VV186 MID2 RC

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Series: Convergence

3.5.1. Definition. Let (a_n) be a sequence in a normed vector space $(V, \|\cdot\|)$. Then we say that (a_n) is ***summable*** with sum $s \in V$ if

$$\lim_{n\to\infty} s_n = s, \qquad s_n := \sum_{k=0}^{n} a_k.$$

We call s_n the ***nth partial sum*** of (a_n). We use the notation

$$\sum_{k=0}^{\infty} a_k \qquad \text{or simply} \qquad \sum a_k \tag{3.5.1}$$

to denote not only s, but also the "procedure of summing the sequence (a_n)." We call (3.5.1) an ***infinite series*** and we say that the series converges if (a_n) is summable. If (s_n) does not converge, we say that $\sum a_k$ diverges.

Cauchy Criterion

3.5.4. Cauchy Criterion. Let $\sum a_k$ be a series in a **complete** vector space $(V, \|\cdot\|)$. Then

$$\begin{aligned}
\sum a_k \text{ converges} \quad &\Leftrightarrow \quad (s_n)_{n\in\mathbb{N}} \text{ converges, } s_n = \sum_{k=0}^{n} a_k \\
&\Leftrightarrow \quad (s_n) \text{ is Cauchy} \\
&\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m>n>N}{\forall}\ \|s_m - s_n\| < \varepsilon \\
&\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{m>n>N}{\forall}\ \left\| \sum_{k=n+1}^{m} a_k \right\| < \varepsilon
\end{aligned}$$

Corollary given by Cauchy Criterion

3.5.5. Corollary. If the series $\sum_{k=0}^{\infty} a_k$ converges, then the sequence $a_k \to 0$ as $k \to \infty$. (Take $m = n + 1$ in the Cauchy Criterion.)

3.5.6. Corollary. If the series $\sum_{k=0}^{\infty} a_k$ converges, then the sequence (A_n) given by

$$A_n := \sum_{k=n}^{\infty} a_k$$

converges to 0 as $n \to \infty$. (Let $m \to \infty$ in the Cauchy Criterion.)

Absolute Convergence

3.5.9. Definition. A series $\sum a_k$ in a normed vector space $(V, \|\cdot\|)$ is called ***absolutely convergent*** if $\sum \|a_k\|$ converges.

A sequence (a_k) in a normed vector space $(V, \|\cdot\|)$ is called ***absolutely summable*** if $\sum a_k$ converges absolutely.

3.5.10. Theorem. An absolutely convergent series $\sum a_k$ in a **complete** vector space $(V, \|\cdot\|)$ is convergent.

Comparison Test

The following criteria are used to establish the absolute convergence of a series.

3.5.13. Comparison Test. Let (a_k) and (b_k) be real-valued sequences with $0 \le a_k \le b_k$ for sufficiently large k. Then

$$\sum b_k \text{ converges} \quad \Rightarrow \quad \sum a_k \text{ converges.}$$

Remark: It is the most general test. You are recommended to try this method with the help of basic inequalities to judge convergence of series first.

the Weierstrass M-Test

3.5.17. Weierstraß M-test. Let $\Omega \subset \mathbb{R}$ and (f_k) be a sequence of functions defined on Ω, $f_k \colon \Omega \to \mathbb{C}$, satisfying

$$\sup_{x\in\Omega} |f_k(x)| \le M_k, \qquad\qquad k \in \mathbb{N} \tag{3.5.9}$$

for a sequence of real numbers (M_k). Suppose that $\sum M_k$ converges. Then the limit

$$f(x) := \sum_{k=0}^{\infty} f_k(x) \qquad\qquad \text{exists for every } x \in \Omega.$$

Furthermore, the sequence (F_n) of partial sums

$$F_n(x) = \sum_{k=0}^{n} f_k(x)$$

converges uniformly to f.

the Root Test

3.5.20. Root Test. Let $\sum a_k$ be a series of positive real numbers $a_k \geq 0$.

(i) Suppose that there exists a $q < 1$ such that

$$\sqrt[k]{a_k} \leq q \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ converges.

(ii) Suppose that

$$\sqrt[k]{a_k} > 1 \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ diverges.

3.5.21. Remark. Note that the existence of a $q < 1$ so that $\sqrt[k]{a_k} < q$ is crucial; this is not the same as requiring $\sqrt[k]{a_k} < 1$.

the Root Test Using Limits

3.5.24. Root Test. Let a_k be a sequence of positive real numbers $a_k \geq 0$. Then

$$\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} < 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{converges,}$$

$$\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} > 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \qquad \text{diverges.}$$

3.5.25. Remarks.

(i) No statement is possible if $\overline{\lim_{k\to\infty}} \sqrt[k]{a_k} = 1$.

