M2R Oral Presentation

Manifolds and homogeneous spaces (M2R2)

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16/06/2021

Outline

Group action on sets

Deifinition

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Examples:

1. The symmetric group $S(n)$ acts on the set $\{1, 2, ..., n\}$ by the various permutation maps;
2. The multiplicative group R^* acts on the Euclidean space by the scalar multiplication map.

Transitive actions

Having defining the group actions, we are interested in a particular type of them.

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$S^n \subset \mathbb{R}^{n+1}$ is defined as the set: $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, and can also be called the **n-sphere**.

Transitive actions

There are 9 transitive actions on a n−sphere...

Transitive actions

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Group	SO(n)	U(n), SU(n)	Sp(n)Sp(1), Sp(n)U(1), Sp(n)
Sphere	S^{n-1}	S^{2n-1}	S^{4n-1}

G_2	Spin(7)	Spin(9)
S^6	S^7	S^{15}

Homeomorphism

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A **diffeomorphism** is a smooth homeomorphism whose inverse is also smooth.

Charts and atlases

Definition

A **chart** (U, φ) for a topological space X is a homeomorphism φ from an open U is a subset of X to an open subset $\tilde{U}$ contained in $\mathbb{R}$, that is $\varphi : U \to \tilde{U}$.

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An **atlas** for a topological space X is the collection of charts for X which covers X.

Example

Consider the simple 2-sphere S^2.

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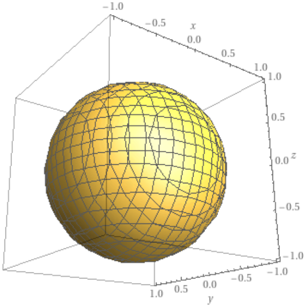

Figure: An atlas of the 2-sphere

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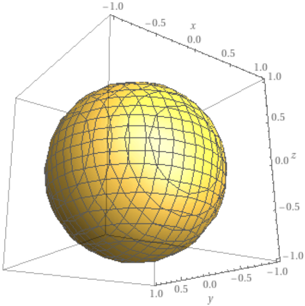

Figure: An atlas of the 2-sphere

$$\varphi_{front}(x, y, z) = (x, z),\ \varphi_{left}(x, y, z) = (y, z),\ \varphi_{top}(x, y, z) = (x, y)\ldots$$

Paracompactness

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For a cover $C = \{U_\alpha : \alpha \in A\}$ of a topological space X, a **refinement** $D = \{V_\beta : \beta \in B\}$ of the cover C is a subcover such that for all V_β in D, exists U_α in C such that $V_\beta \subseteq U_\alpha$.

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An open cover C of X is called **locally finite** if for all x in X, exists $B(x)$ in X where $B(x)$ is a neighbourhood of x such that $B(x)$ intersects a finite number of subsets of C.

Paracompactness

Proposition

Every compact space is paracompact.

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Every compact space is paracompact.

However, the Euclidean space $\mathbb{R}^n$ is paracompact but not compact!

Manifolds

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For $p \in X$, there exists an open neighbourhood of p that is homeomorphic to an open subset of $\mathbb{R}^n$

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Recall definition of charts. Every point in locally Euclidean space is contained in some charts.

Topological manifolds

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2. **Second countable**: M has a countable basis, which means there exists a countable collection $\mathbb{B}$ of open subsets of M such that for any open subset U of M and point p in U, there is an open set $B \in \mathbb{B}$ such that $p \in B \subset U$.

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3. **Locally Euclidean**: For every point $p \in M$, there exists a neighbourhood N of p such that N is homeomorphic to an open subset of .

.

Examples

The trivial $\mathbb{R}^n$ is a topological space since it satisfies the three properties in the definition.

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Proposition

Subspace of Hausdorff and second-countable space is also Hausdorff and second-countable.

Examples

Another example is the **unit n-sphere** $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

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Another example is the **unit n-sphere** $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.
Let $U_i = \{(x_1, , x_{n+1}) \in S^n : x_i > 0\}$ and $V_i = \{(x_1, , x_{n+1}) \in S^n : x_i < 0\}$

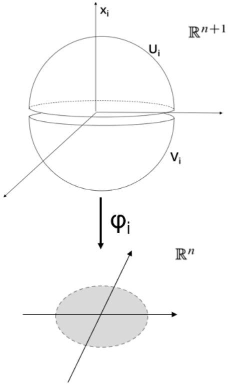

Figure: projection of charts of n-sphere

Transition map

Definition

For two charts (U, φ) and (V, ψ) in a topological manifold, we say if U and V are not disjoint then the composition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$, or $\psi \circ \varphi^{-1} : \varphi(U \cap V) \mapsto \psi(U \cap V)$ defined on the intersection of U and V is a **transition map**.

