luck!