(ii) If $\lim_{k\to\infty} \sqrt[k]{a_k}$ exists, it equals $\overline{\lim_{k\to\infty}} \sqrt[k]{a_k}$. This will be the case in many applications.

the Ratio Test

3.5.26. Ratio Test. Let $\sum a_k$ be a series of strictly positive real numbers $a_k > 0$.

(i) Suppose that there exists a $q < 1$ such that

$$\frac{a_{k+1}}{a_k} \le q \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ converges.

(ii) Suppose that

$$\frac{a_{k+1}}{a_k} \ge 1 \qquad \text{for all sufficiently large } k.$$

Then $\sum a_k$ diverges.

Remark: Every

the Ratio Test Using Limits

3.5.28. Ratio Test. Let (a_k) be a sequence of strictly positive real numbers $a_k > 0$. Then

$$\varlimsup_{k\to\infty} \frac{a_{k+1}}{a_k} < 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \quad \text{converges,}$$

$$\varliminf_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \qquad \Rightarrow \qquad \sum_{k=0}^{\infty} a_k \quad \text{diverges.}$$

Remark: Any problem that can be solved by the ratio test can also be solved by the root test. The root test is a more powerful tool.

the Ratio Comparison Test

3.5.29. Ratio Comparison Test. Let (a_k) and (b_k) be sequences of strictly positive real numbers $a_k, b_k > 0$. Suppose that $\sum b_k$ converges. If

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k} \qquad \text{for sufficiently large } k,$$

then $\sum a_k$ converges.

Remark: if the condition of the ration comparison test is satisfied, then the sequence cannot increase rapidly.

Raabe's test

3.5.32. Raabe's Test. Let $\sum a_k$ be a series of positive real numbers $a_k \geq 0$. Suppose that there exists a number $p > 1$ such that

$$\frac{a_{k+1}}{a_k} \leq 1 - \frac{p}{k} \qquad \text{for sufficiently large } k.$$

Then the series $\sum a_k$ converges.

Remark: For some sequences, if we cannot find a suitable q for the ratio test, we can try to use Raabe's test.

Some comments about the five tests.

Pay attention to the p-Series.

the Leibniz Theorem

3.5.38. Leibniz Theorem. Let $\sum \alpha_k$ be a complex series whose partial sums are bounded but need not converge. Let (a_k) be a decreasing convergent sequence with limit zero, $a_k \searrow 0$. Then the series

$$\sum \alpha_k a_k \qquad \text{converges.}$$

A typical application of the Leibniz Theorem 3.5.38 are alternating series, for which $\alpha_k = (-1)^k$. In particular, the Leibniz series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converges by the Leibniz Theorem.

Cauchy Product

3.5.40. Theorem. Let $\sum a_k$ and $\sum b_k$ be absolutely convergent series. Then the ***Cauchy product*** $\sum c_k$ given by

$$c_k := \sum_{i+j=k} a_i b_j$$

converges absolutely and $\sum c_k = \left(\sum a_k\right)\left(\sum b_k\right)$.

3.5.41. Remark. If $a = (a_k)$ and $b = (b_k)$ are two absolutely summable sequences, the sequence

$$a * b := (c_k), \qquad c_k := \sum_{i+j=k} a_i b_j,$$

is called the ***convolution*** of a and b.

Formal Power Series

3.6.1. Definition. For any complex sequence (a_k), the expression

$$\sum_{k=0}^{\infty} a_k z^k$$

is called a (formal) complex power series.

3.6.4. Definition. A formal power series $\sum a_k z^k$ is said to be ***is (absolutely) convergent at*** $z_0 \in \mathbb{C}$ if the series $\sum_{k=0}^{\infty} a_k z_0^k$ converges (absolutely).

Radius of Convergence

3.6.6. Definition and Theorem. Let $\sum a_k z^k$ be a complex power series. Then there exists a unique number $\varrho \in [0, \infty]$ such that

(i) $\sum a_k z^k$ is absolutely convergent if $|z| < \varrho$,

(ii) $\sum a_k z^k$ diverges if $|z| > \varrho$.

This number is called the ***radius of convergence*** of the power series. It is given by ***Hadamard's formula***,

$$\varrho = \begin{cases} \frac{1}{\overline{\lim}\, \sqrt[k]{|a_k|}}, & 0 < \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} < \infty, \\ 0, & \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} = \infty, \\ \infty, & \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|} = 0. \end{cases} \tag{3.6.1}$$

3.6.7. Remark. No information is given about the convergence of $\sum a_k z^k$ if $|z| = \varrho$. In these cases the series may converge or diverge, depending on the point z.

Radius of Convergence

3.6.9. Lemma. Let $\sum a_k z^k$ be a complex power series with radius of convergence ϱ. Then the series $\sum k a_k z^{k-1}$ has the same radius of convergence ϱ.

3.6.10. Lemma. If $\sum a_k z^k$ is a complex power series with radius of convergence ϱ, then for any $R < \varrho$ the series converges uniformly on $B_R(0) = \{z : |z| < R\}$.