Transition map

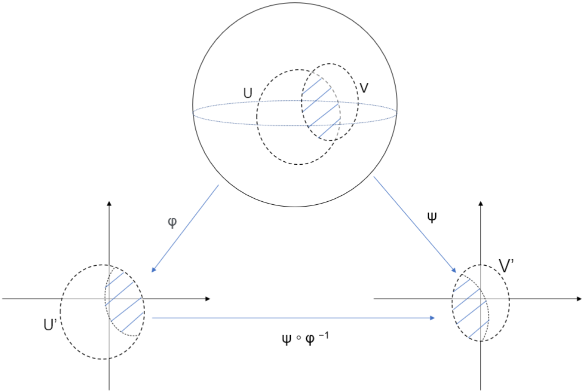

Figure: transition map

Smooth Atlas

We say two charts (U, φ) and (V, ψ) are **smoothly compatible** if either they are disjoint or their transition map is a diffeomorphism.

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Definition

An atlas is a **smooth atlas** if every chart of it is smoothly compatible with each other.

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A **smooth manifold** is a pair $(M, \mathbb{A})$ where $\mathbb{A}$ is a maximal(not contained in any larger smooth atlas) smooth atlas on a topological manifold M.

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Examples: $\mathbb{R}^n$, n-sphere,$\mathrm{GL}_n(\mathbb{R})$,...

Intuition

What are Lie groups?

Lie Groups

Definition

A Lie group is a group G that is also a smooth manifold and the map $G \to G$, $(g,h) \mapsto gh^{-1}$, with $g,h \in G$ is smooth.

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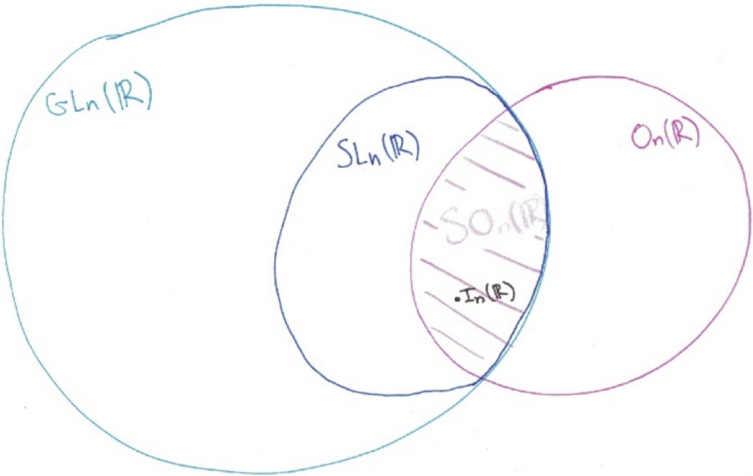
Result from Analysis 2: the composition of smooth maps is itself a smooth map

Some results:

We have a few results already

- $\mathrm{GL}_n(\mathbb{R})$is a smooth manifold
- The determinant map: $\det : GL_n(\mathbb{R}) \to \mathbb{R}$ is a smooth map
- Taking the inverse of a matrix is a smooth map
- Matrix multiplication is a smooth map
- The orthogonal group map $A \mapsto AA^{\top}$ is continuous.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- The preimage of a continuous function on a closed set is also a closed set

E. Cartan Closed subgroup Theorem

Any closed subgroup H of a Lie group G is a Lie subgroup and hence a submanifold of G.

Examples of Lie groups

Example: $\mathrm{GL}_n(\mathbb{R})$

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3. Apply E. Cartan's Closed subgroup theorem
4. **$\mathrm{SL}_n(\mathbb{R})$is a Lie subgroup** and submanifold of $\mathrm{GL}_n(\mathbb{R})$
5. Matrix multiplication and taking inverse are smooth maps in $\mathrm{SL}_n(\mathbb{R}) \implies SL_n(\mathbb{R})$is also a smooth manifold

Examples of Lie groups

Definition: $O_n(\mathbb{R})$

$O_n(\mathbb{R})$is the set of all matrices, that, when multiplied by their own inverse, give the identity matrix

Definition: $\mathrm{SO}_n(\mathbb{R})$

$\mathrm{SO}_n(\mathbb{R})$is the set of all $n \times n$ invertible matrices with det = 1 and the inverse of each matrix is itself.

Examples of Lie groups

Example: $\mathrm{SO}_n(\mathbb{R})$

1. $\mathrm{SO}_n(\mathbb{R})$is a subgroup of $\mathrm{SL}_n(\mathbb{R})$
2. Taking the transverse is a continuous map on O_n.
3. $O : SL_n(\mathbb{R}) \to SL_n(\mathbb{R}),\ O(A) = AA^\top$ also acts on $\mathrm{SO}_n(\mathbb{R})$(as determinant 1 is preserved),
4. $SO_n(\mathbb{R}) = O^{-1}(\{I_n\})$. As $\{I_n\}$ is closed in $SL_n(\mathbb{R})$, the preimage $SO_n(\mathbb{R})$ is also (topologically) closed
5. Apply E. Cartan's Closed subgroup theorem
6. **$\mathrm{SO}_n(\mathbb{R})$is a Lie subgroup** and submanifold of $\mathrm{SL}_n(\mathbb{R})$
7. Matrix multiplication and taking inverse are smooth maps in $\mathrm{SO}_n(\mathbb{R}) \implies SO_n(\mathbb{R})$is also a smooth manifold

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Definition

An **isotropy group** G_x is a subgroup of G, which for all g in G_x, $g \cdot x = x$.

