ECON 133 Global Inequality and Growth
Section 2: Capital concepts, Production functions

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1 What is capital? What is wealth?

- The difference between Capital and Wealth is *sort of* semantic and context dependent.
  - We typically employ the term "capital" to refer to productive assets
  - We typically employ the term wealth in accounting concepts as an account of assets and liabilities
  - But this distinction here is informal and not at all mutually exclusive
- Wealth = Assets - Liabilities
- *Private* Wealth = Assets owned by households - Liabilities owned by households
- *Public* Wealth = Assets owned by gov't - Liabilities owned by gov't
- *National* Wealth = Private wealth + Public wealth = domestic capital + net foreign assets

2 The long-run wealth-income ratio: $\beta = s/g$

- What is s? what is g?
  - Savings is income less spending
  - National income growth is proportion change in national income between years. Where does that come from?
    - Depends on how you think of income. If $Y_t = F(K_t, L_t; A_t)$, growth definitionally comes from growth in either capital inputs, labor inputs, or productivity
- Interpretation: In a steady-state (read: in the long-run), the wealth-to-income ratio is equal to the quotient of the savings rate and the growth rate.
- Intuitively, what does that mean?

$$\uparrow s \implies \uparrow \beta$$
$$\uparrow g \implies \downarrow \beta$$

- What assumptions is this result built on?
  - $W_{t+1} = W_t + s_t Y_t$
  - NB: Central variables are not independent
  - In what sense can we think of this statement as "definitionally true"?
- What do wealth-to-income ratios look like across the world and throughout history?
- How do you prove this result?

*These notes borrow from past notes by José Díaz, Margie Lauter, Cristóbal Otero, Anton Heil, Nina Roussille, Juliana Londoño-Vélez, Jon Schellenberg, and Marcelo Milanello. All mistakes are my own.

Let's go over the proof from lecture again:

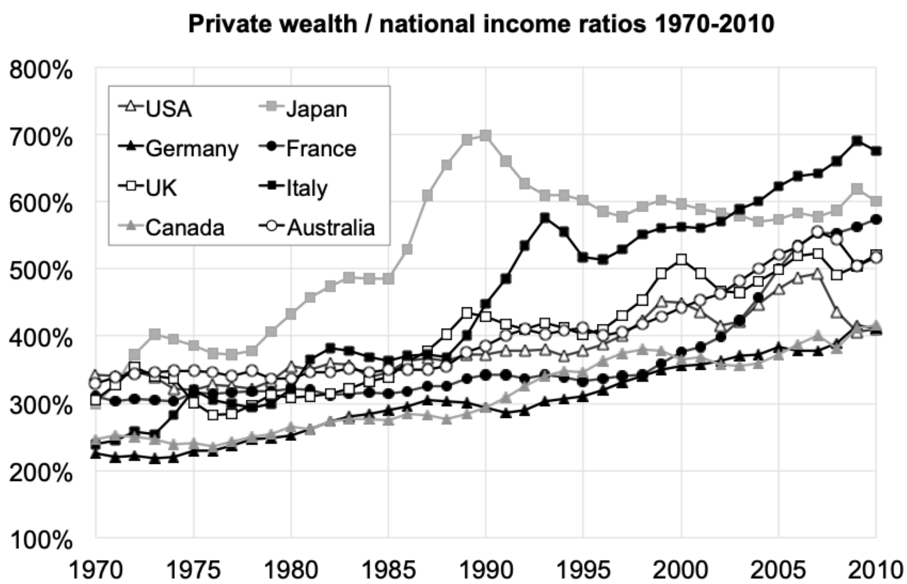
Steps:

1. Start out with the wealth accumulation equation: $W_{t+1} = W_t + s_t Y_t$
    - What is the underlying assumption in this equation?
2. Divide both sides by Y_{t+1}, using the fact that $Y_{t+1} = Y_t(1 + g_t)$
3. Use that in steady-state you can drop the subindices (i.e. $x_t \equiv x_{t+1}$)

$$W_{t+1} = W_t + s_t Y_t \tag{1}$$
$$\frac{W_{t+1}}{Y_{t+1}} = \frac{W_t + s_t Y_t}{Y_{t+1}} \tag{2}$$
$$\frac{W_{t+1}}{Y_{t+1}} = \frac{W_t + s_t Y_t}{Y_t(1 + g_t)} \tag{3}$$
$$\frac{W_{t+1}}{Y_{t+1}} = \frac{W_t}{Y_t(1 + g_t)} + \frac{s_t Y_t}{Y_t(1 + g_t)} \tag{4}$$
$$\beta_{t+1} = \frac{\beta_t + s_t}{1 + g_t} \tag{5}$$