3.6.11. Corollary. A power series $\sum_{k=0}^{\infty} a_k x^k$ with radius of convergence ϱ defines a continuous function

$$f : B_\varrho(0) \to \mathbb{C}, \qquad f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Radius of Convergence

3.6.14. Theorem. The real or complex power series $f(z) := \sum a_k z^k$ with radius of convergence ϱ defines a differentiable function $f\colon B_\varrho(0) \to \mathbb{C}$. Furthermore,

$$f'(z) = \sum k a_k z^{k-1} \tag{3.6.2}$$

where the series has the same radius of convergence as f.

Radius of Convergence

vi) For any $0<R<\rho$, $\sum a_k z^k$ converges uniformly on $[-R,R]$.

Besides $f:(-\rho,\rho)\longrightarrow C$, $f(z)=\sum_{k=0}^{\infty} a_k z^k$ is continuous.

example: $\sum_{k=0}^{\infty}(-1)^k x^k$ converges uniformly on $[-R,R]$ for any $0<R<1$, but it doesn't converge uniformly on $(-1,1)$.

$\Longleftrightarrow$ $f:(-1,1)\longrightarrow R$, $f(x)=\frac{1}{1+x}$ is uniformly continuous on $[-R,R]$ for any $0<R<1$, however, f ~~doesn't con~~ is not uniformly continuous on $(-1,1)$.

Some words

The crucial thing for this part is that you need to know how to analyze series convergence using various kinds of test and calculate the radius of convergence.

Make sure that you know how to do Ex 8.1 and Ex 8.6 in your homework!

Sample

Exercise 1. Test the following series for convergence:

$$\sum_{k=1}^{\infty} \frac{k+5}{5^k}, \qquad \sum_{k=1}^{\infty} \frac{5^k}{4^k+3^k}, \qquad \sum_{k=1}^{\infty} \frac{3^k k^2}{k!}.$$

(3 Marks)

These three problems can be solved by the root test directly.

Nearly all the problems of positive series except those in the form of p − series, like $\sum_{k=1}^{\infty} \frac{g(k)}{h(k)}$, *where $g(k)$ and $h(k)$ are both polynomials or other special ones, can be solved by the root test.*

Exercise

6. If the radius of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ is ρ_1 & ρ_2, respectively. Analyze ~~Find~~ the radius of convergence of the following series.

i) $\sum (a_n + b_n) x^n$

ii) $\sum a_n b_n x^n$ (Optional)

Exercise

i) let $A_n = a_n + b_n$, then

$$\sqrt[n]{|A_n|} = \sqrt[n]{|a_n+b_n|} \le \sqrt[n]{|a_n|+|b_n|} \le \sqrt[n]{2\max(|a_n|,|b_n|)}$$

$$= \sqrt[n]{2}\,\sqrt[n]{\max(|a_n|,|b_n|)} = \sqrt[n]{2}\,\max\left(\sqrt[n]{|a_n|}, \sqrt[n]{|b_n|}\right)$$

Notice that $\overline{\lim_{n\to\infty}} \sqrt[n]{2} = 1$, then

$$\frac{1}{\rho} = \overline{\lim_{n\to\infty}} \sqrt[n]{|A_n|} \le \overline{\lim_{n\to\infty}} \left[\sqrt[n]{2}\max(|a_n|,|b_n|)\right]$$

$$= \overline{\lim_{n\to\infty}} \left[\max(|a_n|,|b_n|)\right]$$

$$= \max\left\{ \overline{\lim_{n\to\infty}} |a_n|,\ \overline{\lim_{n\to\infty}} |b_n| \right\}$$

$$= \max\left(\frac{1}{\rho_1}, \frac{1}{\rho_2}\right)$$

$$\Longrightarrow \rho = \frac{1}{\ }$$

Exercise

ii) let $B_n := a_n b_n$, then $\sqrt[n]{|B_n|} = \sqrt[n]{|a_n b_n|} = \sqrt[n]{|a_n|} \cdot \sqrt[n]{|b_n|}$

$$\Rightarrow \frac{1}{\rho} = \overline{\lim_{n\to\infty}} \sqrt[n]{|B_n|} = \overline{\lim_{n\to\infty}} \left[\sqrt[n]{|a_n|} \cdot \sqrt[n]{|b_n|}\right] \leq \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \cdot \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|}$$

$$= \frac{1}{\rho_1 \rho_2}$$

$$\Longrightarrow \rho \geqslant \rho_1 \rho_2$$

Exercise

8. Application of convergence.

Let $x_n = \frac{n^n}{n! \cdot 3^n}$, find $\lim\limits_{n\to\infty} x_n$.

Solution: Consider $\sum\limits_{n=1}^{\infty} x_n$, since

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1.$$

Hence, $\sum\limits_{n=0}^{\infty} x_n$ is convergent.

By Cauchy Criterion, $\lim\limits_{n\to\infty} x_n = 0$.

Reference

1. VV186 Slide and previous RC Slide