Quotient Containers in Homotopy Type Theory

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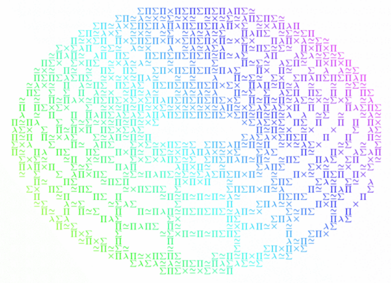
Homotopy Type Theory

- Developed by *Univalent Foundations Program* at the Institute of Advanced Study, Princeton (Steve Awodey, Thierry Coquand and Vladimir Voevodsky et al.)
- Gives homotopical meaning to ideas in Martin-Löf Type Theory
- Same expressive power as Category theory and Type theory



Key Ideas in HoTT

- Terms of a type are points in a space
- A function $f : A \to B$ is a continuous mapping from space A to space B
- If $a = b$ for $a, b : A$ then a path $p : a \rightsquigarrow b$ exists between a and b in A
- Two functions $f, g : A \to B$ can be identified if they are homotopic

Key Ideas in HoTT

Types	Logic	Sets	Homotopy
A	proposition	set	space
$a : A$	proof	element	point
$B(x)$	predicate	family of sets	fibration
$b(x) : B(x)$	conditional proof	family of elements	section
$\mathbf{0},\mathbf{1}$	$\bot, \top$	$\emptyset, \{\emptyset\}$	$\emptyset, *$
$A + B$	$A \vee B$	disjoint union	coproduct

Key Ideas in HoTT

Types	Logic	Sets	Homotopy
$A \times B$	$A \wedge B$	set of pairs	product space
$A \to B$	$A \Rightarrow B$	set of functions	function space
$\Sigma_{(x:A)} B(x)$	$\exists_{x:A} B(x)$	disjoint sum	total space
$\Pi_{(x:A)} B(x)$	$\forall_{x:A} B(x)$	product	space of sections
Id_A	equality $=$	$\{(x,x) \mid x \in A\}$	path space A^I

Containers

- Type theoretic abstraction for collection types (lists, queues, trees etc)
- Allows these types to be represented in a uniform manner
- Unary container is defined as a pair $(S \triangleright P)$ such that $S : Set$ and $P : S \to Set$
- Elements of the set S are *shapes* of the container
- Elements of $P(s)$ are the *positions* of s for $s : S$
- Can be evaluated as an endofunctor

List Container

List Definition:

```
data List (A : Set) : Set where
    Nil : List A
    Cons : A  List A  List A
```

which would have the form $(S \triangleright P)$ such that $S = 1 + A$ and $P = 0$

Category of Containers

Container endofunctor:

$$[\![S \triangleright P]\!] \in Set \to Set$$
$$[\![S \triangleright P]\!]X = \Sigma s \in S.Ps \to X$$

Given containers $(S \triangleright P)$ and $(T \triangleright Q)$, morphism $f \triangleright r$ is given by:

$$f \in S \to T$$
$$r \in \Pi_{s \in S} Q(fs) \to Ps$$

Aims and Objectives

1. To research and understand Homotopy Type Theory
2. To research and understand Quotient Containers and their applications
3. To formalise the notion of Quotient Containers within the context of Homotopy Type Theory
4. To potentially explore applications of Quotient Containers within the context of Homotopy Type Theory
5. To potentially explore interesting or novel theorems surrounding Quotient Containers in Homotopy Type Theory

Project Plan

25 October - 20 November 2013	Ongoing research into HoTT and Quotient Containers
25 October - 20 November 2013	Ongoing learning of Agda
20 November - 26 November 2013	Preparation for presentation
27 November 2013	Presentation of progress
28 November - 31 January 2014	Continued relevant and more focussed research
01 February - 24 February 2014	Writing of dissertation outline and sample chapter
25 February - 27 February 2014	Proof read/copy edit/typeset sample chapter
28 February 2014	Submission of dissertation outline and sample chapter
31 March 2014	Complete first draft of dissertation
01 April - 31 April 2014	Redraft and rewrite dissertation
31 April 2014	Complete dissertation
01 May - 11 May 2014	Final checks and typesetting of dissertation
01 May - 13 May 2014	Preparation for demonstration of project
12 May 2014	Submission of complete dissertation
14 May 2014	Demonstration of project

Progress

Research

- Homotopy Type Theory
  - HoTT Book
  - http://homotopytypetheory.org
- Containers
  - Hakon Robbestad Gylterud - Symmetric Containers
  - Abbott, Altenkirch, Ghani - Constructing Strictly Positive Types
  - Altenkirch, Levy, Staton - Higher Order Containers
- Agda
  - Agda wiki
  - Francesco Mazzoli - Agda by Example: λ-calculus
  - Thorsten Altenkirch - Type Theory in Rosario

Progress

Formalization in Agda

- Containers
- Container morphisms
- Evaluating containers as functors
- Container morphisms as natural transformations

Progress

Written Dissertation

- Abstract
- Introduction
- Background Information

Questions?