Since in steady-state $\beta_t = \beta_{t+1} = \beta$, $s_t = s$ and $g_t = g$ (i.e. constant), then we plug this into equation (5):

$$\beta_{t+1} = \frac{\beta_t + s_t}{1 + g_t} \tag{6}$$
$$\beta = \frac{\beta + s}{1 + g} \tag{7}$$
$$\beta(1 + g) = \beta + s \tag{8}$$
$$\beta + \beta g = \beta + s \tag{9}$$
$$\beta g = s \tag{10}$$
$$\beta = \frac{s}{g} \tag{11}$$



Source: Piketty and Zucman (2014). Authors' computations using country national accounts. Private wealth = non-financial assets + financial assets - financial liabilities (household & non-profit sectors)

- 16 -

3 The link between capital income and wealth

Basic accounting law: $\alpha = r \times \beta$

- What's α?
  - Recall that $Y = Y_K + Y_L$

$$\alpha = Y_K/Y$$

- What's r?
  - Recall that r = average rate of return to wealth

$$r = Y_K/W$$

$\Rightarrow$ Show the basic accounting law

$$\begin{aligned}\alpha &= \frac{Y_K}{Y}\\ &= \underbrace{\frac{Y_K}{W}}_{r}\underbrace{\frac{W}{Y}}_{\beta}\\ &= r \times \beta\end{aligned}$$

How can we use this identity to understand the evolution in the capital income share over time?

4 Production Functions and Competitive Markets

4.1 Production Functions

- A production function is a mathematical function that relates inputs with output
- In economics, we abstract national income Y as a function of the inputs of labor and capital in the economy

$$Y = F(K, L; A)$$

- There are multiple ways in which capital K and labor L could relate to each other to produce output Y
- Think of A as a productivity shifter
- In this class, we focus on two types of functions, which are broadly used by economics to describe the interaction between labor and capital to produce a given product or aggregate output
  - *Cobb-Douglas:*

    $$Y = A \cdot K^{\alpha} L^{1-\alpha},$$

    where α is a parameter (capital share of income in perfect competition), and K and L is the total amount of capital and labor used to produce Y
  - *Constant Elasticity of Substitution (aka CES):*

    $$F(K, L) = \left(a \cdot K^{\frac{\sigma-1}{\sigma}} + (1-a) \cdot L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$

    where a and σ are parameters

4.2 Competitive markets

- In a competitive market, the assumption is that each factor of production is paid it's marginal contribution to output Y
- Recall that the marginal productivity of a production factor is the value of additional outcome (Y) due to an additional unit of the factor (K, L)
- In other words, in perfect competition, the rate of return (r) of capital equals the marginal productivity of capital (K), while wages (w) equal the marginal productivity of labor (L)
- In mathematical terms, competitive markets means that in equilibrium

$$w = \frac{\partial F(K, L)}{\partial L} \equiv F_L; \quad \text{and} \quad r = \frac{\partial F(K, L)}{\partial K} \equiv F_K,$$

  where w are the wages paid to labor and r is the rate of return of capital.
- Capital income is rK and labor income is wL.

4.3 Factor Shares: The Cobb-Douglas Case

- Under competitive markets, α is the capital income share (and $1-\alpha$ goes to workers)

  $\Rightarrow$ Show the above statement

- The share of capital income does not depend on the amount of capital or labor used in production. Note that α is a constant parameter in the production function

- Another way to see the above is using the basic accounting law $\alpha = r\beta$. An increase in capital is completely offset by a decrease of the same magnitude in r