The main result here is the special linear group $SO(n)$ is the isotropy group of $SO(n+1)$.

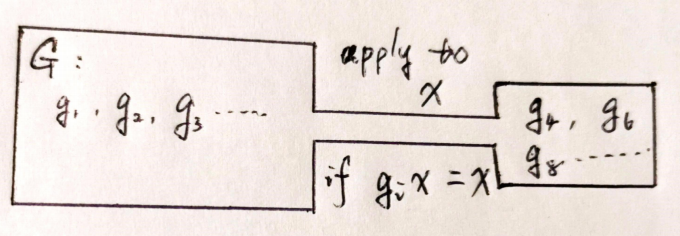

Figure: Isotropy group

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Proof is straight-forward using results from previous slides, but note that it's slightly different from the definition, the ingredients we need:

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An example of homogeneous space would be S^n
Proof is straight-forward using results from previous slides, but note that it's slightly different from the definition, the ingredients we need:

1. Lie group action acts transitively on G.
2. S^n is a smooth manifold.

Homogeneous space

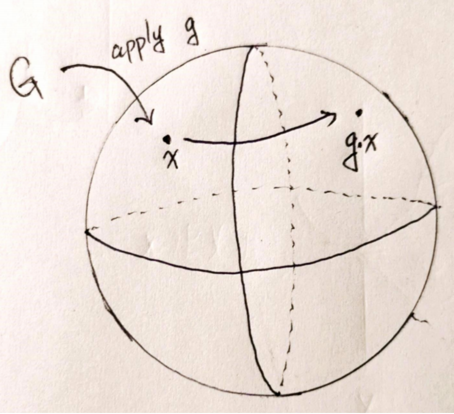

Figure: Homogeneous space

Homogeneous space and principal bundles

First, definition of a fiber bundle:

Definition

For E (total space), B (base fiber), F (fiber) topological spaces and a continuous map $\pi : E \to B$ form a **fiber bundle with fiber F** if:

1. B is a connected topological space.
2. The natural projection map $\pi : E \to B$ is surjective.
3. Each element in the base fiber has an open neighbourhood contained within the base fiber. That is for all $x \in B$, $\exists$ open neighbourhood $U_x \subset B$, there exists a homeomorphism $\varphi : \pi^{-1}(U_x) \to U_x \times F$, that is a topological isomorphism.

Homogeneous space and principal bundles

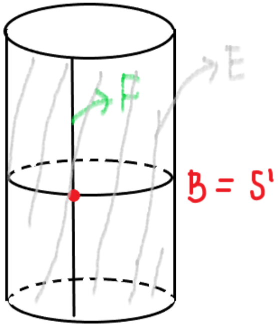

Figure: Fiber bundle

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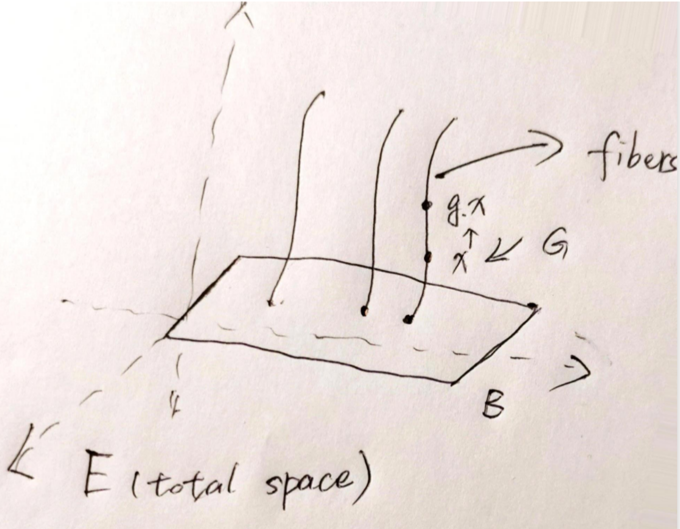
Definition

Principal bundles: A principal G-bundle with a topological space G, is a fiber bundle $\pi: E \to B$, together with a continuous right action $\omega : E \times G \to E$, and that $\tilde{\pi} := E \times G \xrightarrow{\mu} E \xrightarrow{\pi} B$ and $E \times G \xrightarrow{\pi \times C} B \times \{e\} \xrightarrow{\sim} B$ commutes with the map C, a map from everything to identity. And given a point $x \in B$, G acts freely and transitively on the fiber F_x.

Homogeneous space and principal bundles



Applications

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- $G/G_{iso} \times G_{iso}$

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- Take isotropy group to be a fiber
- $G/G_{iso} \times G_{iso}$
- Connections and parallel transport across manifolds
- More applied areas, e.g. Mathematical gauge theory

References

- Xu Jiacheng, Zhang Dingxuan, Ameena Hassan, Jiang Shumin. *M2R2 Written Report 2020-21* Imperial College London [Accessed from 16th June 2021]
- Andreas Cap. *Geometry of homogeneous spaces*[Lecture] University of Vienna. Spring 2019