G53IDS - Project Proposal

Quotient Containers in Homotopy Type Theory

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Part I
Project Title

The agreed title of my dissertation will be '*Quotient Containers in Homotopy Type Theory*'.

Part II
Background and Motivation

1 Homotopy Type Theory

Homotopy type theory is a new branch of mathematics pioneered by *The Univalent Foundations Program* at The Institute for Advanced Study, Princeton. It seeks to marry Martin-Löf's intensional type theory with the topological notion of homotopy theory.

Fundamentally, this means that terms of a type are represented as points of a space and a function $f : A \to B$ represents a continuous mapping from space A to space B. In addition, for $a, b : A$ such that you can logically identify a with b ($a = b$), it is given that in homotopy type theory a path $p : a \rightsquigarrow b$ exists between a and b in the space A. Thusly, it also follows that two functions $f, g : A \to B$ can be identified if they are homotopic, that is, f can be continuously deformed into g. Supplementary to this, a dependent type $x : A \vdash B(x)$ would be represented as a fibration $B \to A$, a mapping from a total space onto a base space.

These novel ways of interpreting type-theoretic constructs lead to theorems and proofs inheriting homotopical meanings and a new logical system can be used.

Not only does homotopy type theory have mathematical implications in terms of the formalisation of homotopy theory and higher-dimensional category theory it can also be used in proof assistants like *Agda* (see below) and *Coq* to allow a more expressive type system and more formal proofs of mathematical theorems.

2 Containers

Containers are a type theoretic abstraction that act as a 'wrapper' around collection types, such as lists, queues, trees etc. allowing them to be represented in a uniform manner. A unary container is formally defined as a pair $(S \rhd P)$ such that $S : Set$ and $P : S \to Set$ where elements of the set S are *shapes* of the container (constructors with non-recursive arguments) and elements of $P(s)$ are the *positions* of s for $s : S$ (constructors with recursive arguments). A simple example is a container representing the List type:

```
data List (A : Set) : Set where
        Nil : List A
        Cons : A  List A  List A
```

which would have the form $(S \rhd P)$ such that $S = 1 + A$, as there two non recursive constructors (namely `Nil` and `Cons`, the latter parameterised with `A`) and $P = 0$, as there are no recursive constructors.

A container represents an endofunctor (a categorical mapping from a category to itself):

$$[\![S \rhd P]\!] \in Set \to Set$$
$$[\![S \rhd P]\!]X = \Sigma s \in S.Ps \to X$$

and given containers $(S \rhd P)$ and $(T \rhd Q)$ morphism $f \rhd r$ is given by:

$$f \in S \to T$$
$$r \in \Pi_{s \in S} Q(fs) \to Ps$$

This constitutes the category of containers.

3 Agda

Agda is a dependently typed functional programming language and formal, interactive proof assistant developed by Ulf Norell at Chalmers University of Technology. Based on Martin-Löf Type Theory, Agda is extensively used in type-theoretic research due to its strong links with type theory and many other programming language features suitable for practical applications.

4 Motivation

While *Quotient Containers in Homotopy Type Theory* may seem a little esoteric, there are several practical applications of a successful outcome to this project — especially in derivative work. The most useful is perhaps formalising quotient containers in a programming language, giving a very expressive type more exposure and documentation. Ultimately though, I see this project as an opportunity to work within the context of a nascent and bleeding edge area of theoretical computer science while building on existing knowledge of functional programming, type theory and category theory.

Part III
Aims and Objectives

1. To research and understand Homotopy Type Theory
2. To research and understand Quotient Containers and their applications
3. To formalise the notion of Quotient Containers within the context of Homotopy Type Theory
4. To potentially explore applications of Quotient Containers within the context of Homotopy Type Theory
5. To potentially explore interesting or novel theorems surrounding Quotient Containers in Homotopy Type Theory

Part IV
Project Plan

25 October - 20 November 2013	Ongoing research into HoTT and Quotient Containers
25 October - 20 November 2013	Ongoing learning of Agda
20 November - 26 November 2013	Preparation for presentation
27 November 2013	Presentation of progress
28 November - 31 January 2014	Continued relevant and more focussed research
01 February - 24 February 2014	Writing of dissertation outline and sample chapter
25 February - 27 February 2014	Proof read/copy edit/typeset sample chapter
28 February 2014	Submission of dissertation outline and sample chapter
31 March 2014	Complete first draft of dissertation
01 April - 31 April 2014	Redraft and rewrite dissertation
31 April 2014	Complete dissertation
01 May - 11 May 2014	Final checks and typesetting of dissertation
01 May - 13 May 2014	Preparation for demonstration of project
12 May 2014	Submission of complete dissertation
14 May 2014	Demonstration of project

References

[1] The Univalent Foundations Program, Institute for Advanced Study, *Homotopy type theory: Univalent foundations of mathematics*. 2013.

[2] Michael Abbott, Thorsten Altenkirch, Neil Ghani, *Constructing Strictly Positive Types*. 2004.

[3] Michael Abbott, Thorsten Altenkirch, Neil Ghani, Connor McBride, *δ for Data, Differentiating Data Structures.* 2005.

[4] Hakon Robbestad Gylterud, *Symmetric Containers (MSc Thesis)*. 2005.

[5] Thorsten Altenkirch, Paul Levy, Sam Staton, *Higher Order Containers* 2010.

[6] http://homotopytypetheory.org/