- In Cobb-Douglas land, we can write

$$w = (1-\alpha)\frac{Y}{L} = (1-\alpha)A\Big(\frac{K}{L}\Big)^{\alpha}$$

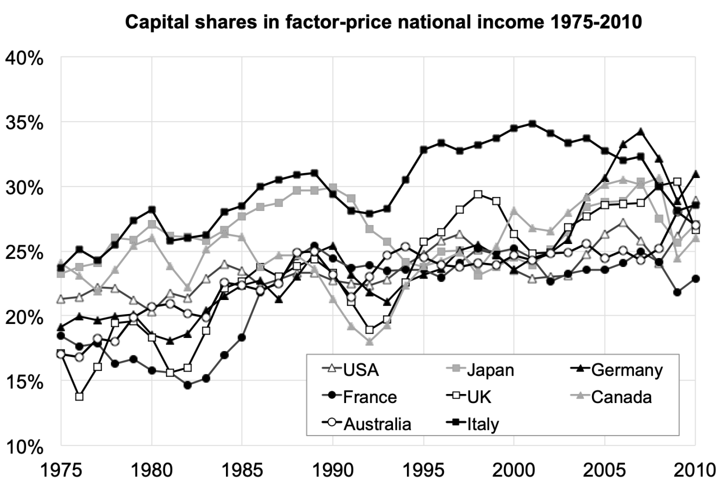
and

$$r = \alpha\frac{Y}{K} = \alpha A\Big(\frac{L}{K}\Big)^{1-\alpha}$$

Since $Y = K^{\alpha}L^{1-\alpha}$ and $r = \frac{\partial Y}{\partial K}$,

$$\begin{aligned} r &= \alpha K^{\alpha-1}L^{1-\alpha} \\ &= \alpha\frac{K^{\alpha}L^{1-\alpha}}{K} \\ &= \alpha\frac{Y}{K} \\ \Rightarrow \frac{rK}{Y} &= \alpha \\ \Rightarrow \frac{Y_K}{Y} &= \alpha \end{aligned}$$

- **Problem**: α has not been constant. In fact, it has been increasing in the last decades ("capital is back!")



Source: Piketty and Zucman (2014)

$\Rightarrow$ Cobb-Douglas is not great to model recent inequality trends.

5 Factor Shares: CES Production

$$F(K, L) = \left(a \cdot K^{\frac{\sigma-1}{\sigma}} + (1-a) \cdot L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$

where σ is the elasticity of substitution, $\sigma \geq 0$.

- Why should we use CES?
  - Might provide a better fit for the data (both α and β are increasing since 1970)
  - NB. "CES" = *constant elasticity of substitution*

5.1 What is the elasticity of substitution?

- The elasticity of substitution σ is the response in the capital-labor ratio (K/L) due to changes in relative factor prices, represented as wages and return to capital.

- I.e. Percent change in input ratio per one percent change in reciprocal of ratio of marginal products

- Mathematically, we can express the elasticity of substitution as:

$$\sigma = \frac{d \ln(K/L)}{d \ln(w/r)} = \frac{\partial K/L}{\partial w/r} \cdot \frac{w/r}{K/L}$$

  where natural logs inform the changes in percentage form, instead of hard-to-interpret units

5.2 Understanding the different values of σ

- $\sigma = 1$ **(Cobb-Douglas)**:
  - A change in relative factor prices w/r will lead to an equal change in the K/L ratio.
  - For example, if the rate of return r rises by 1% relative to w, then firms use 1% less K relative to L, so that capital share in output α remains constant
  - Therefore, following the basic accounting law $\alpha = r \times \beta$, α will remain constant
  - Can you prove that $\sigma = 1$ in Cobb-Douglas land?

- $\sigma = 0$: there is no substitution between K and L, no matter the relative cost of using capital versus labor (Putty-clay Economy).
  - the economy uses a fixed quantity of capital and workers and is completely insensitive to wages and interest rates
  - min production function $F(K, L) = \min(rK, vL)$

- $\sigma = \infty$: firms can perfectly substitute between L and K (Robot Economy)
  - linear production function: $F(K, L, A) = A(rK + wL)$

5.3 The relationship between α and β depends on σ

- Note the following:

$$\begin{aligned}F_K &= \frac{\sigma}{\sigma-1}\left(a\cdot K^{\frac{\sigma-1}{\sigma}} + (1-a)\cdot L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} \times a\frac{\sigma-1}{\sigma}K^{-\frac{1}{\sigma}} \\ &= \left(a\cdot K^{\frac{\sigma-1}{\sigma}} + (1-a)\cdot L^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}\times\frac{1}{\sigma}} \times aK^{-\frac{1}{\sigma}} \\ &= aY^{\frac{1}{\sigma}}K^{-\frac{1}{\sigma}}\end{aligned}$$

- Rearranging, we can see that in a closed economy, we have

$$\alpha = r\cdot\beta = \frac{rK}{Y} = \frac{F_K K}{Y} = \frac{aY^{\frac{1}{\sigma}}K^{-\frac{1}{\sigma}}K}{Y} = a\left(\frac{K}{Y}\right)^{\frac{\sigma-1}{\sigma}} = a\beta^{\frac{\sigma-1}{\sigma}}$$

- If $\sigma > 1$: α is an increasing function of β

- If $\sigma < 1$: α is a decreasing function of β

- What the data says about σ?
  - Since 1970s we are observing an increase in β, from 200-400% to 400-600% among developed countries
  - In the same period, r has remained constant around 6% a year.
  - The capital share α increased from 15-20% to 25-30%

  ⇒ Evidence that elasticity of substitution $\sigma > 1$ for this period

  ⇒ But microeconomics literature suggests otherwise

  ⇒ Maybe the relationship between aggregate output, capital, labor, and income shares is too complicated for a perfectly competitive CES model
    - Remember, all models are wrong, but some are useful
    - To what extent is this one useful?